Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 13313–13335.
DOI: 10.3934/math.2024649
Received: 26 February 2024
Revised: 22 March 2024
Accepted: 27 March 2024
Published: 10 April 2024

Research article

Some new results involving residual Renyi's information measure for k-record values

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Abstract: This article dealt with further properties of the Renyi entropy and the residual Renyi entropy of k-record values. First, we discussed the Renyi entropy order and its connection with the usual stochastic and dispersive orders. We then addressed the monotonicity properties of the residual Renyi entropy of k-records, focusing on the aging properties of the component lifetimes. We also expressed the residual nth upper k-records in terms of Renyi entropy when the first dataset exceeded a certain threshold, and then studied various properties of the given formula. Finally, we conducted a parametric estimation of the Renyi entropy of the nth upper k-records. The estimation was performed using both real COVID-19 data and simulated data.

Keywords: information theory; k-record values; Renyi entropy; residual Renyi entropy; stochastic orders
Mathematics Subject Classification: 94A08, 62P99

1. Introduction

Accurately quantifying uncertainties in the lifetime of systems is of importance for engineers involved in survival analysis. It is widely acknowledged that systems with longer lifetimes and lower uncertainty are considered superior, while system reliability diminishes as uncertainties increase (refer to Ebrahimi and Pellery [1] for further details). Now, let us consider a nonnegative random variable X that is absolutely continuous, characterized by its probability density function (PDF) denoted as f. The Renyi entropy, which measures the uncertainty in a system, can be defined as

$$\mathcal{H}_\gamma(X) = \varepsilon(\gamma) \log \int_0^\infty f^\gamma(x)dx, \ (\gamma > 0, \gamma \neq 1), \tag{1.1}$$

where "log(·)" means natural logarithm and $\varepsilon(\gamma) = 1/(1-\gamma)$. The Shannon differential entropy, introduced by Shannon [2], has been acquired through $\mathcal{H}(X) = \lim_{\gamma\to 1}\mathcal{H}_\gamma(X) = -\mathbb{E}[\log f(X)]$. It is important to note that the Shannon differential entropy serves as a measure of the uniformity of a probability density function. The highest entropy value is achieved when the density function is uniformly distributed. Consequently, large values of entropy indicate higher uncertainty in the probability density function f and less capacity to anticipate future outcomes of X.

In the context of assessing the lifetime of a new system, the Renyi entropy $\mathcal{H}_\gamma(X)$ serves as a valuable metric for quantifying uncertainty. However, there are scenarios where operators possess knowledge about the system's current age. In some situations, it is often necessary to assess the uncertainty associated with system's residual lifetime, denoted as $X_t = [X - t|X > t]$. This implies that the system's lifetime extends beyond a certain time period t, and it raises the question of the remaining uncertainty associated with it. This situation arises when individuals or organizations seek to make informed decisions or predictions based on the remaining system's lifespan. In this case, the conventional Renyi entropy is no longer suitable for measuring uncertainty. To handle this restriction, the notion of the residual Renyi entropy (RRE) has been raised, which is defined as follows (refer to Gupta and Nanda [3] for further details):

$$\mathcal{H}_\gamma(X;t) = \varepsilon(\gamma)\log\int_0^\infty f_t^\gamma(x)dx, = \varepsilon(\gamma)\log\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)^\gamma dx \tag{1.2}$$

$$= \varepsilon(\gamma)\log\mathbb{E}[f_t^{\gamma-1}(X_t)] = \varepsilon(\gamma)\log\int_0^1 f_t^{\gamma-1}(\overline{F}_t^{-1}(u))du,\ \gamma > 0, \tag{1.3}$$

where $f_t(x) = f(x+t)/\overline{F}(t),\ x,t > 0$, denotes the PDF of X_t, $\overline{F}(t) = P(X > t)$ stands for the survival function of X, and $\overline{F}_t^{-1}(u) = \inf\{x; \overline{F}_t(x) \geq u\}$ is known as the quantile function of $\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t),\ x,t > 0$. The concept of $\mathcal{H}_\gamma(X;t)$ holds significant interest among researchers from diverse scientific and engineering disciplines. It stands as a generalization of the classical Shannon differential entropy, offering a range of valuable properties and applications. Scholars such as [3–5] and numerous others have extensively explored the properties of $\mathcal{H}_\gamma(X;t)$. Recently, Mesfioui *et al.* [6] investigated a coherent system's residual lifetime by using the notion of the Renyi entropy when all components of the system are alive until time t using the system signature and provided several findings for it. Their contributions have significantly contributed to the understanding and utilization of this entropy measure.

The study of record values has witnessed a sustained increase in interest since its inception by Chandler [7]. Record value data finds application in diverse practical scenarios, including destructive stress testing, sporting events, meteorological analysis, oil and mining surveys, hydrology, seismology, and more. A specific example can be found in Glick [8], where the breaking strength of wooden beams is tested. For a comprehensive overview of the theory and application of record values, refer to Ahsanullah [9], Arnold *et al.* [10], and the references provided therein.

Statistical inference based on record data encounters significant challenges due to the rarity of record occurrences in practical situations, coupled with the fact that the expected waiting time for subsequent records is infinite after the first record. In certain scenarios, the focus may shift to the second or third largest values, presenting further complexities. For instance, in the field of actuarial science, when examining insurance claims in nonlife insurance (refer to Kamps [11]), the conventional record model proves inadequate. To address these issues, the model of k-record statistics, introduced by Dziubdziela and Kopocinski [12], offers a suitable alternative.

Zarezadeh and Asadi [13] conducted an investigation into the properties of the Renyi entropy pertaining to order statistics and record values. Habibi *et al.* [14] examined the Kullback-Leibler information of such records, while Abbasnejad and Arghami [15] focused on Renyi information. Baratpur *et al.* [16] recently studied information properties of records using Shannon entropy and mutual information, providing entropy bounds. Recently, Jose and Sathar [17] comprehensively discussed the Renyi entropy and important properties of k-records derived from continuous distributions. They also represented noteworthy applications of Renyi entropy for k-records and proposed a simple estimator along with a numerical illustration using real-life data. Moreover, Asha and Chacko [18] explored and investigated several properties of the RRE of k-record values originating from an absolutely continuous distribution including representation, bounds, and stochastic orders. In contrast, Asha and Chacko [19] conducted a study exploring the properties of Verma entropy, which serves as a broader framework encompassing Renyi's entropy. Their investigation focused on k-record values derived from an absolutely continuous distribution, aiming to uncover key characteristics and insights within this context. Furthermore, Shrahili and Kayid [20] conducted a study on the residual Tsallis entropy of upper record values obtained from independent and identically distributed random variables. Building upon this work, this paper delves into further investigations and presents detailed results on the Renyi entropy of k-records derived from continuous distributions.

The result of this paper is organized as follows: In Section 2, we discuss the implications of the Renyi entropy order, considering both the usual stochastic order and dispersive order. We establish these implications under certain sufficient conditions and apply them to the context of k-record values. Section 3 presents additional results focusing on the monotonicity properties of the RRE of k-record values, specifically considering the aging properties of the component lifetimes. Moving to Section 4, we explore and derive various properties of the Renyi entropy for the residual n-th upper k-records when the first record exceeds a specified threshold level. In Section 5, we present a parametric estimator of the Renyi entropy of nth upper k-records. Finally, Section 6 serves as the conclusion of the paper, summarizing the key findings and contributions.

