Research article

Finite-time boundary synchronization of space-time discretized stochastic fuzzy genetic regulatory networks with time delays

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Abstract: This paper presents an investigation into the phenomenon of global mean-squared finite-time synchronization within the context of two distinct schemes: The asymptotic and exponential forms. The subject matter encompasses space-time discrete stochastic fuzzy genetic regulatory networks, wherein Dirichlet controlled boundary values and time delays are taken into account. The findings presented therein pertain to global mean-squared finite-time synchronization for the aforementioned discrete stochastic fuzzy networks, which incorporate the Lyapunov-Krasovskii functional with a double sum representing the delay-dependent components. In addition, this study demonstrates that improved global mean-squared finite-time synchronization of space-time discrete stochastic fuzzy genetic regulatory networks with boundary controls can be achieved by optimizing the small diffusion intensities, the small fuzzy MIN and MAX parameters, and the large degradation rates of mRNA and proteins. It was unexpected to discover that the sizes of the time lags exert a direct influence on the value of the convergent rate of global mean-squared finite-time exponential synchronization of the networks. This paper presents a framework for exploring the issues of global mean-squared finite-time asymptotic and exponential synchronization for space-time discrete stochastic fuzzy genetic regulatory networks via the Dirichlet controlled boundaries. To conclude, an illustrative example is provided to demonstrate the efficacy of the aforementioned method.

Keywords: finite-time synchronization; genetic regulatory network; Dirichlet controlled boundary; time-space discretizations; fuzzy system
Mathematics Subject Classification: 34D06

1. Introduction

Genetic regulatory networks (GRNs) [1–3] describe the complex networks of molecular regulators that control the expression of genes in a cell. They are key in understanding how genes interact with one another and how cells respond to internal and external stimuli. A typical GRN consists of genes, transcription factors, proteins, RNA molecules, and other regulatory components that work together to regulate gene expression, turning genes "on" or "of" in response to various signals. With the advent of high-throughput technologies like DNA microarrays, RNA sequencing (RNA-seq), and chromatin immunoprecipitation sequencing (ChIP-seq), scientists can now measure gene expression levels and map the interactions between regulatory proteins and DNA. This has led to the development of computational models and algorithms that simulate and predict GRN behavior. GRNs have several key applications, particularly in areas like biotechnology, medicine, and evolutionary biology. In particular, GRNs are foundational to synthetic biology, where scientists engineer or design new regulatory networks for specific purposes, such as producing therapeutic proteins, biofuels, or materials. Additionally, the misregulation of gene networks is implicated in various diseases, including cancer, neurodegenerative diseases, and autoimmune disorders. GRNs can help identify which genes are misregulated in these diseases. GRNs are a critical framework for understanding the regulation of gene expression and the interplay between various molecular components in cells. Their study has broad applications across biology and medicine [4], contributing to breakthroughs in disease treatment [5], agricultural improvements [6], synthetic biology [7–9], and personalized medicine [10, 11].

A reaction-diffusion network [12–14] is a theoretical model that is used to describe the change in concentration of a substance or substances distributed in space. These changes occur due to two distinct processes: Local chemical reactions, which create or destroy the substances in question, and diffusion, whereby the substances in question spread out in space [15–17]. These networks play a pivotal role in elucidating the genesis of a multitude of pattern formation phenomena in biological, chemical, and physical systems. Reaction-diffusion networks provide an effective conceptual tool for elucidating the mechanisms by which intricate spatio-temporal patterns arise in natural and synthetic systems. The interdisciplinary applications of these networks make them an invaluable tool in theoretical and applied sciences alike, offering insights into the processes underlying pattern formation and spatial organization. Discrete-time and discrete-space systems (also referred to as DTDSSs) offer a robust framework for the modeling and comprehension of a plethora of dynamic processes across the realms of science and engineering. By decomposing time and space into discrete units, these models streamline the examination and emulation of intricate systems, facilitating insights that are challenging to discern from their continuous counterparts. The applications of these techniques are diverse, encompassing fields as varied as digital signal processing, control systems, ecology, epidemiology, and material science. The state of the system is updated at each discrete time step in accordance with the interactions within the spatial grid, as outlined in reports [9, 18, 19]. Furthermore, it is notable that while previous researches [13, 14] have explored GRNs with reaction-diffusions, there is a dearth of literature examining diffusive GRNs in space-time discretized systems.

Fuzzy systems are a branch of artificial intelligence and control theory that uses fuzzy logic, which is an extension of classical logic. Unlike traditional binary logic (true or false), fuzzy logic allows for reasoning with degrees of truth or uncertainty. This concept was introduced in 1965 by Lotfi A. Zadeh, a professor at the University of California, Berkeley, who recognized the need for a system capable of handling ambiguous or imprecise information. In the real world, most phenomena do not operate strictly within the boundaries of "yes" or "no" but rather in a spectrum of possibilities. Fuzzy logic was created to address such situations. The integration of fuzzy systems into neural networks and genetic algorithms further expanded their capabilities, leading to the development of neuro-fuzzy systems and fuzzy evolutionary systems that combine fuzzy logic with machine learning for enhanced decision-making [10, 20–23]. On the other hand, stochastic systems refer to systems that are inherently influenced by randomness or uncertainty. Unlike deterministic systems, which produce predictable outcomes based on initial conditions and precise equations, stochastic systems exhibit behavior that cannot be predicted with absolute certainty due to the presence of random variables or noise. This randomness can come from external factors (e.g., environmental fluctuations) or internal processes (e.g., thermal noise in physical systems or random mutations in biological processes). As is well-known, stochastic systems are applied in a wide range of fields, from engineering and finance to biology and climate science [24–26].

Synchronization [27–30] in dynamical systems refers to the process where the states (positions, velocities, etc.) of different systems converge and evolve identically over time. It can occur in various forms, e.g., complete synchronization, phase synchronization, and lag synchronization. In finite-time synchronization (FTS), the system reaches synchronization in a definite time, after which the system states remain synchronized [31]. Unlike asymptotic synchronization, this type of synchronization guarantees that the systems will be synchronized within a finite, predetermined time. It refers to a process where two or more dynamical systems, which may start from different initial conditions, achieve synchronization (i.e., their states become identical or follow the same trajectory) in a finite amount of time. This is different from asymptotic synchronization, where systems approach synchronization gradually over an infinite time horizon. Finite-time synchronization ensures that the systems synchronize within a fixed, finite period, regardless of their initial differences. FTS has applications across multiple fields, particularly in systems that require fast and robust synchronization in a specific time frame, such as communication systems, robotics, power systems, and biological networks. In GRNs, the expression of certain genes needs to be synchronized for a cell to maintain its functional state or transition between states, such as during development or differentiation. FTS allows the control of gene expression so that key genes are activated in unison, within a specific time frame, ensuring proper cellular function [7, 8]. While previous works [7, 8] have investigated various forms of GRNs' FTS, there is a noticeable dearth of research on the FTS topic for discrete GRNs within a space-time setting. It is clear that there is a significant research priority demanding investigation into the FTS of space-time discrete GRNs, which has not yet been adequately addressed within the existing literature.

The aforementioned discussions provide a rationale for our investigation into the issue of the

FTS for space-time discrete stochastic fuzzy GRNs with time-varying delays via the boundary controls (BLs). The establishment of a Lyapunov-Krasovskii functional, comprising both delay-free and delay components, allows the FTS of space-time discrete stochastic fuzzy GRNs to be explored in depth, both asymptotically and exponentially. Summarily, the topic of finite-time synchronization of space-time stochastic GRNs with delays include the following difficulties: 1) Space-time variability requires advanced mathematical tools; 2) stochastic noise from biological and environmental factors complicates modeling and analysis; and 3) time delays destabilize systems and demand sophisticated stability criteria. The following part will present the primary findings of this investigation.

1) A framework of discrete stochastic fuzzy GRNs within a space-time context has been developed, which significantly extends the scope of reaction-diffusion GRNs as previously outlined [13, 14]. Besides, this structure also complements recent studies on space-time discrete models [15, 16].

2) By designing the controls at the Dirichlet boundaries of the GRNs, this article discusses the issue of the FTS for the discrete-time and discrete-space GRNs for the first time, which improves the feedback controls in previous works [29, 30]. Additionally, in [18], authors studied the networks with the controls at the Neumann boundaries, but this study shows that the controls can be designed at the Dirichlet boundaries.

3) This shows a significant advance in the field of space-time discrete fuzzy GRNs' FTS, which represents a significant step forward in the controlling fields of biomedical research and medical practice.

The following part outlines the structure of this paper. Section 2 presents the model of discrete stochastic fuzzy GRNs within a space-time context, the concepts of the FTS, and a number of useful lemmas via the designation of the feedback boundary controls in the GRNs. Section 3 discusses the asymptotic and exponential FTS of the time-space discrete stochastic fuzzy GRNs with orderly time-varying delays. Section 4 provides illustrative numerical examples that substantiate the reliability of the main results. The final section of this paper is Section 5, in which the conclusions of this study are presented.

2. Space-time discrete GRNs involving BLs

Denote $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_0 := [0, \infty)$, $\mathbb{Z}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{Z}_0 \setminus \{0\}$, $\mathbb{Z} = \mathbb{Z}_0 \cup (-\mathbb{Z}_+)$. I represents some suitable dimensional identical matrix. For N-order matrix P, $(P)_\vee$ and $(P)_\wedge$ denote a N-dimensional column vector whose element of the ith row is the number computed by making the operations $\vee$ and $\wedge$, respectively, to the elements in the ith row of P, $i = 1, 2, \ldots, N$. Also, denote

$$\Delta_\iota p_n^{[v]} = \frac{p_n^{[v+1]} - p_n^{[v]}}{\iota}, \quad \Delta_\iota^2 p_n^{[v]} = \Delta_\iota(\Delta_\iota p_n^{[v]}) = \frac{p_n^{[v+2]} - 2p_n^{[v+1]} + p_n^{[v]}}{\iota^2} \text{ (\textbf{finite difference})},$$

$$\Delta_{c,\iota}^2 p_n^{[v]} = \Delta_\iota(\Delta_\iota p_n^{[v-1]}) = \frac{p_n^{[v+1]} - 2p_n^{[v]} + p_n^{[v-1]}}{\iota^2} \text{ (\textbf{central finite difference})},$$

where $z = p_n^{[v]} : \mathbb{Z}^2 \to \mathbb{R}^M$, $(v, k) \in \mathbb{Z}^2$, $\iota > 0$.

