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Research article

Application of the q-derivative operator to a specialized class of harmonic functions exhibiting positive real part

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Abstract: This paper introduces a new subclass of harmonic functions with a positive real part, denoted by $HP_q(\beta)$, where $0 \leq \beta < 1$ and $0 < q < 1$. A sufficient coefficient condition is established for functions within this class, which is also necessary when dealing with negative coefficients. In addition, the growth theorem is derived, and the extreme points associated with this subclass are also identified. Finally, the q-integral operator for harmonic functions of the form $f = h + g$ with a positive real part is presented.

Keywords: q-derivative operator, q-integral operator, q-calculus, quantum calculus, harmonic functions, positive real part, growth theorem, extreme points
Mathematics Subject Classification: 30C45

1. Introduction

The study of harmonic functions is deemed an essential part of complex analysis research, which has extensive applications in the fields of mathematical physics, fluid dynamics, and engineering. Harmonic functions are used to model potential flows, enabling the design of streamlined structures. Similarly, in signal processing, the properties of harmonic functions are employed to analyze waveforms and optimize communication systems, particularly when q-calculus provides discrete approximations or deformations [1–3]. Harmonic functions, characterized as solutions to the Laplace equation, are of great significance due to their smoothness and extremum features within specified domains. The subclass of harmonic functions with a positive real part is particularly significant and has been thoroughly examined within the realms of geometric function theory and univalent functions. These functions are associated with applications in conformal mapping and fluid dynamics, making them an important area of research [4–6]. Recent advances have explored these harmonic functions using various operators, among which the q-derivative operator emerged. The q-calculus, developed by Jackson [7] in the early twentieth century, was initially introduced as a generalization of classical calculus, providing a discrete analogue of differentiation and integration calculus. The q-calculus replaces the traditional derivative with a q-difference operator, providing additional flexibility. The q-derivative operator, developed by Jackson [8], has gained considerable attention for its ability to extend classical analytical results to a more generalized framework. The operator has proven useful in various mathematical fields, including applications such as combinatorics, orthogonal polynomials, and quantum theory [9–11]. The application of q-calculus to subclasses of harmonic functions is relatively new but promising; for example, Khan et al. discussed some important applications of the q-difference operator involving a family of meromorphic harmonic functions [12–14]. Moreover, Elhaddad et al. investigated harmonic univalent functions involving q-analogue operators [15, 16]. More recently, researchers have explored the applications of q-analogue operators to harmonic functions [17–19]. The q-derivative has been successfully employed to investigate coefficient bounds, development properties, and distortion theorems in several subclasses of analytic functions [20–22]. Applying these findings to harmonic mappings, particularly those with a positive real part, offers new insights into their geometric and functional properties. This paper aims to explore a subclass of harmonic functions with a positive real part using the q-derivative operator. By utilizing this operator, we will derive new results concerning the geometric properties of these functions, including distortion bounds, coefficient estimates, and extremal properties. Our study contributes to the broader understanding of harmonic mappings and demonstrates the applicability of q-calculus in extending classical function theory into more generalized domains.

2. Materials and methods

Jakubowski et al. [23] studied the class P of all the functions of the form $f = h+\overline{g}$ that are harmonic in $\mathbb{U}$ and such that for h, g, where

$$h(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \tag{2.1}$$

are analytic in $\mathbb{U}$.

The class $HP(\beta)$ of all functions of the form (2.1) with $Re(f) > \beta$, $0 \leq \beta < 1$ and $f(0) = 1$ was studied in [24] and later generalized by [25]. Obviously, $HP(0) = HP$ and $HP(\beta) \subset HP$.

In [7, 8], Jackson defined the q-derivative operator D_q of a function as follows:

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} \quad (z \neq 0, q \neq 0),$$

and $D_q f(z) = f'(0)$. In the case $f(z) = z^n$, the q-derivative of $f(z)$, where n is a positive integer, is given by

$$D_q z^n = \frac{z^n - (zq)^n}{z(1-q)} = [n]_q z^{n-1}.$$

As $q \to 1^-$ and $n \in \mathbb{N}$, we have

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1} \to n. \tag{2.2}$$

3. Results

In this research paper, coefficient inequalities are derived to provide bounds on the coefficients of the functions considered, and distortion theorems are established to describe the geometric behavior and growth of these functions under specific conditions.