Throughout this paper, we consider nonnegative random variables denoted by X and Y. These variables have absolutely continuous cumulative distribution functions (CDFs) denoted by $F(x)$ and $G(x)$, survival functions denoted by $\overline{F}(x)$ and $\overline{G}(x)$, and PDFs denoted by $f(x)$ and $g(x)$, respectively. The terms "increasing" and "decreasing" are used in a non-strict sense. We adopt the following notions: The increasing failure rate (IFR) and decreasing failure rate (DFR), the new worse than used (NWU), the usual stochastic order denoted by $X \leq_{st} Y$, the hazard rate order denoted by $X \leq_{hr} Y$, and the dispersive order denoted by $X \leq_{d} Y$. For informal definitions and properties of these notions, we refer readers to the work of Shaked and Shanthikumar [21].

2. Results on Renyi entropy of k-records

Here, we present additional findings concerning the Renyi entropy of k-records. Let us consider a sequence of independent and identically distributed (i.i.d.) random variables denoted by $\{X_i, i \geq 1\}$, with a CDF of $F(x)$ and a PDF of $f(x)$. An observation X_j is termed an upper record value if it is greater than X_i for every $j > i$. To quantify these upper record values, Dziubdziela and Kopocinski [12] introduced the indices $\{R_k(n), n \geq 1\}$, which represent the times of the n-th upper k-record for the sequence $\{X_i, i \geq 1\}$, defined by

$$R_k(1) = 1,\ R_k(n+1) = \min\{j : j > R_k(n), X_{j:j+k-1} > X_{R_k(n):R_k(n)+k-1}\},$$

where $X_{j:m}$ represents the j-th order statistic coming from i.i.d. random variables of size m. So, one can define $U_{n(k)}$ as a sequence of n-th upper k-record values of the sequence $\{X_i, i \geq 1\}$, given by $U_{n(k)} = X_{R_k(n):R_k(n)+k-1}$. It follows that

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)}[\overline{F}(x)]^{k-1}[-\log \overline{F}(x)]^{n-1} f(x),\ x > 0, \tag{2.1}$$

$$\overline{F}_{n(k)}(x) = [\overline{F}(x)]^k \sum_{i=0}^{n-1} \frac{[-k \log \overline{F}(x)]^i}{i!} = \frac{\Gamma(n, -k \log \overline{F}(x))}{\Gamma(n)},\ \ x \geq 0, \tag{2.2}$$

where

$$\Gamma(a, x) = \int_x^\infty u^{a-1} e^{-u} du,\ a, x > 0, \tag{2.3}$$

is known as the incomplete gamma function and $\Gamma(n) = \Gamma(n, 0)$ is known as the complete gamma function. The Renyi entropy of n-th upper k-record values can be defined as

$$\mathcal{H}_\gamma(U_{n(k)}) = \varepsilon(\gamma) \log \int_0^\infty f^\gamma_{n(k)}(x) dx, \tag{2.4}$$

for all $\gamma > 0, \gamma \neq 1$. In their comprehensive study, Jose and Sathar [17] extensively explored the Renyi entropy of k-records originating from continuous distributions, including deriving expression and bounds for the Renyi entropy of k-records. Furthermore, the study presented several fundamental properties associated with the Renyi entropy of k-records. Building upon the aforementioned research, this study extends the investigation into the Renyi entropy of k-records. Let us first introduce the following theorem.

Theorem 2.1. *Let us suppose nonnegative random variables X and Y, where their PDFs are provided by $f(x)$ and $g(x)$, respectively. Additionally $X \leq_{st} Y$ and Y is DFR, then $\mathcal{H}_\gamma(X) \leq \mathcal{H}_\gamma(Y)$, for all $\gamma > 1$.*

Proof. Let us assume that Y is DFR and X is stochastically smaller than Y, represented as $X \leq_{\mathrm{st}} Y$. Consequently, we can derive the following:

$$\begin{aligned} \int_0^\infty g^\gamma(x) \mathrm{d}x &= \int_0^\infty g(x) g^{\gamma-1}(x) \mathrm{d}x \\ &\leq \int_0^\infty f(x) g^{\gamma-1}(x) \mathrm{d}x \\ &\leq \left(\int_0^\infty f^\gamma(x) \mathrm{d}x\right)^{\frac{1}{\gamma}} \left(\int_0^\infty \left(g^{\gamma-1}(x)\right)^{\frac{\gamma}{\gamma-1}} \mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}} \\ &= \left(\int_0^\infty f^\gamma(x) \mathrm{d}x\right)^{\frac{1}{\gamma}} \left(\int_0^\infty g^\gamma(x) \mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}}. \end{aligned} \tag{2.5}$$

To establish the inequality presented in (2.5), we begin by observing that the stochastic ordering $X \leq_{\rm st} Y$ yields $\mathbb{E}_X[g^{\gamma-1}(X)] \geq \mathbb{E}_Y[g^{\gamma-1}(Y)]$ for $\gamma > 1$. This inequality arises from the fact that $g(x)$ decreases in x since Y is DFR. Applying Hölder's inequality, we derive the second inequality. By utilizing Eqs (1.1) and (2.5), we can conclude that the desired result is obtained. □

It is important to highlight that DFR distributions encompass significant examples such as the gamma and Weibull distributions with shape parameters less than one, the Pareto distribution, and mixtures of exponential distributions. The applicability of Theorem 2.1 extends to these DFR distributions as well as others. Additionally, we present another valuable theorem in the following.

Theorem 2.2. *Let us suppose nonnegative random variables X and Y where their PDFs are provided by $f(x)$ and $g(x)$, respectively. If $f(x)$ is increasing in x and $X \leq_{st} Y$, then $\mathcal{H}_\gamma(X) \geq \mathcal{H}_\gamma(Y)$ for $\gamma > 1$.*

Proof. Assume that the PDF $f(x)$ is increasing in x, and $X \leq_{\rm st} Y$. Utilizing analogous reasoning to the proof of Theorem 2.1, we can establish the following deduction:

$$\int_0^\infty f^\gamma(x)\mathrm{d}x \leq \left(\int_0^\infty g^\gamma(x)\mathrm{d}x\right)^{\frac{1}{\gamma}} \left(\int_0^\infty f^\gamma(x)\mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}}. \tag{2.6}$$

By applying Eqs (1.1) and (2.6), we can derive the desired result. □

The following corollary applies the above theorem to the k-record values.

Corollary 2.1. *Let X be a non negative random variable in which $f(x)$ is increasing in x. For a fixed $n, k \geq 1$, if $X \leq_{st} U_{n(k)}$, then $\mathcal{H}_\gamma(X) \geq \mathcal{H}_\gamma(U_{n(k)})$ for $\gamma > 1$.*

Proof. Since X has an increasing probability density function, we can readily obtain the desired results by applying Theorem 2.2. □

The following example delivers an application of Corollary 2.1 which demonstrates that the Renyi entropy in the original random variable is greater than that of n-th upper k-records derived from the original random variable. Furthermore, we provide a counterexample to demonstrate that Corollary 2.1 does not hold for all values of $0 < \gamma < 1$.