This article mainly discusses the following stochastic fuzzy GRNs (SF-GRNs) with the space-time discrete forms:

$$
\begin{cases}
p_{i,n+1}^{[v]} = e^{-a_i h} p_{i,n}^{[v]} + \dfrac{1 - e^{-a_i h}}{a_i} \Big[\theta_i \Delta_{c,\iota}^2 p_{i,n}^{[v]} + \displaystyle\sum_{j=1}^{N} r_{ij} g_j(m_{j,\tau_n}^{[v]}) \\
\qquad\qquad + \displaystyle\bigvee_{j=1}^{N} \xi_{ij} g_j(m_{j,\tau_n}^{[v]}) + \bigwedge_{j=1}^{N} \zeta_{ij} g_j(m_{j,\tau_n}^{[v]}) + \sum_{j=1}^{N} s_{ij} \sigma_j(m_{j,\eta_n}^{[v]}) \Delta_h \omega_{j,n} \Big], \\
m_{i,n+1}^{[v]} = e^{-b_i h} m_{i,n}^{[v]} + \dfrac{1 - e^{-b_i h}}{b_i} \Big[\vartheta_i \Delta_{c,\iota}^2 m_{i,n}^{[v]} + c_i p_{i,\delta_n}^{[v]} \Big],
\end{cases}
\tag{2.1}
$$

where $(v, n) \in (0, \pounds)_{\mathbb{Z}} \times \mathbb{Z}_0$, $i = 1, 2, \ldots, N$; p_i and m_i signify the concentrations of mRNA and proteins of the ith node; θ_i and ϑ_i denote the diffusion rates; a_i and b_i represent degradation rates of the mRNA and protein, respectively; g_j, σ_j are the activation and noise functions described by the Hill functions with Hill constants $\varkappa_j^g, \varkappa_j^\sigma$, respectively; that is,

$$
g_j(s) = \frac{s^{\varkappa_j^g}}{1 + s^{\varkappa_j^g}}, \quad \sigma_j(s) = \frac{s^{\varkappa_j^\sigma}}{1 + s^{\varkappa_j^\sigma}};
$$

r_{ij}, s_{ij} denote the couplings (see [32]), which can be described by

$$
r_{ij} = \begin{cases}
\tilde{r}_{ij}, & \text{if transcription factor } j \text{ is an activator of gene } i; \\
0, & \text{if there is no link between node } j \text{ and gene } i; \\
-\tilde{r}_{ij}, & \text{if transcription factor } j \text{ is a repressor of gene } i,
\end{cases}
$$

$$
s_{ij} = \begin{cases}
\tilde{s}_{ij}, & \text{if transcription factor } j \text{ is an activator of gene } i; \\
0, & \text{if there is no link between node } j \text{ and gene } i; \\
-\tilde{s}_{ij}, & \text{if transcription factor } j \text{ is a repressor of gene } i,
\end{cases}
$$

where $\tilde{r}_{ij}, \tilde{s}_{ij}$ express basal transcriptional rate; $\bigvee$ and $\bigwedge$ represent the operations of fuzzy OR and fuzzy AND, respectively; ξ_{ij}, ζ_{ij} stand for the state parameters of fuzzy MIN feedback and fuzzy MAX feedback, respectively; $\Delta_h \omega_{j,n} = \frac{\omega_j(nh+h) - \omega_j(nh)}{h}$, ω_j denotes the one-dimensional real-valued Brownian motion on certain probability space satisfying $\mathbb{E}(\omega_j^2) = 1$ and $\mathbb{E}(\omega_i \omega_j) = 0$ for $i \neq j$; $i, j = 1, 2, \ldots, N$; $n - \tau_n, n - \eta_n, n - \delta_n \in [0, \infty)$ for all $n \in \mathbb{Z}_0$ are supposed to be included in set $\mathbb{Z}$ in the sequel. In SF-GRNs (2.1), we employ several highly effective, discrete methodologies that include the Euler time difference and the central finite space difference. For further information, please refer to the relevant papers [18, 19].

In SF-GRNs (2.1), referring to articles [7, 8], the two assumptions below can be provided.

(f_1) The Hill constants $\varkappa_j^g, \varkappa_j^\sigma$ belong to $[1, +\infty)$.

(f_2) $\tau_0 \leq n - \tau_n \leq \tau_\infty < \infty$, $\eta_0 \leq n - \eta_n \leq \eta_\infty < \infty$, $\delta_0 \leq n - \delta_n \leq \delta_\infty < \infty$, $\forall n \in \mathbb{Z}_0$.

In addition, SF-GRNs (2.1) admit the initial and boundary conditions as follows

$$\begin{cases} p_{i,s}^{[v]} = \varphi_{i,s}^{[v]}, m_{i,s}^{[v]} = \tilde{\varphi}_{i,s}^{[v]}, & \forall (v, s) \in [0, \pounds]_{\mathbb{Z}} \times [-\tau_{\diamond}, 0]_{\mathbb{Z}}; \\ p_{i,n}^{[0]} = p_{i,n}^{[\pounds]} = 0 = m_{i,n}^{[0]} = m_{i,n}^{[\pounds]}, & \forall n \in \mathbb{Z}_0, \end{cases} \tag{2.2}$$

where $\tau_{\diamond} = \max\{\tau_{\infty}, \eta_{\infty}, \delta_{\infty}\}$, $i = 1, 2, \ldots, N$.

Owing to assumption (f_1), we can easily verify that the functions g_j, σ_j meet Lipschitz conditions. In fact, it is not loss of generality to assume that x, y are nonnegative. By the mean valued theorem, it gives

$$\begin{aligned} |g_j(x) - g_j(y)| &= \left| \frac{x^{\varkappa_j^g}}{1 + x^{\varkappa_j^g}} - \frac{y^{\varkappa_j^g}}{1 + y^{\varkappa_j^g}} \right| \\ &= \frac{\varkappa_j^g \xi^{\varkappa_j^g - 1}}{(1 + \xi^{\varkappa_j^g})^2} |x - y| \, (\xi \text{ is between } x \text{ and } y) \\ &= \frac{\varkappa_j^g}{(1 + \xi^{\varkappa_j^g})^{\frac{2}{\varkappa_j^g}}} \frac{\xi^{\varkappa_j^g - 1}}{(1 + \xi^{\varkappa_j^g})^{2 - \frac{2}{\varkappa_j^g}}} |x - y| \\ &\leq \frac{\varkappa_j^g}{(1 + 0)^{\frac{2}{\varkappa_j^g}}} \frac{\xi^{\varkappa_j^g - 1}}{2^{2 - \frac{2}{\varkappa_j^g}} \xi^{\varkappa_j^g - 1}} |x - y| = 2^{\frac{2}{\varkappa_j^g} - 2} \varkappa_j^g |x - y|, \end{aligned}$$

similarly, $|\sigma_j(x) - \sigma_j(y)| \leq 2^{\frac{2}{\varkappa_j^\sigma} - 2} \varkappa_j^\sigma |x - y|$, where $x, y \in \mathbb{R}$, $j = 1, 2, \ldots, N$.

Considering SF-GRNs (2.1) as the master system, the corresponding slave system is described as

$$\begin{cases} q_{i,n+1}^{[v]} = e^{-a_i h} q_{i,n}^{[v]} + \dfrac{1 - e^{-a_i h}}{a_i} \Big[\theta_i \Delta_{c,\iota}^2 q_{i,n}^{[v]} + \displaystyle\sum_{j=1}^{N} r_{ij} g_j(t_{j,\tau_n}^{[v]}) \\ \qquad\qquad + \displaystyle\bigvee_{j=1}^{N} \xi_{ij} g_j(t_{j,\tau_n}^{[v]}) + \displaystyle\bigwedge_{j=1}^{N} \zeta_{ij} g_j(t_{j,\tau_n}^{[v]}) + \displaystyle\sum_{j=1}^{N} s_{ij} \sigma_j(t_{j,\eta_n}^{[v]}) \Delta_h \omega_{j,n} \Big], \\ t_{i,n+1}^{[v]} = e^{-b_i h} t_{i,n}^{[v]} + \dfrac{1 - e^{-b_i h}}{b_i} \Big[\vartheta_i \Delta_{c,\iota}^2 t_{i,n}^{[v]} + c_i q_{i,\delta_n}^{[v]} \Big], \end{cases} \tag{2.3}$$

where $(v, n) \in (0, \pounds)_{\mathbb{Z}} \times \mathbb{Z}_0$, $i = 1, 2, \ldots, N$. SF-GRNs (2.3) admit initial and boundary conditions with the control inputs, which are depicted by

$$\begin{cases} q_{i,s}^{[v]} = \phi_{i,s}^{[v]}, t_{i,s}^{[v]} = \tilde{\phi}_{i,s}^{[v]}, & \forall (v, s) \in [0, \pounds]_{\mathbb{Z}} \times [-\tau_{\diamond}, 0]_{\mathbb{Z}}; \\ q_{i,n}^{[0]} = 0 = t_{i,n}^{[0]}, q_{i,n}^{[\pounds]} = \alpha_{i,n}, t_{i,n}^{[\pounds]} = \beta_{i,n}, & \forall n \in \mathbb{Z}_0, \end{cases} \tag{2.4}$$

where $\alpha_{i,n}, \beta_{i,n}$ denote the boundary control inputs in the ith node at time n, $i = 1, 2, \ldots, N$.

Construct synchronized networks by setting $u_i = q_i - p_i$, $\rho_i = t_i - m_i$, which are governed by the

partial difference equations below

$$
\begin{cases}
u_{i,n+1}^{[v]} = e^{-a_i h} u_{i,n}^{[v]} + \dfrac{1 - e^{-a_i h}}{a_i} \Big\{ \theta_i \Delta_{c,\iota}^2 u_{i,n}^{[v]} \\
\qquad + \displaystyle\sum_{j=1}^{N} r_{ij} \tilde{g}_j(\rho_{j,\tau_n}^{[v]}) + \Big[\bigvee_{j=1}^{N} \xi_{ij} g_j(t_{j,\tau_n}^{[v]}) - \bigvee_{j=1}^{N} \xi_{ij} g_j(m_{j,\tau_n}^{[v]}) \Big] \\
\qquad + \Big[\bigwedge_{j=1}^{N} \zeta_{ij} g_j(t_{j,\tau_n}^{[v]}) - \bigwedge_{j=1}^{N} \zeta_{ij} g_j(m_{j,\tau_n}^{[v]}) \Big] + \displaystyle\sum_{j=1}^{N} s_{ij} \tilde{\sigma}_j(\rho_{j,\eta_n}^{[v]}) \Delta_h \omega_{j,n} \Big\}, \\
\rho_{i,n+1}^{[v]} = e^{-b_i h} \rho_{i,n}^{[v]} + \dfrac{1 - e^{-b_i h}}{b_i} \Big[\vartheta_i \Delta_{c,\iota}^2 \rho_{i,n}^{[v]} + c_i u_{i,\delta_n}^{[v]} \Big],
\end{cases}
\tag{2.5}
$$

where $(v, n) \in (0, \pounds)_{\mathbb{Z}} \times \mathbb{Z}_0$, $\tilde{g}_j(\rho_j) = g_j(q_j) - g_j(p_j)$, $\tilde{\sigma}_j(\rho_j) = \sigma_j(q_j) - \sigma_j(p_j)$, $i, j = 1, 2, \ldots, N$.

Via the matrix analysis, the scalar error system (2.5) can be transformed into the vector scheme below

$$
\begin{cases}
\mathbf{u}_{n+1}^{[v]} = e^{-Ah} \mathbf{u}_n^{[v]} + \dfrac{I - e^{-Ah}}{A} \Big[\Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]} \\
\qquad + RG(\boldsymbol{\rho}_{\tau_n}^{[v]}) + G_{\vee}(\boldsymbol{\rho}_{\tau_n}^{[v]}) + G_{\wedge}(\boldsymbol{\rho}_{\tau_n}^{[v]}) + S\sigma_{\Delta}^{\omega}(\boldsymbol{\rho}_{\eta_n}^{[v]}) \Big], \\
\boldsymbol{\rho}_{n+1}^{[v]} = e^{-Bh} \boldsymbol{\rho}_n^{[v]} + \dfrac{I - e^{-Bh}}{B} \Big[\Phi \Delta_{c,\iota}^2 \boldsymbol{\rho}_n^{[v]} + C \mathbf{u}_{\delta_n}^{[v]} \Big],
\end{cases}
\tag{2.6}
$$

where $(v, n) \in (0, \pounds)_{\mathbb{Z}} \times \mathbb{Z}_0$, $\mathbf{u} = (u_1, \ldots, u_N)^T$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_N)^T$, $A = \mathrm{diag}(a_1, \ldots, a_N)^T$, $B = \mathrm{diag}(b_1, \ldots, b_N)^T$, $C = \mathrm{diag}(c_1, \ldots, c_N)$, $R = [r_{ij}]_{i,j=1,2,\ldots,N}$, $\Theta = (\theta_1, \ldots, \theta_N)^T$, $\Phi = (\vartheta_1, \ldots, \vartheta_N)^T$, $S = [s_{ij}]_{i,j=1,2,\ldots,N}$, $G(\boldsymbol{\rho}) = (\tilde{g}_1(\rho_1), \ldots, \tilde{g}_N(\rho_N))^T$,

$$
G_{\vee}(\boldsymbol{\rho}) = \begin{bmatrix} \bigvee_{j=1}^{N} \xi_{1j} g_j(t_j^{[v]}) - \bigvee_{j=1}^{N} \xi_{1j} g_j(m_j^{[v]}) \\ \bigvee_{j=1}^{N} \xi_{2j} g_j(t_j^{[v]}) - \bigvee_{j=1}^{N} \xi_{2j} g_j(m_j^{[v]}) \\ \vdots \\ \bigvee_{j=1}^{N} \xi_{Nj} g_j(t_j^{[v]}) - \bigvee_{j=1}^{N} \xi_{Nj} g_j(m_j^{[v]}) \end{bmatrix}, \quad
G_{\wedge}(\boldsymbol{\rho}) = \begin{bmatrix} \bigwedge_{j=1}^{N} \zeta_{1j} g_j(t_j^{[v]}) - \bigwedge_{j=1}^{N} \zeta_{1j} g_j(m_j^{[v]}) \\ \bigwedge_{j=1}^{N} \zeta_{2j} g_j(t_j^{[v]}) - \bigwedge_{j=1}^{N} \zeta_{2j} g_j(m_j^{[v]}) \\ \vdots \\ \bigwedge_{j=1}^{N} \zeta_{Nj} g_j(t_j^{[v]}) - \bigwedge_{j=1}^{N} \zeta_{Nj} g_j(m_j^{[v]}) \end{bmatrix},
$$

$$
\sigma_{\Delta}^{\omega}(\boldsymbol{\rho}) = \begin{bmatrix} \tilde{\sigma}_1(\rho_1) \Delta_h \omega_1 \\ \tilde{\sigma}_2(\rho_2) \Delta_h \omega_2 \\ \vdots \\ \tilde{\sigma}_N(\rho_N) \Delta_h \omega_N \end{bmatrix}.
$$

