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Research article

Operators and separation axioms within the framework of diving topological spaces

Mesfer H. Alqahtani*

Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322, Saudi Arabia

* **Correspondence:** Email: m_halqahtani@ut.edu.sa; Tel: +966543581663.

Abstract: Operators serve as fundamental tools in the analysis of topological spaces, prompting extensive research and yielding numerous significant results. In this paper, we introduced a new topological framework called the "diving topological space", which was developed based on the diving structure. Within this framework, several operators were introduced, including one that fulfilled the Kuratowski axioms. We examined the core properties of these operators and explored the interrelations among them. Additionally, two new topologies were formulated and investigated with respect to each other and in comparison to classical topology. The study culminated by introducing concepts of fuzzy diving structures and demonstrating applications of fundamental topological properties, all substantiated with illustrative examples.

Keywords: diving structure; diving topological space; diving operator; Kuratowski closure operator
Mathematics Subject Classification: 54A05, 54A10

1. Introduction

Topology constitutes a pivotal framework across diverse domains of mathematics and computer science. Its foundational principles have been extensively harnessed to tackle a myriad of practical challenges, garnering significant attention from scholars in both natural and social sciences. Within classical topology there are filters [1], grills [2], and ideals [3], the latter of which was initially conceptualized by Kuratowski.

Thron [4] was the first to introduce proximity structures within the framework of grills. Later, in 1977, Chattopadhyay and Thron [5] expanded upon these ideas by incorporating grills into closure spaces. Chattopadhyay et al. [6] further extended the concept to explore merotopic spaces. Since then, grill structures have played a significant role in topological studies. Roy and Mukherjee [7–9] investigated various topological properties involving grills, while Roy et al. [10], Nasef and Azzam [11], among others, introduced operators grounded in grill. Modak [12,13] examined grill-filter spaces and their associated properties, and Hosny [14] applied grill structures to δ-sets. Additionally, cluster systems defined via grills were explored in [15].

Acharjee et al. [16] proposed a novel structure termed the "Primal". This framework not only exhibits key properties specific to primals but also reveals connections inter alia primal topological spaces and other similar works. The primal concept is regarded as the dual of grills, just as ideals are the dual of filters. Al-Omari et al. [17–19] employed the primal framework to develop various new primal topological spaces based on operators. Al-Shami et al. [20] introduced a new framework known as the primal soft topology, while Al-Omari et al. [21] explored the concepts of regularity and normality in the context of primal topological spaces. The motivation behind establishing such a distinctive framework lies in its potential to advance the development of novel soft concepts and properties, thereby enriching research within soft set theory. For further findings related to operators within the context of soft primal topology, refer to [22].

Recently, Ismail et al. [23] introduced two innovative structures known as diving and floating. These frameworks hold considerable value due to their prospective applications in multiple fields such as rough set theory, data analysis, fuzzy logic, information systems, and physics. Their capacity to generalize and extend established mathematical ideas makes them a promising foundation for future research in both theoretical mathematics and practical scientific domains. Notably, the concept of a diving represents the dual notion of a floating. Additionally, these constructions are quite different from known studies of topologies, filters, ideals, grills, primals, or any well-known classical branch in set theory.

The motivations for writing this paper are as follows: First, we present this work with the aim of introducing a new topology based on two existing ones, by creating distinctive frameworks that enable us to introduce new operators that satisfy Kuratowski's axioms and, thus, obtain new mathematical results. Second, we devise a new method for generating new topologies and then examine a number of topological properties based on them. Finally, this work serves as a basis for research in several mathematical fields.

The structure of this paper is organized as follows: In the first section, we present fundamental definitions and theoretical concepts essential for providing the reader with a clear understanding of the research context. In the second section, we present a new topological framework known as the diving topological space $(R, \mathfrak{T}, \mathfrak{D})$ and indicated by $\mathcal{DTS}$, developed from the diving structure $\mathfrak{D}$. We introduce a novel operator called the diving ξ-operator with respect to $\mathfrak{D}$ and $\mathfrak{T}$. The fundamental properties of this operator are examined. Based on the ξ-operator, we define a function cl^{ξ} that satisfies Kuratowski's closure axioms. In the third section, a novel topology $\mathfrak{T}^{\xi}$ is proposed and analyzed in connection with classical topology $\mathfrak{T}$. We show that the diving topology $\mathfrak{T}^{\xi}$ is finer than $\mathfrak{T}$. We also define an open base for the diving topology $\mathfrak{T}^{\xi}$. Several basic properties are introduced. In the fourth section, we define a new operator referred to as the "F-operator" and examine its fundamental properties. Building upon this F-operator, a new topology named diving $\mathfrak{T}^{F}$-topology is introduced. The relationships among the $\mathfrak{T}^{F}$-diving topology, $\mathfrak{T}^{\xi}$-diving topology, and classical $\mathfrak{T}$-topology are then analyzed. Finally, we conclude our study by introducing the concepts of fuzzy diving structures and demonstrating applications of the basic topological properties, all supported by illustrative examples.

Throughout this entire paper, spaces imply topological spaces on which no other property is presumed. For a subset Ξ of a space R, we denote the complement (resp., interior, closure, topology) of Ξ in R by Ξ^c (resp., $int(\Xi)$, $cl(\Xi)$, $\mathfrak{T}$). We denote the arbitrary indexing set by Λ and 2^R to be the collection of the whole subsets of a universal set R. T_2 denotes the Hausdorff property. T_4 and T_3 spaces are T_1 norma and T_1 regular spaces, respectively. A subset Ξ of a topological space $(R, \mathfrak{T})$ is called semi-open [24], provided there exists a $\mathfrak{T}$-open set Γ satisfying the inclusion $\Gamma \subseteq \Xi \subseteq cl_{\mathfrak{T}}(\Gamma)$.

2. Definitions and background

We now derive the ensuing concepts and results, which are instrumental for the subsequent sections:

Definition 1. [23] A collection $\mathfrak{D}$ of subsets of a nonempty set R is called a diving structure on R if $\mathfrak{D}$ meets the following requirements:

(D_1) $R \in \mathfrak{D}$;
(D_2) $\emptyset \notin \mathfrak{D}$;
(D_3) if $\Xi \in \mathfrak{D}$ and $\Gamma \in \mathfrak{D}$, then $(\Xi \cup \Gamma) \in \mathfrak{D}$ for all $\Xi, \Gamma \in 2^R$;
(D_4) if $(\Xi \cap \Gamma) \in \mathfrak{D}$, then $\Xi \in \mathfrak{D}$ or $\Gamma \in \mathfrak{D}$ for all $\Xi, \Gamma \in 2^R$.

Corollary 2. [23] A collection $\mathfrak{D}$ of subsets of a nonempty set R is called a diving structure on R if, and only if, $\mathfrak{D}$ meets the following requirements:

(D_1) $R \in \mathfrak{D}$;
(D_2) $\emptyset \notin \mathfrak{D}$;
(D_3) if $\Xi \in \mathfrak{D}$ and $\Gamma \in \mathfrak{D}$, then $(\Xi \cup \Gamma) \in \mathfrak{D}$ for all $\Xi, \Gamma \in 2^R$;
(D_4) if $\Xi \notin \mathfrak{D}$ and $\Gamma \notin \mathfrak{D}$, then $(\Xi \cap \Gamma) \notin \mathfrak{D}$ for all $\Xi, \Gamma \in 2^R$.

Definition 3. Collections $\mathcal{PR}$, $\mathcal{ID}$, $\mathcal{GR}$ and $\mathcal{FI}$ of subsets of a set $R \neq \emptyset$ are, respectively, called a primal [16], an ideal [2], a grill [1], and a filter [3] on R if, and only if, they meet the following requirements:

(i) $R \notin \mathcal{PR}$
(ii) $\Xi \in \mathcal{PR}$ *and* $\Gamma \subseteq \Xi \Rightarrow \Gamma \in \mathcal{PR}$
(iii) $\Xi \cap \Gamma \in \mathcal{PR} \Rightarrow \Xi \in \mathcal{PR}$ *or* $\Gamma \in \mathcal{PR}$

(i) $R \notin \mathcal{ID}$
(ii) $\Xi \in \mathcal{ID}$ *and* $\Gamma \subseteq \Xi \Rightarrow \Gamma \in \mathcal{ID}$
(iii) $\Xi \in \mathcal{ID}$ *and* $\Gamma \in \mathcal{ID} \Rightarrow \Xi \cup \Gamma \in \mathcal{ID}$

(i) $\emptyset \notin \mathcal{GR}$
(ii) $\Xi \in \mathcal{GR}$ *and* $\Xi \subseteq \Gamma \Rightarrow \Gamma \in \mathcal{GR}$
(iii) $\Xi \cup \Gamma \in \mathcal{GR} \Rightarrow \Xi \in \mathcal{GR}$ *or* $\Gamma \in \mathcal{GR}$

(i) $\emptyset \notin \mathcal{FI}$
(ii) $\Xi \in \mathcal{FI}$ *and* $\Xi \subseteq \Gamma \Rightarrow \Gamma \in \mathcal{FI}$
(iii) $\Xi \in \mathcal{FI}$ *and* $\Gamma \in \mathcal{FI} \Rightarrow \Xi \cap \Gamma \in \mathcal{FI}$

Moreover, a topological space $(R, \mathfrak{T})$ endowed with an auxiliary structural entity such as a primal $\mathcal{PR}$ (resp., an ideal $\mathcal{ID}$, a grill $\mathcal{GR}$, or a filter $\mathcal{FI}$) on R is designated as primal topological space (resp., an ideal topological space, a grill topological space, a filter topological space), and is symbolically represented by $(R, \mathfrak{T}, \mathcal{PR})$ (resp., $(R, \mathfrak{T}, \mathcal{ID})$, $(R, \mathfrak{T}, \mathcal{GR})$, or $(R, \mathfrak{T}, \mathcal{FR})$).

Definition 4. [3] The operator $\zeta : 2^R \to 2^R$ is a Kuratowski closure operator provided that:

(1) $\zeta(\emptyset) = \emptyset$;

(2) for each $\Xi \in 2^R$, $\Xi \subseteq \zeta(\Xi)$;
(3) for each $\Xi, \Gamma \in 2^R$, $\zeta(\Xi \cup \Gamma) = \zeta(\Xi) \cup \zeta(\Gamma)$;
(4) for each $\Xi \in 2^R$, $\zeta(\zeta(\Xi)) = \zeta(\Xi)$.

3. New topological structure and operator

In this section, we introduce a novel topological structure, termed the "diving topological space", within which a new operator referred to as the "diving ξ-operator" is defined and its fundamental properties are thoroughly examined.

According to Definition 3, the diving structure $\mathfrak{D}$ is independent of classical structures such as primal $\mathcal{PR}$, ideal $\mathcal{ID}$, grill $\mathcal{GR}$ and filter $\mathcal{FI}$. For instance, if $\mathfrak{D} = \{R, \{\mathfrak{d}_1\}\}$ is a diving structure on $R = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$, but it is neither primal nor ideal because it does not satisfy criterion (i) for them where $R \notin \mathcal{PR}$ and $R \notin \mathcal{ID}$, it is also neither grill nor filter because it does not satisfy criterion (ii) for them where $\{\mathfrak{d}_1\} \in \mathfrak{D}$ and $\{\mathfrak{d}_1\} \subseteq \{\mathfrak{d}_1, \mathfrak{d}_2\}$ but $\{\mathfrak{d}_1, \mathfrak{d}_2\} \notin \mathfrak{D}$.

Definition 5. A diving structure $\mathfrak{D}$ imposed on a set R, when integrated with a topological space $(R, \mathfrak{T})$, constitutes a diving topological space, formally represented as $(R, \mathfrak{T}, \mathfrak{D})$ and herein referred to as a $\mathcal{DTS}$.

Example 6. Let $\mathfrak{D} = \{R, \{\mathfrak{d}_1\}\}$ be a diving structure on $R = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$ and $\mathfrak{T}$ be any topology on R. Then $(R, \mathfrak{T}, \mathfrak{D})$ is a $\mathcal{DTS}$, where $\mathfrak{D} = \{R, \{\mathfrak{d}_1\}\}$. However, $(R, \mathfrak{T}, \mathfrak{D})$ is neither a primal topological space, nor an ideal topological space, nor a grill topological space, nor a filter topological space.

The primal, ideal, grill, and filter structures are all characterized by specific inclusion conditions. However, such constraint is absent in the diving structure, making it a more flexible and nonrestrictive framework. This absence of inclusion requirements highlights the distinctive independence of the diving structure from other structures. As a result, the following definition introduces a novel operator, termed the diving ξ-operator, which is formulated independently of the operators in the previously established structures, thereby ensuring mathematical coherence throughout the remainder of this work.

Definition 7. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. We define a function $\xi : 2^R \to 2^R$ as $\xi(\Xi)(R, \mathfrak{T}, \mathfrak{D}) = \{\mathfrak{d} \in R : (\forall \Omega \in \mathfrak{T}(\mathfrak{d}))(\exists \mho \in \mathfrak{D}^\star : \Xi \cap \Omega \cap \mho \neq \emptyset)\}$, for any subset Ξ of R, $\mathfrak{D}^\star = \mathfrak{D} \setminus \{R\}$, and $\mathfrak{T}(\mathfrak{d}) = \{\Omega \in \mathfrak{T} : \mathfrak{d} \in \Omega\}$. For precision, $\xi(\Xi)(R, \mathfrak{T}, \mathfrak{D})$ is succinctly denoted as $\xi^{\mathfrak{T}}_{\mathfrak{D}}(\Xi)$ or $\xi(\Xi)$ and is formally designated as the diving ξ-operator associated with Ξ in relation to $\mathfrak{T}$ and $\mathfrak{D}$.

