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Research article

The inverse of tails of Riemann zeta function, Hurwitz zeta function and Dirichlet L-function

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Abstract: In this paper, we derive the asymptotic formulas $B^*_{r,s,t}(n)$ such that

$$\lim_{n\to\infty}\left\{\left(\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}\right)^{-1}-B^*_{r,s,t}(n)\right\}=0,$$

where $Re(r+s)>1$ and $t\in\mathbb{C}$. It is evident that the asymptotic formulas for the inverses of the tails of both the Riemann zeta function and the Hurwitz zeta function on the half-plane $Re(s)>1$ are its corollaries. Subsequently we provide the asymptotic formulas for the Riemann zeta function and the Hurwitz zeta function on the half-plane $Re(s)<0$. Finally, we study the asymptotic formulas of the inverse of the tails of the Dirichlet L-function for $Re(s)>1$ and $Re(s)<0$.

Keywords: Riemann zeta function; Hurwitz zeta function; reciprocal sums; asymptotic formulas; Dirichlet L-function
Mathematics Subject Classification: 11B83, 11M06

1. Introduction

The Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ are two important functions in the analytic number theory. The Riemann zeta function has an intimate connection with the distribution of primes, and the Hurwitz zeta function plays an important role in studying the properties of the Dirichlet L-function. The properties of $\zeta(s)$ and $\zeta(s,a)$ have attracted considerable attention from mathematicians.

In the past ten years, many mathematicians have been working on the tails of various zeta functions.

For example, Xin [1] gives the asymptotic formulas of the Riemann zeta function with $s = 2$ and $s = 3$,

$$\begin{aligned} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor &= n - 1, \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor &= 2n(n - 1). \end{aligned} \tag{1.1}$$

Kim and Song [2] studied the inverses of the tails of the Riemann zeta function and derived results for s in the critical strip $0 < s < 1$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\rfloor \sim \begin{cases} 2(1 - 2^{1-s})(n - \frac{1}{2})^s, & \text{if } n \text{ is even;} \\ -2(1 - 2^{1-s})(n - \frac{1}{2})^s, & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

Lee and Park [3] dealt with the inverses of the tails of the Hurwitz zeta function when $s \geq 2$, $s \in \mathbb{N}$, and $0 \leq a < 1$, and derived

$$\left(\sum_{k=n}^{\infty} \frac{1}{(k + a)^s} \right)^{-1} \sim \sum_{j=0}^{s-1} A_j^*(n + a)^j, \tag{1.3}$$

where

$$A_{s-1}^* = s - 1, \quad A_l^s = -\sum_{j=1}^{s-l-1} x_j^s A_{l+j}^s, \quad x_j^s = \binom{s - 2 + j}{j} B_j,$$

and B_j is the Bernoulli numbers.

Ohtsuka and Nakamura [4] derived the asymptotic formulas for the tails of the Fibonacci zeta function with s=1 and s=2,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even, } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd, } n \geq 1; \end{cases} \tag{1.4}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even, } n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd, } n \geq 1. \end{cases} \tag{1.5}$$

More results associated with the asymptotic formulas can be found in [5–12], where Hwang and Song [5] and Xu [6] study the tails related to the Riemann zeta function, Choi and Choo [7], Komatsu [8], Lee and Park [9, 10] and Marques and Trojovsky [11] study the tails of the Fibonacci zeta function, and Xu and Wang [12] studied the tails of Pell zeta function.

The above results reveal the relations between zeta functions and polynomials. From other perspectives, these sums are the First Newton's identity of an infinite polynomial with roots k^{-s} or F_k^{-s} for $k \geq n$; therefore, they also reflect the importance of Newton's identities, which are elaborately introduced in [13].

In general, the consideration is limited to cases where $s \in \mathbb{N}^+$ or $s \in (0, 1)$. Due to the application of the estimate of the tails of the Riemann zeta function in various other problems, such as the upper of the Dirthlet L-function, the average of many well-known arithmetical functions, and so on, we hope to generalize the results to the complex plane. For this proposal, in this paper, the generalized Riemann zeta function $R(r, s, t)$ and its n-th tail $R_n(r, s, t)$ are defined for $Re(r + s) > 1$, $t \in \mathbb{C}$ by series

$$R(r, s, t) = \sum_{k=1}^{\infty} \frac{1}{k^r (k + t)^s} \quad \text{and} \quad R_n(r, s, t) = \sum_{k=n}^{\infty} \frac{1}{k^r (k + t)^s},$$

and then we study the asymptotic formulas of $R_n(r, s, t)$ for r, s in the half-plane $Re(r + s) > 1$ and $t \in \mathbb{C}$. In addition, inspired by the expressions of $\zeta(s)$ with $0 < s < 1$ in [2], we can similarly estimate the inverses of the tails of $\zeta(s)$ for $Re(s) < 0$. Similarly, we can derive the asymptotic formulas for the inverses of the tails of the Hurwitz zeta function and the Dirichlet L-function $L(1, \chi)$ for $Re(s) > 1$ and $Re(s) < 0$. It is worth noting that the inverses of the Hurwitz zeta function and the Dirichlet L-function possess new characteristics, and we obtain their dominant terms.

2. Preliminaries

Theorem 1. *Let the function $R(r, s, t)$ and the n-th tail $R_n(r, s, t)$ be as above, for any $r, s, t \in \mathbb{C}$. If*

$$Re(r + s) - 1 > 0,$$

we have

$$(R_n(r, s, t))^{-1} = B^*_{r,s,t}(n) + o(1),$$

where

$$B^*_{r,s,t}(n) = b^*_{r+s-1} n^{r+s-1} + b^*_{r+s-2} n^{r+s-2} + \cdots + b^*_{\{r+s\}} n^{\{r+s\}}$$

and the coefficients

$$b^*_j \in \mathbb{C} \; (j = r + s - 1, r + s - 2, \cdots, \{r + s\})$$

are constants determined by r, s, t as follows:

$$\begin{cases} b^*_{r+s-1} = r + s - 1; \\ b^*_{r+s-1-k} = \\ \dfrac{\sum\limits_{\substack{k_1+k_2=k \\ k_1 \geq 0, k_2 \geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i \geq \{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1} \binom{s}{k_2} t^{k_2} - \sum\limits_{\substack{k_1+k_2=k \\ k_1, k_2 \geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i \geq \{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k_1} b^*_{r+s-1-k_2}}{r+s-1+k} + \dfrac{\sum\limits_{\substack{i=r+s-k \\ i \geq \{r+s-1\}}}^{r+s-1} b^*_i \left[\binom{i}{i-r-s+2+k} - (r+s-1) \binom{i}{i-r-s+1+k} \right]}{r+s-1+k}, \\ \text{for } [r + s - 1] \geq k \geq 1; \end{cases} \tag{2.1}$$

where $[x]$ denotes the integer part of the real part of x, and $\{x\} = x - [x]$.