Example 2.1. Let's consider a random variable X that follows a uniform distribution on $(2, 7)$, characterized by a PDF given by $f(x) = \frac{1}{5}$ for $2 < x < 7$. We utilize the Renyi entropy as a measure of uncertainty associated with the random variable X. Applying Eq (1.1), we find that $\mathcal{H}_\gamma(X) = \log 5$. Next, we determine the Renyi entropy of $U_{n(k)}$ using Eq (2.4), which can be calculated as follows:

$$\mathcal{H}_\gamma(U_{n(k)}) = \varepsilon(\gamma) \log\left[\frac{k^{n\gamma}}{\Gamma^\gamma(n)5^{\gamma-1}} \frac{\Gamma(\gamma(n-1)+1)}{(\gamma(k-1)+1)^{\gamma(n-1)+1}}\right].$$

It is obvious that the Renyi entropy of X does not depend on the parameter γ. In this case, since $f(x)$ is an increasing function of x, we can proceed with the analysis. Assuming $n = 10$, we plotted the graph of $\overline{F}_{n(k)}(x) - \overline{F}(x)$ for various values of $k = 1, 2, \ldots, 8$ in Figure 1. The graph shows that $\overline{F}_{10(k)}(x) - \overline{F}(x) \geq 0$ for all values of $2 < x < 7$ when $k = 1, 2, 3, 4$. However, this inequality does not hold for all values of $2 < x < 7$ when $k = 5, 6, 7, 8$. Hence, we can conclude that $X \leq_{\rm st} U_{10(k)}$ for $k = 1, 2, 3, 4$. Consequently, the assumption of Corollary 2.1 holds true, allowing us to conclude that

$\mathcal{H}_\gamma(X) \geq \mathcal{H}_\gamma(U_{10(k)})$ for $k = 1, 2, 3, 4$ and $\gamma > 1$. We further plotted the graph of $\mathcal{H}_\gamma(X) - \mathcal{H}_\gamma(U_{10(k)})$, for $k = 1, 2, 3, 4$ in the left panel and for $k = 5, 6, 7, 8$ in the right panel in Figure 2. It is evident from the graph that $\mathcal{H}_\gamma(X) - \mathcal{H}_\gamma(U_{10(k)}) \geq 0$, for $k = 1, 2, 3, 4$ and $\gamma > 1$, while the same inequality does not hold for $k = 5, 6, 7, 8$. So, one can result that the uncertainty of X, as measured by the Renyi entropy, is greater than that of $U_{10(k)}$ for $k = 1, 2, 3, 4$ and $\gamma > 1$. Furthermore, as depicted in Figure 3, we can observe that the inequality $\mathcal{H}_\gamma(X) - \mathcal{H}_\gamma(U_{10(k)}) \geq 0$, does not hold for $k = 1, 2, 3, 4$ when $0 < \gamma < 1$, despite the fact that $X \leq_{\text{st}} U_{10(k)}$ holds true for $k = 1, 2, 3, 4$.

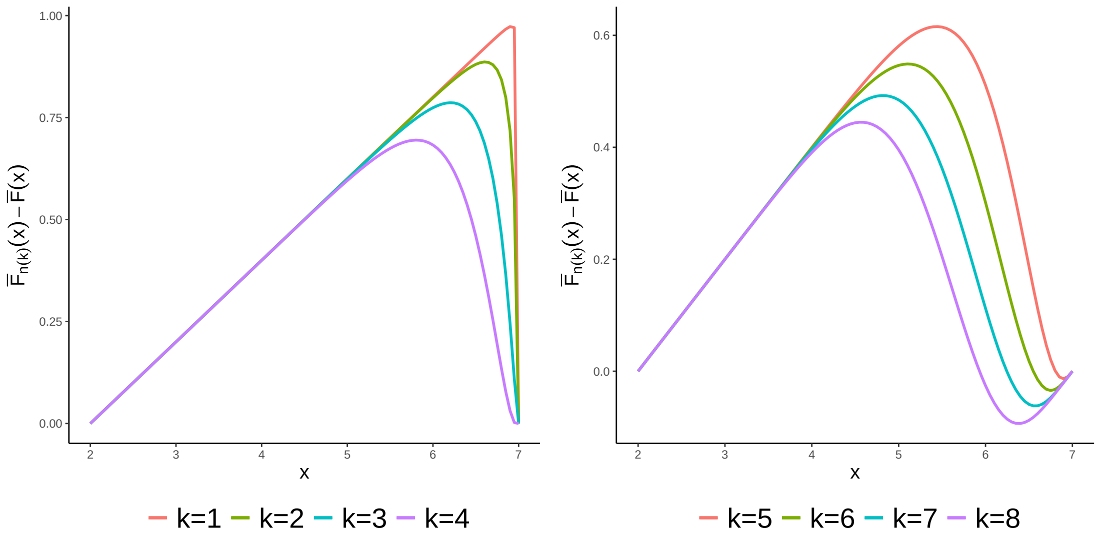

Figure 1. The $\overline{F}_{10(k)}(x) - \overline{F}(x)$ for $k = 1, 2, 3, 4$ (left panel) and $k = 5, 6, 7, 8$ (right panel) with respect to $2 < x < 7$.

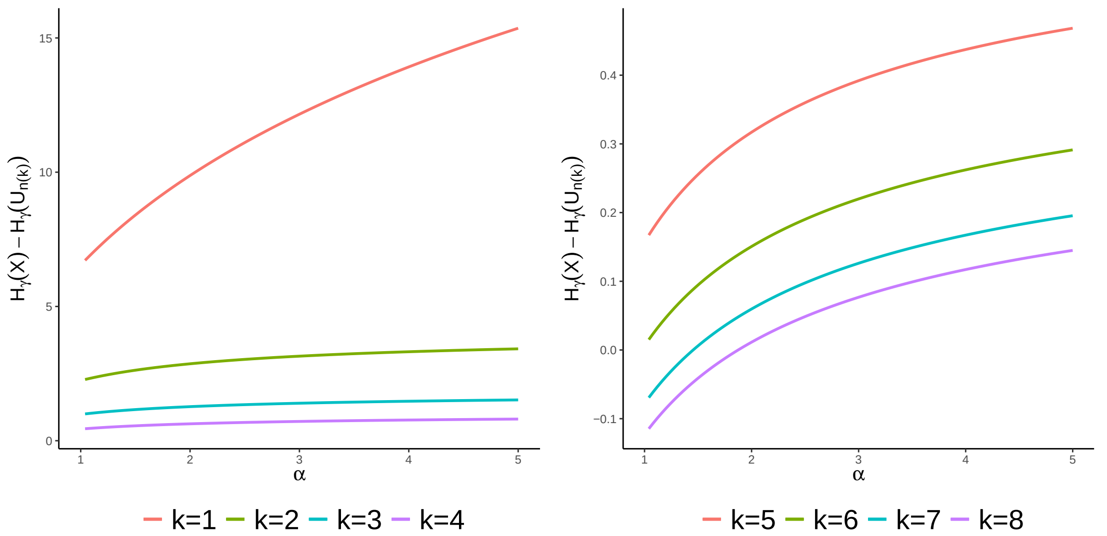

Figure 2. The values of $\mathcal{H}_\gamma(X) - \mathcal{H}_\gamma(U_{10(k)})$ for $k = 1, 2, 3, 4$ (left panel) and $k = 5, 6, 7, 8$ (right panel) when $\gamma > 1$.

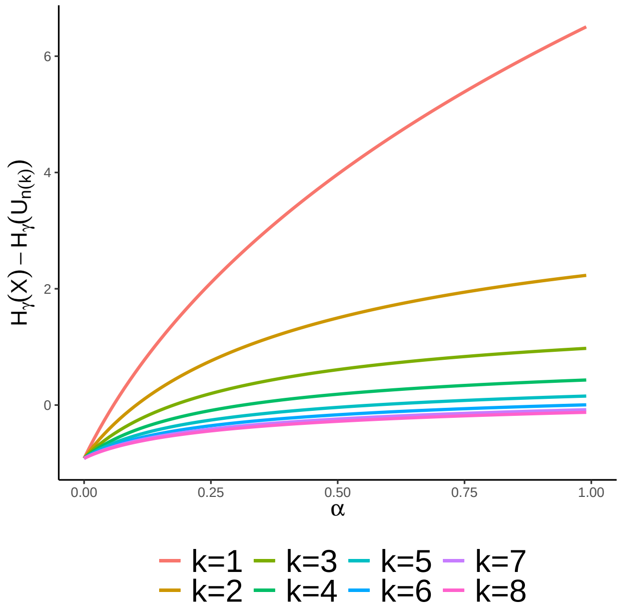

Figure 3. The values of $\mathcal{H}_\gamma(X) - \mathcal{H}_\gamma(U_{10(k)})$ for $k = 1, 2, \ldots, 8$ when $0 < \gamma < 1$.