Define $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^T$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)^T$. From initial and boundary conditions (2.2) and (2.4), the initial and boundary conditions of the error system (2.6) are given by

$$
\begin{cases}
\mathbf{u}_s^{[v]} = \psi_s^{[v]}, \boldsymbol{\rho}_s^{[v]} = \tilde{\psi}_s^{[v]}, \quad \forall (v, s) \in [0, \pounds]_{\mathbb{Z}} \times [-\tau_\diamond, 0]_{\mathbb{Z}}; \\
\mathbf{u}_n^{[0]} = 0 = \boldsymbol{\rho}_n^{[0]}, \mathbf{u}_n^{[\pounds]} = \boldsymbol{\alpha}_n, \boldsymbol{\rho}_n^{[\pounds]} = \boldsymbol{\beta}_n, \quad \forall n \in \mathbb{Z}_0,
\end{cases}
\tag{2.7}
$$

where $\psi = (\phi_1 - \varphi_1, \ldots, \phi_N - \varphi_N)^T$, $\tilde{\psi} = (\tilde{\phi}_1 - \tilde{\varphi}_1, \ldots, \tilde{\phi}_N - \tilde{\varphi}_N)^T$.

Definition 2.1. *(FTS) The master-slave networks (2.1) and (2.3) are called exponentially mean-squared finite-time synchronized (**EMS-FTS**) with respect to $\{\epsilon, \varsigma, \mathcal{N}, \kappa\}$ if*

$$\max_{s \in [-\tau_\diamond, 0]_{\mathbb{Z}}} \sum_{v=0}^{\pounds} \mathbb{E}\left[\left\|\psi_s^{[v]}\right\|^2 + \left\|\tilde{\psi}_s^{[v]}\right\|^2\right] \leq \varsigma$$

induces

$$\sum_{v=0}^{\pounds} \mathbb{E}\left[\left\|\mathbf{u}_n^{[v]}\right\|^2 + \left\|\rho_n^{[v]}\right\|^2\right] \leq \{(\varsigma)e^{-\kappa n} \leq \epsilon, \quad \forall n \in (0, \mathcal{N}]_{\mathbb{Z}},$$

*where $0 < \varsigma < \epsilon$, $\kappa > 0$ and $\{$ is a nonnegative function. If $\{ \equiv 0$, then the master-slave networks (2.1) and (2.3) are called asymptotically mean-squared finite-time synchronized (**AMS-FTS**) with respect to $\{\epsilon, \varsigma, \mathcal{N}\}$.*

Remark 2.1. *In [31,33,34], the finite-time controls of various continuous-time systems were discussed. This study considers the finite-time dynamics of the GRNs in the space-time discrete frame, which aims to complement the works of papers [31, 33, 34].*

In the lemmas below, Δ_ι and Δ_ι^2 denote the first- and second-order finite difference operators with respect to $v \in [0, \pounds]_{\mathbb{Z}}$, which are displayed at the beginning of Section 2.

Lemma 2.1. *[35, Theorem 25, discrete formula of integration by parts] Let $\{p^{[v]} \in \mathbb{R}^m : v \in [0, \pounds]_{\mathbb{Z}}\}$ be the real-valued sequence and W be a m-order matrix, then*

$$\sum_{v=1}^{\pounds-1} p^{[v]T} W \Delta_\iota^2 p^{[v-1]} = \frac{1}{\iota} p^{[v]T} W \Delta_\iota p^{[v-1]} \Big|_1^{\pounds} - \sum_{v=1}^{\pounds-1} \Delta_\iota p^{[v]T} W \Delta_\iota p^{[v]}.$$

Lemma 2.2. *[36, Corollary 2, discrete inequalities of Wirtinger's type] The inequalities below are valid for the m-dimensional real-valued sequence $\{p^{[v]} \in \mathbb{R}^m : v \in [0, \pounds]_{\mathbb{Z}}\}$ and m-order positive definitive matrix W. If $p^{[0]} = 0$, then*

$$4 \sin^2 \frac{\pi}{2(2\pounds + 1)} \sum_{v=0}^{\pounds} p^{[v]T} W p^{[v]} \leq \iota^2 \sum_{v=0}^{\pounds-1} (\Delta_\iota p^{[v]})^T W \Delta_\iota p^{[v]} \leq 4 \cos^2 \frac{\pi}{2\pounds + 1} \sum_{v=0}^{\pounds} p^{[v]T} W p^{[v]}.$$

If $p^{[0]} = 0$ may not be valid, then

$$\iota^2 \sum_{v=1}^{\pounds-1} (\Delta_\iota p^{[v]})^T W \Delta_\iota p^{[v]} \leq 4 \cos^2 \frac{\pi}{2\pounds} \sum_{v=1}^{\pounds} p^{[v]T} W p^{[v]}.$$

Lemma 2.3. *[37] Supposing $u, v \in \mathbb{R}$, then*

$$\left| \bigvee_{j=1}^{N} \xi_{ij} g_j(u) - \bigvee_{j=1}^{N} \xi_{ij} g_j(v) \right| \leq \sum_{j=1}^{N} |\xi_{ij}| |g_j(u) - g_j(v)|,$$

$$\left| \bigwedge_{j=1}^{N} \zeta_{ij} g_j(u) - \bigwedge_{j=1}^{N} \zeta_{ij} g_j(v) \right| \leq \sum_{j=1}^{N} |\zeta_{ij}| |g_j(u) - g_j(v)|,$$

where $i = 1, 2, \ldots, N$.

3. Boundary controlling AMS-FTS and EMS-FTS

The objective of this section is to achieve the AMS-FTS and EMS-FTS of SF-GRNs (2.1) and (2.3) through the integration by parts formula, the inequalities of Wirtinger's type, the designations of the boundary feedback controls in the SF-GRNs (2.3), and the establishment of the Lyapunov-Krasovskii functional involving the delayed component.

Let $\tau_* = \tau_\infty - \tau_0 + 1$, $\delta_* = \delta_\infty - \delta_0 + 1$, $\eta_* = \eta_\infty - \eta_0 + 1$, $A_* = \frac{I - e^{-Ah}}{A}$, $B_* = \frac{I - e^{-Bh}}{B}$. Design the feedback controls in the boundary (2.7) by

$$\alpha_n = \mathbf{u}_n^{[\pounds-1]} + \sum_{v=1}^{\pounds-1} \mathbb{G}_{\mathbf{u}} \mathbf{u}_n^{[v]}, \quad \beta_n = \rho_n^{[\pounds-1]} + \sum_{v=1}^{\pounds-1} \mathbb{G}_\rho \rho_n^{[v]}, \quad \forall n \in \mathbb{Z}_0. \tag{3.1}$$

Hereby, $\mathbb{G}_{\mathbf{u}}$ and $\mathbb{G}_\rho$ are two N-order matrices denoting the gains of the controls.

Theorem 3.1. *Let conditions* (f_1) *and* (f_2) *be fulfilled. Furthermore, it holds that*

(f_3) *there are matrices* $P > 0$ *and* $Q > 0$ *so that*

$$16 \cos^2 \frac{\pi}{2\pounds} \Theta A_* P A_* \Theta + \frac{\iota^2}{4 \sin^2 \frac{\pi}{4\pounds-2}} I \leq \iota^2 \mathrm{sym}(e^{-Ah} P A_* \Theta),$$

$$8 \cos^2 \frac{\pi}{2\pounds} \Phi B_* Q B_* \Phi + \frac{\iota^2}{4 \sin^2 \frac{\pi}{4\pounds-2}} I \leq \iota^2 \mathrm{sym}(e^{-Bh} Q B_* \Phi);$$

(f_4) *there are matrices* $\Pi_1 > 0, \Pi_2 > 0, \Pi_3 > 0$ *and positive constants* $\pi_1, \pi_2, \pi_3, \pi_4$ *so that*

$$(\pi_1 + \pi_2 \xi_\infty N^2 + \pi_3 \zeta_\infty N^2) H^2 \leq \Pi_1, \quad \pi_4 K^2 \leq \Pi_2,$$

$$(2 + \frac{1}{\varepsilon}) C^T B_* Q B_* C \leq \Pi_3, \quad S^T A_* P A_* S \leq \pi_4 I,$$

where $\varepsilon > 0$, $\xi_\infty = \max_{i,j} \xi_{i,j}^2$, $\zeta_\infty = \max_{i,j} \zeta_{i,j}^2$, $H = \mathrm{diag}(2^{\frac{2}{\varkappa_1^g}-2} \varkappa_1^g, \ldots, 2^{\frac{2}{\varkappa_N^g}-2} \varkappa_N^g)$, $K = \frac{1}{h} \mathrm{diag}(2^{\frac{2}{\varkappa_1^\sigma}-2} \varkappa_1^\sigma, \ldots, 2^{\frac{2}{\varkappa_N^\sigma}-2} \varkappa_N^\sigma)$;

(f_5) *there are symmetric matrices* $\mathbb{E}_1, \mathbb{E}_2$ *and constant* $\lambda_\Omega \geq 0$ *so that*

$$\Omega = \begin{pmatrix} \Omega_{11} & \frac{1}{\iota^2}\mathbb{E}_1 & 0 & 0 & e^{-Ah}PA_*R & e^{-Ah}PA_* & e^{-Ah}PA_* \\ * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & \frac{1}{\iota^2}\mathbb{E}_2 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & R^T A_* P A_* & R^T A_* P A_* \\ * & * & * & * & * & \Omega_{66} & A_* P A_* \\ * & * & * & * & * & * & \Omega_{77} \end{pmatrix} < 0,$$

where $\Omega_{11} = -P + e^{-Ah}Pe^{-Ah} + \frac{2}{\iota^2}\mathbb{E}_1 + \delta_*\Pi_3 - \lambda_\Omega I$, $\Omega_{33} = -Q + (1+\varepsilon)e^{-Bh}Qe^{-Bh} + \frac{2}{\iota^2}\mathbb{E}_2 + \tau_*\Pi_1 + \eta_*\Pi_2 - \lambda_\Omega I$, $\Omega_{55} = -\pi_1 I + 2R^T A_* P A_* R$, $\Omega_{66} = -\pi_2 I + 2A_* P A_*$, $\Omega_{77} = -\pi_3 I + 2A_* P A_*$.