Theorem 2. *Let*

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s}$$

be the n-th tail of the Riemann zeta function, then for all $Re(s) < 0$, we have

$$(\zeta_n(s))^{-1} = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} B^*_{0,1-s,0}(n) + o(1),$$

where

$$B^*_{0,1-s,0}(n) = b^*_{-s}n^{-s} + b^*_{-s-1}n^{-s-1} + \cdots + b^*_{\{-s\}}n^{\{-s\}}$$

and the coefficients

$$b^*_j \in \mathbb{C}\ (j = -s, -s-1, \cdots, \{-s\})$$

are constants determined by s as follows:

$$\begin{cases} b^*_{-s} = -s; \\ b^*_{-s-k} = \dfrac{\sum\limits_{\substack{k_1+k_2=k \\ k_1\geq 0, k_2\geq 1}} \sum\limits_{\substack{i=-s-k_1 \\ i\geq\{-s\}}}^{-s} b^*_i \binom{i}{i+s+1+k_1}\binom{s}{k_2} t^{k_2} - \sum\limits_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \sum\limits_{\substack{i=-s-k_1 \\ i\geq\{-s\}}}^{-s} b^*_i \binom{i}{i+s+k_1} b^*_{-s-k_2}}{-s+k} + \dfrac{\sum\limits_{\substack{i=1-s-k \\ i\geq\{-s\}}}^{-s} b^*_i \left[\binom{i}{i+s+1+k} - (-s)\binom{i}{i+s+k}\right]}{-s+k}, \\ \text{for } [-s] \geq k \geq 1; \end{cases} \tag{2.2}$$

and $[x]$ *denotes the integer part of the real part of* x*, and* $\{x\} = x - [x]$.

Corollary 1. *Let*

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s}$$

be the n-th tail of the Riemann zeta function, then for $Re(s) > 1$*, we have*

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s} = B^*_{0,s,0}(n) + o(1), \tag{2.3}$$

where

$$B^*_{0,s,0}(n) = b^*_{s-1}n^{s-1} + b^*_{s-2}n^{s-2} + \cdots + b^*_{\{s-1\}}n^{\{s-1\}}$$

and

$$\begin{cases} b^*_{s-1} = s-1; \\ b^*_{s-1-k} = \dfrac{\sum\limits_{\substack{i=s-k \\ i\geq\{s-1\}}}^{s-1} b^*_i \left[\binom{i}{i-s+2+k} - (s-1)\binom{i}{i-s+1+k}\right]}{s-1+k} - \dfrac{\sum\limits_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \sum\limits_{\substack{i=s-1-k_1 \\ i\geq\{s-1\}}}^{s-1} b^*_i \binom{i}{i-s+1+k_1} b^*_{s-1-k_2}}{s-1+k}, \quad \text{for } [s-1] \geq k \geq 1. \end{cases} \tag{2.4}$$

Corollary 2. *If* $r = 0$*,* $0 < t = a \leq 1$*, and* $|s| > 1$*, we have*

$$\zeta_n(s,a) = \sum_{k=n}^{\infty} \frac{1}{(k+a)^s} = B^*_{0,s,a}(n) + o(1), \tag{2.5}$$

where

$$B^*_{0,s,a}(n) = b^*_{s-1,a}n^{s-1} + b^*_{s-2,a}n^{s-2} + \cdots + b^*_{\{s-1\},a}n^{\{s-1\}}$$

and

$$\begin{cases} b^*_{s-1} = s-1; \\ b^*_{s-1-k} = \dfrac{\sum\limits_{\substack{k_1+k_2=k \\ k_1\geq 0, k_2\geq 1}} \sum\limits_{\substack{i=s-1-k_1 \\ i\geq\{s-1\}}}^{s-1} b^*_i \binom{i}{i-s+2+k_1}\binom{s}{k_2} a^{k_2} - \sum\limits_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \sum\limits_{\substack{i=s-1-k_1 \\ i\geq\{s-1\}}}^{s-1} b^*_i \binom{i}{i-s+1+k_1} b^*_{s-1-k_2}}{s-1+k} + \dfrac{\sum\limits_{\substack{i=s-k \\ i\geq\{s-1\}}}^{s-1} b^*_i \left[\binom{i}{i-s+2+k} - (s-1)\binom{i}{i-s+1+k}\right]}{s-1+k}; \\ \text{for } [s-1] \geq k \geq 1. \end{cases} \tag{2.6}$$

Now we start considering the asymptotic formulas for inverses of the tails of the Hurwitz zeta function $\zeta(s,a)$ for $Re(s)<0$ and $a=\frac{p}{q}\in\mathbb{Q}, p,q\in\mathbb{N}, (p,q)=1$.

In combination with the following functional equation related to the Hurwitz zeta function [14]:

$$\zeta\left(1-s,\frac{p}{q}\right)=\frac{2\Gamma(s)}{(2\pi q)^s}\sum_{m=1}^{q}\cos\left(\frac{\pi s}{2}-\frac{2\pi mp}{q}\right)\zeta\left(s,\frac{m}{q}\right). \tag{2.7}$$

For any positive integer n, write

$$n=q(l-1)+r$$

for some $q\geq r\geq 1$ and $l,r\in\mathbb{N}_+$. The n-th tails of the Hurwitz zeta function for $Re(s)<0$ are defined by

$$\zeta_n\left(s,\frac{p}{q}\right)=\frac{2\Gamma(1-s)}{(2\pi q)^{1-s}}\left(\sum_{m=1}^{r-1}a_m\zeta_{l+1}\left(1-s,\frac{m}{q}\right)+\sum_{m=r}^{q}a_m\zeta_l\left(1-s,\frac{m}{q}\right)\right),$$

where

$$a_m=\cos\left(\frac{\pi(1-s)}{2}-\frac{2\pi mp}{q}\right)$$

for $m=1,2,\cdots,q$.

When

$$n=q(l-1)+1,$$

rewrite above n-th tails of the Hurwitz zeta function for $Re(s)<0$ as

$$\zeta_n\left(s,\frac{p}{q}\right)=\frac{2\Gamma(1-s)}{(2\pi q)^{1-s}}\sum_{m=1}^{q}a_m\zeta_l\left(1-s,\frac{m}{q}\right).$$

Theorem 3. *Let function $\zeta(s,\frac{p}{q})$ and the n-th tail $\zeta_n(s,\frac{p}{q})$ be as above for $Re(s)<0$ and $p,q\in\mathbb{N}_+$, $(p,q)=1$.*
(i) Let $n=q(l-1)+1$ for some $l\in\mathbb{N}_+$, if

$$\sum_{m=1}^{q}ma_m\neq 0$$

holds, we have

$$\left(\zeta_n\left(s,\frac{p}{q}\right)\right)^{-1}=\frac{-\pi q^2}{(2q\pi)^s\,\Gamma(1-s)\sum\limits_{m=1}^{q}ma_m}l^{1-s}+O_{s,q}\left(l^{-s}\right),$$

where

$$a_m=\cos\left(\frac{\pi(1-s)}{2}-\frac{2\pi mp}{q}\right)$$

for $m=1,2,\cdots,q$.
(ii) Let

$$n=q(l-1)+r$$

for some $q \geq r \geq 1$ and $l, r \in \mathbb{N}_+$, if

$$\sum_{m=1}^{q} ma_m \neq 0 \quad \textit{and} \quad \sum_{m=1}^{q} ma_m - q\sum_{m=r}^{q} a_m \neq 0$$

hold, we have

$$\left(\zeta_n\left(s, \frac{p}{q}\right)\right)^{-1} = \frac{-\pi q^2}{(2\pi q)^s\Gamma(1-s)\left(\sum\limits_{m=1}^{q} ma_m - q\sum\limits_{m=r}^{q} a_m\right)} l^{1-s} + O_{s,q}(l^{-s}),$$

where

$$a_m = \cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mp}{q}\right)$$

for $m = 1, 2, \cdots, q$.