Let us consider the random variable $U^X_{n(k_1)}$, which represents the n-th upper k_1-record value made of i.i.d. sequence of $\{X_i, i \geq 1\}$ with a common CDF F. Additionally, we define $U^Y_{n(k_2)}$ as the n-th upper k_2-record value in the i.i.d. sequence $\{Y_i, i \geq 1\}$ having the common CDF G, satisfying the condition $k_1 \geq k_2$. In the next theorem, we establish sufficient conditions for ordering the Renyi entropy of two records in relation to the dispersive order of the parent distribution and under sufficient conditions.

Theorem 2.3. *In the case where either X or Y is DFR and if $X \leq_d Y$, then $\mathcal{H}_\gamma(U^X_{n(k_1)}) \leq \mathcal{H}_\gamma(U^Y_{n(k_2)})$ for $\gamma > 0$ when $k_1 \geq k_2$.*

Proof. The result can be derived from Part (c) of Theorem 2.1 in the work of Khaledi and Shojaei [22] and Theorem 2.1 presented by Abbasnejad and Arghami [23]. □

Example 2.2. Consider a sequence of i.i.d. random variables $\{X_i, i \geq 1\}$ following the Makeham distribution. The survival function of X is given by $\overline{F}(x) = e^{-x+\theta(x+e^{-x}-1)}$, where $x > 0$, and $\theta > 0$. The hazard rate of X is $\lambda_X(x) = f(x)/\overline{F}(x) = 1 + \theta(1 - e^{-x})$. Similarly, let $\{Y_i, i \geq 1\}$ be another sequence of i.i.d. random variables following an exponential distribution with the survival function $\overline{G}(x) = g(x)/\overline{G}(x) = e^{-x}$, where $x > 0$, and the hazard rate function $\lambda_Y(x) = 1$. It can be shown that $X \leq_{hr} Y$ for $\theta > 0$, and Y is DFR. Therefore, we conclude that $X \leq_d Y$ based on the result established by Bagai and Kochar [24]. As a consequence, Theorem 2.3 becomes applicable, implying that $\mathcal{H}_\gamma(U^X_{n(k_1)}) \leq \mathcal{H}_\gamma(U^Y_{n(k_2)})$ for $\gamma > 0$ when $k_1 \geq k_2$.

3. Results on RRE of k-record values

In the subsequent analysis, we focus shifts toward investigating the RRE of the random variable $U_{n(k)}$. This quantity quantifies the uncertainty present in the density of $[U_{n(k)} - t | U_{n(k)} > t]$ and provides insights into residual lifetime's predictability of n-th upper k-records. Asha and Chacko [18] studied several properties of the RRE of k-record values arising from an absolutely continuous distribution. In this regard, they obtained the following representation:

$$\mathcal{H}_\gamma(U_{n(k)};t) = \mathcal{H}_\gamma(U^\star_{n(k)};F(t)) + \varepsilon(\gamma)\log\mathbb{E}[f^{\gamma-1}(F^{-1}(1-e^{-V_{n(k)}}))],\ t>0, \tag{3.1}$$

where $U^\star_{n(k)}$ denotes the nth upper k-record of uniform distribution over (0, 1) and $V_{n(k)} \sim \Gamma_{-\log\overline{F}(t)}(\gamma(n-1)+1, \gamma(k-1)+1)$. Here, the notation $V \sim \Gamma_\tau(b,d)$ indicates that the random variable V has a truncated Gamma distribution with the following PDF:

$$f_V(z) = \frac{d^b}{\Gamma(b,\tau)} z^{b-1}e^{-dz},\ z>\tau>0, \tag{3.2}$$

such that $b > 0$ and $d > 0$. The aging property of the random variable X has a vital role in shaping the behavior of its RRE for a given order $\gamma > 0$. To establish this connection, we present the following theorem, which demonstrates how the IFR (DFR) property of the parent distribution impacts the characteristics of the RRE for record values.

Theorem 3.1. *Let X have IFR(DFR) property, then $\mathcal{H}_\gamma(X;t)$ is decreasing (increasing) in t.*

Proof. We specifically concentrate on the scenario where X has IFR property, although a similar approach can be applied when X is DFR. It is worth noting that $f_t(\overline{F}_t^{-1}(u)) = u\lambda_t(\overline{F}_t^{-1}(u))$ for $0 < u < 1$, where $\lambda_t(x) = f_t(x)/\overline{F}_t(x)$ represents the hazard rate of the conditional distribution $[X - t|X > t]$. This relationship allows us to express Eq (1.3) as follows:

$$e^{(1-\gamma)\mathcal{H}_\gamma(X;t)} = \int_0^1 u^{\gamma-1}\left(\lambda_t(\overline{F}_t^{-1}(u))\right)^{\gamma-1} du,\ t>0, \tag{3.3}$$

for all $\gamma > 0$. We can easily verify that

$$\lambda_t(\overline{F}_t^{-1}(u)) = \lambda(\overline{F}_t^{-1}(u)+t) = \lambda(\overline{F}^{-1}(u\overline{F}(t))), 0<u<1. \tag{3.4}$$

For $t_1 \le t_2$, we have $\overline{F}^{-1}(u\overline{F}(t_1)) \le \overline{F}^{-1}(u\overline{F}(t_2))$. Consequently, in the case where X exhibits an IFR property, for all $\gamma > 1$ $(0 < \gamma \le 1)$, we obtain the following inequality:

$$\begin{aligned}
\int_0^1 u^{\gamma-1}\left(\lambda_{t_1}(\overline{F}_{t_1}^{-1}(u))\right)^{\gamma-1} du &= \int_0^1 u^{\gamma-1}\left(\lambda(\overline{F}^{-1}(u\overline{F}(t_1)))\right)^{\gamma-1} du \\
&\le (\ge) \int_0^1 u^{\gamma-1}\left(\lambda(\overline{F}^{-1}(u\overline{F}(t_2)))\right)^{\gamma-1} du \\
&= \int_0^1 u^{\gamma-1}\left(\lambda_{t_2}(\overline{F}_{t_2}^{-1}(u))\right)^{\gamma-1} du,
\end{aligned}$$

for all $t_1 \le t_2$. Using (3.3), we get

$$e^{(1-\gamma)\mathcal{H}_\gamma(X;t_1)} \le (\ge) e^{(1-\gamma)\mathcal{H}_\gamma(X;t_2)},$$

when $\gamma > 1(0 < \gamma \le 1)$, and this completes the proof by recalling (1.2). □

The subsequent theorem establishes the impact of the IFR property on the behavior of the RRE of k-record values.

Theorem 3.2. *Assume X to be IFR. Thus, for all $\gamma > 0$, $\mathcal{H}_\gamma(U_{n(k)};t)$ is decreasing in t.*

Proof. The assumption that X is IFR leads to the conclusion that $U_{n(k)}$ also possesses an IFR property, as indicated in the remark provided by Raqab and Amin [25]. Consequently, we can deduce that the RRE $\mathcal{H}_\gamma(U_{n(k)};t)$, for all $\gamma > 0$, exhibits a decreasing behavior in t in accordance with the findings presented in Theorem 3.1. □

The next example demonstrates how to use Theorem 3.2.