Consider $(R, \mathfrak{T}, \mathfrak{D})$ as a diving topological space ($\mathcal{DTS}$). For an arbitrary subset Ξ of R, the inclusions $\xi(\Xi) \subseteq \Xi$ or $\Xi \subseteq \xi(\Xi)$ do not necessarily hold, as exemplified in the following example.

Example 8. Let $R = \{\epsilon, \theta, \kappa\}$ and $\mathfrak{T} = \{\phi, \{\epsilon\}, \{\theta\}, \{\epsilon, \theta\}, R\}$. We consider a diving structure $\mathfrak{D} = \{\{\epsilon\}, \{\epsilon, \theta\}, R\}$ on R.

Ξ	$\{\epsilon\}$	$\{\theta\}$	$\{\kappa\}$	$\{\epsilon, \theta\}$	$\{\epsilon, \kappa\}$	$\{\theta, \kappa\}$	R	$\emptyset$
$\xi(\Xi)$	R	$\{\theta, \kappa\}$	$\emptyset$	R	R	$\{\theta, \kappa\}$	R	$\emptyset$

From the table, we have $\xi(\{\epsilon\}) = R$, then $\xi(\{\epsilon\}) \nsubseteq \{\epsilon\}$. We also have $\xi(\{\kappa\}) = \emptyset$, then $\{\kappa\} \nsubseteq \xi(\{\kappa\})$.

Remark 9. Since the diving ξ-operator is formulated relative to the collections $\mathfrak{T}$ and $\mathfrak{D}$ on a diving topological space $(R, \mathfrak{T}, \mathfrak{D})$, and as illustrated in Example 6, we conclude that the ξ-operator is independent of any operators defined within other frameworks, such as a primal topological space $(R, \mathfrak{T}, \mathcal{PR})$, an ideal topological space $(R, \mathfrak{T}, \mathcal{ID})$, a grill topological space $(R, \mathfrak{T}, \mathcal{GR})$, a filter topological space $(R, \mathfrak{T}, \mathcal{FL})$.

Now, we present the main theorem that plays an important role in this work.

Theorem 10. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the following statements hold for any two subsets Ξ and Γ of R.

(1) $\xi(\emptyset) = \emptyset$;
(2) if $\Gamma^c \in \mathfrak{T}$, then $\xi(\Gamma) \subseteq \Gamma$;
(3) if $\Xi \subseteq \Gamma$, then $\xi(\Xi) \subseteq \xi(\Gamma)$;
(4) if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two diving structures on R and $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $\xi_{\mathfrak{D}_1}(\Gamma) \subseteq \xi_{\mathfrak{D}_2}(\Gamma)$;
(5) $cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) \subseteq cl_{\mathfrak{T}}(\Gamma)$;
(6) $\xi(\xi(\Gamma)) \subseteq \xi(\Gamma)$;
(7) $\xi(\Xi \cup \Gamma) = \xi(\Xi) \cup \xi(\Gamma)$;
(8) $\xi(\Xi \cap \Gamma) \subseteq \xi(\Xi) \cap \xi(\Gamma)$.

Proof. (1) Let $\mathfrak{d} \in \xi(\emptyset)$. Then, for any $\Omega \in \mathfrak{T}(\mathfrak{d})$ there exists $\mho \in \mathfrak{D}^\star$ such that $\emptyset \cap \Omega \cap \mho \neq \emptyset$, which is impossible. Hence, $\xi(\emptyset) = \emptyset$.

(2) Let $\mathfrak{d} \in \xi(\Gamma)$. Suppose that $\mathfrak{d} \notin \Gamma$, and this implies $\Gamma^c \in \mathfrak{T}(\mathfrak{d})$. Hence, there exists $\mho \in \mathfrak{D}^\star$ such that $\Gamma \cap \Gamma^c \cap \mho \neq \emptyset$, which is a contradiction since $\Gamma \cap \Gamma^c = \emptyset$. Hence, $\mathfrak{d} \in \Gamma$, and so $\xi(\Gamma) \subseteq \Gamma$.

(3) Let $\mathfrak{d} \in \xi(\Xi)$. Then, for any $\Omega \in \mathfrak{T}(\mathfrak{d})$ there exists $\mho \in \mathfrak{D}^\star$ such that $\Xi \cap \Omega \cap \mho \neq \emptyset$. Since $\Xi \subseteq \Gamma$, then $\Xi \cap \Omega \subseteq \Gamma \cap \Omega$, and so $\Gamma \cap \Omega \cap \mho \neq \emptyset$. Thus, $\mathfrak{d} \in \xi(\Gamma)$. Therefore, $\xi(\Xi) \subseteq \xi(\Gamma)$.

(4) Let $\mathfrak{d} \in \xi_{\mathfrak{D}_1}(\Gamma)$ and $\Omega \in \mathfrak{T}(\mathfrak{d})$. Then, there exists $\mho \in \mathfrak{D}_1^\star$ such that $\Gamma \cap \Omega \cap \mho \neq \emptyset$. Since $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $\mho \in \mathfrak{D}_2^\star$ and $\Gamma \cap \Omega \cap \mho \neq \emptyset$. Hence, $\mathfrak{d} \in \xi_{\mathfrak{D}_2}(\Gamma)$. Therefore, $\xi_{\mathfrak{D}_1}(\Gamma) \subseteq \xi_{\mathfrak{D}_2}(\Gamma)$.

(5) Let $\mathfrak{d} \notin cl_{\mathfrak{T}}(\Gamma)$. Then, there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$ such that $\Omega \cap \Gamma = \emptyset$. Let $\mho \in \mathfrak{D}^\star$. Then, $\Gamma \cap \Omega \cap \mho = \emptyset$, hence, $\mathfrak{d} \notin \xi(\Gamma)$. Thus, $\xi(\Gamma) \subseteq cl_{\mathfrak{T}}(\Gamma)$. Now, let $\mathfrak{d} \in cl_{\mathfrak{T}}(\xi(\Gamma))$ and $\Omega \in \mathfrak{T}(\mathfrak{d})$. Then, $(\Omega \cap \xi(\Gamma)) \neq \emptyset$. Let $\mathfrak{s} \in \Omega \cap \xi(\Gamma)$, and this implies $\mathfrak{s} \in \Omega$ and $\mathfrak{s} \in \xi(\Gamma)$. Hence, there exists $\mho \in \mathfrak{D}^\star$ such that $\Gamma \cap \Omega \cap \mho \neq \emptyset$ and so $\mathfrak{d} \in \xi(\Gamma)$. Thus, $cl_{\mathfrak{T}}(\xi(\Gamma)) \subseteq \xi(\Gamma)$. Since $\xi(\Gamma) \subseteq cl_{\mathfrak{T}}(\xi(\Gamma))$, then $cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma)$. Therefore, $cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) \subseteq cl_{\mathfrak{T}}(\Gamma)$.

(6) It is clear from (2) and (5).

(7) Since $\Xi \subseteq (\Xi \cup \Gamma)$, $\Gamma \subseteq (\Xi \cup \Gamma)$, and by part (3), $\xi(\Xi) \subseteq \xi(\Xi \cup \Gamma)$ and $\xi(\Gamma) \subseteq \xi(\Xi \cup \Gamma)$. Hence, $\xi(\Xi) \cup \xi(\Gamma) \subseteq \xi(\Xi \cup \Gamma)$. On the other side, suppose that $\mathfrak{d} \notin \xi(\Xi) \cup \xi(\Gamma)$. Then, for any $\mho \in \mathfrak{D}^\star$ there exist $\Omega_1, \Omega_2 \in \mathfrak{T}(\mathfrak{d})$ such that $\Xi \cap \Omega_1 \cap \mho = \emptyset$ and $\Gamma \cap \Omega_2 \cap \mho = \emptyset$. Hence, $(\Xi \cap \Omega_1 \cap \mho) \cup (\Gamma \cap \Omega_2 \cap \mho) = \emptyset$. Now, there exists $\Omega = (\Omega_1 \cap \Omega_2) \in \mathfrak{T}(\mathfrak{d})$, and for any $\mho \in \mathfrak{D}^\star$, we have $(\Xi \cup \Gamma) \cap \Omega \cap \mho = \emptyset$. Hence, $\mathfrak{d} \notin \xi(\Xi \cup \Gamma)$. Thus, $\xi(\Xi \cup \Gamma) \subseteq \xi(\Xi) \cup \xi(\Gamma)$. Therefore, $\xi(\Xi \cup F) = \xi(\Xi) \cup \xi(\Gamma)$.

(8) It is clear from (3).

□

The inclusion presented in (8) of Theorem 10, needs not to be reversible as shown in the following example:

Example 11. Let $\mathfrak{T} = \{\phi, R\}$ be an indiscrete topology on $R = \{\epsilon, \theta, \kappa\}$ and $\mathfrak{D} = \{R, \{\epsilon\}, \{\epsilon, \theta\}\}$. Then,

Ξ	$\{\epsilon\}$	$\{\theta\}$	$\{\kappa\}$	$\{\epsilon,\theta\}$	$\{\epsilon,\kappa\}$	$\{\theta,\kappa\}$	R	$\emptyset$
$\xi(\Xi)$	R	R	$\emptyset$	R	R	R	R	$\emptyset$

From the table, we have $\xi(\{\epsilon,\kappa\})\cap\xi(\{\theta,\kappa\}) = R \nsubseteq \emptyset = \xi(\{\epsilon,\kappa\}\cap\{\theta,\kappa\})$.

Definition 12. Let $(R,\mathfrak{T},\mathfrak{D})$ be a $\mathcal{DTS}$. We define a function $cl^{\xi}: 2^R \to 2^R$ by $cl^{\xi}(\Gamma) = \Gamma\cup\xi(\Gamma)$ for all $\Gamma\in 2^R$.

Theorem 13. Let $(R,\mathfrak{T},\mathfrak{D})$ be a $\mathcal{DTS}$. Then, the following statements hold for any two subsets Ξ and Γ of R.

(1) $cl^{\xi}(\emptyset)=\emptyset$;
(2) $cl^{\xi}(R)=R$;
(3) $\Gamma\subseteq cl^{\xi}(\Gamma)$;
(4) if $\Xi\subseteq\Gamma$, then $cl^{\xi}(\Xi)\subseteq cl^{\xi}(\Gamma)$;
(5) $cl^{\xi}(\Xi\cup\Gamma)=cl^{\xi}(\Xi)\cup cl^{\xi}(\Gamma)$;
(6) $cl^{\xi}(cl^{\xi}(\Gamma))=cl^{\xi}(\Gamma)$.

Proof. (1) By (1) of Theorem 10, $\xi(\emptyset)=\emptyset$, then $cl^{\xi}(\emptyset)=\emptyset\cup\xi(\emptyset)=\emptyset$.

(2) Since $\xi(R)\subseteq R$, then $cl^{\xi}(R)=R\cup\xi(R)=R$.

(3) Obvious.

(4) Let $\Xi\subseteq\Gamma$. By (3) of Theorem 10, $\xi(\Xi)\subseteq\xi(\Gamma)$. Then, $(\Xi\cup\xi(\Xi))\subseteq(\Gamma\cup\xi(\Gamma))$, hence, $cl^{\xi}(\Xi)\subseteq cl^{\xi}(\Gamma)$.

(5)

$$\begin{aligned}
cl^{\xi}(\Xi\cup\Gamma) &= (\Xi\cup\Gamma)\cup\xi(\Xi\cup\Gamma)\\
&= (\Xi\cup\Gamma)\cup(\xi(\Xi)\cup\xi(\Gamma))\\
&= (\Xi\cup\xi(\Xi))\cup(\Gamma\cup\xi(\Gamma))\\
&= cl^{\xi}(\Xi)\cup cl^{\xi}(\Gamma).
\end{aligned}$$

(6)

$$\begin{aligned}
cl^{\xi}(cl^{\xi}(\Gamma)) &= cl^{\xi}(\Gamma\cup\xi(\Gamma))\\
&= (\Gamma\cup\xi(\Gamma))\cup\xi((\Gamma\cup\xi(\Gamma)))\\
&= (\Gamma\cup\xi(\Gamma))\cup(\xi(\Gamma)\cup\xi(\xi(\Gamma)))\\
&= \Gamma\cup\xi(\Gamma)\cup\xi(\xi(\Gamma))\\
&= \Gamma\cup\xi(\Gamma)\\
&= cl^{\xi}(\Gamma).
\end{aligned}$$

□

Corollary 14. The function cl^{ξ} satisfies Kuratowski's closure axioms.

Theorem 15. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ and $\Xi, \Gamma \in 2^R$. If $\Xi \in \mathfrak{T}$, then $\Xi \cap \xi(\Gamma) = \Xi \cap \xi(\Xi \cap \Gamma)$.

Proof. By part (5) of Theorem 10, we have $\Xi \cap \xi(\Xi \cap \Gamma) \subseteq \Xi \cap \xi(\Gamma)$. Let $\mathfrak{d} \in (\Xi \cap \xi(\Gamma))$. Then, $\mathfrak{d} \in \Xi$ and $\mathfrak{d} \in \xi(\Gamma)$. Hence, for any $\Omega \in \mathfrak{T}(\mathfrak{d})$ there exists $\mho \in \mathfrak{D}^\star$ such that $\Gamma \cap \Omega \cap \mho \neq \emptyset$. Since $\Xi \in \mathfrak{T}$ and $\mathfrak{d} \in \Xi$, then $(\Xi \cap \Omega) \in \mathfrak{T}(\mathfrak{d})$. Since $\mathfrak{d} \in \xi(\Gamma)$, then there exists $\mathbb{K} \in \mathfrak{D}^\star$ such that $(\Xi \cap \Omega) \cap \Gamma \cap \mathbb{K} \neq \emptyset$. Hence, $(\Xi \cap \Gamma) \cap \Omega \cap \mathbb{K} \neq \emptyset$. Thus, $\mathfrak{d} \in \xi(\Xi \cap \Gamma)$. Therefore, $\Xi \cap \xi(\Gamma) = \Xi \cap \xi(\Xi \cap \Gamma)$. □

Corollary 16. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ and $\Xi, \Gamma \in 2^R$. If $\Xi \in \mathfrak{T}$, then $\Xi \cap \xi(\Gamma) \subseteq \xi(\Xi, \cap \Gamma)$.