Remark 1. *Since $\sum_{m=1}^{q} a_m = 0$ (see the proof of Theorem 3 in this paper), we observe that the result (i) of Theorem 3 follows (ii) when $r = 1$.*

Similarly, according to the following functional equation related to the Dirichlet L-function [14]:

$$L(s,\chi) = k^{-s}\sum_{j=1}^{k}\chi(j)\zeta\left(s, \frac{j}{k}\right), \tag{2.8}$$

where χ is a character module k.

The asymptotic formula of the inverse of the Dirichlet L-function $L(s,\chi)$ is trivial when $\chi = \chi_1$ is the principal character module k. Therefore, we only consider the case for $\chi \neq \chi_1$. Similar to the Hurwitz zeta function, the n-th tail of $L(s,\chi)$ could be defined for $Re(s) > 1$ by

$$L_n(s,\chi) = k^{-s}\left(\sum_{j=1}^{r-1}\chi(j)\zeta_{l+1}\left(s, \frac{j}{k}\right) + \sum_{j=r}^{k}\chi(j)\zeta_l\left(s, \frac{j}{k}\right)\right),$$

where

$$n = k(l-1) + r$$

for some $k \geq r \geq 1$ and $l, r \in \mathbb{N}_+$.

When $r = 1$, we have

$$L_n(s,\chi) = k^{-s}\left(\sum_{j=1}^{k}\chi(j)\zeta_l\left(s, \frac{j}{k}\right)\right).$$

In combination with (2.7) and (2.8), for $Re(s) < 0$, we have

$$\begin{aligned} L(s,\chi) &= k^{-s}\sum_{j=1}^{k}\chi(j)\zeta\left(s, \frac{j}{k}\right) \\ &= k^{-s}\sum_{j=1}^{k}\chi(j)\frac{2\Gamma(1-s)}{(2\pi k)^{1-s}}\sum_{m=1}^{k}\cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mj}{k}\right)\zeta\left(1-s, \frac{m}{k}\right) \qquad (2.9) \\ &= \frac{\Gamma(1-s)}{2^{-s}\pi^{1-s}k}\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\zeta\left(1-s, \frac{m}{k}\right), \end{aligned}$$

where

$$a_{j,m} = \cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mj}{k}\right)$$

for $j = 1, 2, \cdots, k$ and $m = 1, 2, \cdots, k$.

The n-th tail of $L(s,\chi)$ could be defined for $Re(s) < 0$ by

$$L_n(s,\chi) = \frac{\Gamma(1-s)}{2^{-s}\pi^{1-s}k}\left(\left(\sum_{j=1}^{u-1}\sum_{m=1}^{k} + \sum_{\substack{m=1\\ j=u}}^{r-1}\right)\chi(j)a_{j,m}\zeta_{l+1}\left(1-s, \frac{m}{k}\right) + \left(\sum_{\substack{m=r\\ j=u}}^{k} + \sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\chi(j)a_{j,m}\zeta_{l}\left(1-s, \frac{m}{k}\right)\right),$$

where

$$n = k^2(l-1) + k(u-1) + r$$

for some $k \geq u \geq 1$, $k \geq r \geq 1$, and $l, u, r \in \mathbb{N}_+$.

When $r = 1$, we have

$$L_{k^2+1}(s,\chi) = \frac{\Gamma(1-s)}{2^{-s}\pi^{1-s}k}\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\zeta_l\left(1-s, \frac{m}{k}\right).$$

Theorem 4. *Let the function $L(s,\chi)$ and the n-th tail $L_n(s,\chi)$ be as above for $\chi \neq \chi_1$ is the nonprincipal character module k.*
(i) When $Re(s) > 1$, let $n = k(l-1) + 1$ for some $l \in \mathbb{N}_+$, if

$$\sum_{j=1}^{k} j\chi(j) \neq 0$$

holds, we have

$$(L_n(s,\chi))^{-1} = \frac{-k^{s+1}}{\sum_{j=1}^{k} j\chi(j)} l^s + O_{s,k}\left(l^{s-1}\right).$$

(ii) When $Re(s) > 1$, let $n = k(l-1) + r$ for some $k \geq r \geq 1$, and $l, r \in \mathbb{N}_+$, if

$$\sum_{j=1}^{k} j\chi(j) - k\sum_{j=r}^{k}\chi(j) \neq 0 \quad and \quad \sum_{j=1}^{k} j\chi(j) \neq 0$$

hold, we have

$$(L_n(s,\chi))^{-1} = \frac{-k^{1+s}}{\sum_{j=1}^{k} j\chi(j) - k\sum_{j=r}^{k}\chi(j)} l^s + O_{s,k}(l^{s-1}).$$

(iii) When $Re(s) < 0$, let $n = k^2(l-1) + 1$ for some $l \in \mathbb{N}_+$, if

$$\sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m} \neq 0$$

holds, we have

$$(L_n(s,\chi))^{-1} = \frac{-\pi k^2}{(2\pi)^s\Gamma(1-s)\sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m}} l^{1-s} + O_{s,k}(l^{-s}),$$

where

$$a_{j,m} = \cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mj}{k}\right)$$

for $j = 1, 2, \cdots, k$ *and* $m = 1, 2, \cdots, k$.
(iv) When $Re(s) < 0$*, let*

$$n = k^2(l-1) + k(u-1) + r$$

for some $k \geq u \geq 1$, $k \geq r \geq 1$ *and* $l, u, r \in \mathbb{N}_+$*, if*

$$\sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m} - k\left(\sum_{\substack{m=r\\ j=u}}^{k} + \sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\chi(j)a_{j,m} \neq 0 \quad and \quad \sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m} \neq 0$$

hold, we have

$$(L_n(s,\chi))^{-1} = \frac{-\pi k^2}{(2\pi)^s\Gamma(1-s)\left(\sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m} - k\left(\sum_{\substack{m=r\\ j=u}}^{k} + \sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\chi(j)a_{j,m}\right)} l^{1-s} + O_{s,k}(l^{-s}),$$

where

$$a_{j,m} = \cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mj}{k}\right)$$

for $j = 1, 2, \cdots, k$ *and* $m = 1, 2, \cdots, k$.

3. The proof of Theorem 1

Before our proof, we first propose several lemmas.

Lemma 1. *For all* $s, t \in \mathbb{C}$ *and* $n \in \mathbb{N}$*, if* $n > |t|$*, we have*

$$(n+t)^s = \sum_{k=0}^{\infty}\binom{s}{k} n^{s-k}t^k.$$

Proof. Let

$$f(s) = (n+t)^s = n^s(1+\frac{t}{n})^s = n^s\sum_{k=0}^{\infty}\binom{s}{k}(\frac{t}{n})^k = \sum_{k=0}^{\infty}\binom{s}{k} n^{s-k}t^k, \tag{3.1}$$

and the third equation holds for $|\frac{t}{n}| < 1$. □

Lemma 2. *For all $r, s, t \in \mathbb{C}$ and $Re(r+s) > 1$, there exists a unique generalized complex coefficient polynomial*

$$B^*_{r,s,t}(n) = b^*_{r+s-1}n^{r+s-1} + b^*_{r+s-2}n^{r+s-2} + \cdots + b^*_{\{r+s\}}n^{\{r+s\}},$$

where

$$b^*_j \in \mathbb{C}\ (j = r+s-1, r+s-2, \cdots, \{l\})$$

is determined by r, s, and t, subject to the real part of its order being greater than 0 *and*

$$[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)]\, n^r(n+t)^s - B^*_{r,s,t}(n+1)B^*_{r,s,t}(n) = O(n^{r+s+\{r+s\}-2}).$$

Proof. First, we assume

$$B^*_{r,s,t}(n) = b^*_l n^l + b^*_{l-1}n^{l-1} + \cdots + b^*_{\{l\}+1}n^{\{l\}+1} + b^*_{\{l\}}n^{\{l\}}, \quad Re(l) > 0$$

satisfies

$$[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)]\, n^r(n+t)^s - B^*_{r,s,t}(n+1)B^*_{r,s,t}(n) = O(n^{r+s+\{r+s\}-2}). \tag{3.2}$$

Since the absolute value of the order of $[B^*_{r,s,t}(n+1)-B^*_{r,s,t}(n)]\, n^r(n+t)^s$ is greater than $|r+s+\{r+s\}-2|$, then we have the necessary condition (1): The order of $[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)]\, n^r(n+t)^s$ is equal to the order of $B^*_{r,s,t}(n+1)B^*_{r,s,t}(n)$.