Definition 3.1. *For $0 \leq \beta < 1$, and $0 < q < 1$, we denote by $HP_q(\beta)$ the class of all functions of the form (2.1) that satisfy the condition*

$$Re\Big\{qz\Big(D_q(h(z)) + D_q(g(z))\Big) + h(z) + g(z)\Big\} > \beta. \tag{3.1}$$

Clearly, $HP_{q\to 1}(0) = HP$ and $HP_{q\to 1}(\beta) = HP(\beta)$. It is obvious that both classes $HP_q(\beta)$ and $HP(\beta)$ deal with harmonic functions with a positive real part and share properties such as coefficient bounds and distortion theorems. Also, the q-analogue $HP_q(\beta)$ introduces a deformation parameter q, allowing for generalized representations and extending classical results. The geometric and analytical behaviors of functions in $HP_q(\beta)$ vary from $HP(\beta)$.

We further denote by $\overline{HP}_q(\beta)$ the subclass of $HP_q(\beta)$ such that the functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = 1 - \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = -\sum_{k=1}^{\infty} b_k z^k \tag{3.2}$$

with $a_k \geq 0$ and $b_k \geq 0$ for all $k \geq 1$.

In the following theorem, we prove that the condition (3.1) is sufficient for a harmonic function f of the form (2.1) to be in the class $HP_q(\beta)$.

Theorem 3.2. *Let $f = h + \overline{g}$ be given by (2.1). Additionally, let*

$$\sum_{k=1}^{\infty} \frac{\sum_{i=0}^{k} q^i}{1-\beta}(|a_k| + |b_k|) \leq 1, \tag{3.3}$$

where $0 < q < 1$ and $0 \leq \beta < 1$. Then, $f \in HP_q(\beta)$.

Proof. We show that the inequality (3.3) is a sufficient condition for f to belong to $HP_q(\beta)$. Per the requirement (3.1), we must demonstrate that if (3.3) is satisfied, then

$$|1-\beta + qz(D_q(h(z)) + D_q(g(z))) + h(z) + g(z)| - |1+\beta - qz(D_q(h(z)) + D_q(g(z))) - h(z) - g(z)| > 0. \tag{3.4}$$

Substituting h and g in (3.4) yields by (3.3),

$$|1-\beta + qz(D_q(h(z)) + D_q(g(z))) + h(z) + g(z)| - |1+\beta - qz(D_q(h(z)) + D_q(g(z))) - h(z) - g(z)|$$

$$\begin{aligned}
&= \left|2-\beta + \sum_{k=1}^{\infty}\left(\sum_{i=0}^{k} q^i\right)(a_k + b_k)z^k\right| - \left|\beta - \sum_{k=1}^{\infty}\left(\sum_{i=0}^{k} q^i\right)(a_k + b_k)z^k\right| \\
&\geq 2(1-\beta) - 2\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k} q^i\right)(|a_k| + |b_k|)|z|^k
\end{aligned}$$

$$> \quad 2(1-\beta)\left\{1-\sum_{k=1}^{\infty}\frac{\sum_{i=0}^{k}q^i}{1-\beta}(|a_k|+|b_k|)\right\} \geq 0.$$

The harmonic mappings

$$f(z) = 1 + \sum_{k=1}^{\infty}\frac{1-\beta}{\sum_{i=0}^{k}q^i}(x_k z^k + \overline{y_k z^k}), \tag{3.5}$$

where

$$\sum_{k=1}^{\infty}(|x_k|+|y_k|) = 1,$$

show that the coefficient bound given by (3.3) is sharp. Therefore, the functions f of the form (3.5) are in the class $HP_q(\beta)$ because

$$\sum_{k=1}^{\infty}\frac{\sum_{i=0}^{k}q^i}{1-\beta}(|a_k|+|b_k|) = \sum_{k=1}^{\infty}(|x_k|+|y_k|) = 1.$$

□

Next, we obtain the coefficient estimate for a function f of the form (3.2).