Theorem 17. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Then, $\Gamma \subseteq \xi(\Gamma)$ for all $\Gamma \in \mathfrak{T}$.

Proof. If $\Gamma = \emptyset$, then by part (1) of Theorem 10, $\xi(\emptyset) = \emptyset$. Suppose that $\Gamma \neq \emptyset$. Now, let $\Gamma = R$ and $\mathfrak{d} \in R$. Let $\Omega \in \mathfrak{T}(\mathfrak{d})$. If $\Omega = R$, then one can assert the existence of $\mho \in \mathfrak{D}^\star$ ($|\mathfrak{D}| > 1$) where $R \cap \Omega \cap \mho \neq \emptyset$. Assume that $\Omega \neq R$, then there exists $\Omega = \mho \in \mathfrak{D}^\star$ ($\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$) such that $R \cap \Omega \cap \mho \neq \emptyset$. Thus, $\mathfrak{d} \in \xi(R)$. Hence, $R \subseteq \xi(R)$. Since $\xi(R) \subseteq R$, then $R = \xi(R)$. Now, assume that, $\Gamma \neq R$. Since $\Gamma \in \mathfrak{T}$ and by Theorem 15, $\Gamma = \Gamma \cap R = \Gamma \cap \xi(R) = \Gamma \cap \xi(\Gamma \cap R) = \Gamma \cap \xi(\Gamma)$, and this means $\Gamma \subseteq \xi(\Gamma)$. Therefore, $\Gamma \subseteq \xi(\Gamma)$ for all $\Gamma \in \mathfrak{T}$. □

Theorem 17 is important for the rest of the paper, as $\Gamma \subseteq \xi(\Gamma)$ can be satisfied for all $\Gamma \in \mathfrak{T}$ only if $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$.

Remark. If $|\mathfrak{D}| = 1$, then according to Example 53, we have $|\mathfrak{D}_1| = 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}_1$, and $R \in \mathfrak{T}_1$, but $R \nsubseteq \xi(R)$. Also, if $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \nsubseteq \mathfrak{D}$, then according to Example 51, set $\mathfrak{D} = \{R, \{\theta\}\}$. Hence, $\{\epsilon\} \in \mathfrak{T}$ and $\{\epsilon\} \nsubseteq \xi(\{\epsilon\}) = \emptyset$.

Theorem 18. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, $\xi(\Xi) \setminus \xi(\Gamma) = \xi(\Xi \setminus \Gamma) \setminus \xi(\Gamma)$ for all $\Xi, \Gamma \in 2^R$.

Proof. By parts (3), (7) of Theorem 10, $\xi(\Xi) = \xi((\Xi \setminus \Gamma) \cup (\Xi \cap \Gamma)) = \xi((\Xi \setminus \Gamma)) \cup \xi((\Xi \cap \Gamma)) \subseteq \xi((\Xi \setminus \Gamma)) \cup \xi(\Gamma)$. Hence, $\xi(\Xi) \setminus \xi(\Gamma) \subseteq \xi(\Xi \setminus \Gamma) \setminus \xi(\Gamma)$. On the other side, since $(\Xi \setminus \Gamma) \subseteq \Xi$, then $\xi(\Xi \setminus \Gamma) \subseteq \xi(\Xi)$. Hence, $\xi(\Xi \setminus \Gamma) \setminus \xi(\Gamma) \subseteq \xi(\Xi) \setminus \xi(\Gamma)$. Therefore, $\xi(\Xi) \setminus \xi(\Gamma) = \xi(\Xi \setminus \Gamma) \setminus \xi(\Gamma)$ for all $\Xi, \Gamma \in 2^R$. □

Lemma 19. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ and $\Xi \in 2^R$. Let $\Xi \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^\star$. Then, $\xi(\Xi) = \emptyset$.

Proof. Let $\mathfrak{d} \in \xi(\Xi)$. Then for all $\Omega \in \mathfrak{T}(\mathfrak{d})$ there exists $\mho \in \mathfrak{D}^\star$ such that $\Xi \cap \Omega \cap \mho \neq \emptyset$, which is a contradiction since $\Xi \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^\star$. Hence $\xi(\Xi) = \emptyset$. □

Corollary 20. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ and $\Xi, \Gamma \in 2^R$. Let $\Gamma \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^\star$. Then $\xi(\Xi \cup \Gamma) = \xi(\Xi) = \xi(\Xi \setminus \Gamma)$.

Theorem 21. Let $\zeta : R_1 \to R_2$ be a function. If $\mathfrak{D}_1$ is a diving structure on R_1 and Ψ is bijection, then $\mathfrak{D}_2 = \{\zeta(\Gamma) : \Gamma \in \mathfrak{D}_1\}$ is a diving structure on R_2.

Proof. (D_1) Since ζ is onto and $R_1 \in \mathfrak{D}_1$, then $R_2 \in \mathfrak{D}_2$.

(D_2) Suppose that $\emptyset \in \mathfrak{D}_2$. Then, there exists $\Gamma \in \mathfrak{D}_1$ such that $\zeta(\Gamma) = \emptyset$ and $\Gamma \neq \emptyset$. However, ζ is one-to-one, and this is a contradiction. Therefore, $\emptyset \notin \mathfrak{D}_2$.

(D_3) Let $\Xi_1 \in \mathfrak{D}_2$ and $\Xi_2 \in \mathfrak{D}_2$. Then, there exist $\Gamma_1 \in \mathfrak{D}_1$ and $\Gamma_2 \in \mathfrak{D}_1$ such that $\Xi_1 = \zeta(\Gamma_1)$ and $\Xi_2 = \zeta(\Gamma_2)$. Since ζ is one-to-one, $\zeta^{-1}(\Xi_1) = \Gamma_1$ and $\zeta^{-1}(\Xi_2) = \Gamma_2$, and this means $\zeta^{-1}(\Xi_1) \in \mathfrak{D}_1$ and $\zeta^{-1}(\Xi_2) \in \mathfrak{D}_1$. Thus, $\zeta^{-1}(\Xi_1 \cup \Xi_2) \in \mathfrak{D}_1$. By surjectivity, $\zeta(\zeta^{-1}(\Xi_1 \cup \Xi_2)) = \Xi_1 \cup \Xi_2 \in \mathfrak{D}_2$.

(D_4) Let $\Xi_1 \cup \Xi_2 \in \mathfrak{D}_2$. Then, there exists $\Gamma \in \mathfrak{D}_1$ such that $\Xi_1 \cup \Xi_2 = \zeta(\Gamma)$. Since ζ is one-to-one, $\zeta^{-1}(\Xi_1) \cup \zeta^{-1}(\Xi_2) = \zeta^{-1}(\Xi_1 \cup \Xi_2) = \Gamma$, and this means $\zeta^{-1}(\Xi_1) \cup \zeta^{-1}(\Xi_2) \in \mathfrak{D}_1$. Thus, $\zeta^{-1}(\Xi_1) \in \mathfrak{D}_1$ or $\zeta^{-1}(\Xi_2) \in \mathfrak{D}_1$. By surjectivity, $\zeta(\zeta^{-1}(\Xi_1)) = \Xi_1 \in \mathfrak{D}_2$ or $\zeta(\zeta^{-1}(\Xi_2)) = \Xi_2 \in \mathfrak{D}_2$.

□

4. Diving topology and its basic properties

In this section, we present a new topology referred to as the diving $\mathfrak{T}^{\xi}$-topology. An open base for this topology is established, its connection to classical topology is identified, and its fundamental properties are analyzed in detail.

Theorem 22. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the family $\mathfrak{T}^{\xi} = \{\Gamma \subseteq R : cl^{\xi}(\Gamma^c) = \Gamma^c\}$ constitutes a topology on R, induced by the original topology $\mathfrak{T}$ and the diving structure $\mathfrak{D}$. This topology is referred to as the diving topology on R. We can also write $\mathfrak{T}^{\xi}_{\mathfrak{D}}$ instead of $\mathfrak{T}^{\xi}$ to specify the diving structure as per our requirements.

Proof. (1) Since $cl^{\xi}(\emptyset^c) = cl^{\xi}(R) = R \cup \xi(R) = R = \emptyset^c$, then $\emptyset \in \mathfrak{T}^{\xi}$. Also, by Theorem 10, part (1), $cl^{\xi}(R^c) = cl^{\xi}(\emptyset) = \emptyset \cup \xi(\emptyset) = \emptyset = R^c$, then $R \in \mathfrak{T}^{\xi}$. Now, let $\Gamma, \Xi \in \mathfrak{T}^{\xi}$. Then, $cl^{\xi}(\Gamma^c) = \Gamma^c$ and $cl^{\xi}(\Xi^c) = \Xi^c$. Hence, $cl^{\xi}((\Gamma \cap \Xi)^c) = (\Gamma \cap \Xi)^c \cup \xi((\Gamma \cap \Xi)^c) = (\Gamma^c \cup \Xi^c) \cup \xi((\Gamma^c \cup \Xi^c))$. By Theorem 10, part (7), $cl^{\xi}((\Gamma \cap \Xi)^c) = \Gamma^c \cup \Xi^c \cup \xi(\Gamma^c) \cup \xi(\Xi^c)$. Thus, $cl^{\xi}((\Gamma \cap \Xi)^c) = (\Gamma^c \cup \xi(\Gamma^c)) \cup (\Xi^c \cup \xi(\Xi^c)) = cl^{\xi}(\Gamma^c) \cup cl^{\xi}(\Xi^c) = \Gamma^c \cup \Xi^c = (\Gamma \cap \Xi)^c$. Thus, $(\Gamma \cap \Xi) \in \mathfrak{T}^{\xi}$. Let $\{\Gamma_\nu : \nu \in \Lambda\} \subseteq \mathfrak{T}^{\xi}$. Then, $cl^{\xi}((\bigcup_{\nu\in\Lambda}(\Gamma_\nu))^c) = (\bigcup_{\nu\in\Lambda}(\Gamma_\nu))^c \cup \xi((\bigcup_{\nu\in\Lambda}(\Gamma_\nu))^c) = \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c \cup \xi(\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c)$.
Claim. $\xi(\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c) \subseteq \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c$. Suppose that $\mathfrak{d} \notin \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c$. Then, there exists $\nu_1 \in \Lambda$ such that $\mathfrak{d} \notin \Gamma^c_{\nu_1}$. Since $\Gamma^c_{\nu_1} \in \mathfrak{T}^{\xi}$, hence, $\xi(\Gamma^c_{\nu_1}) \subseteq \Gamma^c_{\nu_1}$, this implies $\mathfrak{d} \notin \xi(\Gamma^c_{\nu_1})$. Hence, there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$, and for any $\mho \in \mathfrak{D}^\star$ we have $\Gamma^c_{\nu_1} \cap \Omega \cap \mho = \emptyset$. Since $\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c \subseteq \Gamma^c_{\nu_1}$, then $\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c \cap \Omega \cap \mho = \emptyset$, and this implies $\mathfrak{d} \notin \xi(\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c)$. Therefore, $\xi(\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c) \subseteq \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c$. Hence, $cl^{\xi}((\bigcup_{\nu\in\Lambda}(\Gamma_\nu))^c) = \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c \cup \xi(\bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c) = \bigcap_{\nu\in\Lambda}(\Gamma_\nu)^c = (\bigcup_{\nu\in\Lambda}(\Gamma_\nu))^c$. Hence, $\bigcup_{\nu\in\Lambda}(\Gamma_\nu) \in \mathfrak{T}^{\xi}$. Therefore, $\mathfrak{T}^{\xi}$ is a topology on R.

□

Here we give an example of $\mathfrak{T}^{\xi}$.