In other words, we have $l - 1 + r + s = 2l$, which implies

$$l = r + s - 1. \tag{3.3}$$

Then we rewrite

$$B^*_{r,s,t}(n) = b^*_{r+s-1}n^{r+s-1} + b^*_{r+s-2}n^{r+s-2} + \cdots + b^*_{\{l\}+1}n + b^*_{\{r+s-1\}}$$

with $b^*_{r+s-1} \neq 0$, by Lemma 1, we have

$$(n+t)^s = \sum_{k=0}^{\infty} \binom{s}{k} n^{s-k}t^k \quad \text{for } n > |t|.$$

Let $n > 1$, we have

$$\begin{aligned}
[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)]\, n^r(n+t)^s &= \left[\sum_{i=\{r+s-1\}}^{r+s-1} b^*_i(n+1)^i - \sum_{i=\{r+s-1\}}^{r+s-1} b^*_i n^i\right] n^r \sum_{j=0}^{\infty}\binom{s}{j}n^{s-j}t^j \\
&= \sum_{i=\{r+s-1\}}^{r+s-1} b^*_i \sum_{k=1}^{\infty}\binom{i}{k}n^{i-k}n^r \sum_{j=0}^{\infty}\binom{s}{j}t^j n^{s-j} \\
&= \sum_{k=0}^{\infty} \sum_{\substack{i=r+s-1-k \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-(r+s-2-k)} n^{r+s-2-k} \sum_{j=0}^{\infty}\binom{s}{j}t^j n^{s+r-j} \\
&= \sum_{k=0}^{\infty} \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1}\binom{s}{k_2} t^{k_2} n^{2r+2s-2-k}
\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
B^*_{r,s,t}(n+1)B^*_{r,s,t}(n) &= \sum_{i=\{r+s-1\}}^{r+s-1} b^*_i \sum_{k=0}^{\infty} \binom{i}{k} n^{i-k} \sum_{j=\{r+s-1\}}^{r+s-1} b^*_j n^j \\
&= \sum_{k=0}^{\infty} \sum_{\substack{i=r+s-1-k \\ i\geq\{r+s-1\}}}^{\infty} b^*_i \binom{i}{i-(r+s-1-k)} n^{r+s-1-k} \sum_{j=0}^{[r+s-1]} b^*_{r+s-1-j} n^{r+s-1-j} \\
&= \sum_{k=0}^{\infty} \left(\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-(r+s-1-k_1)} b^*_{r+s-1-k_2} \right) n^{2r+2s-2-k},
\end{aligned} \tag{3.5}
$$

where $[x]$ denotes the integer part of the real part of x.

Now we derive the necessary condition (2): The following system of equations has at least one solution,

$$
\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1} \binom{s}{k_2} t^{k_2} = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k_1} b^*_{r+s-1-k_2}, \tag{3.6}
$$

where $k = 0, 1, 2, \cdots, [r+s-2]$ and $[r+s-1]$.

By calculation, the system of Eq (3.6) is equivalent with

$$
(r+s-1)b^*_{r+s-1} = [b^*_{r+s-1}]^2 \quad \text{for} \ \ k = 0,
$$

and

$$
\begin{aligned}
&(r+s-1-k)b^*_{r+s-1-k} + \sum_{\substack{k_1+k_2=k \\ k_1\geq 0,k_2\geq 1}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1} \binom{s}{k_2} t^{k_2} + \sum_{\substack{i=r+s-k \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k} \\
&= (2r+2s-2)b^*_{r+s-1-k} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \sum_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k_1} b^*_{r+s-1-k_2} + \sum_{\substack{i=r+s-k \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k} b^*_{r+s-1}
\end{aligned} \tag{3.7}
$$

for $k = 1, 2, \cdots, [r+s-1]$. Combining with $Re(r+s-1) > 0$, we can then derive the unique solution of the system of Eq (3.6) as follows:

$$
b^*_{r+s-1} = (r+s-1) \neq 0
$$

and

$$
\begin{aligned}
b^*_{r+s-1-k} &= \frac{\sum\limits_{\substack{k_1+k_2=k \\ k_1\geq 0,k_2\geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1}\binom{s}{k_2} t^{k_2} - \sum\limits_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k_1} b^*_{r+s-1-k_2}}{r+s-1+k} \\
&\quad + \frac{\sum\limits_{\substack{i=r+s-k \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \left[\binom{i}{i-r-s+2+k} - (r+s-1)\binom{i}{i-r-s+1+k}\right]}{r+s-1+k},
\end{aligned} \tag{3.8}
$$

where $k = 1, 2, \cdots, [r+s-1]$.

So we derive the necessary condition of (3.2):

$$B^*_{r,s,t}(n) = b^*_{r+s-1}n^{r+s-1} + b^*_{r+s-2}n^{r+s-2} + \cdots + b^*_{\{r+s\}}n^{\{r+s\}} \tag{3.9}$$

with

$$\begin{cases} b^*_{r+s-1} = r+s-1; \\ b^*_{r+s-1-k} = \\ \dfrac{\sum\limits_{\substack{k_1+k_2=k \\ k_1\geq 0, k_2\geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+2+k_1}\binom{s}{k_2} t^{k_2} - \sum\limits_{\substack{k_1+k_2=k \\ k_1, k_2\geq 1}} \sum\limits_{\substack{i=r+s-1-k_1 \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \binom{i}{i-r-s+1+k_1} b^*_{r+s-1-k_2}}{r+s-1+k} + \dfrac{\sum\limits_{\substack{i=r+s-k \\ i\geq\{r+s-1\}}}^{r+s-1} b^*_i \left[\binom{i}{i-r-s+2+k} - (r+s-1)\binom{i}{i-r-s+1+k}\right]}{r+s-1+k}; \\ \text{for } [r+s-1] \geq k \geq 1, \end{cases} \tag{3.10}$$

where $[x]$ denotes the integer part of the real part of x, $\{x\} = x - [x]$.

Now we show that (3.9) and (3.10) are also sufficient conditions. If Eq (3.10) holds, the functions $[B^*(n+1, r, s, t) - B^*(n, r, s, t)]\, n^r(n+t)^s$ and $B^*(n+1, r, s, t)B^*(n, r, s, t)$ are analytic functions of n, which means the series of k in formulas (3.4) and (3.5) are their Taylor expansion.