Example 3.1. Assume a sequence of i.i.d. random variables $\{X_i, i \geq 1\}$ that follows a common Weibull distribution with the CDF as

$$F(x) = 1 - e^{-x^3},\ x > 0. \tag{3.5}$$

It is not hard to see that $f(F^{-1}(1-e^{-u})) = 3u^{\frac{2}{3}}e^{-u},\ 0 < u < 1$, then, we can calculate

$$\mathbb{E}[f^{\gamma-1}(F^{-1}(1-e^{-V_{n(k)}}))] = \frac{3^{\gamma-1}(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma(\gamma(n-\frac{1}{3})+\frac{1}{3},\gamma kt^3)}{(\gamma k)^{\gamma(n-\frac{1}{3})+\frac{1}{3}}\Gamma(\gamma(n-1)+1,t^3(\gamma(k-1)+1))},$$

and

$$\mathcal{H}_\gamma(U^\star_{n(k)};F(t)) \quad = \quad \varepsilon(\gamma)\log\frac{k^{n\gamma}\Gamma(\gamma(n-1)+1,t^3(\gamma(k-1)+1)))}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^\gamma(n,kt^3)}.$$

Using (3.1), we get

$$\mathcal{H}_\gamma(U_{n(k)};t) = \varepsilon(\gamma)\log\left[\frac{3^{\gamma-1}k^{n\gamma}\Gamma(\gamma(n-\frac{1}{3})+\frac{1}{3},\gamma kt^3)}{(\gamma k)^{\gamma(n-\frac{1}{3})+\frac{1}{3}}\Gamma^\gamma(n,kt^3)}\right],\ n \geq 1. \tag{3.6}$$

To investigate the behavior of the RRE $\mathcal{H}_\gamma(U_{n(k)};t)$, we consider the case where $n = 5$. We plotted $\mathcal{H}_\gamma(U_{5(k)};t)$, with respect to t for $\gamma = 0.5$ and $\gamma = 2$ and considering different values of $k = 1,2,\cdots,5$. The resulting plots are presented in Figure 4. The observed trends in the plots align with the findings of Theorem 3.2, which establishes that the RRE decreases as t increases when the random variable X exhibits an IFR property.

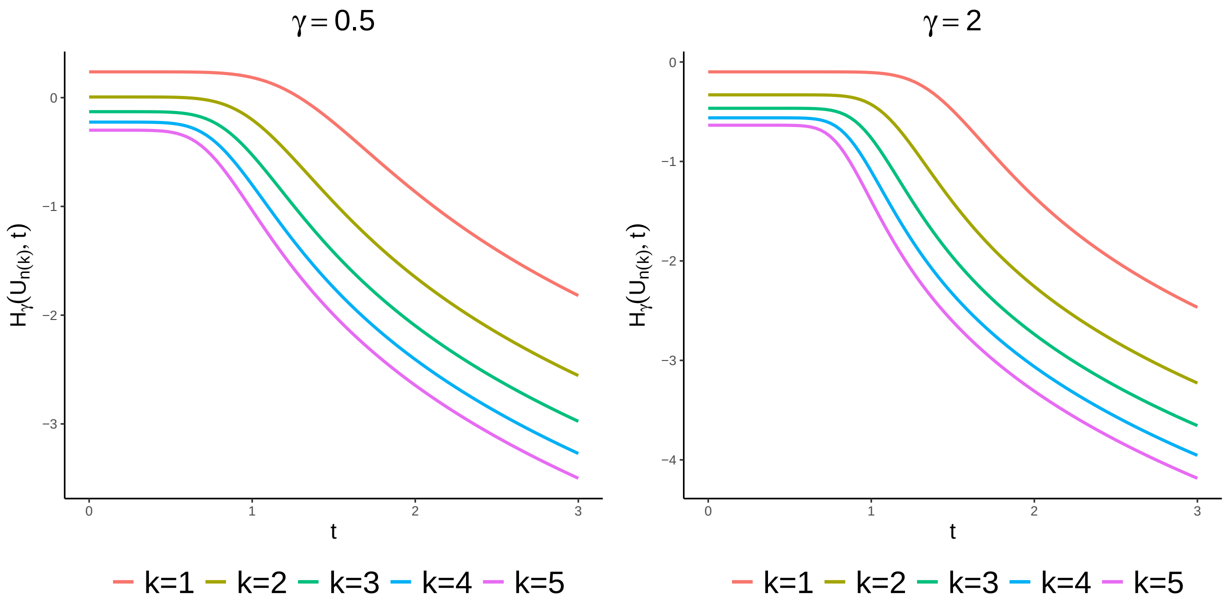

Figure 4. The graph of $\mathcal{H}_\gamma(U_{5(k)};t)$ for $\gamma = 0.5$ (left panel) and $\gamma = 2$ (right panel) as a function of t.

The following theorem explores the relationship between the RRE functions of two random variables and the proportional hazard rates model.

Theorem 3.3. *Assume that X and Y are two absolutely continuous nonnegative random variables with survival functions $\overline{F}(t)$ and $\overline{G}(t)$, and hazard rate functions $\lambda_X(t)$ and $\lambda_Y(t)$, respectively. Additionally, let $\theta(t)$ be a nonnegative increasing function satisfying the condition $\lambda_Y(t) = \theta(t)\lambda_X(t)$ for $t > 0$, where $0 \le \theta(t) \le 1$. If $\mathcal{H}_\gamma(X;t)$ is a decreasing function of t, then $\mathcal{H}_\gamma(Y;t)$ is also decreasing in t for all $\gamma > 0$, provided that $\lim_{t\to\infty} \frac{\overline{G}(t)}{\overline{F}(t)} < \infty$.*

Proof. We provide a proof specifically for the case where $\gamma > 1$, noting that the case $0 < \gamma < 1$ follows a similar reasoning. Considering Eq (1.3), it is evident that $\mathcal{H}_\gamma(Y;t)$ decreases with t if, and only if, $\mathbb{E}[g_t^{\gamma-1}(Y_t)]$ increases with t. Let us set $\delta_1(t) = \mathbb{E}[f_t^{\gamma-1}(X_t)]$ and $\delta_2(t) = \mathbb{E}[g_t^{\gamma-1}(Y_t)]$. From the derivatives $\delta_1'(t) = \lambda_X(t)[\gamma\delta_1(t) - \lambda_X^{\gamma-1}(t)]$ and $\delta_2'(t) = \lambda_Y(t)[\gamma\delta_2(t) - \lambda_Y^{\gamma-1}(t)]$, it follows that $\delta_2(t)$ increases with t if

$$\gamma\delta_2(t) \ge \lambda_Y^{\gamma-1}(t) = \theta^{\gamma-1}(t)\lambda_X^{\gamma-1}(t),$$

which holds if $\delta_2(t) \ge \theta^{\gamma-1}(t)\delta_1(t),\ t > 0$. Let us define the function $\zeta(t)$ as

$$\zeta(t) = \overline{G}_\gamma(t)\left[\theta^{\gamma-1}(t)\delta_1(t) - \delta_2(t)\right],$$

where $\overline{G}_\gamma(t) = [\overline{G}(t)]^\gamma$. Next, we will demonstrate that $\zeta(t) \le 0$. To begin, let us differentiate $\zeta(t)$ with respect to t and carry out some algebraic manipulations, resulting in

$$\begin{aligned}\zeta'(t) &= -g_\gamma(t)\left(\theta^{\gamma-1}(t)\delta_1(t) - \delta_2(t)\right) + \overline{G}_\gamma(t)\left\{(\gamma-1)\theta'(t)\theta^{\gamma-2}(t)\delta_1(t) + \theta^{\gamma-1}(t)\delta_1'(t) - \delta_2'(t)\right\}\\ &= \overline{G}_\gamma(t)\left\{(\gamma-1)\theta'(t)\theta^{\gamma-2}(t)\delta_1(t) + (1-\theta(t))\theta^{\gamma-1}(t)\delta_1'(t)\right\}.\end{aligned}$$

Since $0 \le \theta(t) \le 1$ and $\delta_1(t)$ is an increasing function of t, we can conclude that for $\gamma > 1$, $\zeta'(t) > 0$, which means that $\zeta(t)$ increases when t increases. Now, considering that $\lim \delta_{t\to\infty} \frac{\overline{G}(t)}{\overline{F}(t)} < \infty$, we obtain

$$\begin{aligned}\lim_{t\to\infty}\zeta(t) &= \lim_{t\to\infty}\left\{\left(\frac{\overline{G}(t)}{\overline{F}(t)}\right)^\gamma \int_t^\infty \theta^{\gamma-1}(t) f^\gamma(x)dx\right\}\\ &- \lim_{t\to\infty}\left\{\int_t^\infty g^\gamma(x)dx\right\} = 0.\end{aligned}$$

Consequently, we have $\zeta(t) \le 0$ for all t, implying that $\theta^{\gamma-1}(t)\delta_1(t) \le \delta_2(t)$. This completes the proof of the theorem. □

The following theorem gives an important result concerning the closure property of decreasing RRE of distributions under the formation of k-record values. Hereafter, we assume that $\{X_i, i \ge 1\}$ is a sequence of i.i.d. random variables with CDF F and PDF f, and $U_{n(k)}$ denotes the n-th upper k-record values.