Example 23. Let $\mathbb{R}$ be the set of all real numbers. Let $\mathfrak{T}$ be the particular point topology on $\mathbb{R}$ at 7. Define $\mathfrak{D} = \{\mathbb{R}, \{5\}\}$. Let $\mathbb{S} \subseteq \mathbb{R}$. If $\mathbb{S} = \emptyset$, then $\xi(\emptyset) = \emptyset$. Suppose that $\mathbb{S} \neq \emptyset$, then we have three cases:
Case 1. If $\mathfrak{d} = 5$ and $5 \in \mathbb{S}$, then $\xi(\mathbb{S}) = \{5\}$. Assume that $\xi(\mathbb{S}) \neq \{5\}$, then there exists $\mathfrak{s} \neq 5$ and $\mathfrak{s} \in \xi(\mathbb{S})$. Hence, there exists $\Omega = \{\mathfrak{s}, 7\} \in \mathfrak{T}(\mathfrak{s})$ such that for any $\mho \in \mathfrak{D}^\star$, we have $\mathbb{S} \cap \Omega \cap \mho = \emptyset$. Thus, $\mathfrak{s} \notin \xi(\mathbb{S})$, which is a contradiction. Therefore, $\xi(\mathbb{S}) = \{5\}$.
Case 2. If $\mathfrak{d} = 5$ and $5 \notin \mathbb{S}$, then for any $\mathfrak{d} \in \mathbb{R}$ there exists $\Omega = \{\mathfrak{d}, 7\} \in \mathfrak{T}(\mathfrak{d})$, such that for any $\mho \in \mathfrak{D}^\star = \{\{5\}\}$ we have $\mathbb{S} \cap \Omega \cap \mho = \emptyset$. Thus, $\mathfrak{d} \notin \xi(\mathbb{S})$. Therefore, $\xi(\mathbb{S}) = \emptyset$.
Case 3. If $\mathfrak{d} \neq 5$ and $\mathfrak{d} \in \mathbb{S}$ or $\mathfrak{d} \notin \mathbb{S}$, then there exists $\Omega = \{\mathfrak{d}, 7\} \in \mathfrak{T}(\mathfrak{d})$, such that for any $\mho \in \mathfrak{D}^\star = \{\{5\}\}$ we have $\mathbb{S} \cap \Omega \cap \mho = \emptyset$. Thus, $\mathfrak{d} \notin \xi(\mathbb{S})$. Therefore, $\xi(\mathbb{S}) = \emptyset$.
By Cases 1–3, we have

$$\xi(\mathbb{S}) = \begin{cases} \{5\} & if\ \mathfrak{d} = 5 \text{ and } 5 \in \mathbb{S}, \\ \emptyset & if\ \mathfrak{d} = 5 \text{ and } 5 \notin \mathbb{S}, \\ \emptyset & if\ \mathfrak{d} \neq 5 \text{ and } \mathfrak{d} \in \mathbb{S} \text{ or } \mathfrak{d} \notin \mathbb{S}. \end{cases}$$

Now, for any $\mathbb{S} \subseteq \mathbb{R}$, we have $\xi(\mathbb{S}) \subseteq \mathbb{S}$. Then, $\mathfrak{T}^{\xi} = \{\mathbb{S} : \mathbb{S} \subseteq \mathbb{R}\} = \mathcal{P}(\mathbb{R})$ is a discrete topology.

Theorem 24. Let $(R, \mathfrak{T}, \mathfrak{D}_1)$, $(R, \mathfrak{T}, \mathfrak{D}_2)$ be two $\mathcal{DTS}s$. If $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $\mathfrak{T}^{\xi}_{\mathfrak{D}_2} \subseteq \mathfrak{T}^{\xi}_{\mathfrak{D}_1}$.

Proof. Let $\Gamma \in \mathfrak{T}^{\xi}_{\mathfrak{D}_2}$. Then, $cl^{\xi}_{\mathfrak{D}_2}(\Gamma) = \Gamma \cup \xi_{\mathfrak{D}_2}(\Gamma) = \Gamma$. Hence, $\xi_{\mathfrak{D}_2}(\Gamma) \subseteq \Gamma$. By (4) of Theorem 10, $\xi_{\mathfrak{D}_1}(\Gamma) \subseteq \Gamma$. Thus, $cl^{\xi}_{\mathfrak{D}_1}(\Gamma) = \Gamma \cup \xi_{\mathfrak{D}_1}(\Gamma) = \Gamma$. Therefore, $\Gamma \in \mathfrak{T}^{\xi}_{\mathfrak{D}_1}$.

□

Theorem 25. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the diving topology $\mathfrak{T}^{\xi}$ is finer than $\mathfrak{T}$.

Proof. Let $\Gamma \in \mathfrak{T}$. Then, Γ^c is $\mathfrak{T}$-closed in R, and by part (1) of Theorem 10, $\xi(\Gamma^c) \subseteq \Gamma^c$. Thus $cl^{\xi}(\Gamma^c) = \Gamma^c \cup \xi(\Gamma^c) = \Gamma^c$, and this implies $\Gamma \in \mathfrak{T}^{\xi}$. Therefore, $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$.

□

The following theorem defines an open base for the diving topology $\mathfrak{T}^{\xi}$.

Theorem 26. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, $\mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}} = \{\Gamma \setminus \Xi : \Gamma, \Omega \in \mathfrak{T}, \Xi \cap \Omega \cap \mho = \emptyset\ for\ all\ \mho \in \mathfrak{D}^{\star}\}$ is an open base for the diving topology $\mathfrak{T}^{\xi}$.

Proof. Let $\Theta \in \mathfrak{T}^{\xi}$ and $\mathfrak{d} \in \Theta$. Then, $R \setminus \Theta$ is $\mathfrak{T}^{\xi}$-closed, hence, $cl^{\xi}(R \setminus \Theta) = R \setminus \Theta$, and $(R \setminus \Theta) \cup \xi(R \setminus \Theta) = (R \setminus \Theta)$, which implies $\xi(R \setminus \Theta) \subseteq (R \setminus \Theta)$. Then, $\mathfrak{d} \notin \xi(R \setminus \Theta)$, implying the existence of $\Omega \in \mathfrak{T}(\mathfrak{d})$ where $(R \setminus \Theta) \cap \Omega \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^{\star}$. Put $\Xi = (R \setminus \Theta) \cap \Omega$, then $\mathfrak{d} \notin \Xi$ and $\Xi \cap \Omega \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^{\star}$. Thus, $\mathfrak{d} \in (\Omega \setminus \Xi) = \Omega \setminus [(R \setminus \Theta) \cap \Omega] = \Omega \setminus (R \setminus \Theta) \subseteq \Theta$, where $(\Omega \setminus \Xi) \in \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$. It now suffices to observe that $\mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$ is closed under finite intersections. Let $(\Gamma_1 \setminus \Xi_1), (\Gamma_2 \setminus \Xi_2) \in \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$, which means $\Gamma_1, \Gamma_2 \in \mathfrak{T}$, $\Xi_1 \cap \Omega_1 \cap \mho = \emptyset$ and $\Xi_2 \cap \Omega_2 \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^{\star}$. Hence, $(\Gamma_1 \cap \Gamma_2) \in \mathfrak{T}$ and $(\Xi_1 \cup \Xi_2) \cap \Omega \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^{\star}$ and $\Omega = \Omega_1 \cap \Omega_2$. Thus, $(\Gamma_1 \setminus \Xi_1) \cap (\Gamma_2 \setminus \Xi_2) = (\Gamma_1 \cap \Gamma_2) \setminus (\Xi_1 \cup \Xi_2) \in \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$. Therefore, $\mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$ is an open base for the diving topology $\mathfrak{T}^{\xi}$.

□

Corollary 27. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, $\mathfrak{T} \subseteq \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}} \subseteq \mathfrak{T}^{\xi}$.

Theorem 28. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$, $\Gamma \in 2^R$ and $\Gamma \subseteq \xi(\Gamma)$. Then, $cl_{\mathfrak{T}^{\xi}}(\Gamma) = cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) = cl_{\mathfrak{T}}(\Gamma)$.

Proof. By (5) of Theorem 10, $cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) \subseteq cl_{\mathfrak{T}}(\Gamma)$. Now, since $\Gamma \subseteq \xi(\Gamma)$, then $cl_{\mathfrak{T}}(\Gamma) \subseteq cl_{\mathfrak{T}}(\xi(\Gamma))$. Therefore, $cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) = cl_{\mathfrak{T}}(\Gamma)$. Now, since $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$, then $cl_{\mathfrak{T}^{\xi}}(\Gamma) \subseteq cl_{\mathfrak{T}}(\Gamma)$. Let $\mathfrak{d} \notin cl_{\mathfrak{T}^{\xi}}(\Gamma)$. Then, there exists $\mathbb{K} = (\Xi \setminus \Theta) \in \mathcal{H}^{\xi}_{\mathfrak{D}}$ such that $\mathfrak{d} \in \mathbb{K}$ and $\mathbb{K} \cap \Gamma = (\Xi \setminus \Theta) \cap \Gamma = \emptyset$. Then $\xi((\Xi \setminus \Theta) \cap \Gamma) = \emptyset$, hence $\xi((\Xi \cap \Gamma) \setminus \Theta) = \emptyset$. By Corollary 20, $\xi(\Xi \cap \Gamma) = \emptyset$. Since $\Xi \in \mathfrak{T}$, then by Theorem 15, $\Xi \cap \xi(\Gamma) = \Xi \cap \xi(\Xi \cap \Gamma) \subseteq \xi(\Xi \cap \Gamma) = \emptyset$, which implies $\Xi \cap \xi(\Gamma) = \emptyset$. Since $\Gamma \subseteq \xi(\Gamma)$, then $\Xi \cap \Gamma = \emptyset$, hence, $\mathfrak{d} \notin cl_{\mathfrak{T}}(\Gamma)$. Thus, $cl_{\mathfrak{T}}(\Gamma) \subseteq cl_{\mathfrak{T}^{\xi}}(\Gamma)$. Thus, $cl_{\mathfrak{T}}(\Gamma) = cl_{\mathfrak{T}^{\xi}}(\Gamma)$. Therefore, $cl_{\mathfrak{T}^{\xi}}(\Gamma) = cl_{\mathfrak{T}}(\xi(\Gamma)) = \xi(\Gamma) = cl_{\mathfrak{T}}(\Gamma)$.

□

Theorem 29. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the following are equivalent:

(1) For any $\mathfrak{T}$-open subset Ξ of R, $\Xi \subseteq \xi(\Xi)$;
(2) for any semi-open set Ξ of R, $\Xi \subseteq \xi(\Xi)$.

Proof. (2) $\Rightarrow$ (1) : It is clear. For (1) $\Rightarrow$ (2) : Let Ξ be a semi-open set in R. There exists a $\mathfrak{T}$-open set Γ such that $\Gamma \subseteq \Xi \subseteq cl_{\mathfrak{T}}(\Gamma)$. Since Γ is $\mathfrak{T}$-open, then by part (1), $\Gamma \subseteq \xi(\Gamma)$. Since $\Gamma \subseteq \xi(\Gamma)$, then by Theorem 28, $cl_{\mathfrak{T}}(\Gamma) = \xi(\Gamma)$. Furthermore, by part (3) of Theorem 10, we obtain $\Xi \subseteq cl_{\mathfrak{T}}(\Gamma) = \xi(\Gamma) \subseteq \xi(\Xi)$. □

Theorem 30. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the below properties hold:

(1) If $\mathfrak{D} = \{R\}$, then Γ is $\mathfrak{T}^{\xi}$-closed for any $\Gamma \in 2^R$;
(2) if $\Gamma \in 2^R$ and $\Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^{\star}$, then Γ is $\mathfrak{T}^{\xi}$-closed;
(3) if $\Gamma \in 2^R$ and $\Gamma^c \in \mathfrak{T}$, then Γ is $\mathfrak{T}^{\xi}$-closed;
(4) if $\Gamma \in 2^R$, then $\xi(\Gamma)$ is $\mathfrak{T}^{\xi}$-closed.

Proof. (1) Let $\Gamma \in 2^R$. Since $\mathfrak{D} = R$, then $\xi(\Gamma) = \emptyset$. Hence, $cl^{\xi}(\Gamma) = \Gamma \cup \xi(\Gamma) = \Gamma$, and this means $\Gamma^c \in \mathfrak{T}^{\xi}$. Therefore, Γ is $\mathfrak{T}^{\xi}$-closed.
(2) Let $\Gamma \in 2^R$. Since $\Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^{\star}$, then $\xi(\Gamma) = \emptyset$. Hence, $cl^{\xi}(\Gamma) = \Gamma \cup \xi(\Gamma) = \Gamma$, and this means $\Gamma^c \in \mathfrak{T}^{\xi}$. Therefore, Γ is $\mathfrak{T}^{\xi}$-closed.
(3) Let $\Gamma^c \in \mathfrak{T}$. Then, by (2) of Theorem 10, $\xi(\Gamma) \subseteq \Gamma$. Hence, $cl^{\xi}(\Gamma) = \Gamma \cup \xi(\Gamma) = \Gamma$, and this means $\Gamma^c \in \mathfrak{T}^{\xi}$. Therefore, Γ is $\mathfrak{T}^{\xi}$-closed.
(4) Let $\Gamma \in 2^R$. Then, by part (6) of Theorem 10, $\xi(\xi(\Gamma)) \subseteq \xi(\Gamma)$. Hence, $cl^{\xi}(\xi(\Gamma)) = \xi(\Gamma) \cup \xi(\xi(\Gamma)) = \xi(\Gamma)$, and this means $(\xi(\Gamma))^c \in \mathfrak{T}^{\xi}$. Therefore, $\xi(\Gamma)$ is $\mathfrak{T}^{\xi}$-closed.

□

Theorem 31. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ and $\Gamma \in 2^R$. Then, for any $\Gamma \in \mathfrak{T}^{\xi}$ if, and only if, for all $\mathfrak{d} \in \Gamma$, there exist $\Omega \in \mathfrak{T}(\mathfrak{d})$ and $\Omega \subseteq \Gamma$;

Proof.

$$\begin{aligned}
\Gamma \in \mathfrak{T}^{\xi} &\Leftrightarrow cl^{\xi}(\Gamma^c) = \Gamma^c \\
&\Leftrightarrow \Gamma^c \cup \xi(\Gamma^c)) = \Gamma^c \\
&\Leftrightarrow \xi(\Gamma^c) \subseteq \Gamma^c \\
&\Leftrightarrow \Gamma \subseteq (\xi(\Gamma^c))^c \\
&\Leftrightarrow (\forall \mathfrak{d} \in \Gamma)(\mathfrak{d} \notin \xi(\Gamma^c)) \\
&\Leftrightarrow (\forall \mathfrak{d} \in \Gamma)(\exists \Omega \in \mathfrak{T}(\mathfrak{d}))(\forall \mho \in \mathfrak{D}^{\star})(\Gamma^c \cap \Omega \cap \mho = \emptyset) \\
&\Leftrightarrow (\forall \mathfrak{d} \in \Gamma)(\exists \Omega \in \mathfrak{T}(\mathfrak{d}))(\Gamma^c \cap \Omega = \emptyset) \\
&\Leftrightarrow (\forall \mathfrak{d} \in \Gamma)(\exists \Omega \in \mathfrak{T}(\mathfrak{d}))(\Omega \subseteq \Gamma).
\end{aligned}$$

□

Theorem 32. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the below properties hold:

(1) If $\mathfrak{D} = \{R\}$, then $\mathfrak{T}^{\xi}$ is the discrete topology.