Then, the master-slave networks (2.1) *and* (2.3) *are* **AMS-FTS** *with respect to* $\{\epsilon, \varsigma, \mathcal{N}\}$, *viz., for any* $\epsilon > 0$, *there exists* $\varsigma > 0$ *so that*

$$\max_{s \in [-\tau_\diamond, 0]_{\mathbb{Z}}} \sum_{v=0}^{\pounds} \mathbb{E}\left[\left\|\psi_s^{[v]}\right\|^2 + \left\|\tilde{\psi}_s^{[v]}\right\|^2\right] \leq \varsigma \Rightarrow \sum_{v=0}^{\pounds} \mathbb{E}\left[\left\|\mathbf{u}_n^{[v]}\right\|^2 + \left\|\rho_n^{[v]}\right\|^2\right] \leq \epsilon$$

for all $n \in (0, \mathcal{N}]_{\mathbb{Z}}$. *Hereby, the gains of the boundary controls are determined by*

$$\mathfrak{G}_{\mathbf{u}} = \mathrm{inv}(e^{-Ah} PA_* \Theta)\mathrm{Œ}_1, \quad \mathfrak{G}_\rho = \mathrm{inv}(e^{-Bh} QB_* \Phi)\mathrm{Œ}_2.$$

Proof. Define a Lyapunov-Krasovskii functional, which is described by

$$V_n = V_{1,n} + V_{2,n} + V_{3,n} + V_{4,n} + V_{5,n},$$

where

$$V_{1,n} = \sum_{v=1}^{\pounds-1} (\mathbf{u}_n^{[v]})^T P \mathbf{u}_n^{[v]}, \quad V_{2,n} = \sum_{v=1}^{\pounds-1} (\rho_n^{[v]})^T Q \rho_n^{[v]}, \quad V_{3,n} = \sum_{l=\tau_0}^{\tau_\infty} \sum_{s=n-l}^{n-1} \sum_{v=1}^{\pounds-1} (\rho_s^{[v]})^T \Pi_1 \rho_s^{[v]},$$

$$V_{4,n} = \sum_{l=\eta_0}^{\eta_\infty} \sum_{s=n-l}^{n-1} \sum_{v=1}^{\pounds-1} (\rho_s^{[v]})^T \Pi_2 \rho_s^{[v]}, \quad V_{5,n} = \sum_{l=\delta_0}^{\delta_\infty} \sum_{s=n-l}^{n-1} \sum_{v=1}^{\pounds-1} (\mathbf{u}_s^{[v]})^T \Pi_3 \mathbf{u}_s^{[v]},$$

where $n \in \mathbb{Z}_0$.

Calculating the first-order forward difference of V_1 with respect to time n, it gains

$$\mathbb{E}(\Delta V_{1,n}) = \mathbb{E} V_{1,n+1} - \mathbb{E} V_{1,n} = \underbrace{\mathbb{E} \sum_{v=1}^{\pounds-1} (\mathbf{u}_{n+1}^{[v]})^T P \mathbf{u}_{n+1}^{[v]}}_{U_{1,n}} - \mathbb{E} \sum_{v=1}^{\pounds-1} (\mathbf{u}_n^{[v]})^T P \mathbf{u}_n^{[v]}, \tag{3.2}$$

where $n \in \mathbb{Z}_0$. Compute the first term in the right side of the last equation, it follows

$$U_{1,n} = \mathbb{E} \sum_{v=1}^{\pounds-1} (\mathbf{u}_n^{[v]})^T e^{-Ah} P e^{-Ah} \mathbf{u}_n^{[v]} + \underbrace{\mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* PA_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}}_{U_{2,n}}$$

$$+ \mathbb{E} \sum_{v=1}^{\pounds-1} (G(\rho_{\tau_n}^{[v]}))^T R^T A_* PA_* R G(\rho_{\tau_n}^{[v]}) + \mathbb{E} \sum_{v=1}^{\pounds-1} (G_\vee(\rho_{\tau_n}^{[v]}))^T A_* PA_* G_\vee(\rho_{\tau_n}^{[v]})$$

$$+ \mathbb{E} \sum_{v=1}^{\pounds-1} (G_\wedge(\rho_{\tau_n}^{[v]}))^T A_* PA_* G_\wedge(\rho_{\tau_n}^{[v]}) + \mathbb{E} \sum_{v=1}^{\pounds-1} (\sigma_\Delta^\omega(\rho_{\eta_n}^{[v]}))^T S^T A_* PA_* S \sigma_\Delta^\omega(\rho_{\eta_n}^{[v]})$$

$$+ \underbrace{\mathbb{E} \sum_{v=1}^{\pounds-1} \mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah} PA_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}]}_{U_{3,n}} + \mathbb{E} \sum_{v=1}^{\pounds-1} \mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah} PA_* R G(\rho_{\tau_n}^{[v]})]$$

$$+\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]})] + \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})]$$

$$+\underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\mathbf{u}_n^{[v]})^T\Theta A_*PA_*RG(\boldsymbol{\rho}_{\tau_n}^{[v]})]}_{U_{4,k}}$$

$$+\underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\mathbf{u}_n^{[v]})^T\Theta A_*PA_*G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]})]}_{U_{5,k}}$$

$$+\underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\mathbf{u}_n^{[v]})^T\Theta A_*PA_*G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})]}_{U_{6,k}}$$

$$+\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T R^T A_*PA_*G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]})]$$

$$+\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T R^T A_*PA_*G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})]$$

$$+\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T A_*PA_*G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})], \quad \forall n \in \mathbb{Z}_0. \tag{3.3}$$

According to the discrete formula of integration by parts in Lemma 2.1, it gets

$$U_{3,n} = \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*\Theta\Delta_{c,\iota}^2\mathbf{u}_n^{[v]}]$$

$$= \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*\Theta\Delta_\iota^2\mathbf{u}_n^{[v-1]}]$$

$$= \frac{1}{\iota}\mathbb{E}\mathrm{sym}[(\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*\Theta\Delta_\iota\mathbf{u}_n^{[v-1]}]\Big|_{v=1}^{v=\pounds} - \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_\iota\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*\Theta\Delta_\iota\mathbf{u}_n^{[v]}]$$

$$= \underbrace{\frac{1}{\iota}\mathbb{E}\mathrm{sym}[(\mathbf{u}_n^{[\pounds]})^T e^{-Ah}PA_*\Theta\Delta_\iota\mathbf{u}_n^{[\pounds-1]}]}_{\P_{1,n}} - \frac{1}{\iota^2}\mathbb{E}\mathrm{sym}[(\mathbf{u}_n^{[1]})^T e^{-Ah}PA_*\Theta\mathbf{u}_n^{[1]}]$$

$$-\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_\iota\mathbf{u}_n^{[v]})^T e^{-Ah}PA_*\Theta\Delta_\iota\mathbf{u}_n^{[v]}], \quad \forall n \in \mathbb{Z}_0. \tag{3.4}$$

Additionally,

$$U_{4,k} = \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\mathbf{u}_n^{[v]})^T\Theta A_*PA_*RG(\boldsymbol{\rho}_{\tau_n}^{[v]})]$$

$$\leq \mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]} + \mathbb{E} \sum_{v=1}^{\pounds-1} (G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T R^T A_* P A_* R G(\boldsymbol{\rho}_{\tau_n}^{[v]}), \tag{3.5}$$

$$\underbrace{\phantom{\mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}}}_{U_{2,n}}$$

$$U_{5,k} = \mathbb{E} \sum_{v=1}^{\pounds-1} \operatorname{sym}[(\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]})]$$

$$\leq \mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]} + \mathbb{E} \sum_{v=1}^{\pounds-1} (G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]})^T A_* P A_* G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}), \tag{3.6}$$

$$\underbrace{\phantom{\mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}}}_{U_{2,n}}$$

as well

$$U_{6,k} = \mathbb{E} \sum_{v=1}^{\pounds-1} \operatorname{sym}[(\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})] \tag{3.7}$$

$$\leq \mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]} + \mathbb{E} \sum_{v=1}^{\pounds-1} (G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T A_* P A_* G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}), \tag{3.8}$$

$$\underbrace{\phantom{\mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}}}_{U_{2,n}}$$

where $n \in \mathbb{Z}_0$.

Via Lemma 2.2, it computes

$$4U_{2,n} = 4\mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_{c,\iota}^2 \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_{c,\iota}^2 \mathbf{u}_n^{[v]}$$

$$= 4\mathbb{E} \sum_{v=1}^{\pounds-1} [\Delta_\iota(\Delta_\iota \mathbf{u}_n^{[v-1]})]^T \Theta A_* P A_* \Theta [\Delta_\iota(\Delta_\iota \mathbf{u}_n^{[v-1]})]$$

$$= 4\mathbb{E} \sum_{v=0}^{\pounds-2} [\Delta_\iota(\Delta_\iota \mathbf{u}_n^{[v]})]^T \Theta A_* P A_* \Theta [\Delta_\iota(\Delta_\iota \mathbf{u}_n^{[v]})]$$

$$\leq \frac{16}{\iota^2} \cos^2 \frac{\pi}{2\pounds} \mathbb{E} \sum_{v=0}^{\pounds-1} (\Delta_\iota \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_\iota \mathbf{u}_n^{[v]}$$

$$= \frac{16}{\iota^4} \cos^2 \frac{\pi}{2\pounds} \mathbb{E}(\mathbf{u}_n^{[1]})^T \Theta A_* P A_* \Theta \mathbf{u}_n^{[1]}$$

$$+ \frac{16}{\iota^2} \cos^2 \frac{\pi}{2\pounds} \mathbb{E} \sum_{v=1}^{\pounds-1} (\Delta_\iota \mathbf{u}_n^{[v]})^T \Theta A_* P A_* \Theta \Delta_\iota \mathbf{u}_n^{[v]}, \quad \forall n \in \mathbb{Z}_0. \tag{3.9}$$

By the definition of V_2, it calculates

$$\mathbb{E}(\Delta V_{2,n}) = \mathbb{E} V_{2,n+1} - \mathbb{E} V_{2,n} = \mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{n+1}^{[v]})^T Q \boldsymbol{\rho}_{n+1}^{[v]} - \mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_n^{[v]})^T Q \boldsymbol{\rho}_n^{[v]}, \tag{3.10}$$

$$\underbrace{\phantom{\mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{n+1}^{[v]})^T Q \boldsymbol{\rho}_{n+1}^{[v]}}}_{\S_{1,n}}$$

where

$$\S_{1,n} = \mathbb{E}\sum_{v=1}^{\pounds-1}(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}Qe^{-Bh}\boldsymbol{\rho}_n^{[v]} + \underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}(\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]})^T\Phi B_*QB_*\Phi\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]}}_{\S_{2,n}}$$

$$+\mathbb{E}\sum_{v=1}^{\pounds-1}(\mathbf{u}_{\delta_n}^{[v]})^T C^T B_*QB_*C\mathbf{u}_{\delta_n}^{[v]} + \underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]}]}_{\S_{3,n}}$$

$$+\underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*C\mathbf{u}_{\delta_n}^{[v]}]}_{\S_{4,n}} + \underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]})^T\Phi B_*QB_*C\mathbf{u}_{\delta_n}^{[v]}]}_{\S_{5,n}}, \qquad (3.11)$$

where $n \in \mathbb{Z}_0$.