Theorem 3.4. *If $H_\gamma(X;t)$ is decreasing in t, then $H_\gamma(U_{n(k)};t)$ is also decreasing in t for all $\gamma > 0$.*

Proof. From (2.1) and (2.2), the hazard rate function of $U_{n(k)}$ can be written as

$$\lambda_{n(k)}(t) = \frac{f_{n(k)}(t)}{\overline{F}_{n(k)}(t)} = \Psi_{n(k)}(t)\lambda(t) \tag{3.7}$$

where

$$\Psi_{n(k)}(t) = \frac{[-\log \overline{F}(t)]^{n-1}/\Gamma(n)}{\sum_{i=0}^{n-1} \frac{[-k\log \overline{F}(t)]^{i}}{i!}}.$$

It is evident that $\Psi_{n(k)}(t)$ increases monotonically with respect to t and takes values within the range $(0, 1)$. Moreover, it can be readily observed that

$$\lim_{t\to\infty} \frac{\overline{F}_{n(k)}(t)}{\overline{F}(t)} = 0.$$

Consequently, the assumptions of Theorem 3.3 are satisfied, thereby establishing that $\mathcal{H}_\gamma(U_{n(k)}; t)$ decreases with respect to t for all $\gamma > 0$. □

The following example illustrates the application of Theorem 3.4.

Example 3.2. Let us consider a sequence of i.i.d. random variables $\{X_i, i \geq 1\}$ that follow a uniform distribution on the interval $[0, 1]$. It is straightforward to observe that

$$\mathcal{H}_\gamma(X; t) = \log(1 - t), \quad 0 < t < 1,$$

which clearly decreases with increasing t. Additionally, employing (3.1), we can derive the following expression

$$\mathcal{H}_\gamma(U_{n(k)}; t) = \varepsilon(\gamma) \log \frac{k^{n\gamma}\Gamma(\gamma(n-1)+1, -\log(1-t)(\gamma(k-1)+1)))}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^{\gamma}(n, -k\log(1-t))}, \ 0 < t < 1.$$

To investigate the behavior of the RRE $\mathcal{H}_\gamma(U_{n(k)}; t)$, we consider the case where $n = 5$. We plot the graph of $\mathcal{H}_\gamma(U_{n(k)}; t)$ for different values of $\gamma = 0.5$ and $\gamma = 2$, while varying t, and considering different values of $k = 1, 2, \cdots, 5$. The resulting plots are presented in Figure 5. It shows that the RRE decreases as t increases.

The following example shows that if $\mathcal{H}_\gamma(X; t)$ is increasing in t, then $\mathcal{H}_\gamma(U_{n(k)}; t)$ need not to be increasing in t for all $\gamma > 0$.

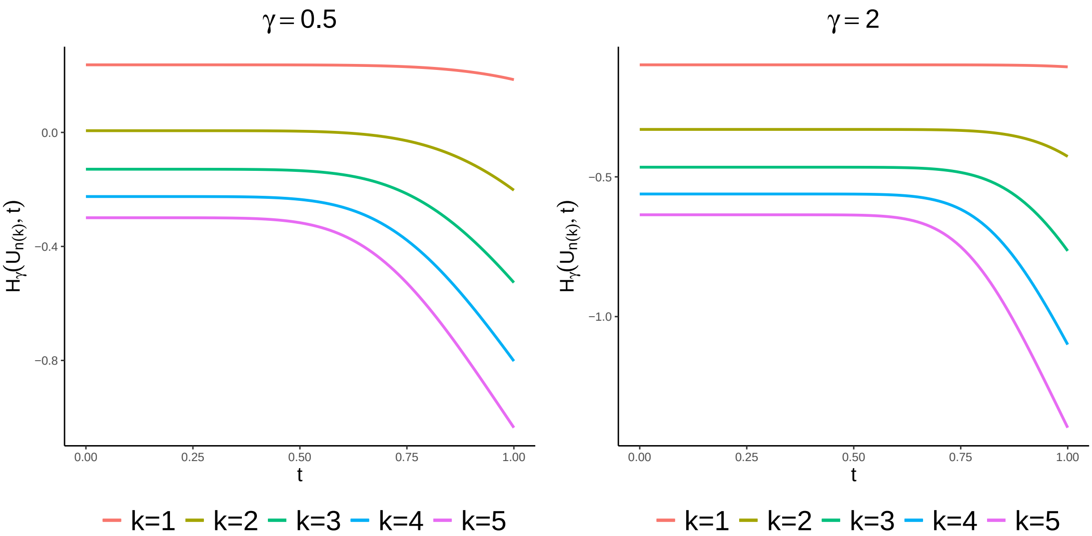

Figure 5. The graph of $\mathcal{H}_\gamma(U_{n(k)}; t)$ for $\gamma = 0.5$ (left panel) and $\gamma = 2$ (right panel) as a function of t.

Example 3.3. Let us consider a sequence of i.i.d. random variables $\{X_i, i \geq 1\}$ that follow a Pareto distribution with the survival function $\overline{F}(x) = 1/(1+x),\ x > 0$. The hazard rate of X is $\lambda_X(x) = 1/(1+x)$, which is a decreasing function of x. So, X is DFR and, hence, $H_\gamma(X; t)$ is an increasing function of t. Additionally, employing (3.1), we can derive the following expression:

$$\mathcal{H}_\gamma(U_{n(k)}; t) = \varepsilon(\gamma) \log \frac{k^{n\gamma}\Gamma(\gamma(n-1)+1, \log(1+t)(\gamma(k+1)-1))}{(\gamma(k+1)-1)^{\gamma(n-1)+1}\Gamma^\gamma(n, k\log(1+t))},\ t > 0.$$

We plotted the RRE $\mathcal{H}_\gamma(U_{n(k)}; t)$ for $n = 5$ and values of $\gamma = 0.5$ and $\gamma = 2$, while varying t and considering different values of $k = 1, 2, \cdots, 5$ in Figure 6. Although the function $H_\gamma(X; t)$ is an increasing function of t, $\mathcal{H}_\gamma(U_{n(k)}; t)$ need not be an increasing function of t, as shown in Figure 6.