(2) If $\mathfrak{D} = 2^R \setminus \{\emptyset\}$, then $\mathfrak{T} = \mathfrak{T}^{\xi}$.

Proof. (1) It is clear that $\mathfrak{T}^{\xi} \subseteq 2^R$. Now, let $\Gamma \in 2^R$. Since $\mathfrak{D} = R$, then $\xi(\Gamma) = \emptyset$ for any $\Gamma \in 2^R$. Now, $cl^{\xi}(\Gamma^c) = \Gamma^c \cup \xi(\Gamma^c) = \Gamma^c$, and this implies $\Gamma \in \mathfrak{T}^{\xi}$. Thus, $2^R = \mathfrak{T}^{\xi}$.
(2) By Theorem 25, $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$. Let $\Gamma \in \mathfrak{T}^{\xi}$. Then, $\Gamma^c \cup \xi(\Gamma^c) = \Gamma^c$, and this implies $\xi(\Gamma^c) \subseteq \Gamma^c$. Now, let $\mathfrak{d} \notin \xi(\Gamma^c)$. Then, there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$, and for any $\mho \in \mathfrak{D}^\star$ we have $\Gamma^c \cap \Omega \cap \mho = \emptyset$. Put $\mho = 2^R \setminus \{\emptyset\}$, and this means $\Gamma^c \cap \Omega = \emptyset$. Hence, $\mathfrak{d} \notin cl_{\mathfrak{T}}(\Gamma^c)$. Thus, $cl_{\mathfrak{T}}(\Gamma^c) \subseteq \xi(\Gamma^c) \subseteq \Gamma^c$. Hence, $cl_{\mathfrak{T}}(\Gamma^c) = \Gamma^c$, and this implies Γ^c is $\mathfrak{T}$-closed in R. Thus, $\Gamma \in \mathfrak{T}$. Therefore, $\mathfrak{T} = \mathfrak{T}^{\xi}$. □

Theorem 33. Suppose that $(R, \mathfrak{T}, \mathfrak{D})$ is a $\mathcal{DTS}$. Then, the subsequent propositions are mutually equivalent:

(1) For any $\Gamma \in 2^R$, $\Gamma \setminus \xi(\Gamma) \notin \mathfrak{D}$.
(2) For any $\mathfrak{T}^{\xi}$-closed subset Γ of R, $\Gamma \setminus \xi(\Gamma) \notin \mathfrak{D}$.
(3) For any subset Γ of R and each $\mathfrak{d} \in \Gamma$, there corresponds some $\Omega \in \mathfrak{T}(\mathfrak{d})$ with $\Omega \cap \Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$, then $\Gamma \notin \mathfrak{D}$.
(4) If $\Gamma \in 2^R$ and $\Gamma \cap \xi(\Gamma) = \emptyset$, then $\Gamma \notin \mathfrak{D}$.

Proof. (1) $\Rightarrow$ (2) : It is obvious. For (2) $\Rightarrow$ (3) : Let $\Gamma \subseteq R$, and presuming that for each $\mathfrak{d} \in \Gamma$, one can find $\Omega \in \mathfrak{T}(\mathfrak{d})$ where $\Omega \cap \Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$. Then, $\mathfrak{d} \notin \xi(\Gamma)$, hence, $\Gamma \cap \xi(\Gamma) = \emptyset$. Since $\Gamma \cup \xi(\Gamma)$ is $\mathfrak{T}^{\xi}$-closed, then by (2), $(\Gamma \cup \xi(\Gamma)) \setminus \xi(\Gamma \cup \xi(\Gamma)) \notin \mathfrak{D}$. Hence by part (7) of Theorem 10, $(\Gamma \cup \xi(\Gamma)) \setminus (\xi(\Gamma) \cup \xi(\xi(\Gamma))) \notin \mathfrak{D}$. Since $\Gamma \cap \xi(\Gamma) = \emptyset$ and by part (6) of Theorem 10, $(\Gamma \cup \xi(\Gamma)) \setminus (\xi(\Gamma) \cup \xi(\xi(\Gamma))) = (\Gamma \cup \xi(\Gamma)) \setminus \xi(\Gamma) = \Gamma \notin \mathfrak{D}$.
(3) $\Rightarrow$ (4) : Let $\Gamma \in 2^R$ and $\Gamma \cap \xi(\Gamma) = \emptyset$. Then, $\Gamma \subseteq (R \setminus \xi(\Gamma))$. Let $\mathfrak{d} \in \Gamma$. Then, $\mathfrak{d} \notin \xi(\Gamma)$, which implies there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$ with $\Omega \cap \Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$. Then, by part (3), $\Gamma \notin \mathfrak{D}$;
(4) $\Rightarrow$ (1) : Let $\Gamma \subseteq R$. We claim: $(\Gamma \setminus \xi(\Gamma)) \cap \xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Suppose that $\mathfrak{d} \in (\Gamma \setminus \xi(\Gamma)) \cap \xi(\Gamma \setminus \xi(\Gamma))$. Then, $\mathfrak{d} \in (\Gamma \setminus \xi(\Gamma))$, which implies $\mathfrak{d} \notin \xi(\Gamma)$. Then, one can find $\Omega \in \mathfrak{T}(\mathfrak{d})$ where $\Omega \cap \Gamma \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$. Since $(\Gamma \setminus \xi(\Gamma)) \subseteq \Gamma$, then $\Omega \cap (\Gamma \setminus \xi(\Gamma)) \cap \mho = \emptyset$, which means $\mathfrak{d} \notin \xi(\Gamma \setminus \xi(\Gamma))$, and this contradicts our assumption. Hence, by (4), $\Gamma \setminus \xi(\Gamma) \notin \mathfrak{D}$. □

We frequently utilize the condition $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \in 2^R$ in the forthcoming results, as it ensures that the induced topology $\mathfrak{T}^{\xi}$ aligns more effectively with fundamental topological properties.

Theorem 34. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Let $\Gamma \subseteq R$ and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Then, the following are equivalent:

(1) For any $\Gamma \subseteq R$, $\Gamma \cap \xi(\Gamma) = \emptyset$, then $\xi(\Gamma) = \emptyset$.
(2) For any $\Gamma \subseteq R$, $\xi(\Gamma \cap \xi(\Gamma)) = \xi(\Gamma)$.

Proof. For (1) $\Rightarrow$ (2) : Let $\Gamma \subseteq R$. Then, $\Gamma = (\Gamma \setminus (\Gamma \cap \xi(\Gamma))) \cup (\Gamma \cap \xi(\Gamma))$. By part (7) of Theorem 10, $\xi(\Gamma) = \xi(\Gamma \setminus (\Gamma \cap \xi(\Gamma))) \cup \xi(\Gamma \cap \xi(\Gamma)) = \xi(\Gamma \setminus \xi(\Gamma)) \cup \xi(\Gamma \cap \xi(\Gamma))$. Since $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$, then $\xi(\Gamma) = \xi(\Gamma \setminus \xi(\Gamma)) \cup \xi(\Gamma \cap \xi(\Gamma)) = \xi(\Gamma \cap \xi(\Gamma))$. For (2) $\Rightarrow$ (1) : Let $\Gamma \subseteq R$ and $\Gamma \cap \xi(\Gamma) = \emptyset$. Then, by (2), $\xi(\Gamma) = \xi(\Gamma \cap \xi(\Gamma)) = \xi(\emptyset) = \emptyset$.

□

Corollary 35. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Let $\Gamma \subseteq R$ and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Then, ξ is an idempotent operator, which implies that $\xi(\xi(\Gamma)) = \xi(\Gamma)$ for any $\Gamma \subseteq R$.

Proof. By part (6) of Theorem 10, $\xi(\xi\Gamma)) \subseteq \xi(\Gamma)$. On the other side, since $(\Gamma \cap \xi(\Gamma)) \subseteq \xi(\Gamma)$, then by part (3) of Theorem 10, $\xi(\Gamma \cap \xi(\Gamma)) \subseteq \xi(\xi(\Gamma))$. Also, by part (2) of Theorem 34, $\xi(\Gamma) = \xi(\Gamma \cap \xi(\Gamma)) \subseteq \xi(\xi(\Gamma))$. Therefore, $\xi(\xi(\Gamma)) = \xi(\Gamma)$ for any $\Gamma \subseteq R$. □

Theorem 36. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Let $\Gamma \subseteq R$ and $(\Gamma \setminus \xi(\Gamma)) \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^\star$. Then, Γ is $\mathfrak{T}^{\xi}$-closed, if and only, if Γ can be written as a union of a set Ξ which is $\mathfrak{T}$-closed in $(R, \mathfrak{T})$ and a set Θ such that $\xi(\Theta) = \emptyset$, where $\Xi, \Theta \subseteq R$.

Proof. Let Γ be a $\mathfrak{T}^{\xi}$-closed. Then, $\Gamma \cup \xi(\Gamma) = \Gamma$, which implies $\xi(\Gamma) \subseteq \Gamma$. Now, $\Gamma = \xi(\Gamma) \cup (\Gamma \setminus \xi(\Gamma))$. By part (5) of Theorem 10, $\xi(\Gamma)$ is $\mathfrak{T}$-closed. and by given, $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Conversely, let $\Gamma = \Xi \cup \Theta$, where Ξ is $\mathfrak{T}$-closed and $\xi(\Theta) = \emptyset$. Then $\xi(\Gamma) = \xi(\Xi \cup \Theta) = \xi(\Xi) \cup \xi(\Theta) = \xi(\Xi)$. By part (5) of Theorem 10, $\xi(\Gamma) = \xi(\Xi) \subseteq cl_{\mathfrak{T}}(\Xi)$. Since Ξ is $\mathfrak{T}$-closed, then $\xi(\Gamma) = \xi(\Xi) \subseteq cl_{\mathfrak{T}}(\Xi) = \Xi \subseteq \Gamma$. Thus, $\Gamma \cup \xi(\Gamma) = \Gamma$, which implies Γ is $\mathfrak{T}^{\xi}$-closed. □

Corollary 37. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Let $\Gamma \subseteq R$ and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Then, $\mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$ is a topology in R and $\mathfrak{T}^{\xi} = \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$.

Proof. By Corollary 27, $\mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}} \subseteq \mathfrak{T}^{\xi}$. Let $\Gamma \in \mathfrak{T}^{\xi}$. Then, by Theorem 36, $(R \setminus \Gamma) = \Theta \cup \Xi$, where Θ is $\mathfrak{T}$-closed and $\xi(\Xi) = \emptyset$. Then, $\Gamma = R \setminus (\Theta \cup \Xi) = (R \setminus \Theta) \cap (R \setminus \Xi) = (R \setminus \Theta) \setminus \Xi = \mathbb{K} \setminus \Xi$, where $\mathbb{K} = (R \setminus \Theta) \in \mathfrak{T}$, and since $\xi(\Xi) = \emptyset$, then for any $\mathfrak{d} \in R$ there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$ such that $\Xi \cap \Omega \cap \mho = \emptyset$ for all $\mho \in \mathfrak{D}^\star$. Thus, $\Gamma \in \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$. Therefore, $\mathfrak{T}^{\xi} = \mathcal{H}^{\mathfrak{T}}_{\mathfrak{D}}$. □

Theorem 38. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Let $\mathbb{K}$ be a $\mathfrak{T}^{\xi}$-open set such that $\mathbb{K} = \Xi \setminus \Theta$, where $\Xi \in \mathfrak{T}$ and $\xi(\Theta) = \emptyset$. Then, $cl^{\xi}_{\mathfrak{T}}(\mathbb{K}) = cl_{\mathfrak{T}}(\mathbb{K}) = \xi(\mathbb{K}) = \xi(\Xi) = cl_{\mathfrak{T}}(\Xi) = cl^{\xi}_{\mathfrak{T}}(\Xi)$.

Proof. Let $\mathbb{K} = \Xi \setminus \Theta$, where $\Xi \in \mathfrak{T}$ and $\xi(\Theta) = \emptyset$. Since $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, then by Theorem 17, $\Xi \subseteq \xi(\Xi)$. Also by Theorem 28,

$$\xi(\Xi) = cl_{\mathfrak{T}}(\Xi) = cl_{\mathfrak{T}}(\xi(\Xi)) = cl_{\mathfrak{T}^{\xi}}(\Xi). \tag{4.1}$$

Since $\mathbb{K}$ is $\mathfrak{T}^{\xi}$-open, we claim that $\mathbb{K} \subseteq \xi(\mathbb{K})$, hence, $cl^{\xi}(R \setminus \mathbb{K}) = R \setminus \mathbb{K}$, which implies $\xi(R \setminus \mathbb{K}) \subseteq (R \setminus \mathbb{K})$. By Theorem 18, $\xi(R) \setminus \xi(\mathbb{K}) \subseteq R \setminus \mathbb{K}$. Since $|\mathfrak{D}| > 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$ and $R \in \mathfrak{T}$, then by Theorem 17, $R \subseteq \xi(R)$, which implies $R = \xi(R)$, hence, $\xi(R) \setminus \xi(\mathbb{K}) = R \setminus \xi(\mathbb{K}) \subseteq R \setminus \mathbb{K}$. Hence, $\mathbb{K} \subseteq \xi(\mathbb{K})$. Again by Theorem 17, we get

$$\xi(\mathbb{K}) = cl_{\mathfrak{T}}(\mathbb{K}) = cl_{\mathfrak{T}}(\xi(\mathbb{K})) = cl_{\mathfrak{T}^{\xi}}(\mathbb{K}). \tag{4.2}$$

Also, since $\mathbb{K} \subseteq \Xi$, then by part (3) of Theorem 10, $\xi(\mathbb{K}) \subseteq \xi(\Xi)$. Now, $\xi(\mathbb{K}) = \xi(\Xi \setminus \Theta)$. By Theorem 18, $\xi(\Xi) \setminus \xi(\Theta) = \xi(\Xi \setminus \Theta) \setminus \xi(\Theta)$, then $\xi(\Xi) \setminus \xi(\Theta) \subseteq \xi(\Xi \setminus \Theta) = \xi(\mathbb{K})$. Since $\xi(\Theta) = \emptyset$, then $\xi(\Xi) \subseteq \xi(\mathbb{K})$. Thus,

$$\xi(\mathbb{K}) = \xi(\Xi). \tag{4.3}$$

From 4.1–4.3, we reach the requirement $cl^{\xi}_{\mathfrak{T}}(\mathbb{K}) = cl_{\mathfrak{T}}(\mathbb{K}) = \xi(\mathbb{K}) = \xi(\Xi) = cl_{\mathfrak{T}}(\Xi) = cl^{\xi}_{\mathfrak{T}}(\Xi)$.