Then consider the following function

$$[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)\,]\, n^r(n+t)^s - B^*_{r,s,t}(n+1)B^*_{r,s,t}(n),$$

the coefficients of the first [r+s] terms are equal to zero, and the tail term is convergent, which proves the Eq (3.2). □

Lemma 3. *If for all $r, s, t \in \mathbb{C}$ and $Re(r+s) > 1$, there exists a generalized complex coefficient polynomial*

$$B^*_{r,s,t}(n) = b^*_{r+s-1}n^{r+s-1} + b^*_{r+s-2}n^{r+s-2} + \cdots + b^*_{\{l\}+1}n + b^*_{\{l\}},$$

where

$$b^*_j\ (j = r+s-1, r+s-2, \cdots, \{l\}) \in \mathbb{C}$$

subjects to the real part of its order is greater than 0*, and*

$$[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)\,]\, n^r(n+t)^s - B^*_{r,s,t}(n+1)B^*_{r,s,t}(n) = O(n^{r+s+\{r+s\}-2}).$$

Then, we have

$$\left(\sum_{k=n}^{\infty} \frac{1}{k^r(k+t)^s}\right)^{-1} = B^*_{r,s,t}(n) + o(1).$$

Proof. Using the equation

$$[B^*_{r,s,t}(n+1) - B^*_{r,s,t}(n)\,]\, n^r(n+t)^s - B^*_{r,s,t}(n+1)B^*_{r,s,t}(n) = O(n^{r+s+\{r+s\}-2}). \tag{3.11}$$

Then, we have

$$\begin{aligned}\frac{1}{B^*_{r,s,t}(n)}-\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}&=\sum_{k=n}^{\infty}\left[\frac{1}{B^*_{r,s,t}(k)}-\frac{1}{B^*_{r,s,t}(k+1)}-\frac{1}{k^r(k+t)^s}\right]\\&=\sum_{k=n}^{\infty}\frac{[B^*_{r,s,t}(k+1)-B^*_{r,s,t}(k)]k^r(k+t)^s-B^*_{r,s,t}(k+1)B^*_{r,s,t}(k)}{B^*_{r,s,t}(k)B^*_{r,s,t}(k+1)k^r(k+t)^s}\\&=\sum_{k=n}^{\infty}\frac{O(k^{r+s+\{r+s\}-2})}{k^{3r+3s-2}+O(k^{3r+3s-3})}\\&=O\left(\frac{1}{n^{[r+s]}}\sum_{k=n}^{\infty}\frac{1}{k^{r+s}}\right)\\&=O(\frac{1}{n^{[r+s]}}).\end{aligned}\tag{3.12}$$

Hence, we have

$$\frac{1}{B^*_{r,s,t}(n)}=\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}=O(\frac{1}{n^{r+s-1}}),$$

and then

$$\begin{aligned}\lim_{n\to\infty}\left|\left(\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}\right)^{-1}-B^*_{r,s,t}(n)\right|&=\lim_{n\to\infty}\left|\frac{1}{\sum\limits_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}}-\frac{1}{\frac{1}{B^*_{r,s,t}(n)}}\right|\\&=\lim_{n\to\infty}\left|\frac{\frac{1}{B^*_{r,s,t}(n)}-\sum\limits_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}}{\frac{1}{B^*_{r,s,t}(n)}\sum\limits_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}}\right|\\&=\lim_{n\to\infty}\left|O\left(\frac{\frac{1}{n^{[r+s]}}}{\frac{1}{n^{r+s-1}}}\right)\right|\\&=\lim_{n\to\infty}\left|O\left(\frac{1}{n^{1-\{r+s\}}}\right)\right|\\&=0,\end{aligned}$$

which is equivalent with

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}\right)^{-1}=B^*_{r,s,t}(n)+o(1).\tag{3.13}$$

This completes the proof. □

Proof of Theorem 1. Since $Re(r+s)>1$ and series

$$\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}$$

absolutely converges and is analytic in the half-plane $Re(r+s)>1$. Then we have

$$\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}\to 0 \quad \text{as} \quad \mathrm{n}\to\infty,$$

Let $n>|t|$, by Lemmas 1 and 2, these are unique generalized complex coefficient polynomials

$$B^*_{r,s,t}(n)=b^*_{r+s-1}n^{r+s-1}+b^*_{r+s-2}n^{r+s-2}+\cdots+b^*_{\{r+s\}}n^{\{r+s\}},$$

where

$$b^*_j\in\mathbb{C}\;(j=r+s-1,r+s-2,\cdots,\{l\})$$

is determined by r,s,t and

$$\begin{cases} b^*_{r+s-1}=r+s-1; \\ b^*_{r+s-1-k}= \\ \dfrac{\sum\limits_{\substack{k_1+k_2=k\\k_1\geq 0,k_2\geq 1}}\sum\limits_{\substack{i=r+s-1-k_1\\i\geq\{r+s-1\}}}^{r+s-1} b^*_i\binom{i}{i-r-s+2+k_1}\binom{s}{k_2}t^{k_2}-\sum\limits_{\substack{k_1+k_2=k\\k_1,k_2\geq 1}}\sum\limits_{\substack{i=r+s-1-k_1\\i\geq\{r+s-1\}}}^{r+s-1} b^*_i\binom{i}{i-r-s+1+k_1}b^*_{r+s-1-k_2}}{r+s-1+k}+\dfrac{\sum\limits_{\substack{i=r+s-k\\i\geq\{r+s-1\}}}^{r+s-1} b^*_i\left[\binom{i}{i-r-s+2+k}-(r+s-1)\binom{i}{i-r-s+1+k}\right]}{r+s-1+k}, \\ \text{for}\;\;[r+s-1]\geq k\geq 1; \end{cases} \tag{3.14}$$

subject to the real part of its order being greater than 0, and

$$[B^*_{r,s,t}(n+1)-B^*_{r,s,t}(n)\,]\,n^r(n+t)^s-B^*_{r,s,t}(n+1)B^*_{r,s,t}(n)=O(n^{r+s+\{r+s\}-2}).$$

Then, in combination with Lemma 3, we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^r(k+t)^s}\right)^{-1}=B^*_{r,s,t}(n)+o(1).$$

This completes thhe proof. □

4. Proof of Corollaries 1 and 2 and Theorem 2

Proof of Corollary 1. If $r=0$ and $t=0$, we have

$$R(r,s,t)|_{r=0,t=0}=R(0,s,0)=\zeta(s) \quad \text{and} \quad \mathrm{R_n(r,s,t)|_{r=0,t=0}=R_n(0,s,0)=\zeta_n(s)}.$$

By Theorem 1, we have

$$\zeta_n(s)=\sum_{k=n}^{\infty}\frac{1}{k^s}=B^*_{0,s,0}(n)+o(1),$$

where

$$B^*_{0,s,0}(n)=b^*_{s-1}n^{s-1}+b^*_{s-2}n^{s-2}+\cdots+b^*_{\{s-1\}}n^{\{s-1\}}$$

and

$$\begin{cases} b^*_{s-1}=s-1; \\ b^*_{s-1-k}=\dfrac{\sum\limits_{\substack{i=s-k\\i\geq\{s-1\}}}^{s-1} b^*_i\left[\binom{i}{i-s+2+k}-(s-1)\binom{i}{i-s+1+k}\right]}{s-1+k}-\dfrac{\sum\limits_{\substack{k_1+k_2=k\\k_1,k_2\geq 1}}\sum\limits_{\substack{i=s-1-k_1\\i\geq\{s-1\}}}^{s-1} b^*_i\binom{i}{i-s+1+k_1}b^*_{s-1-k_2}}{s-1+k},\;\text{for}\;\;[s-1]\geq k\geq 1. \end{cases} \tag{4.1}$$

□

Proof of Corollary 2. The proof of Corollary 2 follows from Corollary 1. □

Before the proof of Theorem 2, we introduce some basic preliminary properties that could be found in [15].