Theorem 3.3. *Let $f = h + \overline{g}$ be given by (3.2). Then $f \in \overline{HP}_q(\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{\sum_{i=0}^{k}q^i}{1-\beta}(|a_k|+|b_k|) \leq 1. \tag{3.6}$$

Proof. The "if part" follows from Theorem 3.2 upon noting that if $f = h + \overline{g} \in HP_q(\beta)$ are of the form (3.2), then $f \in \overline{HP}_q(\beta)$.

Suppose that $f \in \overline{HP}_q(\beta)$. Then, we find from (3.1) that

$$Re\left\{1-\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k}q^i\right)(a_k+b_k)z^k\right\} > \beta, z \in \mathbb{U}, q \in (0,1), 0 \leq \beta < 1.$$

If we choose z to be real and let $z \to 1^-$, we get

$$1-\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k}q^i\right)(a_k+b_k) \geq \beta,$$

or equivalently,

$$\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k}q^i\right)(a_k+b_k) \leq 1-\beta,$$

which is precisely assertion (3.6) of the theorem. □

Theorem 3.4. *Let $f \in \overline{HP}_q(\beta)$, then*

$$1-\frac{1-\beta}{1+q}r \leq |f(z)| \leq 1+\frac{1-\beta}{1+q}r, \quad |z| = r < 1. \tag{3.7}$$

Proof. We prove that the right-hand inequality and the left-hand inequality are similar. Let $f \in \overline{HP}_q(\beta)$, taking the absolute value, f we obtain

$$\begin{aligned} |f(z)| &\leq 1 + \sum_{k=1}^{\infty}(a_k + b_k)|z|^k \\ &\leq 1 + \sum_{k=1}^{\infty}(a_k + b_k)r \\ &\leq 1 + \frac{1-\beta}{1+q}\sum_{k=1}^{\infty}\frac{\sum_{i=0}^{k} q^i}{1-\beta}(a_k + b_k)r \\ &\leq 1 + \frac{1-\beta}{1+q}r. \end{aligned}$$

□

The following result follows from Theorem 3.2.

Corollary 3.5. *If $f \in \overline{HP}_q(\beta)$, then*

$$\left\{w : |w| < \frac{1-\beta}{1+q}\right\} \subset f(\mathbb{U}), \tag{3.8}$$

where $0 < q < 1$ and $0 \leq \beta < 1$.

Theorem 3.6. *Set*

$$h_k(z) = 1 - \frac{1-\beta}{\sum_{i=0}^{k} q^i} z^k \quad \text{and} \quad g_k(z) = 1 - \frac{1-\beta}{\sum_{i=0}^{k} q^i}\bar{z}^k, \quad \text{for} \quad k = 1, 2, \dots .$$

Then, $f \in \overline{HP}_q(\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k + y_k g_k), \tag{3.9}$$

where $x_k \geq 0$, $y_k \geq 0$ and $\sum_{k=1}^{\infty}(x_k + y_k) = 1$. In particular, the extreme points of $\overline{HP}_q(\beta)$ are h_k and g_k.

Proof. For functions f of the form (3.9), we have

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k + y_k g_k) = 1 - \sum_{k=1}^{\infty}\frac{1-\beta}{\sum_{i=0}^{k} q^i}(x_k z^k + y_k \bar{z}^k).$$

Then

$$\sum_{k=1}^{\infty}\frac{\sum_{i=0}^{k} q^i}{1-\beta}\left[\frac{1-\beta}{\sum_{i=0}^{k} q^i}(x_k + y_k)\right] = \sum_{k=1}^{\infty}(x_k + y_k) = 1,$$

and so $f \in \overline{HP}_q(\beta)$. Conversely, suppose $f \in \overline{HP}_q(\beta)$. Set

$$x_n = \frac{\sum_{i=0}^{k} q^i}{1-\beta}a_k \quad \text{and} \quad y_n = \frac{\sum_{i=0}^{k} q^i}{1-\beta}b_k.$$