□

Theorem 39. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Let $\Gamma \subseteq R$ and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$. Then, for every $\mathfrak{T}$-clopen set Θ and any subset Ξ of R, $\xi(\Theta \cap \Xi) = \xi(\Theta \cap \xi(\Xi)) = cl_{\mathfrak{T}}(\Theta \cap \xi(\Xi))$.

Proof. Let $\Theta \in \mathfrak{T}$. Then, by Theorem 15, $\Theta \cap \xi(\Xi) = \Theta \cap \xi(\Theta \cap \xi(\Xi)) \subseteq \xi(\Theta \cap \xi(\Xi))$, hence, by part (3) of Theorem 10 and Corollary 35,

$$\xi(\Theta \cap \xi(\Xi)) \subseteq \xi(\xi(\Theta \cap \Xi)) = \xi(\Theta \cap \Xi). \tag{4.4}$$

Now, since $\Theta \in \mathfrak{T}$, then by Theorem 15, $\Theta \cap \xi(\Theta \cap (\Xi \setminus \xi(\Xi))) = \Theta \cap \xi(\Xi \setminus \xi(\Xi))$. Since $\xi(\Xi \setminus \xi(\Xi)) = \emptyset$, then $\Theta \cap \xi(\Theta \cap (\Xi \setminus \xi(\Xi))) = \Theta \cap \xi(\Xi \setminus \xi(\Xi)) = \Theta \cap \emptyset = \emptyset$. Then, by Theorem 17, $\xi(\Theta \cap \Xi) \setminus (\xi(\Theta \cap \xi(\Xi))) = \xi[(\Theta \cap \Xi) \setminus (\Theta \cap \xi(\Xi))] \setminus \xi(\Theta \cap \xi(\Xi)) \subseteq \xi[(\Theta \cap \Xi) \setminus (\Theta \cap \xi(\Xi))] = \xi[\Theta \cap (\Xi \setminus \xi(\Xi))]$. By parts (8), (2) of Theorem 10 and $\mathfrak{T}$-closedness, $\xi[\Theta \cap (\Xi \setminus \xi(\Xi))] \subseteq \xi(\Theta) \cap \xi(\Xi \setminus \xi(\Xi)) = \xi(\Theta) \subseteq \Theta$. Hence, $\xi(\Theta \cap \Xi) \setminus \xi(\Theta \cap \xi(\Xi)) \subseteq \Theta \cap \xi[\Theta \cap (\Xi \setminus \xi(\Xi))] = \xi[\Theta \cap (\Xi \setminus \xi(\Xi))] = \emptyset$. Thus,

$$\xi(\Theta \cap \Xi) \subseteq \xi(\Theta \cap \xi(\Xi)). \tag{4.5}$$

From 4.4 and 4.5, we obtain the result, $\xi(\Theta \cap \Xi) = \xi(\Theta \cap \xi(\Xi))$.
Now, by part (5) of Theorem 10, $\xi(\Theta \cap \Xi) = \xi(\Theta \cap \xi(\Xi)) \subseteq cl_{\mathfrak{T}}(\Theta \cap \xi(\Xi))$. Also, by Theorem 15 and part (5) of Theorem 10, $\Theta \cap \xi(\Xi) = \Theta \cap \xi(\Theta \cap \Xi) \subseteq \xi(\Theta \cap \Xi)$, hence, $cl_{\mathfrak{T}}(\Theta \cap \xi(\Xi)) \subseteq cl_{\mathfrak{T}}(\xi(\Theta \cap \Xi)) = \xi(\Theta \cap \Xi)$. Hence, $\xi(\Theta \cap \Xi) = cl_{\mathfrak{T}}(\Theta \cap \xi(\Xi))$. Therefore, $\xi(\Theta \cap \Xi) = \xi(\Theta \cap \xi(\Xi)) = cl_{\mathfrak{T}}(\Theta \cap \xi(\Xi))$. □

5. Further results

Within this section, we delineate a novel operator referred to as the "F-operator" and examine its fundamental properties. Building upon this F-operator, a new topology named diving $\mathfrak{T}^F$-topology is introduced. The relationships among the $\mathfrak{T}^F$-diving topology, $\mathfrak{T}^{\xi}$-diving topology, and classical $\mathfrak{T}$-topology are then analyzed. Finally, several applications are presented to demonstrate the significance of the associated topological properties.

Definition 40. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. We define a function $F : 2^R \to 2^R$ by $F(\Xi) = [\xi(\Xi^c)]^c$ for any subset $\Xi \in 2^R$. For clarity, $F(\Xi)(R, \mathfrak{T}, \mathfrak{D})$is represented as $F^{\mathfrak{T}}_{\mathfrak{D}}(\Xi)$ or simply $F(\Xi)$ for conciseness, and is termed the diving F-operator of Ξ relative to $\mathfrak{T}$ and $\mathfrak{D}$.

It follows from part (5) of Theorem 10, for any $\Xi \in 2^R$, $F(\Xi)$ is $\mathfrak{T}$-open in $(R, \mathfrak{T})$.

Theorem 41. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Then, the following statements hold for any two subsets Ξ and Θ of R.

(1) if $\Xi \subseteq \Theta$, then $F(\Xi) \subseteq F(\Theta)$;
(2) if $\Xi \in \mathfrak{T}^{\xi}$, then $\Xi \subseteq F(\Xi)$;
(3) $F(\Xi \cap \Theta) = F(\Xi) \cap F(\Theta)$;
(4) if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two diving structures on R and $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $F_{\mathfrak{D}_2}(\Theta) \subseteq F_{\mathfrak{D}_1}(\Theta)$;
(5) if $\xi(\Xi) = \emptyset$, then $F(\Xi) = (\xi(R))^c$;
(6) if $\xi(\Theta) = \emptyset$, then $F(\Xi) = F(\Xi \setminus \Theta) = F(\Xi \cup \Theta)$;
(7) $F(\Xi) \cup F(\Theta) \subseteq F(\Xi \cup \Theta)$;
(8) if $\Xi, \Theta \subseteq R$ with $\xi(\Xi \vartriangle \Theta) = \emptyset$, then $\xi(\Xi) = \xi(\Theta)$ (where $\Xi \vartriangle \Theta$ denotes, as usual, the symmetric difference of Ξ and Θ).

Proof. (1) Let $\eth \in F(\Xi)$. Then, $\eth \in (\xi(\Xi^c))^c$, this implies $\eth \notin \xi(\Xi^c)$. Hence, there exists $\Omega \in \mathfrak{T}(\eth)$ such that $\Xi \cap \Omega \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$. Since $\Xi \subseteq \Theta$, then $\Theta^c \subseteq \Xi^c$, and by part (3) of Theorem 10, $\xi(\Theta^c) \subseteq \xi(\Xi^c)$. Now, we have $\eth \notin \xi(\Theta^c)$, which implies $\eth \in (\xi(\Theta^c))^c$. Therefore, $F(\Xi) \subseteq F(\Theta)$.

(2) Let $\Xi \in \mathfrak{T}^{\xi}$. Then, $\Xi^c \cup \xi(\Xi^c) = \Xi^c$, which means $\xi(\Xi^c) \subseteq \Xi^c$, hence, $\Xi \subseteq (\xi(\Xi^c))^c = F(\Xi)$.

(3) We have $F(\Xi \cap \Theta) = [\xi((\Xi \cap \Theta)^c)]^c = [\xi((\Xi^c \cup \Theta^c))]^c$, and by part (7) of Theorem 10, $\xi(\Xi^c \cup \Theta^c) = \xi(\Xi^c) \cup \xi(\Theta^c)$, hence, $F(\Xi \cap \Theta) = [\xi((\Xi^c) \cup \xi(\Theta^c))]^c = (\xi((\Xi^c))^c \cap (\xi((\Theta^c))^c = F(\Xi) \cap F(\Theta)$.

(4) Let $\eth \in F_{\mathfrak{D}_2}(\Theta)$. Then, $\eth \in (\xi_{\mathfrak{D}_2}(\Theta^c))^c$, which implies $\eth \notin \xi_{\mathfrak{D}_2}(\Theta^c)$. Hence, there exists $\Omega \in \mathfrak{T}(\eth)$ such that $\Theta^c \cap \Omega \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}_2^\star$. Since $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, then $\Theta^c \cap \Omega \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}_1^\star$, which implies $\eth \in (\xi_{\mathfrak{D}_1}(\Theta^c))^c$. Hence, $\eth \in F_{\mathfrak{D}_1}(\Theta)$. Therefore, $F_{\mathfrak{D}_2}(\Theta) \subseteq F_{\mathfrak{D}_1}(\Theta)$.

(5) Let $\xi(\Xi) = \emptyset$. Then, $F(\Xi) = (\xi(\Xi^c))^c = R \backslash \xi(R \backslash \Xi)$, and by Theorem 18, $\xi(R) \backslash \xi(\Xi) = \xi(R \backslash \Xi) \backslash \xi(\Xi)$. Since $\xi(\Xi) = \emptyset$, then $\xi(R) \setminus \xi(\Xi) = \xi(R \setminus \Xi)$. Thus, $F(\Xi) = R \setminus [\xi(R) \setminus \xi(\Xi)] = R \setminus [\xi(R)] = [\xi(R)]^c$.

(6) It is clear from Corollary 20.

(7) It is clear from (1).

(8) Let $\xi(\Xi \triangle \Theta) = \emptyset$ so that $\xi(\Xi \setminus \Theta) = \emptyset$ and $\xi(\Theta \setminus \Xi) = \emptyset$. Then, by Corollary 20, $F(\Xi) = F[(\Theta \setminus (\Theta \setminus \Xi)) \cup (\Xi \setminus \Theta)] = F(\Theta \setminus (\Theta \setminus \Xi)) = F(\Theta)$.

□

The inclusion given in (1) and (7) of Theorem 41 needs not to be reversible. It is shown in the following example:

Example 42. Let $\mathfrak{T} = \{\phi, \{\epsilon\}, \{\epsilon, \theta\}, R\}$ be a topology on $R = \{\epsilon, \theta, \kappa\}$ and $\mathfrak{D} = \{R, \{\epsilon\}, \{\epsilon, \theta\}\}$ be a diving structure on R. Then,

Table 1. Diving F-operator of Ξ relative to $\mathfrak{T}$ and $\mathfrak{D}$.

Ξ	$\xi(\Xi)$	$\xi(\Xi^c)$	$F(\Xi)$
$\{\epsilon\}$	R	$\{\theta, \kappa\}$	$\{\epsilon\}$
$\{\theta\}$	$\{\theta, \kappa\}$	R	$\emptyset$
$\{\kappa\}$	$\emptyset$	R	$\emptyset$
$\{\epsilon, \theta\}$	R	$\emptyset$	R
$\{\epsilon, \kappa\}$	R	$\{\theta, \kappa\}$	$\{\epsilon\}$
$\{\theta, \kappa\}$	$\{\theta, \kappa\}$	R	$\emptyset$
R	R	$\emptyset$	R
$\emptyset$	$\emptyset$	R	$\emptyset$

From the Table 1, we have $F(\{\epsilon\}) = \{\epsilon\}$ and $F(\{\epsilon, \theta\}) = R$, hence, $F(\{\epsilon, \theta\}) = R \not\subseteq \{\epsilon\} = F(\{\epsilon\})$, where $\{\epsilon\} \subseteq \{\epsilon, \theta\}$. Also, $F(\{\epsilon, \kappa\}) = \{\epsilon\}$, $F(\{\theta\}) = \emptyset$, and $F(R) = R$, hence, $F(\{\epsilon, \kappa\} \cup \{\theta\}) = R \not\subseteq \{\epsilon\} = F(\{\epsilon, \kappa\}) \cup F(\{\theta\})$.

Theorem 43. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Then, the following statements hold:

(1) $F(\emptyset) = \emptyset$.

(2) For any subset Ξ of R, $F(\Xi) \setminus \Xi = \emptyset$.

(3) For any subset Ξ of R, $int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) = F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)))$.

(4) If Ξ is regular open in R, then $\Xi = F(\Xi)$.
(5) if $\Xi \subseteq \xi(\Xi)$ and $\Xi \in \mathfrak{T}$, then $F(\Xi) \subseteq int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) \subseteq \xi(\Xi)$.

Proof. (1) We have $F(\emptyset) = (\xi(\emptyset^c))^c = (\xi(R))^c$. Since $R \in \mathfrak{T}$, $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, then by Theorem 17, $R \subseteq \xi(R)$. Hence, $R = \xi(R)$, which implies $F(\emptyset) = (\xi(R))^c = R^c = \emptyset$.