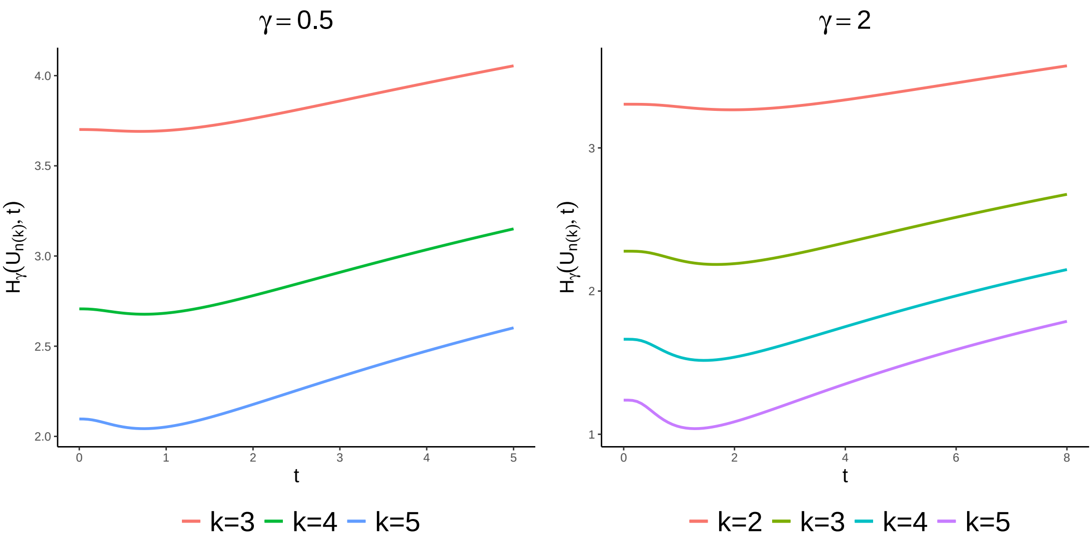

Figure 6. The graph of $\mathcal{H}_\gamma(U_{n(k)}; t)$ for $\gamma = 0.5$ (left panel) and $\gamma = 2$ (right panel) as a function of t.

Stochastic comparisons of record values have been considered in Kochar [26], Raqab and Amin [25], Khaledi [27], and Khaledi and Shojaei [22], among others. Gupta and Kirmani [28] have proved that the record values from an IFR distribution have IFR distribution. Kochar [26] has shown that $U_{n+1(1)}$ is IFR if $U_{n(1)}$ is IFR, and $U_{n(1)}$ is DFR if $U_{n-1(1)}$ is DFR. Kamps [29] has also observed that $U_{n+1(k)}$ is IFR if $U_{n(k)}$ is IFR, and that $U_{n(k)}$ is DFR whenever $U_{n-1(k)}$ is DFR. Now, using Theorem 3.3, we compare the n-th upper k-record values in the sense of decreasing property of RRE. Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with CDF F and PDF f. Let $U_{n(k_j)},\ j = 1, 2$, denote the n-th upper k_j-record values. Let $\lambda_{n(k_1)}(t)$ and $\lambda_{n(k_2)}(t)$ denote the hazard rates of $U_{n(k_1)}$ and $U_{n(k_2)}$, respectively. It can be shown that

$$\lambda_{n(k_2)}(t) = R(t)\lambda_{n(k_1)}(t), \tag{3.8}$$

where

$$R(t) = \left(\frac{k_2}{k_1}\right)^n \frac{\sum_{i=0}^{n-1} \frac{[-k_1 \log \overline{F}(t)]^i}{i!}}{\sum_{i=0}^{n-1} \frac{[-k_1 \log \overline{F}(t)]^i}{i!}},\ t > 0.$$

In this case, Raqaband and Amin [25] proved that for $k_1 > k_2$, the function $R(t)$ is increasing in t and its range is a subset of $(0, 1)$. So, we have the following theorem.

Theorem 3.5. *If $\mathcal{H}_\gamma(U_{n(k_1)}; t)$ is decreasing in t, then $\mathcal{H}_\gamma(U_{n(k_2)}; t)$ is also decreasing for all $\gamma > 0$ when $k_1 > k_2$.*

4. Conditional Renyi entropy of k-records

In this section, we focus on the evaluation of the residual nth upper k-records denoted as $U_{n(k)}-t,\ t \geq 0$, under the premise that all units exceed the threshold $t > 0$. We define $U_{1(1)}$ as U_1, representing the first records. Consequently, the survival function of $U^t_{n(k)} = [U_{n(k)} - t | U_1 > t]$ can be expressed as (refer to Raqab and Asadi [30])

$$\begin{aligned} \overline{F}_{n(k),t}(x) &= P(U_{n(k)} - t > x | U_1 > t), \\ &= \frac{\Gamma(n, -k \log \overline{F}_t(x))}{\Gamma(n)}, \quad x, t \geq 0, \end{aligned} \tag{4.1}$$

and it follows that

$$f_{n(k),t}(x) = \frac{k^n}{\Gamma(n)} [\overline{F}_t(x)]^{k-1} [-\log \overline{F}_t(x)]^{n-1} f_t(x), \quad x, t \geq 0. \tag{4.2}$$

In the subsequent analysis, the primary objective is to investigate the Renyi entropy associated with the random variable $U^t_{n(k)}$, which quantifies the level of uncertainty inherent in the density of $[U_{n(k)} - t | U_1 > t]$. To achieve this aim, the probability integral transformation $V_{n(k)} = \overline{F}_t(U^t_{n(k)})$ assumes a pivotal role in this approach. The transformation $V_{n(k)} = \overline{F}_t(U^t_{n(k)})$ is of significant importance, which has the PDF as follows:

$$g_{n(k)}(u) = \frac{k^n}{\Gamma(n)} u^{k-1} (-\log u)^{n-1}, \quad 0 < u < 1,\ n \geq 1. \tag{4.3}$$

The next theorem presents a derived expression for the Renyi entropy of $U^t_{n(k)}$ utilizing the aforementioned transforms.

Theorem 4.1. *The Renyi entropy of $U^t_{n(k)}$ can be expressed as follows:*

$$\mathcal{H}_\gamma(U^t_{n(k)}) = \omega(\gamma)\log\left[\int_0^1 g^\gamma_{n(k)}(u) f_t^{\gamma-1}(\overline{F}_t^{-1}(u))du\right],\ t > 0, \tag{4.4}$$

for all $\gamma > 0$.

Proof. Applying $u = \overline{F}_t(x)$ and recalling relations (1.2) and (4.2), it holds that

$$\begin{aligned}
\mathcal{H}_\gamma(U^t_{n(k)}) &= \omega(\gamma)\log\left[\int_0^\infty \left(f_{U^t_{n(k)}}(x)\right)^\gamma dx\right] \\
&= \omega(\gamma)\log\left[\int_0^\infty \left(\frac{k^n}{\Gamma(n)}[\overline{F}_t(x)]^{k-1}[-\log\overline{F}_t(x)]^{n-1} f_t(x)\right)^\gamma dx\right] \\
&= \omega(\gamma)\log\left[\int_0^1 \left(\frac{k^n}{\Gamma(n)}(1-u)^{k-1}(-\log(1-u))^{n-1}\right)^\gamma \left(f_t(\overline{F}_t^{-1}(u))\right)^{\gamma-1} dx\right] \\
&= \omega(\gamma)\log\left[\int_0^1 g^\gamma_{n(k)}(u)\left(f_t(\overline{F}_t^{-1}(u))\right)^{\gamma-1} du\right].
\end{aligned}$$

The final equality is established by considering $g_{n(k)}(u)$ as the PDF of $V_{n(k)}$, as given in Eq (4.3). By incorporating this result, the proof is successfully concluded. □

The forthcoming theorem delves into the examination of how the RRE of k-record values is influenced by the aging effects of their components.