Proposition 5. *The Riemann zeta function $\zeta(s)$ satisfies*

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{4.2}$$

Proposition 6. *(1) The function $\frac{1}{\Gamma(s)}$ is entire and only possesses zeros at $s = 0, -1, -2, \cdots$.*
(2) The function $\Gamma(s)$ is holomorphic for $s \in \mathbb{C}$ except poles at $s = 0, -1, -2, \cdots$ and $\Gamma(s)$ possesses no zero.

Proof of Theorem 2. By Proposition 5, we have

$$\begin{aligned}\zeta(s) &= \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})\frac{1}{\Gamma(\frac{s}{2})}\zeta(1-s)\\ &= \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})\frac{1}{\Gamma(\frac{s}{2})}\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}.\end{aligned}$$

By Proposition 6, we have $\zeta(s)$ is entire in half-plane $Re(s) < 0$. Hence, for $Re(s) < 0$, we have

$$\begin{aligned}\zeta_n(s) &= \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})\frac{1}{\Gamma(\frac{s}{2})}\zeta_n(1-s)\\ &= \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})\frac{1}{\Gamma(\frac{s}{2})}\sum_{k=n}^{\infty}\frac{1}{k^{1-s}}.\end{aligned}$$

By Theorem 1, we derive

$$\zeta_n(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}B^{*}_{0,1-s,0}(n) + o(1),$$

which completes the proof. □

5. Proof of Theorems 3 and 4

Proof of Theorem 3. (i) We first prove

$$a_1 + a_2 + \cdots + a_q = 0, \tag{5.1}$$

where

$$a_m = \cos\left(\frac{\pi(1-s)}{2} - \frac{2\pi mp}{q}\right)$$

for $m = 1, 2, \cdots, q$.

For any complex variable $s = \sigma + it$, we have

$$\cos s = \frac{e^{is} + e^{-is}}{2},$$

and then

$$\begin{aligned}a_1+a_2+\cdots+a_q &= \sum_{m=1}^{q}\cos\left(\frac{\pi(1-s)}{2}-\frac{2\pi mp}{q}\right)\\ &= \frac{1}{2}\sum_{m=1}^{q}\left(e^{\frac{\pi}{2}(t+i(1-\sigma))-\frac{2\pi mpi}{q}}+e^{-\frac{\pi}{2}(t+i(1-\sigma))+\frac{2\pi mpi}{q}}\right)\\ &= \frac{1}{2}e^{\frac{\pi}{2}(t+i(1-\sigma))}\sum_{m=1}^{q}e^{-\frac{2\pi mpi}{q}}+\frac{1}{2}e^{-\frac{\pi}{2}(t+i(1-\sigma))}\sum_{m=1}^{q}e^{\frac{2\pi mpi}{q}}\\ &= 0.\end{aligned}\tag{5.2}$$

Since

$$\frac{1}{1\pm O(\varepsilon)}=1\pm O(\varepsilon),$$

Lemma 1 and Theorem 1, let $n=q(l-1)+1$, then for any sufficiently large positive integer $l\geq 1$ and $Re(s)<0$, if

$$\sum_{m=1}^{q}ma_m\neq 0$$

holds, we have

$$\begin{aligned}\left(\zeta_n\left(s,\frac{p}{q}\right)\right)^{-1} &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left(\sum_{m=1}^{q}a_m\zeta_l\left(1-s,\frac{m}{q}\right)\right)^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left(\sum_{m=1}^{q}a_m\sum_{k=l}^{\infty}\frac{1}{(k+\frac{m}{q})^{1-s}}\right)^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left[\sum_{k=l}^{\infty}\left(\frac{a_1}{(k+\frac{1}{q})^{1-s}}+\frac{a_2}{(k+\frac{2}{q})^{1-s}}+\cdots+\frac{a_q}{(k+1)^{1-s}}\right)\right]^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left[\sum_{k=l}^{\infty}\frac{1}{k^{1-s}}\left(\frac{a_1}{(1+\frac{1}{qk})^{1-s}}+\frac{a_2}{(1+\frac{2}{qk})^{1-s}}+\cdots+\frac{a_q}{(1+\frac{1}{k})^{1-s}}\right)\right]^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left[\sum_{k=l}^{\infty}\frac{1}{k^{1-s}}\left(\sum_{m=1}^{q}a_m\sum_{l_1=0}^{\infty}\binom{s-1}{l_1}\left(\frac{m}{qk}\right)^{l_1}\right)\right]^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left[\sum_{k=l}^{\infty}\frac{1}{k^{1-s}}\left(a_1+a_2+\cdots+a_q+\frac{s-1}{qk}\sum_{m=1}^{q}ma_m+O_{q,s}\left(\frac{1}{k^2}\right)\right)\right]^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\left[\sum_{k=l}^{\infty}\frac{1}{k^{2-s}}\left(\frac{s-1}{q}\sum_{m=1}^{q}ma_m+O_{q,s}\left(\frac{1}{k}\right)\right)\right]^{-1}\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\times\left(\sum_{k=l}^{\infty}\frac{1}{k^{2-s}}\right)^{-1}\left(\frac{s-1}{q}\sum_{m=1}^{q}ma_m\right)^{-1}\left(1+O_{s,q}\left(\frac{1}{n}\right)\right)\\ &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)}\frac{q}{(s-1)\sum_{m=1}^{q}ma_m}\times(1-s)l^{1-s}+O(l^{-s})\\ &= \frac{-\pi q^2}{(2q\pi)^s\,\Gamma(1-s)\sum_{m=1}^{q}ma_m}l^{1-s}+O_{s,q}\left(l^{-s}\right).\end{aligned}\tag{5.3}$$

(ii) Let $n = q(l-1) + r$ for some $q \geq r \geq 1$ and $l, r \in \mathbb{N}_+$, if