Then by Theorem 3.2, $0 \le x_k \le 1$ and $0 \le y_k \le 1$, $(k = 1, 2, ...)$. Consequently, we obtain

$$f(z) = \sum_{k=1}^{\infty}(x_k h_k + y_k g_k)$$

as required. □

Definition 3.7. *For a function $f \in \overline{PH_q}(\beta)$, the $\delta - q$- neighborhood of f is defined by*

$$N_{\delta,q}(f) = \Big\{F : F(z) = 1 - \sum_{k=1}^{\infty}(|A_k|z^k + |B_k|\bar{z}^k) \quad and \quad \sum_{k=1}^{\infty}[k]_q(|a_k - A_k| + |b_k - B_k|) \le \delta\Big\}.$$

In particular, for the constant function $\mathbb{I}(z) = 1$, we immediately have

$$N_{\delta,q}(\mathbb{I}) = \Big\{f : f(z) = 1 - \sum_{k=1}^{\infty}(|a_k|z^k + |b_k|\bar{z}^k) \quad \text{and} \quad \sum_{k=1}^{\infty}[k]_q(|a_k| + |b_k|) \le \delta\Big\}.$$

Theorem 3.8. *Let $\delta = (1-\beta)\backslash q$. Then $\overline{HP}_q(\beta) \subset N_{\delta,q}(\mathbb{I})$.*

Proof. Suppose that $f \in \overline{HP}_q(\beta)$, then we have

$$\begin{aligned}
\sum_{k=1}^{\infty}[k]_q(|a_k| + |b_k|) &= \frac{1}{q}\sum_{k=1}^{\infty} q[k]_q(|a_k| + |b_k|) \\
&\le \frac{1}{q}\sum_{k=1}^{\infty}(\sum_{i=0}^{k} q^i)(|a_k| + |b_k|) \\
&\le \frac{1}{q}(1-\beta) = \delta.
\end{aligned}$$

Hence, $f \in N_{\delta,q}(\mathbb{I})$. □

We now examine the convex combination of $\overline{HP}_q(\beta)$.

Theorem 3.9. *Let the functions f_j defined as*

$$f_j(z) = 1 - \sum_{k=1}^{\infty} a_{k,j}z^k - \sum_{k=1}^{\infty} b_{k,j}\bar{z}^k, a_{k,j} \ge 0, b_{k,j} \ge 0 \tag{3.10}$$

be in the class $\overline{HP}_q(\beta)$ for every $j = 1, 2, \cdots, \ell.$, then the function

$$\xi(z) = \sum_{j=1}^{\ell}\mu_j f_j(z)$$

is also in the class $\overline{HP}_q(\beta)$, where $\sum_{j=1}^{\ell}\mu_i = 1$.

Proof. According to the definition of ξ, we can write

$$\xi(z) = 1 - \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\ell} \mu_j a_{k,j} \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\ell} \mu_j b_{k,j} \right) \overline{z^k}.$$

Further, since f_j are in $\overline{HP}_q(\beta)$ for every $j = 1, 2, \cdots, \ell$. Then by (3.6),we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{i=0}^{k} q^i \left\{ \left(\sum_{j=1}^{\ell} \mu_j a_{k,j} \right) + \left(\sum_{j=1}^{\ell} \mu_j b_{k,j} \right) \right\} \\ = & \sum_{j=1}^{\ell} \mu_j \left\{ \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k} q^i \right) \left(a_{k,j} + b_{k,j} \right) \right\} \\ \leq & \ 1 - \beta. \end{aligned}$$

Hence, the proof is complete. □

Corollary 3.10. *The class $\overline{HP}_q(\beta)$ is closed under convex combination.*

In the following definition, we define the q-integral operator for the harmonic functions $f = h + \overline{g}$ with a positive real part.

Definition 3.11. *Let $f = h + \overline{g}$ be defined by (3.2); then the q-integral operator $F_q : \overline{HP}(\beta) \to \overline{HP}(\beta)$ is defined by the relation*

$$F_q(z) = \frac{[c+1]_q}{z^{c+1}} \int_0^z t^c h(t) d_q t + \overline{\frac{[c+1]_q}{z^{c+1}} \int_0^z t^c g(t) d_q t}, (c > -1), z \in \mathbb{U}, \tag{3.11}$$

where $[a]_q$ is the q-number defined by (2.2).