(2) Let $\mathfrak{d} \in F(\Xi) \setminus \Xi$. Then, $\mathfrak{d} \in F(\Xi)$ and $\mathfrak{d} \notin \Xi$, hence, $\mathfrak{d} \notin \xi(\Xi^c)$, which implies there exists $\Omega \in \mathfrak{T}(\mathfrak{d})$ such that $\Omega \cap \Xi^c \cap \mho = \emptyset$ for any $\mho \in \mathfrak{D}^\star$. Since $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, we take $\mho = \Omega$, which implies $\mathfrak{d} \notin \Xi^c$, hence, $\mathfrak{d} \in \Xi$, a contradiction to $\mathfrak{d} \notin \Xi$. Thus, $F(\Xi) \setminus \Xi = \emptyset$;

(3) Let $\Xi \subseteq R$. Since $int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi))$ is $\mathfrak{T}$-open and $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$, then from part (2) of Theorem 41, $int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) \subseteq F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)))$. By (2), $F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi))) \setminus int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) = \emptyset$, then $F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi))) \subseteq int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi))$. Hence, $int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) = F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)))$;

(4) It is clear from (3);

(5) Let $\Xi \subseteq \xi(\Xi)$. Then, from part (5) of Theorem 10, $\xi(\Xi) = cl_{\mathfrak{T}}(\Xi)$, hence, $\Xi \subseteq int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) \subseteq cl_{\mathfrak{T}}(\Xi) = \xi(\Xi)$. Then, $F(\Xi) \subseteq F(int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi))) = int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) \subseteq \xi(\Xi)$. Therefore, $F(\Xi) \subseteq int_{\mathfrak{T}}(cl_{\mathfrak{T}}(\Xi)) \subseteq \xi(\Xi)$.

□

We now demonstrate that the operator F gives rise to another topology, denoted by $\mathfrak{T}_{F(\mathcal{H})}$, which is constructed in a natural way from any given base $\mathcal{H}$ of the topology $\mathfrak{T}$ along with a diving structure $\mathfrak{D}$ on the space R. Moreover, it is observed that $\mathfrak{T}_{F(\mathcal{H})}$ is coarser than the original topology $\mathfrak{T}$ on the space R, while the topology $\mathfrak{T}^{\xi}$ is finer than $\mathfrak{T}$.

Now, if $\mathcal{H}$ is a base for some topology $\mathfrak{T}$ on R, then the collection $F(\mathcal{H}) = \{F(H) : H \in \mathcal{H}\}$ also forms a base for a topology on R, which is weaker than the original topology. We denote this new topology by $\mathfrak{T}_{F(\mathcal{H})}$ and refer to it as the F-topology generated by $\mathcal{H}$.

We begin by demonstrating that, starting from any base $\mathcal{H}$ of the topology $\mathfrak{T}$ on R (and thus from $\mathfrak{T}$ itself), we consistently arrive at the same topology $\mathfrak{T}_{F(\mathcal{H})}$. In other words, this resulting topology is uniquely determined, regardless of the particular base chosen for $\mathfrak{T}$.

Theorem 44. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$. Suppose that $\mathcal{H}$ be a base for $\mathfrak{T}$. Then, $\mathfrak{T}_{F(\mathfrak{T})} = \mathfrak{T}_{F(\mathcal{H})}$.

Proof. Since $\mathcal{H} \subseteq \mathfrak{T}$, then $\mathfrak{T}_{F(\mathcal{H})} \subseteq \mathfrak{T}_{F(\mathfrak{T})}$. On the other side, let $\mathbb{D} \in \mathfrak{T}$ and $\mathfrak{d} \in F(\mathbb{D})$. Then, $\mathbb{D} = \cup_{\gamma \in \wedge} H_\gamma$ where $H_\gamma \in \mathcal{H}$ for each γ. Now, we have two cases:

Case 1. If $\mathfrak{d} \in \mathbb{D}$, then there exists some $H_\gamma \in \mathcal{H}$ such that $\mathfrak{d} \in H_\gamma \subseteq \mathbb{D}$. Since $H_\gamma \in \mathfrak{T}$ and $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$, then by parts (1,2) of Theorem 41, $\mathfrak{d} \in F(H_\gamma) \subseteq F(\Xi)$.

Case 2. If $\mathfrak{d} \notin \mathbb{D}$, then there exists some $H_\gamma \in \mathcal{H}$ such that $\mathfrak{d} \in H_\gamma \setminus \mathbb{D}$. Since $\mathfrak{d} \in F(\mathbb{D})$ and $F(\mathbb{D}) = (\xi(\mathbb{D}^c))^c$, then $\mathfrak{d} \notin \xi(\mathbb{D}^c)$. Hence, there exists $\mathbb{N} \in \mathfrak{T}(\mathfrak{d})$ such that $\mathbb{D}^c \cap \mathbb{N} \cap \mathbb{S} = \emptyset$ for any $\mathbb{S} \in \mathfrak{D}^\star$. Since $H_\gamma \setminus \mathbb{D} \subseteq \mathbb{D}^c$, then $\xi(H_\gamma \setminus \mathbb{D}) = \emptyset$. Now, $\xi[(\mathbb{D} \cup H_\gamma) \triangle \mathbb{D}] = \xi(H_\gamma \setminus \mathbb{D}) = \emptyset$, so by parts (1,2,8) of Theorem 41, we have $\mathfrak{d} \in H_\gamma \subseteq F(H_\gamma) \subseteq F(\mathbb{D} \cup H_\gamma) = F(\mathbb{D})$, and, hence, $\mathfrak{T}_{F(\mathfrak{T})} \subseteq \mathfrak{T}_{F(\mathcal{H})}$. Therefore, $\mathfrak{T}_{F(\mathfrak{T})} = \mathfrak{T}_{F(\mathcal{H})}$. □

As established in [25], the family of all regular open sets in a topological space $(R, \mathfrak{T})$ forms a basis for a topology $\mathfrak{T}_s$, known as the semi-regularization topology on R, which satisfies the inclusion $\mathfrak{T}_s \subseteq \mathfrak{T}$. By part (4) of Theorem 43, we have $\mathfrak{T}_{F(\mathfrak{T})} \subseteq \mathfrak{T}$.

Theorem 45. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then, $(R, \mathfrak{T})$ is T_2-space if, and only if, $(R, \mathfrak{T}^{\xi})$ is T_2-space.

Proof. According to Theorem 25, $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$, we have $(R, \mathfrak{T}^{\xi})$ is T_2-space, since $(R, \mathfrak{T})$ is T_2-space. Conversely, let $(R, \mathfrak{T}^{\xi})$ be T_2-space and $\mathfrak{d}_1, \mathfrak{d}_2$ be any two distinct points of R. Then there exist two disjoint $\mathfrak{T}^{\xi}$-open sets Ξ and Γ such that $\mathfrak{d}_1 \in \Xi$ and $\mathfrak{d}_2 \in \Gamma$. By Corollary 37, $\Xi = \Xi_1 \setminus \Gamma_1$ and $\Gamma = \Xi_2 \setminus \Gamma_2$ such that $\Xi_1, \Xi_2 \in \mathfrak{T}$ and $\xi(\Gamma_1) = \xi(\Gamma_2) = \emptyset$. Hence, Ξ_1 and Ξ_2 are $\mathfrak{T}$-open sets containing $\mathfrak{d}_1$ and $\mathfrak{d}_2$, respectively. Now, since $\Xi \cap \Gamma = \emptyset$, then $(\Xi_1 \setminus \Gamma_1) \cap (\Xi_2 \setminus \Gamma_2) = \emptyset$. Hence, $(\Xi_1 \cap \Xi_2) \setminus (\Gamma_1 \cup \Gamma_2) = \emptyset$, which means $(\Xi_1 \cap \Xi_2) \subseteq (\Gamma_1 \cup \Gamma_2)$. By part (3) of Theorem 10, $\xi(\Xi_1 \cap \Xi_2) \subseteq \xi(\Gamma_1 \cup \Gamma_2) = \xi(\Gamma_1) \cup \xi(\Gamma_2) = \emptyset$. Since $\Xi_1 \cap \Xi_2 \in \mathfrak{T}$, $|\mathfrak{D}| > 1$, and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, then by Theorem 17, $\Xi_1 \cap \Xi_2 \subseteq \xi(\Xi_1 \cap \Xi_2) = \emptyset$, hence, $\Xi_1 \cap \Xi_2 = \emptyset$. Therefore, $(R, \mathfrak{T})$ is T_2-space. □

Example 46. Let $(\mathbb{R}, \mathfrak{T})$ be a discrete topological space and the diving structure $\mathfrak{D} = \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$. Then, by part (2) of Theorem 32, $\mathfrak{T} = \mathfrak{T}^{\xi}$. Hence, the diving topological space $(\mathbb{R}, \mathfrak{T}^{\xi}, \mathfrak{D})$ is T_2-space.

Theorem 47. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then, $(R, \mathfrak{T})$ is $T_{2\frac{1}{2}}$-space (Urysohn) if, and only if, $(R, \mathfrak{T}^{\xi})$ is $T_{2\frac{1}{2}}$-space (Urysohn).

Proof. Let $(R, \mathfrak{T})$ is $T_{2\frac{1}{2}}$-space. Then, for any $\mathfrak{d}_1, \mathfrak{d}_2$ two distinct points of R, there exist two disjoint $\mathfrak{T}$-open sets, Ξ, Γ containing $\mathfrak{d}_1, \mathfrak{d}_2$, respectively, and $cl_{\mathfrak{T}}(\Xi) \cap cl_{\mathfrak{T}}(\Gamma) = \emptyset$. By Theorem 25, $\Xi, \Gamma \in \mathfrak{T}^{\xi}$ hence, by Corollary 37 and Theorem 38, $cl_{\mathfrak{T}^{\xi}}(\Xi) = cl_{\mathfrak{T}}(\Xi)$ and $cl_{\mathfrak{T}^{\xi}}(\Gamma) = cl_{\mathfrak{T}}(\Gamma)$. Thus, $(R, \mathfrak{T}^{\xi})$ is $T_{2\frac{1}{2}}$-space. Conversely, let $(R, \mathfrak{T}^{\xi})$ be $T_{2\frac{1}{2}}$-space and $\mathfrak{d}_1, \mathfrak{d}_2$ be any two distinct points of R. Then, there exist two disjoint $\mathfrak{T}^{\xi}$-open sets Ξ and Γ such that $\mathfrak{d}_1 \in \Xi$, $\mathfrak{d}_2 \in \Gamma$, and $cl_{\mathfrak{T}^{\xi}}(\Xi) \cap cl_{\mathfrak{T}^{\xi}}(\Gamma) = \emptyset$. By Corollary 37, $\Xi = \Xi_1 \setminus \Gamma_1$ and $\Gamma = \Xi_2 \setminus \Gamma_2$ such that $\Xi_1, \Xi_2 \in \mathfrak{T}$ and $\xi(\Gamma_1) = \xi(\Gamma_2) = \emptyset$. Hence, Ξ_1 and Ξ_2 are two disjoint $\mathfrak{T}$-open sets containing $\mathfrak{d}_1$ and $\mathfrak{d}_1$, respectively. Now, by Theorem 38, $cl_{\mathfrak{T}^{\xi}}(\Xi) = cl_{\mathfrak{T}}(\Xi_1)$ and $cl_{\mathfrak{T}^{\xi}}(\Gamma) = cl_{\mathfrak{T}}(\Xi_2)$. Hence, $cl_{\mathfrak{T}}(\Xi_1) \cap cl_{\mathfrak{T}}(\Xi_2) = \emptyset$. Therefore, $(R, \mathfrak{T})$ is $T_{2\frac{1}{2}}$-space. □

Corollary 48. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then, $(R, \mathfrak{T})$ is T_i-space if, and only if, $(R, \mathfrak{T}^{\xi})$ is T_i-space, where $i \in \{0, 1\}$.

In the following diagram, we show a diving topological space $(R, \mathfrak{T}, \mathfrak{D})$ with $|\mathfrak{D}| > 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$.

$$\begin{array}{ccccc}
\mathfrak{T} - T_2 - space & \longrightarrow & \mathfrak{T} - T_1 - space & \longrightarrow & \mathfrak{T} - T_0 - space \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathfrak{T}^{\xi} - T_2 - space & \longrightarrow & \mathfrak{T}^{\xi} - T_1 - space & \longrightarrow & \mathfrak{T}^{\xi} - T_0 - space
\end{array}$$

Theorem 49. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then $(R, \mathfrak{T})$ is normal if $(R, \mathfrak{T}^{\xi})$ is normal.