Via the similar argument as that in (3.4), it contains

$$\S_{3,n} = \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]}]$$

$$= \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{\iota}^2\boldsymbol{\rho}_n^{[v-1]}]$$

$$= \frac{1}{\iota}\mathbb{E}\,\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{\iota}\boldsymbol{\rho}_n^{[v-1]}]\Big|_{v=1}^{v=\pounds} - \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{\iota}\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{\iota}\boldsymbol{\rho}_n^{[v]}]$$

$$= \underbrace{\frac{1}{\iota}\mathbb{E}\,\mathrm{sym}[(\boldsymbol{\rho}_n^{[\pounds]})^T e^{-Bh}QB_*\Phi\Delta_{\iota}\boldsymbol{\rho}_n^{[\pounds-1]}]}_{\P_{2,n}} - \frac{1}{\iota^2}\mathbb{E}\,\mathrm{sym}[(\boldsymbol{\rho}_n^{[1]})^T e^{-Bh}QB_*\Phi\boldsymbol{\rho}_n^{[1]}]$$

$$-\mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{\iota}\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*\Phi\Delta_{\iota}\boldsymbol{\rho}_n^{[v]}], \qquad (3.12)$$

and similar to (3.5),

$$\S_{4,n} = \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}QB_*C\mathbf{u}_{\delta_n}^{[v]}]$$

$$\leq \varepsilon\mathbb{E}\sum_{v=1}^{\pounds-1}(\boldsymbol{\rho}_n^{[v]})^T e^{-Bh}Qe^{-Bh}\boldsymbol{\rho}_n^{[v]} + \frac{1}{\varepsilon}\mathbb{E}\sum_{v=1}^{\pounds-1}(\mathbf{u}_{\delta_n}^{[v]})^T CB_*QB_*C\mathbf{u}_{\delta_n}^{[v]}, \qquad (3.13)$$

$$\S_{5,n} = \mathbb{E}\sum_{v=1}^{\pounds-1}\mathrm{sym}[(\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]})^T\Phi B_*QB_*C\mathbf{u}_{\delta_n}^{[v]}]$$

$$\leq \underbrace{\mathbb{E}\sum_{v=1}^{\pounds-1}(\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]})^T\Phi B_*QB_*\Phi\Delta_{c,\iota}^2\boldsymbol{\rho}_n^{[v]}}_{\S_{2,n}} + \mathbb{E}\sum_{v=1}^{\pounds-1}(\mathbf{u}_{\delta_n}^{[v]})^T CB_*QB_*C\mathbf{u}_{\delta_n}^{[v]}, \qquad (3.14)$$

and similar to (3.9),

$$2\S_{2,n} = 2\mathbb{E} \sum_{v=1}^{£-1} (\Delta_{c,\iota}^2 \boldsymbol{\rho}_n^{[v]})^T \Phi B_* Q B_* \Phi \Delta_{c,\iota}^2 \boldsymbol{\rho}_n^{[v]}$$

$$= 2\mathbb{E} \sum_{v=1}^{£-1} [\Delta_\iota(\Delta_\iota \boldsymbol{\rho}_n^{[v-1]})]^T \Phi B_* Q B_* \Phi [\Delta_\iota(\Delta_\iota \boldsymbol{\rho}_n^{[v-1]})]$$

$$= 2\mathbb{E} \sum_{v=0}^{£-2} [\Delta_\iota(\Delta_\iota \boldsymbol{\rho}_n^{[v]})]^T \Phi B_* Q B_* \Phi [\Delta_\iota(\Delta_\iota \boldsymbol{\rho}_n^{[v]})]$$

$$\leq \frac{8}{\iota^2} \cos^2 \frac{\pi}{2£} \mathbb{E} \sum_{v=0}^{£-1} (\Delta_\iota \boldsymbol{\rho}_n^{[v]})^T \Phi B_* Q B_* \Phi \Delta_\iota \boldsymbol{\rho}_n^{[v]}$$

$$= \frac{8}{\iota^4} \cos^2 \frac{\pi}{2£} \mathbb{E} (\boldsymbol{\rho}_n^{[1]})^T \Phi B_* Q B_* \Phi \boldsymbol{\rho}_n^{[1]}$$

$$+ \frac{8}{\iota^2} \cos^2 \frac{\pi}{2£} \mathbb{E} \sum_{v=1}^{£-1} (\Delta_\iota \boldsymbol{\rho}_n^{[v]})^T \Phi B_* Q B_* \Phi \Delta_\iota \boldsymbol{\rho}_n^{[v]}, \quad \forall n \in \mathbb{Z}_0. \tag{3.15}$$

Set $\hat{\mathbf{u}}_n^{[v]} = \mathbf{u}_n^{[£]} - \mathbf{u}_n^{[v]}$ and $\hat{\boldsymbol{\rho}}_n^{[v]} = \boldsymbol{\rho}_n^{[£]} - \boldsymbol{\rho}_n^{[v]}$, $\forall (v, n) \in [0, £]_{\mathbb{Z}} \times \mathbb{Z}_0$. So

$$\P_{1,n} = \frac{1}{\iota} \mathbb{E} \mathrm{sym}[(\mathbf{u}_n^{[£]})^T e^{-Ah} P A_* \Theta \Delta_\iota \mathbf{u}_n^{[£-1]}]$$

$$= \frac{1}{\iota^2} \mathbb{E} \mathrm{sym}[(\mathbf{u}_n^{[£]})^T e^{-Ah} P A_* \Theta \sum_{v=1}^{£-1} \mathfrak{G}_\mathbf{u} \mathbf{u}_n^{[v]}]$$

$$= \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\hat{\mathbf{u}}_n^{[v]} + \mathbf{u}_n^{[v]})^T e^{-Ah} P A_* \Theta \mathfrak{G}_\mathbf{u} \mathbf{u}_n^{[v]}]$$

$$= \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\hat{\mathbf{u}}_n^{[v]})^T Œ_1 \mathbf{u}_n^{[v]}] + \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\mathbf{u}_n^{[v]})^T Œ_1 \mathbf{u}_n^{[v]}], \tag{3.16}$$

as well

$$\P_{2,n} = \frac{1}{\iota} \mathbb{E} \mathrm{sym}[(\boldsymbol{\rho}_n^{[£]})^T e^{-Bh} Q B_* \Phi \Delta_\iota \boldsymbol{\rho}_n^{[£-1]}]$$

$$= \frac{1}{\iota^2} \mathbb{E} \mathrm{sym}[(\boldsymbol{\rho}_n^{[£]})^T e^{-Bh} Q B_* \Phi \sum_{v=1}^{£-1} \mathfrak{G}_\rho \boldsymbol{\rho}_n^{[v]}]$$

$$= \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\boldsymbol{\rho}_n^{[£]})^T e^{-Bh} Q B_* \Phi \mathfrak{G}_\rho \boldsymbol{\rho}_n^{[v]}]$$

$$= \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\hat{\boldsymbol{\rho}}_n^{[v]} + \boldsymbol{\rho}_n^{[v]})^T e^{-Bh} Q B_* \Phi \mathfrak{G}_\rho \boldsymbol{\rho}_n^{[v]}]$$

$$= \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\hat{\boldsymbol{\rho}}_n^{[v]})^T Œ_2 \boldsymbol{\rho}_n^{[v]}] + \frac{1}{\iota^2} \sum_{v=1}^{£-1} \mathbb{E} \mathrm{sym}[(\boldsymbol{\rho}_n^{[v]})^T Œ_2 \boldsymbol{\rho}_n^{[v]}], \quad \forall n \in \mathbb{Z}_0. \tag{3.17}$$

Through Lemma 2.2, it achieves

$$
\begin{aligned}
0 \;=\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} \\
=\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \sum_{v=1}^{\pounds}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} \\
\xlongequal{v'=\pounds-v}\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \sum_{v'=0}^{\pounds-1}(\hat{\mathbf{u}}_n^{[\pounds-v']})^T\hat{\mathbf{u}}_n^{[\pounds-v']} \\
\leq\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \frac{\iota^2}{4\sin^2\frac{\pi}{4\pounds-2}}\sum_{v'=0}^{\pounds-2}(\Delta_\iota\hat{\mathbf{u}}_n^{[\pounds-v']})^T\Delta_\iota\hat{\mathbf{u}}_n^{[\pounds-v']} \\
\xlongequal{v=\pounds-v'}\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \frac{\iota^2}{4\sin^2\frac{\pi}{4\pounds-2}}\sum_{v=1}^{\pounds-1}(\Delta_\iota\hat{\mathbf{u}}_n^{[v]})^T\Delta_\iota\hat{\mathbf{u}}_n^{[v]} \\
=\;& -\sum_{v=1}^{\pounds-1}(\hat{\mathbf{u}}_n^{[v]})^T\hat{\mathbf{u}}_n^{[v]} + \frac{\iota^2}{4\sin^2\frac{\pi}{4\pounds-2}}\sum_{v=1}^{\pounds-1}(\Delta_\iota\mathbf{u}_n^{[v]})^T\Delta_\iota\mathbf{u}_n^{[v]},
\end{aligned}
\tag{3.18}
$$

and similarly,

$$
\begin{aligned}
0 =\;& -\sum_{v=1}^{\pounds-1}(\hat{\boldsymbol{\rho}}_n^{[v]})^T\hat{\boldsymbol{\rho}}_n^{[v]} + \sum_{v=1}^{\pounds-1}(\hat{\boldsymbol{\rho}}_n^{[v]})^T\hat{\boldsymbol{\rho}}_n^{[v]} \\
\leq\;& -\sum_{v=1}^{\pounds-1}(\hat{\boldsymbol{\rho}}_n^{[v]})^T\hat{\boldsymbol{\rho}}_n^{[v]} + \frac{\iota^2}{4\sin^2\frac{\pi}{4\pounds-2}}\sum_{v=1}^{\pounds-1}(\Delta_\iota\boldsymbol{\rho}_n^{[v]})^T\Delta_\iota\boldsymbol{\rho}_n^{[v]}, \quad \forall n \in \mathbb{Z}_0.
\end{aligned}
\tag{3.19}
$$

Besides,

$$
\begin{aligned}
0 =\;& -\pi_1\sum_{v=1}^{\pounds-1}(G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_1\sum_{v=1}^{\pounds-1}(G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G(\boldsymbol{\rho}_{\tau_n}^{[v]}) \\
\leq\;& -\pi_1\sum_{v=1}^{\pounds-1}(G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_1\sum_{v=1}^{\pounds-1}(\boldsymbol{\rho}_{\tau_n}^{[v]})^T H^2\boldsymbol{\rho}_{\tau_n}^{[v]},
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
0 =\;& -\pi_2\sum_{v=1}^{\pounds-1}(G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_2\sum_{v=1}^{\pounds-1}(G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}) \\
\leq\;& -\pi_2\sum_{v=1}^{\pounds-1}(G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_2\sum_{v=1}^{\pounds-1}\left[\left|\bigvee_{j=1}^{N}\xi_{1j}\tilde{g}_j(\rho_{j,\tau_n}^{[v]})\right|^2 + \cdots + \left|\bigvee_{j=1}^{N}\xi_{Nj}\tilde{g}_j(\rho_{j,\tau_n}^{[v]})\right|^2\right] \\
\leq\;& -\pi_2\sum_{v=1}^{\pounds-1}(G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_2\xi_\infty N^2\sum_{v=1}^{\pounds-1}(\boldsymbol{\rho}_{\tau_n}^{[v]})^T H^2\boldsymbol{\rho}_{\tau_n}^{[v]},
\end{aligned}
\tag{3.21}
$$

$$
0 = -\pi_3\sum_{v=1}^{\pounds-1}(G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_3\sum_{v=1}^{\pounds-1}(G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]})
$$

$$\leq -\pi_3 \sum_{v=1}^{\pounds-1} (G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}) + \pi_3 \zeta_\infty N^2 \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{\tau_n}^{[v]})^T H^2 \boldsymbol{\rho}_{\tau_n}^{[v]}, \tag{3.22}$$

and

$$0 = -\pi_4 \mathbb{E} \sum_{v=1}^{\pounds-1} (\sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]}))^T \sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]}) + \pi_4 \mathbb{E} \sum_{v=1}^{\pounds-1} (\sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]}))^T \sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]})$$

$$\leq -\pi_4 \mathbb{E} \sum_{v=1}^{\pounds-1} (\sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]}))^T \sigma_\Delta^\omega(\boldsymbol{\rho}_{\eta_n}^{[v]}) + \pi_4 \mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{\eta_n}^{[v]})^T K^2 \boldsymbol{\rho}_{\eta_n}^{[v]}, \quad \forall n \in \mathbb{Z}_0. \tag{3.23}$$

Additionally,

$$\mathbb{E}(\Delta V_{3,n}) \leq \tau_* \sum_{v=1}^{\pounds-1} \mathbb{E}(\boldsymbol{\rho}_n^{[v]})^T \Pi_1 \boldsymbol{\rho}_n^{[v]} - \sum_{v=1}^{\pounds-1} \mathbb{E}(\boldsymbol{\rho}_{\tau_n}^{[v]})^T \Pi_1 \boldsymbol{\rho}_{\tau_n}^{[v]}, \tag{3.24}$$

$$\mathbb{E}(\Delta V_{4,n}) \leq \eta_* \sum_{v=1}^{\pounds-1} \mathbb{E}(\boldsymbol{\rho}_n^{[v]})^T \Pi_2 \boldsymbol{\rho}_n^{[v]} - \sum_{v=1}^{\pounds-1} \mathbb{E}(\boldsymbol{\rho}_{\eta_n}^{[v]})^T \Pi_2 \boldsymbol{\rho}_{\eta_n}^{[v]}, \tag{3.25}$$

and

$$\mathbb{E}(\Delta V_{5,n}) \leq \delta_* \sum_{v=1}^{\pounds-1} \mathbb{E}(\mathbf{u}_n^{[v]})^T \Pi_3 \mathbf{u}_n^{[v]} - \sum_{v=1}^{\pounds-1} \mathbb{E}(\mathbf{u}_{\delta_n}^{[v]})^T \Pi_3 \mathbf{u}_{\delta_n}^{[v]}, \quad \forall n \in \mathbb{Z}_0. \tag{3.26}$$