Definition 3.11 leads us to

$$\begin{aligned} F_q(z) &= \frac{[c+1]_q}{z^{c+1}} \int_0^z \left\{ t^c - \sum_{k=1}^{\infty} a_k t^{c+k} \right\} d_q t - \overline{\frac{[c+1]_q}{z^{c+1}} \int_0^z \left\{ b_k t^{c+k} \right\} d_q t} \\ &= \frac{[c+1]_q}{z^{c+1}} \left[\left((1-q) z \sum_{n=0}^{\infty} (z q^n)^c q^n \right) - \sum_{k=1}^{\infty} a_k \left((1-q) z \sum_{n=0}^{\infty} (z q^n)^{c+k} q^n \right) \right] \\ &\quad - \overline{\frac{[c+1]_q}{z^{c+1}} \sum_{k=1}^{\infty} b_k \left((1-q) z \sum_{n=0}^{\infty} (z q^n)^{c+k} q^n \right)} \\ &= \frac{[c+1]_q}{z^{c+1}} \left[\frac{z^{c+1}}{[c+1]_q} - \sum_{k=1}^{\infty} \frac{1}{[c+k+1]_q} a_k z^{c+k+1} \right] - \overline{\frac{[c+1]_q}{z^{c+1}} \sum_{k=1}^{\infty} \frac{1}{[c+k+1]_q} b_k z^{c+k+1}} \\ &= 1 - \sum_{k=1}^{\infty} \frac{[c+1]_q}{[c+k+1]_q} a_k z^k - \sum_{k=1}^{\infty} \frac{[c+1]_q}{[c+k+1]_q} b_k \overline{z}^k, \quad c > -1,\ 0 < q < 1,\ |z| < 1. \end{aligned} \tag{3.12}$$

Next, we show that the class $\overline{HP}_q(\beta)$ is closed under the q-integral operator defined by (3.11).

Theorem 3.12. *Let $f = h + \overline{g}$ be given by (3.2) and $f \in \overline{HP}_q(\beta)$, then F_q is defined by (3.11) and also belongs to $\overline{HP}_q(\beta)$.*

Proof. From the series representation of F_q defined by (3.12), we see that

$$[c+k+1]_q - [c+1]_q = \sum_{i=0}^{c+k} q^i - \sum_{i=0}^{c} q^i = \sum_{i=c+1}^{c+k} q^i > 0,$$

then $\frac{[c+1]_q}{[c+k+1]_q} \leq 1$. Therefore,

$$\sum_{k=1}^{\infty} \frac{\sum_{i=0}^{k} q^i}{1-\beta} \left(\frac{[c+1]_q}{[c+k+1]_q} |a_k| + \frac{[c+1]_q}{[c+k+1]_q} |b_k| \right)$$

$$\leq \sum_{k=1}^{\infty} \frac{\sum_{i=0}^{k} q^i}{1-\beta} (|a_k| + |b_k|) \leq 1.$$

Hence, $F_q \in \overline{HP}_q(\beta)$. □

4. Conclusions

Using the q-derivative operator, we thoroughly investigated a subclass of harmonic functions with a positive real part, denoted as $HP_q(\beta)$. A rigorous mathematical characterization of this class was achieved by developing sufficient and necessary coefficient conditions, especially for negative coefficients. The results expand on previous findings and apply them to q-calculus. The growth theorem and extreme points of $HP_q(\beta)$ were established, providing greater insight into the geometric and analytical features of these harmonic functions. Introducing the q-integral operator for functions $f = h + g$ with a positive real part showcases the versatility of q-calculus in studying harmonic functions due to its wide applications in mathematical physics and dynamics. The q-integral operator simplifies certain calculations, such as evaluating q-series and solving q-deformed differential equations, by providing closed-form solutions that would otherwise require tedious summation or numerical methods. Furthermore, it provides new analytical tools for exploring geometric and structural properties in q-deformed systems, highlighting its potential in advancing research in complex dynamics and quantum mechanics.

Use of Generative-AI tools declaration

The author declares that Artificial Intelligence (AI) tools were not used in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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