$$\sum_{m=1}^{q} m a_m \neq 0$$

and

$$\sum_{m=1}^{q} m a_m - q \sum_{m=r}^{q} a_m \neq 0$$

hold, we have

$$\begin{aligned}
\left(\zeta_n\left(s, \frac{p}{q}\right)\right)^{-1} &= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\sum_{m=1}^{r-1} a_m \zeta_{l+1}\left(1-s, \frac{m}{q}\right) + \sum_{m=r}^{q} a_m \zeta_l\left(1-s, \frac{m}{q}\right) \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\sum_{m=1}^{q} a_m \zeta_{l+1}\left(1-s, \frac{m}{q}\right) + \frac{a_r}{\left(l+\frac{r}{q}\right)^{1-s}} + \frac{a_{r+1}}{\left(l+\frac{r+1}{q}\right)^{1-s}} + \cdots + \frac{a_q}{(l+1)^{1-s}} \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\sum_{k=l+1}^{\infty} \sum_{m=1}^{q} \frac{a_m}{(k+\frac{m}{q})^{1-s}} + \frac{a_r}{\left(l+\frac{r}{q}\right)^{1-s}} + \frac{a_{r+1}}{\left(l+\frac{r+1}{q}\right)^{1-s}} + \cdots + \frac{a_q}{(l+1)^{1-s}} \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\sum_{k=l+1}^{\infty} \sum_{m=1}^{q} \frac{a_m}{(k+\frac{m}{q})^{1-s}} \right)^{-1} \left(1 + \frac{\sum_{m=r}^{q} \frac{a_m}{\left(l+\frac{m}{q}\right)^{1-s}}}{\sum_{k=l+1}^{\infty} \sum_{m=1}^{q} \frac{a_m}{(k+\frac{m}{q})^{1-s}}} \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\frac{-q}{\sum_{m=1}^{q} m a_m} l^{1-s} + O_{s,q}(l^{-s}) \right) \left(1 + \left(\frac{-q}{\sum_{m=1}^{q} m a_m} l^{1-s} + O_{s,q}(l^{-s}) \right) l^{s-1} \sum_{m=r}^{q} \frac{a_m}{\left(1+\frac{m}{ql}\right)^{1-s}} \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\frac{-q}{\sum_{m=1}^{q} m a_m} l^{1-s} + O_{s,q}(l^{-s}) \right) \left(1 + \left(\frac{-q}{\sum_{m=1}^{q} m a_m} + O_{s,q}(l^{-1}) \right) \left(\sum_{m=r}^{q} a_m + O_{q,s}\left(l^{-1}\right) \right) \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\frac{-q}{\sum_{m=1}^{q} m a_m} l^{1-s} + O_{s,q}(l^{-s}) \right) \left(1 - \frac{q \sum_{m=r}^{q} a_m}{\sum_{m=1}^{q} m a_m} + O(l^{-1}) \right)^{-1} \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \left(\frac{-q}{\sum_{m=1}^{q} m a_m} l^{1-s} + O_{s,q}(l^{-s}) \right) \frac{\sum_{m=1}^{q} m a_m}{\sum_{m=1}^{q} m a_m - q \sum_{m=r}^{q} a_m} \left(1 + O(l^{-1})\right) \\
&= \frac{(2\pi q)^{1-s}}{2\Gamma(1-s)} \frac{-q}{\sum_{m=1}^{q} m a_m - q \sum_{m=r}^{q} a_m} l^{1-s} + O(l^{-s}) \\
&= \frac{-\pi q^2}{(2\pi q)^s \Gamma(1-s) \left(\sum_{m=1}^{q} m a_m - q \sum_{m=r}^{q} a_m \right)} l^{1-s} + O(l^{-s}).
\end{aligned} \tag{5.4}$$

□

Proof of Theorem 4. (i) First of all, since χ is a nonprincipal character mod k, we have

$$\chi(1)+\chi(2)+\cdots+\chi(k)=0.$$

Let

$$n=k(l-1)+1,$$

if

$$\sum_{j=1}^{k} j\chi(j)\neq 0$$

holds, and the asymptotic formula of the inverse of the n-th tail of the L-function $L(s,\chi)$ for $Re(s)>1$ is as follows:

$$\begin{aligned}
(L_n(s,\chi))^{-1} &= k^s\left(\sum_{j=1}^{k}\chi(j)\zeta_n\left(s,\frac{j}{k}\right)\right)^{-1}\\
&= k^s\left(\sum_{j=1}^{k}\chi(j)\sum_{i=l}^{\infty}\frac{1}{(i+\frac{j}{k})^s}\right)^{-1}\\
&= k^s\left[\sum_{i=l}^{\infty}\left(\frac{\chi(1)}{(i+\frac{1}{k})^s}+\frac{\chi(2)}{(i+\frac{2}{k})^s}+\cdots+\frac{\chi(k)}{(i+1)^s}\right)\right]^{-1}\\
&= k^s\left[\sum_{i=l}^{\infty}\frac{1}{i^s}\left(\frac{\chi(1)}{(1+\frac{1}{ki})^s}+\frac{\chi(2)}{(1+\frac{2}{ki})^s}+\cdots+\frac{\chi(k)}{(1+\frac{1}{i})^s}\right)\right]^{-1}\\
&= k^s\left(\sum_{i=l}^{\infty}\frac{1}{i^s}\left(\chi(1)+\chi(2)+\cdots+\chi(k)-\frac{s}{ki}\sum_{j=1}^{k}j\chi(j)+O_{s,k}\left(\frac{1}{i^2}\right)\right)\right)^{-1} \qquad (5.5)\\
&= k^s\left(\sum_{i=l}^{\infty}\frac{1}{i^{s+1}}\right)^{-1}\left(-\frac{s}{k}\sum_{j=1}^{k}j\chi(j)+O_{s,k}\left(\frac{1}{l}\right)\right)^{-1}\\
&= k^s\times\frac{k\left(sl^s+O\left(l^{s-1}\right)\right)}{-s\sum_{j=1}^{k}j\chi(j)}\left(1+O_{s,k}\left(\frac{1}{l}\right)\right)\\
&= \frac{-k^{s+1}}{\sum_{j=1}^{k}j\chi(j)}l^s+O_{s,k}\left(l^{s-1}\right).
\end{aligned}$$

(ii) Let

$$n=k(l-1)+r$$

for some $k\geq r\geq 1$ and $l,r\in\mathbb{N}_+$, if

$$\sum_{j=1}^{k}j\chi(j)-k\sum_{j=r}^{k}\chi(j)\neq 0$$

and

$$\sum_{j=1}^{k} j\chi(j) \neq 0$$

hold, we have

$$\begin{aligned}
(L_n(s,\chi))^{-1} &= \left[k^{-s}\left(\sum_{j=1}^{r-1}\chi(j)\zeta_{l+1}\left(s,\frac{j}{k}\right)+\sum_{j=r}^{k}\chi(j)\zeta_{l}\left(s,\frac{j}{k}\right)\right)\right]^{-1}\\
&= k^{s}\left(\sum_{j=1}^{r-1}\chi(j)\sum_{i=l+1}^{\infty}\frac{1}{(i+\frac{j}{k})^{s}}+\sum_{j=r}^{k}\chi(j)\sum_{i=l}^{\infty}\frac{1}{(i+\frac{j}{k})^{s}}\right)^{-1}\\
&= k^{s}\left(\sum_{i=l+1}^{\infty}\sum_{j=1}^{k}\frac{\chi(j)}{(i+\frac{j}{k})^{s}}+\left(\frac{\chi(r)}{(l+\frac{r}{k})^{s}}+\frac{\chi(r+1)}{(l+\frac{r+1}{k})^{s}}+\cdots+\frac{\chi(k)}{(l+1)^{s}}\right)\right)^{-1}\\
&= k^{s}\left[\sum_{i=l+1}^{\infty}\sum_{j=1}^{k}\frac{\chi(j)}{(i+\frac{j}{k})^{s}}\left(1+\frac{\sum\limits_{j=r}^{k}\frac{\chi(j)}{\left(l+\frac{j}{k}\right)^{s}}}{\sum\limits_{i=l+1}^{\infty}\sum\limits_{j=1}^{k}\frac{\chi(j)}{(i+\frac{j}{k})^{s}}}\right)\right]^{-1}\\
&= k^{s}\left(\frac{-k}{\sum\limits_{j=1}^{k} j\chi(j)}l^{s}+O_{s,k}\left(l^{s-1}\right)\right)\left(1+\left(\frac{-k}{\sum\limits_{j=1}^{k} j\chi(j)}l^{s}+O_{s,k}\left(l^{s-1}\right)\right)\left(\sum_{j=r}^{k}\frac{\chi(j)}{\left(1+\frac{j}{kl}\right)^{s}}\right)\right)^{-1} \qquad (5.6)\\
&= k^{s}\left(\frac{-k}{\sum\limits_{j=1}^{k} j\chi(j)}l^{s}+O_{s,k}\left(l^{s-1}\right)\right)\left(1-\frac{k\sum\limits_{j=r}^{k}\chi(j)}{\sum\limits_{j=1}^{k} j\chi(j)}+O_{s,k}\left(l^{-1}\right)\right)^{-1}\\
&= k^{s}\times\frac{-k}{\sum\limits_{j=1}^{k} j\chi(j)-k\sum\limits_{j=r}^{k}\chi(j)}l^{s}+O_{s,k}(l^{s-1})\\
&= \frac{-k^{1+s}}{\sum\limits_{j=1}^{k} j\chi(j)-k\sum\limits_{j=r}^{k}\chi(j)}l^{s}+O_{s,k}(l^{s-1}).
\end{aligned}$$