Define a vector as follows

$$\varpi_n^{[v]} = \left((\mathbf{u}_n^{[v]})^T, (\hat{\mathbf{u}}_n^{[v]})^T, (\boldsymbol{\rho}_n^{[v]})^T, (\hat{\boldsymbol{\rho}}_n^{[v]})^T, (G(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T, (G_\vee(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T, (G_\wedge(\boldsymbol{\rho}_{\tau_n}^{[v]}))^T \right)^T,$$

where $(v,n) \in (0,\pounds)_\mathbb{Z} \times \mathbb{Z}_0$. By (3.2)–(3.26) and assumptions (f_3)–(f_5), it follows

$$\mathbb{E}(\Delta V_n) \leq \sum_{v=1}^{\pounds-1} \mathbb{E}(\varpi_n^{[v]})^T \Omega \varpi_n^{[v]} + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\mathbf{u}_n^{[v]}\|^2 + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\boldsymbol{\rho}_n^{[v]}\|^2$$

$$\leq \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\mathbf{u}_n^{[v]}\|^2 + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\boldsymbol{\rho}_n^{[v]}\|^2, \quad \forall n \in \mathbb{Z}_0. \tag{3.27}$$

Let λ_+ denote the maximum eigenvalue of the matrices $P, Q, \Pi_1, \Pi_2, \Pi_3$, λ_- denote the minimum eigenvalue of the matrices P, Q, and λ_G denote the maximum eigenvalue of the matrices $\mathfrak{G}_\mathbf{u}^T \mathfrak{G}_\mathbf{u}$ and $\mathfrak{G}_\rho^T \mathfrak{G}_\rho$. For any $\epsilon > 0$, taking

$$\varsigma = \left\{ \frac{3 + 2\pounds \lambda_G}{\lambda_-} (1 + \frac{\lambda_\Omega}{\lambda_-})^{N-1} [\lambda_\Omega + (3\tau_\diamond^2 + 2)\lambda_+] \right\}^{-1} \epsilon.$$

Assume that

$$\max_{s \in [-\tau_\diamond, 0]_\mathbb{Z}} \sum_{v=0}^{\pounds} \mathbb{E}\left[\|\psi_s^{[v]}\|^2 + \|\tilde{\psi}_s^{[v]}\|^2 \right] \leq \varsigma.$$

Taking $n = 0, 1, 2$ orderly in the inequality (3.27), it results in

$$\lambda_- \sum_{v=1}^{£-1} \mathbb{E}[\|\mathbf{u}_1^{[v]}\|^2 + \|\rho_1^{[v]}\|^2] \leq \mathbb{E}V_1 \leq [\lambda_\Omega + (3\tau_\diamond^2 + 2)\lambda_+]\varsigma,$$

$$\lambda_- \sum_{v=1}^{£-1} \mathbb{E}[\|\mathbf{u}_2^{[v]}\|^2 + \|\rho_2^{[v]}\|^2] \leq \mathbb{E}V_2 \leq (1 + \frac{\lambda_\Omega}{\lambda_-})\mathbb{E}V_1,$$

$$\lambda_- \sum_{v=1}^{£-1} \mathbb{E}[\|\mathbf{u}_3^{[v]}\|^2 + \|\rho_3^{[v]}\|^2] \leq \mathbb{E}V_3 \leq (1 + \frac{\lambda_\Omega}{\lambda_-})^2\mathbb{E}V_1.$$

Via the technique of mathematical induction, it directly gains

$$\sum_{v=1}^{£-1} \mathbb{E}[\|\mathbf{u}_n^{[v]}\|^2 + \|\rho_n^{[v]}\|^2] \leq \frac{1}{\lambda_-}(1 + \frac{\lambda_\Omega}{\lambda_-})^{n-1}[\lambda_\Omega + (3\tau_\diamond^2 + 2)\lambda_+]\varsigma, \quad \forall n \in \mathbb{Z}_0 \setminus \{0\}.$$

Together with (2.7) and (3.1), it gets

$$\sum_{v=0}^{£} \mathbb{E}[\|\mathbf{u}_n^{[v]}\|^2 + \|\rho_n^{[v]}\|^2] \leq \frac{3 + 2£\lambda_G}{\lambda_-}(1 + \frac{\lambda_\Omega}{\lambda_-})^{\mathcal{N}-1}[\lambda_\Omega + (3\tau_\diamond^2 + 2)\lambda_+]\varsigma$$

$$= \epsilon, \quad \forall n \in (0, \mathcal{N}]_{\mathbb{Z}_0}.$$

Thus, the master-slave networks (2.1) and (2.3) have finite-time asymptotic synchronization in the mean-square sense with respect to $\{\epsilon, \varsigma, \mathcal{N}\}$. This completes the proof. $\qquad\square$

Theorem 3.2. *Let the gains be defined as those in* Theorem 3.1, *conditions* (f_1), (f_2), (f_3), *and* (f_4) *be fulfilled. Furthermore, it holds that*

(f_6) *there exists* $\kappa \in (0, \frac{1}{\tau_\diamond+1}]$ *and constant* $\lambda_\Omega \geq 0$ *so that*

$$\Omega_* = \Omega + \begin{pmatrix} \kappa P + \delta_\infty \delta_* \Pi_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \kappa Q + \tau_\infty \tau_* \Pi_1 + \eta_\infty \eta_* \Pi_2 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{pmatrix} < 0,$$

where Ω *is defined as that in* Theorem 3.1.

Then, the master-slave networks (2.1) *and* (2.3) *are* **EMS-FTS** *with respect to* $\{\epsilon, \varsigma, \mathcal{N}, \kappa_*\}$, *viz., for any* $\epsilon > 0$, *it exists* $\varsigma > 0$, $\kappa_* > 0$ *and function* $\{(\varsigma) > 0$ *so that*

$$\max_{s \in [-\tau_\diamond, 0]_{\mathbb{Z}}} \sum_{v=0}^{£} \mathbb{E}\left[\|\psi_s^{[v]}\|^2 + \|\tilde{\psi}_s^{[v]}\|^2\right] \leq \varsigma \Rightarrow \sum_{v=0}^{£} \mathbb{E}\left[\|\mathbf{u}_n^{[v]}\|^2 + \|\rho_n^{[v]}\|^2\right] \leq \{(\varsigma)e^{-\kappa_* n} \leq \epsilon$$

for all $n \in (0, \mathcal{N}]_{\mathbb{Z}}$.

Proof. Let us build a Lyapunov-Krasovskii functional, which is given by

$$V_n = V_{1,n} + V_{2,n} + V_{3,n} + V_{4,n} + V_{5,n} + V_{6,n} + V_{7,n} + V_{8,n},$$

where $V_{1,n}, \ldots, V_{5,n}$ are defined as those in Theorem 3.1, and

$$V_{6,n} = \sum_{l=\tau_0}^{\tau_\infty} \sum_{s=n-l}^{n-1} \sum_{s'=s}^{n-1} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{s'}^{[v]})^T \Pi_1 \boldsymbol{\rho}_{s'}^{[v]},$$

$$V_{7,n} = \sum_{l=\eta_0}^{\eta_\infty} \sum_{s=n-l}^{n-1} \sum_{s'=s}^{n-1} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_{s'}^{[v]})^T \Pi_2 \boldsymbol{\rho}_{s'}^{[v]}, \quad V_{8,n} = \sum_{l=\delta_0}^{\delta_\infty} \sum_{s=n-l}^{n-1} \sum_{s'=s}^{n-1} \sum_{v=1}^{\pounds-1} (\mathbf{u}_{s'}^{[v]})^T \Pi_3 \mathbf{u}_{s'}^{[v]}, \quad \forall n \in \mathbb{Z}_0.$$

It follows

$$
\begin{cases}
\mathbb{E}(\Delta V_{6,n}) \leq \tau_\infty \tau_* \mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_n^{[v]})^T \Pi_1 \boldsymbol{\rho}_n^{[v]} - \mathbb{E} V_{3,n}, \\[2ex]
\mathbb{E}(\Delta V_{7,n}) \leq \eta_\infty \eta_* \mathbb{E} \sum_{v=1}^{\pounds-1} (\boldsymbol{\rho}_n^{[v]})^T \Pi_2 \boldsymbol{\rho}_n^{[v]} - \mathbb{E} V_{4,n}, \\[2ex]
\mathbb{E}(\Delta V_{8,n}) \leq \delta_\infty \delta_* \mathbb{E} \sum_{v=1}^{\pounds-1} (\mathbf{u}_n^{[v]})^T \Pi_3 \mathbf{u}_n^{[v]} - \mathbb{E} V_{5,n}, \quad \forall n \in \mathbb{Z}_0,
\end{cases}
\tag{3.28}
$$

and

$$V_{6,n} \leq \tau_\infty V_{3,n}, \quad V_{7,n} \leq \eta_\infty V_{4,n}, \quad V_{8,n} \leq \delta_\infty V_{5,n}, \quad \forall n \in \mathbb{Z}_0. \tag{3.29}$$

By (3.27)–(3.29), it has $\kappa \in (0, \frac{1}{\tau_\diamond+1}]$ such that

$$
\begin{aligned}
\mathbb{E}(\Delta V_n) + \kappa \mathbb{E}(V_n) &\leq \sum_{v=1}^{\pounds-1} \mathbb{E}(\boldsymbol{\varpi}_n^{[v]})^T \Omega_* \boldsymbol{\varpi}_n^{[v]} + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\mathbf{u}_n^{[v]}\|^2 + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\boldsymbol{\rho}_n^{[v]}\|^2 \\
&\quad - (1 - \kappa - \kappa\tau_\infty) V_{3,n} - (1 - \kappa - \kappa\eta_\infty) V_{4,n} - (1 - \kappa - \kappa\delta_\infty) V_{5,n} \\
&\leq \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\mathbf{u}_n^{[v]}\|^2 + \lambda_\Omega \sum_{v=1}^{\pounds-1} \mathbb{E}\|\boldsymbol{\rho}_n^{[v]}\|^2, \quad \forall n \in \mathbb{Z}_0.
\end{aligned}
\tag{3.30}
$$

For any $\epsilon > 0$, taking

$$\varsigma = \left\{ \frac{3 + 2\pounds\lambda_G}{\lambda_-(1-\kappa)} (1 + \frac{\lambda_\Omega}{\lambda_-})^N [\lambda_\Omega + (6\tau_\diamond^3 + 2)\lambda_+] \right\}^{-1} \epsilon.$$

Assume that

$$\max_{s \in [-\tau_\diamond, 0]_\mathbb{Z}} \sum_{v=0}^{\pounds} \mathbb{E}\left[\left\| \psi_s^{[v]} \right\|^2 + \left\| \tilde{\psi}_s^{[v]} \right\|^2 \right] \leq \varsigma.$$

Taking $n = 0, 1, 2$ orderly in the inequality (3.30), it addresses

$$\lambda_- \sum_{v=1}^{\pounds-1} \mathbb{E}[\|\mathbf{u}_1^{[v]}\|^2 + \|\boldsymbol{\rho}_1^{[v]}\|^2] \leq \mathbb{E} V_1 \leq (1 - \kappa)[\lambda_\Omega + (6\tau_\diamond^3 + 2)\lambda_+]\varsigma,$$

$$\lambda_- \sum_{v=1}^{\pounds-1} \mathbb{E}[\|\mathbf{u}_2^{[v]}\|^2 + \|\boldsymbol{\rho}_2^{[v]}\|^2] \le \mathbb{E}V_2 \le (1 - \kappa + \frac{\lambda_\Omega}{\lambda_-})\mathbb{E}V_1,$$

$$\lambda_- \sum_{v=1}^{\pounds-1} \mathbb{E}[\|\mathbf{u}_3^{[v]}\|^2 + \|\boldsymbol{\rho}_3^{[v]}\|^2] \le \mathbb{E}V_3 \le (1 - \kappa + \frac{\lambda_\Omega}{\lambda_-})^2\mathbb{E}V_1.$$