Proof. Let Ξ and Γ be two disjoint $\mathfrak{T}$-closed sets. By closedness, $cl_{\mathfrak{T}}(\Xi) = \Xi$ and $cl_{\mathfrak{T}}(\Gamma) = \Gamma$. Also by Theorem 38, $cl_{\mathfrak{T}^{\xi}}(\Xi) = cl_{\mathfrak{T}}(\Xi) = \Xi$ and $cl_{\mathfrak{T}^{\xi}}(\Gamma) = cl_{\mathfrak{T}}(\Gamma) = \Gamma$. By normality of $(R, \mathfrak{T}^{\xi})$, there exist two disjoint $\mathfrak{T}^{\xi}$-open sets $\mho$ and $\mathbb{K}$ containing Ξ and Γ, respectively. By Corollary 37, $\mho = \Gamma_1 \setminus \Gamma_2$ and $\mathbb{K} = \Theta_1 \setminus \Theta_2$ with $\Gamma_1, \Theta_1 \in \mathfrak{T}$ and $\xi(\Gamma_2) = \xi(\Theta_2) = \emptyset$. Now, since $\mho \cap \mathbb{K} = \emptyset$, then $(\Gamma_1 \setminus \Gamma_2) \cap (\Theta_1 \setminus \Theta_2) = \emptyset$. Hence, $(\Gamma_1 \cap \Theta_1) \setminus (\Gamma_2 \cup \Theta_2) = \emptyset$, which means $(\Gamma_1 \cap \Theta_1) \subseteq (\Gamma_2 \cup \Theta_2)$. By part (3) of Theorem 10, $\xi(\Gamma_1 \cap \Theta_1) \subseteq \xi(\Gamma_2 \cup \Theta_2) = \xi(\Gamma_2) \cup \xi(\Theta_2) = \emptyset$. Since $\Gamma_1 \cap \Theta_1 \in \mathfrak{T}$, $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, then by Theorem 17, $\Gamma_1 \cap \Theta_1 \subseteq \xi(\Gamma_1 \cap \Theta_1) = \emptyset$. Thus, Γ_1 and Θ_1 are two disjoint $\mathfrak{T}$-open sets containing Ξ and Γ, respectively. Therefore, $(R, \mathfrak{T})$ is normal. □

Corollary 50. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then, $(R, \mathfrak{T})$ is regular (resp., T_3-space, T_4-space) if $(R, \mathfrak{T}^{\xi})$ is regular (resp., T_3-space, T_4-space).

The converse of Theorem 49 may fail as shown in the following example.

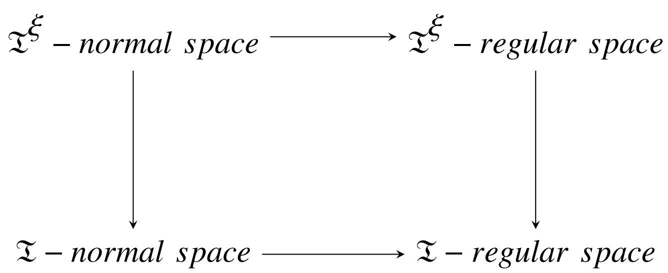
Example 51. Let $\mathfrak{T} = \{\phi, R, \{\epsilon\}\}$ be a topology on $R = \{\epsilon, \theta, \kappa\}$ and $\mathfrak{D} = \{R, \{\epsilon\}\}$. Then, we can see the Table 2:

Table 2. Diving topology on R.

Ξ	$\xi(\Xi)$	$\xi(\Xi \setminus \xi(\Xi)) = \emptyset$	$\mathfrak{T}^{\xi}$	$\mathfrak{T}$
$\{\epsilon\}$	R	Yes	$\in$	$\in$
$\{\theta\}$	$\emptyset$	Yes	$\notin$	$\notin$
$\{\kappa\}$	$\emptyset$	Yes	$\notin$	$\notin$
$\{\epsilon, \theta\}$	R	Yes	$\in$	$\notin$
$\{\epsilon, \kappa\}$	R	Yes	$\in$	$\notin$
$\{\theta, \kappa\}$	$\emptyset$	Yes	$\notin$	$\notin$
R	R	Yes	$\in$	$\in$
$\emptyset$	$\emptyset$	Yes	$\in$	$\in$

We have $|\mathfrak{D}| > 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$ and $\xi(\Xi \setminus \xi(\Xi)) = \emptyset$ for all $\Xi \subseteq R$. However, $(R, \mathfrak{T})$ is normal, while $(R, \mathfrak{T}^{\xi})$ is not normal.

In the following diagram, we show a diving topological space $(R, \mathfrak{T}, \mathfrak{D})$ with $|\mathfrak{D}| > 1$, $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, T_1-space and $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$.



The converses of these implications are not true in general; see Example 51.

Theorem 52. Let $(R, \mathfrak{T}, \mathfrak{D})$ be a $\mathcal{DTS}$ with $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$. Let $\xi(\Gamma \setminus \xi(\Gamma)) = \emptyset$ for all $\Gamma \subseteq R$. Then, $(R, \mathfrak{T}^{\xi})$ is connected if, and only if, $(R, \mathfrak{T})$ is connected.

Proof. According to Theorem 25, $\mathfrak{T} \subseteq \mathfrak{T}^{\xi}$, we have $(R, \mathfrak{T})$ is connected, since $(R, \mathfrak{T}^{\xi})$ is connected. Conversely, let $(R, \mathfrak{T}^{\xi})$ be disconnected. Then there exists a nonempty $\mathfrak{T}^{\xi}$-clopen set $\Xi \subsetneq R$, such that $R = \Xi \cup \Xi^c$. Hence, by part (7) of Theorem 10, we get $\xi(R) = \xi(\Xi \cup \Xi^c) = \xi(\Xi) \cup \xi(\Xi^c)$. Since Ξ and Ξ^c are $\mathfrak{T}^{\xi}$-closed, then by part (2) of Theorem 10, $\xi(\Xi) \cap \xi(\Xi^c) \subseteq \Xi \cap \Xi^c = \emptyset$, which implies $\xi(\Xi) \cap \xi(\Xi^c) = \emptyset$. Now, since Ξ is $\mathfrak{T}^{\xi}$-open, $|\mathfrak{D}| > 1$ and $\mathfrak{T} \setminus \{\emptyset\} \subseteq \mathfrak{D}$, then by Theorem 38, $\xi(\Xi) = cl_{\mathfrak{T}}(\Xi) \neq \emptyset$. Also, $\xi(\Xi^c) = cl_{\mathfrak{T}}(\Xi^c) \neq \emptyset$. Thus, $R = cl_{\mathfrak{T}}(\Xi) \cup cl_{\mathfrak{T}}(\Xi^c)$, $cl_{\mathfrak{T}}(\Xi) \cap cl_{\mathfrak{T}}(\Xi^c) = \emptyset$, $cl_{\mathfrak{T}}(\Xi) \neq \emptyset$, and $cl_{\mathfrak{T}}(\Xi^c) \neq \emptyset$. Therefore, $(R, \mathfrak{T})$ is disconnected. □

Example 53. Let $\mathfrak{T}_1 = \{\phi, R\}$, $\mathfrak{T}_2 = \{\phi, \{\epsilon\}, \{\theta, \kappa\}, R\}$ be two topologies on $R = \{\epsilon, \theta, \kappa\}$ and $\mathfrak{D}_1 = \{R\}$, $\mathfrak{D}_2 = \{R, \{\epsilon\}, \{\theta\}, \{\epsilon, \theta\}, \{\epsilon, \kappa\}, \{\theta, \kappa\}\}$ be two diving structures on R. Then,
(Note: $*$ means for any $\Xi \subseteq R$, $\xi(\Xi \setminus \xi(\Xi)) = \emptyset$.)

Table 3. Diving topologies on R.

Ξ	$\xi^{\mathfrak{T}_1}_{\mathfrak{D}_1}(\Xi)$	$*$	$\mathfrak{T}^{\xi}_{\mathfrak{D}_1,\mathfrak{T}_1}$	$\mathfrak{T}_1$	$\xi^{\mathfrak{T}_2}_{\mathfrak{D}_2}(\Xi)$	$*$	$\mathfrak{T}^{\xi}_{\mathfrak{D}_2,\mathfrak{T}_2}$	$\mathfrak{T}_2$
$\{\epsilon\}$	$\emptyset$	Yes	$\in$	$\notin$	$\{\epsilon\}$	Yes	$\in$	$\in$
$\{\theta\}$	$\emptyset$	Yes	$\in$	$\notin$	$\{\theta, \kappa\}$	Yes	$\notin$	$\notin$
$\{\kappa\}$	$\emptyset$	Yes	$\in$	$\notin$	$\{\theta, \kappa\}$	Yes	$\notin$	$\notin$
$\{\epsilon, \theta\}$	$\emptyset$	Yes	$\in$	$\notin$	R	Yes	$\notin$	$\notin$
$\{\epsilon, \kappa\}$	$\emptyset$	Yes	$\in$	$\notin$	R	Yes	$\notin$	$\notin$
$\{\theta, \kappa\}$	$\emptyset$	Yes	$\in$	$\notin$	$\{\theta, \kappa\}$	Yes	$\in$	$\in$
R	$\emptyset$	Yes	$\in$	$\in$	R	Yes	$\in$	$\in$
$\emptyset$	$\emptyset$	Yes	$\in$	$\in$	$\emptyset$	Yes	$\in$	$\in$

From the Table 3, we see that this condition $|\mathfrak{D}_1| > 1$ of Theorems 45 and 47 is necessary. For example, $\mathfrak{T}^{\xi}_{\mathfrak{D}_1,\mathfrak{T}_1}$ is a discrete topology in R, while $\mathfrak{T}_1$ is an indiscrete topology in R, even if $\mathfrak{T}_1 \setminus \{\emptyset\} \subseteq \mathfrak{D}_1$ and $\xi^{\mathfrak{T}_1}_{\mathfrak{D}_1}(\Xi \setminus \xi^{\mathfrak{T}_1}_{\mathfrak{D}_1}(\Xi)) = \emptyset$ for all $\Xi \subseteq R$. Also, $(R, \mathfrak{T}^{\xi}_{\mathfrak{D}_2,\mathfrak{T}_2})$ is disconnected if, and only if, $(R, \mathfrak{T}_y)$ is disconnected, such that $|\mathfrak{D}_2| > 1$, $\mathfrak{T}_2 \setminus \{\emptyset\} \subseteq \mathfrak{D}_2$ and $\xi^{\mathfrak{T}_2}_{\mathfrak{D}_2}(\Xi \setminus \xi^{\mathfrak{T}_2}_{\mathfrak{D}_2}(\Xi)) = \emptyset$ for all $\Xi \subseteq R$.

Zadeh [26] initially promulgated the seminal construct of a fuzzy set, conceptualized as a sophisticated and nuanced generalization transcending the boundaries of classical set theory. All terminologies not explicitly defined within this manuscript are comprehensively elucidated in references [26–28].

Let R represent a universal domain, I denote the closed unit interval $[0, 1]$, and I^R signify the entire class of fuzzy sets constituted over the set R.

Definition 54. [26] A function $\varsigma : R \to I$ is referred to as a fuzzy set on R, where $\varsigma(\mathfrak{d})$ represents the membership degree of the element $\mathfrak{d} \in R$ in the fuzzy set ς.

Definition 55. [27] A subcollection $\mathfrak{T}$ of I^R is said to be a fuzzy topology on R if

(i) $1_R, 0_R \in \mathfrak{T}$;
(ii) $\varsigma_1 \cap \varsigma_2 \in \mathfrak{T}$ whenever $\varsigma_1, \varsigma_2 \in \mathfrak{T}$;
(iii) $\bigcup \varsigma_\alpha(\mathfrak{d}) \in \mathfrak{T}$ whenever $\{\varsigma_\alpha(\mathfrak{d}) : \alpha \in \Lambda\} \subseteq \mathfrak{T}$.

The pair $(R, \mathfrak{T})$ is designated as a fuzzy topological space.

Definition 56. A subcollection $\mathfrak{D}$ of I^R is designated as a fuzzy diving structure on R provided it conforms to the ensuing axiomatic conditions:

(i) $1_R \in \mathfrak{D}$;
(ii) $0_R \notin \mathfrak{D}$;
(iii) if $\varsigma \in \mathfrak{D}$ and $\varrho \in \mathfrak{D}$, then $\varsigma \cup \varrho \in \mathfrak{D}$ for all $\varsigma, \varrho \in I^R$;
(iv) if $\varsigma \cap \varrho \in \mathfrak{D}$, then $\varsigma \in \mathfrak{D}$ or $\varrho \in \mathfrak{D}$ for all $\varsigma, \varrho \in I^R$.

Example 57. Let $\mathfrak{D} = I^R \setminus \{0_R\}$, then $\mathfrak{D}$ is a fuzzy diving structure on R.

Definition 58. Let $(R, \mathfrak{T})$ be a fuzzy topological space and $\mathfrak{D}$ be a fuzzy diving structure on R. Then, the triplet $(R, \mathfrak{T}, \mathfrak{D})$ is referred to as a diving fuzzy topological space (abbreviated as $\mathcal{DFTS}$)).

Example 59. Consider the co-finite fuzzy topology, defined as the collection $\mathfrak{F}$ of all functions $\varsigma : R \rightarrow [0, 1]$ such that $supp(\overline{\varsigma})$ is finite, along with the zero fuzzy set 0_R, and let the fuzzy diving structure be $\mathfrak{D} = I^R \setminus \{0_R\}$. Then, the triplet $(R, \mathfrak{F}, \mathfrak{D})$ constitutes a diving fuzzy topological space ($\mathcal{DFTS}$).

Remark 60. Recently, in [23], the notion of roughness within the framework of diving spaces was introduced. Specifically, when Z denotes an equivalence relation on a nonempty set R, the concept of rough sets is formulated in the context of diving spaces. Consequently, the structure $(R, Z, \mathfrak{D})$ is referred to as a diving approximation space. Within this framework, several essential properties and theoretical results have been rigorously developed and substantiated through illustrative examples.

6. Conclusions

We have introduced a new topological structure, known as the diving topological space. Based on this framework, we have defined several operators and conducted a detailed investigation of their properties. Two different topologies were then constructed based on these operators, and their interrelationships, as well as their connections to classical topology, were thoroughly studied. Finally, we demonstrated practical applications of the proven topological properties, supported by illustrative examples.

Such work opens the way for researchers to learn more about topological operators and will contribute to providing more fruitful results in various mathematical fields. Moreover, extending these operators to the fuzzy context presents a compelling direction for future research, with significant potential applications in artificial intelligence and decision-making processes.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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