(iii) Let

$$n = k^{2}(l-1)+1$$

for some $l \in \mathbb{N}_{+}$, if

$$\sum_{j=1}^{k}\sum_{m=1}^{k} m\chi(j)a_{j,m} \neq 0$$

holds, in combination with

$$a_{j,m} = \cos\left(\frac{\pi(1-s)}{2}-\frac{2\pi mj}{k}\right),$$

we obtain

$$\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}=\sum_{j=1}^{k-1}\sum_{m=1}^{k}\chi(j)a_{j,m}=0.$$

Then, we have

$$\begin{aligned}
(L_n(s,\chi))^{-1} &= \left(\frac{\Gamma(1-s)}{2^{-s}\pi^{1-s}k}\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\zeta_l\left(1-s,\frac{m}{k}\right)\right)^{-1}\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left[\sum_{i=l}^{\infty}\left(\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\right)\frac{1}{\left(i+\frac{m}{k}\right)^{1-s}}\right]^{-1}\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left[\sum_{i=l}^{\infty}\frac{1}{i^{1-s}}\left(\sum_{j=1}^{k}\sum_{m=1}^{k}\frac{\chi(j)a_{j,m}}{\left(1+\frac{m}{ki}\right)^{1-s}}\right)\right]^{-1}\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left[\sum_{i=l}^{\infty}\frac{1}{i^{1-s}}\left(\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}+\frac{(s-1)}{ki}\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}+O_{s,k}\left(\frac{1}{i^2}\right)\right)\right]^{-1}\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left[\sum_{i=l}^{\infty}\frac{1}{i^{2-s}}\left(\frac{(s-1)}{k}\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}+O_{s,k}\left(\frac{1}{i}\right)\right)\right]^{-1}\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left(\sum_{i=l}^{\infty}\frac{1}{i^{2-s}}\right)^{-1}\left(\frac{(s-1)}{k}\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}\right)^{-1}\left(1+O_{s,k}(l^{-1})\right)\\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\times\left((1-s)l^{1-s}+O(l^{-s})\right)\frac{k}{(s-1)\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}}\left(1+O_{s,k}(l^{-1})\right)\\
&= \frac{-\pi k^2}{(2\pi)^s\Gamma(1-s)\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}}l^{1-s}+O_{s,k}(l^{-s}).
\end{aligned} \tag{5.7}$$

(iv) Let

$$n=k^2(l-1)+k(u-1)+r$$

for some $k\geq u\geq 1$, $k\geq r\geq 1$, and $l,u,r\in\mathbb{N}_+$, if

$$\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}\neq 0$$

and

$$\sum_{j=1}^{k}\sum_{m=1}^{k}m\chi(j)a_{j,m}-k\left(\sum_{\substack{m=r\\ j=u}}^{k}+\sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\chi(j)a_{j,m}\neq 0$$

hold, we have

$$
\begin{aligned}
(L_n(s,\chi))^{-1} &= \left[\frac{\Gamma(1-s)}{2^{-s}\pi^{1-s}k}\left(\left(\sum_{j=1}^{u-1}\sum_{m=1}^{k}+\sum_{\substack{m=1\\j=u}}^{r-1}\right)\chi(j)a_{j,m}\zeta_{l+1}\left(1-s,\frac{m}{k}\right)+\left(\sum_{\substack{m=r\\j=u}}^{k}+\sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\chi(j)a_{j,m}\zeta_{l}\left(1-s,\frac{m}{k}\right)\right)\right]^{-1} \\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left(\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\zeta_{l+1}\left(1-s,\frac{m}{k}\right)+\left(\sum_{\substack{m=r\\j=u}}^{k}+\sum_{j=u+1}^{k}\sum_{m=1}^{k}\right)\frac{\chi(j)a_{j,m}}{\left(l+\frac{m}{k}\right)^{1-s}}\right)^{-1} \\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left(\sum_{j=1}^{k}\sum_{m=1}^{k}\chi(j)a_{j,m}\zeta_{l+1}\left(1-s,\frac{m}{k}\right)\right)^{-1}\left(1+\frac{\left(\sum\limits_{\substack{m=r\\j=u}}^{k}+\sum\limits_{j=u+1}^{k}\sum\limits_{m=1}^{k}\right)\frac{\chi(j)a_{j,m}}{\left(l+\frac{m}{k}\right)^{1-s}}}{\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{k}\chi(j)a_{j,m}\zeta_{l+1}\left(1-s,\frac{m}{k}\right)}\right)^{-1} \\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\left(\frac{-k}{\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{k}m\chi(j)a_{j,m}}l^{1-s}+O_{s,k}(l^{-s})\right)\left(1-\frac{k\left(\sum\limits_{\substack{m=r\\j=u}}^{k}+\sum\limits_{j=u+1}^{k}\sum\limits_{m=1}^{k}\right)\chi(j)a_{j,m}}{\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{k}m\chi(j)a_{j,m}}+O_{s,k}\left(\frac{1}{l}\right)\right)^{-1} \\
&= \frac{2^{-s}\pi^{1-s}k}{\Gamma(1-s)}\times\frac{-k}{\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{k}m\chi(j)a_{j,m}-k\left(\sum\limits_{\substack{m=r\\j=u}}^{k}+\sum\limits_{j=u+1}^{k}\sum\limits_{m=1}^{k}\right)\chi(j)a_{j,m}}l^{1-s}+O_{s,k}(l^{-s}) \\
&= \frac{-\pi k^2}{(2\pi)^s\Gamma(1-s)\left(\sum\limits_{j=1}^{k}\sum\limits_{m=1}^{k}m\chi(j)a_{j,m}-k\left(\sum\limits_{\substack{m=r\\j=u}}^{k}+\sum\limits_{j=u+1}^{k}\sum\limits_{m=1}^{k}\right)\chi(j)a_{j,m}\right)}l^{1-s}+O_{s,k}(l^{-s}).
\end{aligned} \tag{5.8}
$$

□

6. Conclusions

In this paper, we derive the asymptotic formulas for the reciprocal sums of the Riemann zeta function, the Hurwitz zeta function, and the Dirchlet L-function for $Re(s) > 1$ and $Re(s) < 0$.

Author contributions

Zhenjiang Pan: writing-review and editing, writing-original draft, validation, resources, methodology, formal analysis, conceptualization.

Zhengang Wu: writing-review and editing, resources, methodology, supervision, validation, formal analysis, funding acquisition.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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