Via the technique of mathematical induction, it directly gains

$$\sum_{v=1}^{\pounds-1} \mathbb{E}[\|\mathbf{u}_n^{[v]}\|^2 + \|\boldsymbol{\rho}_n^{[v]}\|^2] \le \frac{1}{\lambda_-}(1 - \kappa + \frac{\lambda_\Omega}{\lambda_-})^{n-1}[\lambda_\Omega + (6\tau_\diamond^3 + 2)\lambda_+]\varsigma, \quad \forall n \in \mathbb{Z}_0 \backslash \{0\}.$$

Together with (2.7) and (3.1), it gets

$$\sum_{v=0}^{\pounds} \mathbb{E}[\|\mathbf{u}_n^{[v]}\|^2 + \|\boldsymbol{\rho}_n^{[v]}\|^2] \le \underbrace{\frac{3 + 2\pounds\lambda_G}{\lambda_-(1 - \kappa)}(1 + \frac{\lambda_\Omega}{\lambda_-})^{\mathcal{N}}[\lambda_\Omega + (6\tau_\diamond^3 + 2)\lambda_+]\varsigma}_{\{(\varsigma)} \, e^{-\kappa_* n}$$

$$\le \epsilon, \quad \forall n \in (0, \mathcal{N}]_{\mathbb{Z}_0},$$

where $\kappa_* = -\ln \frac{1 - \kappa + \frac{\lambda_\Omega}{\lambda_-}}{1 + \frac{\lambda_\Omega}{\lambda_-}}$. Thus, the master-slave networks (2.1) and (2.3) are **EMS-FTS** with respect to $\{\epsilon, \varsigma, \mathcal{N}, \kappa_*\}$. This completes the proof. □

Remark 3.1. *In [13, 14], authors examined the diverse control issues associated with reaction-diffusion GRNs, including finite-time stability, state observer, and other pertinent aspects. Nevertheless, to date, there has been a paucity of published literature on discrete GRNs in space and time. In order to address this gap, this paper proposes a discrete-time stochastic fuzzy GRNs framework with discrete spatial diffusions. This constitutes a meaningful contribution to the research field.*

4. Illustration example

This section considers a numerical example of the space-time discretized SF-GRNs with the time delays as noted below

$$\begin{cases} p_{1,n+1}^{[v]} = e^{-0.2h}p_{1,n}^{[v]} + \dfrac{1 - e^{-0.2h}}{0.2}\Big[0.1\Delta_{c,\iota}^2 p_{1,n}^{[v]} + 0.1g_2(m_{2,n-1}^{[v]}) + \bigwedge_{j=1}^{2} \zeta_{1j}g_j(m_{j,n-1}^{[v]})\Big], \\[3mm] p_{2,n+1}^{[v]} = e^{0.1h}p_{2,n}^{[v]} + \dfrac{e^{0.1h} - 1}{0.1}\Big[0.2\Delta_{c,\iota}^2 p_{2,n}^{[v]} + 0.2g_1(m_{1,n-1}^{[v]}) + 0.15\sigma(m_{1,n}^{[v]})\Delta_h\omega_n\Big], \\[3mm] m_{1,n+1}^{[v]} = e^{-3h}m_{1,n}^{[v]} + \dfrac{1 - e^{-3h}}{3}\Big[0.2\Delta_{c,\iota}^2 m_{1,n}^{[v]} + 0.15p_{1,n-2}^{[v]}\Big], \\[3mm] m_{2,n+1}^{[v]} = e^{-5h}m_{2,n}^{[v]} + \dfrac{1 - e^{-5h}}{5}\Big[0.1\Delta_{c,\iota}^2 m_{2,n}^{[v]} + 0.12p_{2,n-2}^{[v]}\Big], \end{cases} \quad (4.1)$$

where $(v, n) \in (0, 3)_{\mathbb{Z}} \times \mathbb{Z}_0$; $h = 0.05$, $\iota = 1$; $g_1(s) = \frac{s^{1.1}}{1+s^{1.1}} = \sigma(s)$, $g_2(s) = \frac{s^{1.2}}{1+s^{1.2}}$, $\forall s \in \mathbb{R}$; $\zeta_{11} = 0.1$, $\zeta_{12} = 0.3$. Besides, the initial boundary values of SF-GRNs (4.1) are described by

$$\begin{cases} p^{[v]}_{1,s} = s'|\sin s'|, \ p^{[v]}_{2,s} = 2s'|\sin s'|, \\ m^{[v]}_{1,s} = s''|\sin s''|, \ m^{[v]}_{2,s} = 2s''|\sin s''|, \quad \forall s \in [-2, 0]_{\mathbb{Z}}, v \in (0, 3)_{\mathbb{Z}}; \\ p^{[0]}_{i,n} = 0 = p^{[3]}_{i,n}, \ m^{[0]}_{i,n} = 0 = m^{[3]}_{i,n}, \quad \forall n \in \mathbb{Z}_0, i = 1, 2, \end{cases}$$

where $s' = (s + 3)(v + 1)$ and $s'' = s + 3 + v + 1$, $\forall s \in [-2, 0]_{\mathbb{Z}}, v \in (0, 3)_{\mathbb{Z}}$.

In line with the master SF-GRNs (4.1), the slave SF-GRNs are described by

$$\begin{cases} q^{[v]}_{1,n+1} = e^{-0.2h}q^{[v]}_{1,n} + \dfrac{1 - e^{-0.2h}}{0.2}\Big[0.1\Delta^2_{c,\iota}q^{[v]}_{1,n} + 0.1g_2(t^{[v]}_{2,n-1}) + \bigwedge\limits_{j=1}^{2} \zeta_{1j}g_j(t^{[v]}_{j,n-1})\Big], \\ q^{[v]}_{2,n+1} = e^{0.1h}q^{[v]}_{2,n} + \dfrac{e^{0.1h} - 1}{0.1}\Big[0.2\Delta^2_{c,\iota}q^{[v]}_{2,n} + 0.2g_1(t^{[v]}_{1,n-1}) + 0.15\sigma(t^{[v]}_{1,n})\Delta_h\omega_n\Big], \\ t^{[v]}_{1,n+1} = e^{-3h}t^{[v]}_{1,n} + \dfrac{1 - e^{-3h}}{3}\Big[0.2\Delta^2_{c,\iota}t^{[v]}_{1,n} + 0.15q^{[v]}_{1,n-2}\Big], \\ t^{[v]}_{2,n+1} = e^{-5h}t^{[v]}_{2,n} + \dfrac{1 - e^{-5h}}{5}\Big[0.1\Delta^2_{c,\iota}t^{[v]}_{2,n} + 0.12q^{[v]}_{2,n-2}\Big], \end{cases} \quad (4.2)$$

where $(v, n) \in (0, 3)_{\mathbb{Z}} \times \mathbb{Z}_0$. Additionally, the initial and boundary values of the slave SF-GRNs (4.2) are depicted as

$$\begin{cases} q^{[v]}_{1,s} = s'|\sin s'| + 0.1|\text{randn}(1)|, \\ q^{[v]}_{2,s} = 2s'|\sin s'| + 0.1|\text{randn}(1)|, \\ t^{[v]}_{1,s} = s''|\sin s''| + 0.1|\text{randn}(1)|, \\ t^{[v]}_{2,s} = 2s''|\sin s''| + 0.1|\text{randn}(1)|, \quad \forall s \in [-2, 0]_{\mathbb{Z}}, v \in (0, 3)_{\mathbb{Z}}; \\ q^{[0]}_{i,n} = 0 = t^{[0]}_{i,n}, \ q^{[3]}_{i,n} = \alpha_{i,n}, t^{[3]}_{i,n} = \beta_{i,n}, \quad \forall n \in \mathbb{Z}_0, i = 1, 2. \end{cases}$$

Hereby, randn(1) represents the normally distributed random number with the means of 0 and variance of 1.

Taking $\varepsilon = 0.1$, by computing the linear matrix inequalities in Theorem 3.1, it attains

$$P = \begin{bmatrix} 119.0654 & 0 \\ 0 & 53.6820 \end{bmatrix}, \quad Q = \begin{bmatrix} 1836.6232 & 0 \\ 0 & 803.0303 \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} 46.4409 & 0 \\ 0 & 52.7829 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 8.9593 & 0 \\ 0 & 2.5156 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} 1.1278 & 0 \\ 0 & 0.2857 \end{bmatrix}, \quad \text{Œ}_1 = -\begin{bmatrix} 0.9923 & 0 \\ 0 & 0.9925 \end{bmatrix},$$

$$\text{Œ}_2 = -\begin{bmatrix} 0.9887 & 0 \\ 0 & 0.9906 \end{bmatrix}, \quad \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_3 & \pi_4 \end{bmatrix} = \begin{bmatrix} 5.5006 & 1817.8828 \\ 25.2990 & 0.0161 \end{bmatrix}, \quad \lambda_\Omega = 0.4789.$$

So, the assumptions in Theorem 3.1 are valid, and the gains of the controls are calculated by

$$\mathbb{G}_{\mathbf{u}} = -\begin{bmatrix} 1.6920 & 0 \\ 0 & 1.8351 \end{bmatrix}, \quad \mathbb{G}_{\rho} = -\begin{bmatrix} 0.0674 & 0 \\ 0 & 0.3580 \end{bmatrix}.$$

According to Theorem 3.1, the master-slave SF-GRNs (4.1) and (4.2) are asymptotically mean-squared finite-time synchronized under the boundary controls below

$$\begin{cases} q_{1,n}^{[3]} = -0.6920u_{1,n}^{[2]} - 1.6920u_{1,n}^{[1]}, \\ q_{2,n}^{[3]} = -0.8351u_{2,n}^{[2]} - 1.8351u_{2,n}^{[1]}, \\ t_{1,n}^{[3]} = 0.9326\rho_{1,n}^{[2]} - 0.0674\rho_{1,n}^{[1]}, \\ t_{2,n}^{[3]} = 0.6420\rho_{2,n}^{[2]} - 0.3580\rho_{2,n}^{[1]}, \quad \forall n \in \mathbb{Z}_0. \end{cases} \tag{4.3}$$

Hereby, $u_i = q_i - p_i$ and $\rho_i = t_i - m_i$, $i = 1, 2$.

Figures 1 and 2 illustrate the 3D surfaces of the states in the master GRNs (4.1). The states of the slave GRNs (4.2) are shown in Figures 3 and 4 under the boundary feedback controller (4.3). In Figures 5 and 6, under the boundary feedback controller (4.3), we can see that the errors tend to be small as the time n increases in a finite-time interval. That is, the master-slave SF-GRNs (4.1) and (4.2) achieve global asymptotic finite-time synchronization in the mean-square sense via the boundary feedback controller (4.3). When the spatial variable $v = 2$, Figures 7 and 8 show the 2D curves of the errors, which instinctively display the variations of the errors for the readers.
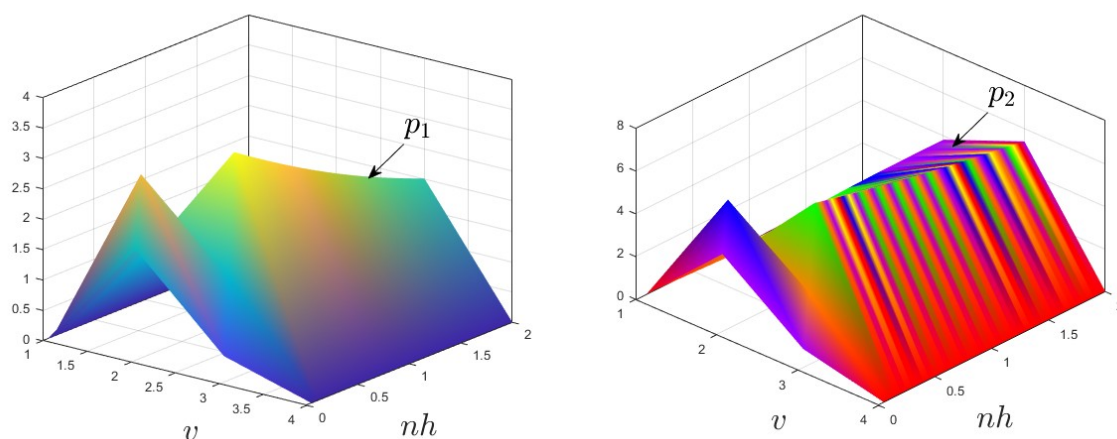
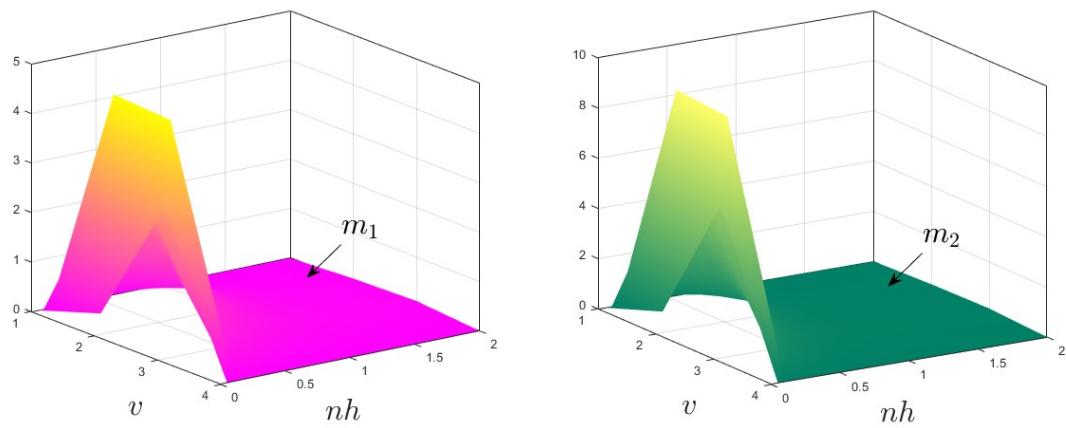**Figure 1.** The states p_1, p_2 in the master GRNs (4.1).

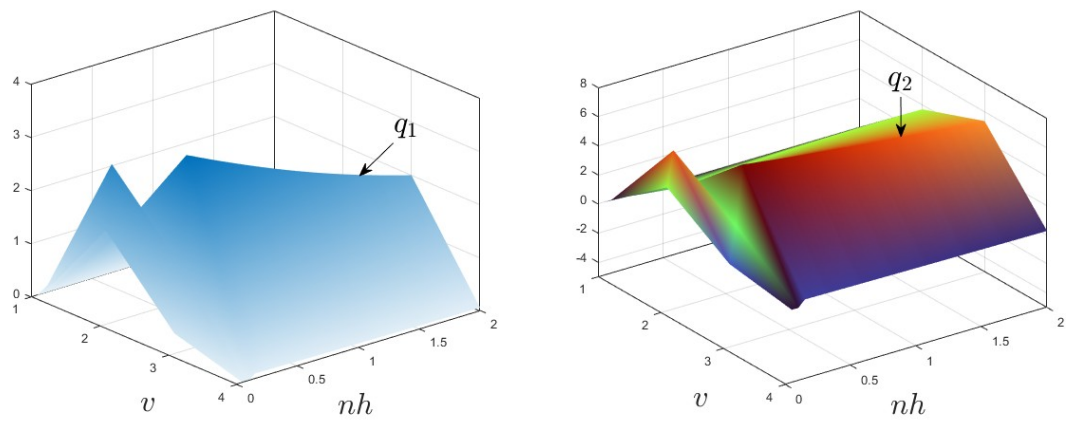**Figure 2.** The states m_1, m_2 in the master GRNs (4.1).

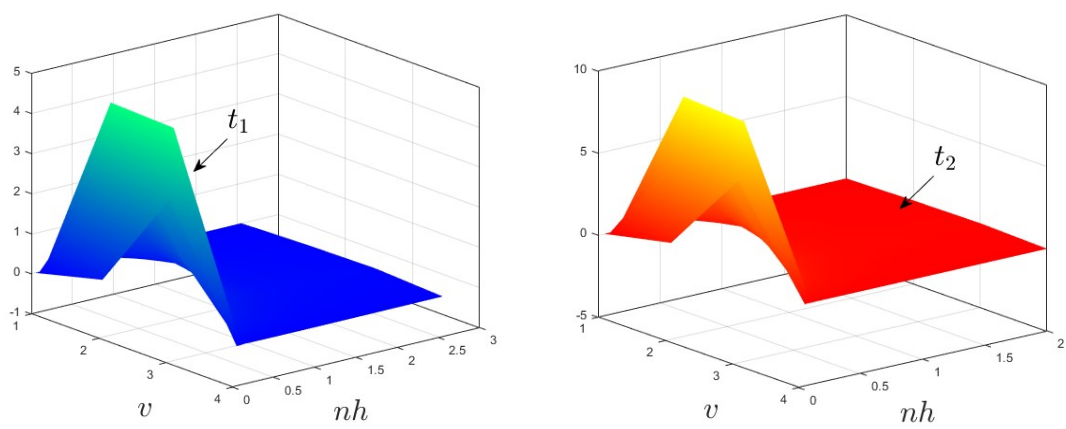**Figure 3.** The states q_1, q_2 in the slave GRNs (4.2).
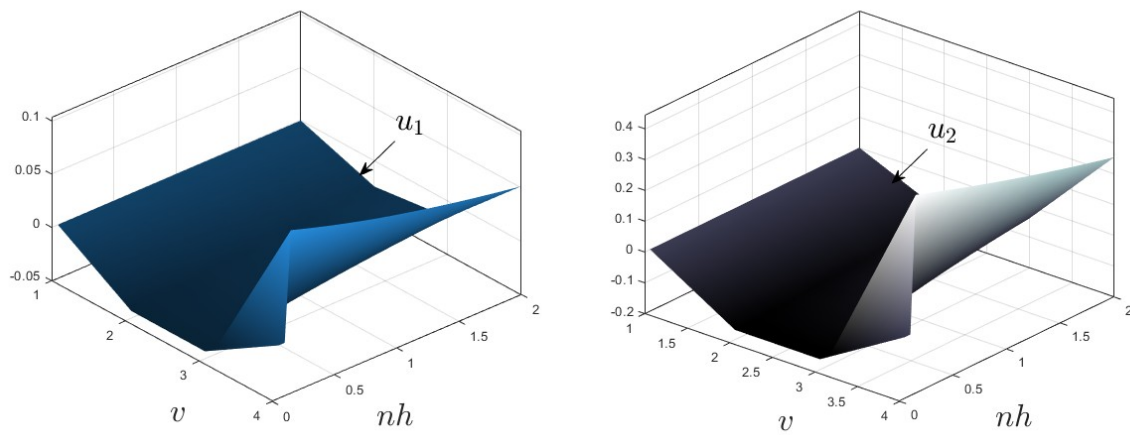

Figure 4. The states t_1, t_2 in the slave GRNs (4.2).
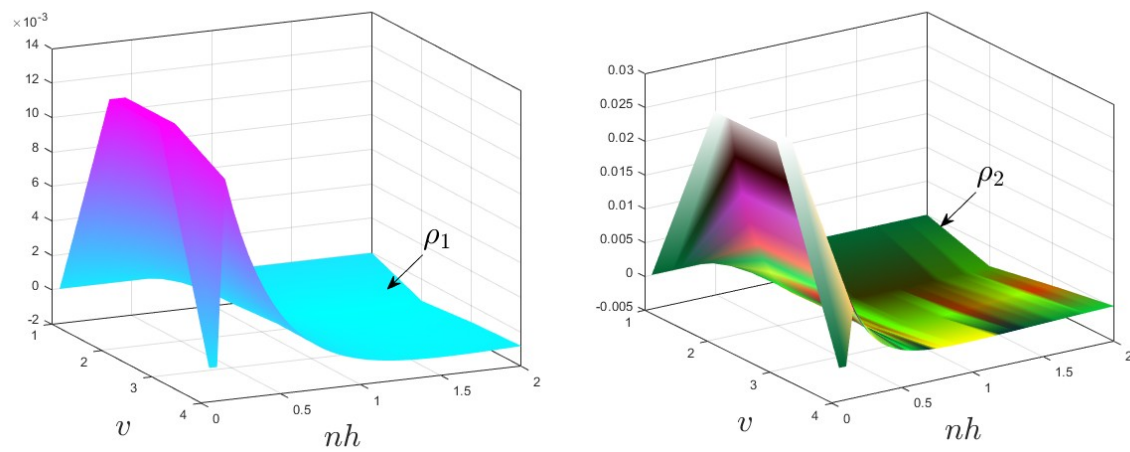
Figure 5. The error surfaces u_1, u_2.

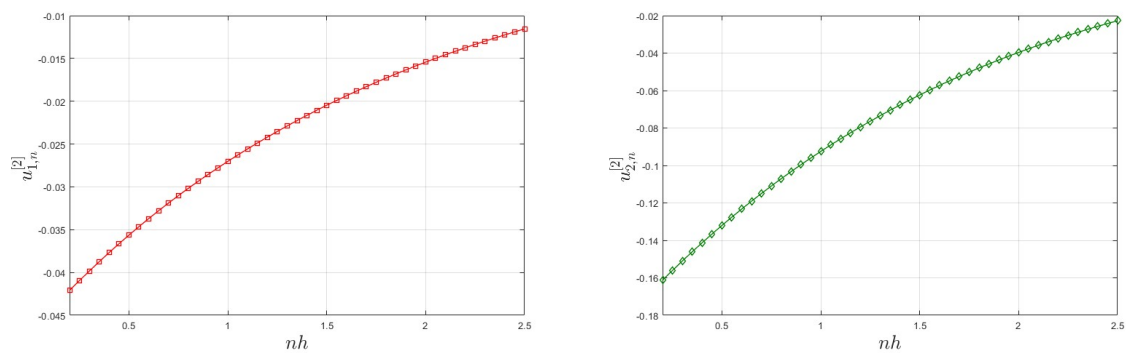**Figure 6.** The error surfaces ρ_1, ρ_2.
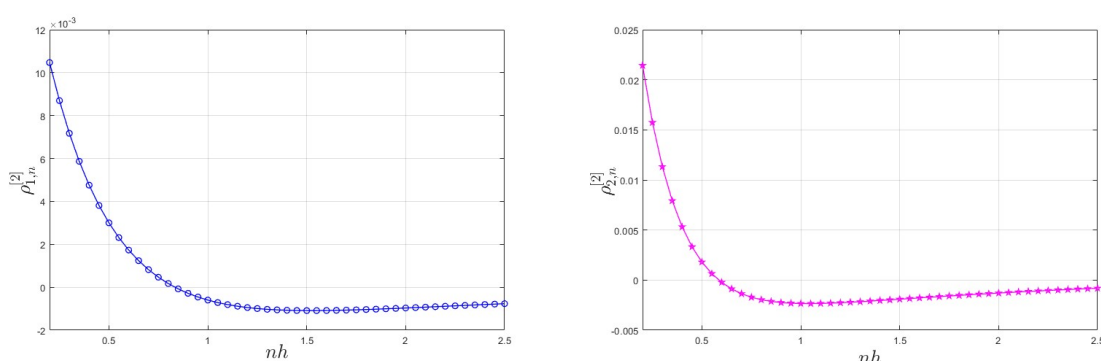

Figure 7. The error curves $u_{1,n}^{[2]}, u_{2,n}^{[2]}$.

Figure 8. The error curves $\rho_{1,n}^{[2]}, \rho_{2,n}^{[2]}$.

5. Conclusions

This article examines the subject of global asymptotic or exponential finite-time synchronization in the mean-squared sense of space-time discrete stochastic fuzzy genetic oscillator networks with Dirichlet controlled boundary conditions, accounting for the impact of time delays. By applying the appropriate boundary feedback controls in the slave genetic oscillator networks and utilizing the stochastic theory, this article successfully achieves global asymptotic or exponential finite-time synchronization in the mean-square sense of the space-time discrete genetic oscillator networks. The present study establishes a theoretical framework that will facilitate the global mean-squared finite-time synchronization of genetic oscillator networks within discrete schemes, taking into account both space and time domains. In future works, some interesting topics are worthy of further study, such as fixed/preassigned time stabilization/synchronization [38, 39].

Author contributions

Dong Pan: Formal analysis, investigation, conceptualization, methodology, writing-original draft; Huizhen Qu: Funding acquisition, validation, writing-review and editing. All the authors have agreed and given their consent for the publication of this research paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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