Theorem 4.2. *Let X have IFR(DFR) property. So, $\mathcal{H}_\gamma(U^t_{n(k)})$ is decreasing (increasing) in t for all $\gamma > 0$.*

Proof. Let X be IFR. Applying analogous reasoning to that of Theorem 3.1, we can conclude that

$$\begin{aligned}
\int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\lambda_{t_1}(\overline{F}_{t_1}^{-1}(u))\right)^{\gamma-1} du &= \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\lambda(\overline{F}^{-1}(u\overline{F}(t_1)))\right)^{\gamma-1} du \\
&\leq (\geq) \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\lambda(\overline{F}^{-1}(u\overline{F}(t_2)))\right)^{\gamma-1} du \\
&= \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\lambda_{t_2}(\overline{F}_{t_2}^{-1}(u))\right)^{\gamma-1} du,
\end{aligned}$$

for all $\gamma > 1 (0 < \gamma \leq 1)$, and $t_1 \leq t_2$. Using (4.4), we get

$$e^{(1-\gamma)\mathcal{H}_\gamma(U^{t_1}_{n(k)})} \leq (\geq) e^{(1-\gamma)\mathcal{H}_\gamma(U^{t_2}_{n(k)})},$$

when $\gamma > 1 (0 < \gamma \leq 1)$. Consequently, for all $\gamma > 0$, it holds that $\mathcal{H}_\gamma(U^{t_1}_{n(k)}) \geq \mathcal{H}_\gamma(U^{t_2}_{n(k)})$. Hence, the theorem holds. □

The following example demonstrates the outcomes derived from Theorems 4.1 and 4.2.

Example 4.1. Suppose we have a sequence of random variables $\{X_i, i \geq 1\}$ that are i.i.d. with a common power function distribution. The survival function of this distribution is

$$\overline{F}(x) = 1 - x^2,\ 0 < x < 1. \tag{4.5}$$

Using this, we can show that

$$\mathcal{H}_\gamma(U^t_{n(k)}) = \log\left(\frac{1-t^2}{2}\right) + \varepsilon(\gamma)\left[\int_0^1 (1-u(1-t^2))^{\frac{\gamma-1}{2}} g^\gamma_{n(k)}(u)du\right],\ t > 0.$$

Figure 7 illustrates the plot of the RRE $\mathcal{H}_\gamma(U_{n(k)}; t)$ for $n = 5$, with γ values set to 0.5 and 2. The plot considers various values of k ranging from 1 to 5 and varying values of t. The results demonstrate that the Renyi entropy of $U^t_{n(k)}$ decreases as time t increases. It is noteworthy that the distribution under consideration exhibits the property of DFR.

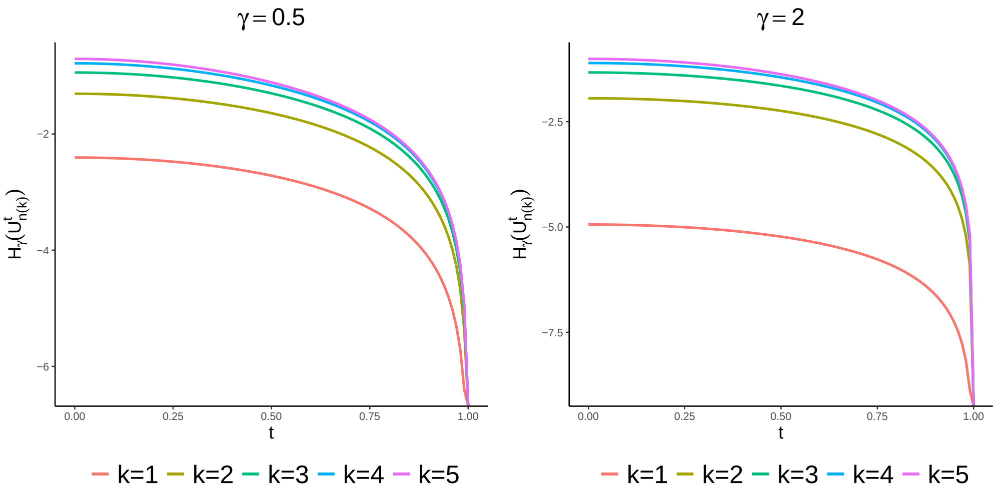

Figure 7. The values of $\mathcal{H}_\gamma(U^t_{5(k)})$ for the power distribution with respect to t for $\gamma = 0.5$ and $\gamma = 2$ for $k = 1, 2, 3, 4, 5$.

Theorem 4.3. *Let X have IFR(DFR) property, so* $\mathcal{H}_\gamma(U^t_{n(k)}) \leq (\geq)\mathcal{H}_\gamma(U_{n(k)})$ *for all* $\gamma > 0$.

Proof. By utilizing Theorem 4.2, we can establish that if X possesses the IFR or DFR property, then $\mathcal{H}_\gamma(U^t_{n(k)})$ exhibits a decreasing or increasing behavior, respectively, with respect to the time t for any $\gamma > 0$. Consequently, we can infer that $\mathcal{H}_\gamma(U^t_{n(k)}) \leq (\geq)\mathcal{H}_\gamma(U^0_{n(k)}) = \mathcal{H}_\gamma(U_{n(k)})$ for any $t \geq 0$. This observation concludes the proof. □

In the following theorem, we establish the upper and lower bounds of RRE of the lifetime of the n-th upper k-records expressed regarding the RRE of the parent distribution denoted as $\mathcal{H}_\gamma(X; t)$.

Theorem 4.4. *When* $\gamma > 1$ $(0 < \gamma < 1)$, *we have*

$$\mathcal{H}_\gamma(U^t_{n(k)}) \geq (\leq)\frac{\gamma}{1-\gamma}\log g_{n(k)}(v^\star) + \mathcal{H}_\gamma(X; t), \tag{4.6}$$

where $g_{n(k)}(v^\star)$ *and* $v^\star = e^{-\frac{n-1}{k-1}}$.

Proof. The mode of $g_{n(k)}(v)$ can be readily observed to be $v^\star = e^{-\frac{n-1}{k-1}}$, then, we obtain $g_V(v) \leq g_{n(k)}(v^\star),\ 0 < v < 1$. Thus, for $\gamma > 1$ $(0 < \gamma < 1)$, we have

$$\begin{aligned}\mathcal{H}_\gamma(U^t_{n(k)}) &= \varepsilon(\gamma)\log\int_0^1 g_V^\gamma(v)\left(f_t(\overline{F}_t^{-1}(u))\right)^{\gamma-1}dv\\ &\geq (\leq)\varepsilon(\gamma)\log\int_0^1 (g_{n(k)}(v^\star))^\gamma\left(f_t(\overline{F}_t^{-1}(u))\right)^{\gamma-1}dv\\ &= \frac{\gamma}{1-\gamma}\log g_{n(k)}(v^\star) + \mathcal{H}_\gamma(X;t).\end{aligned}$$

The final equality is obtained by utilizing (1.3), leading to the desired result. □

Theorem 4.5. *A lower bound for $\mathcal{H}_\gamma(U^t_{n(k)})$ can be given as*

$$\mathcal{H}_\gamma(U^t_{n(k)}) \geq \log\overline{F}(t) + \mathcal{H}_\gamma(U_{n(k)}), \tag{4.7}$$

for all $\gamma > 0$, provided that X is DFR.

Proof. Since X possesses the DFR property, it can be inferred that it is NWU, which means that $\overline{F}_t(x) \geq \overline{F}(x),\ x, t \geq 0$. Consequently, we can deduce the inequality

$$\overline{F}_t^{-1}(u) + t \geq \overline{F}^{-1}(u),\ t \geq 0,\ 0 < u < 1.$$

Furthermore, it is known that for DFR random variables, the PDF f is decreasing. This property of the PDF implies that

$$f^{\gamma-1}(\overline{F}_t^{-1}(u) + t) \leq (\geq) f^{\gamma-1}(\overline{F}^{-1}(u)),\ 0 < u < 1,$$

for all $\gamma > 1$ $(0 < \gamma < 1)$. Now, Eq (4.4) yields

$$\begin{aligned}\mathcal{H}_\gamma(U^t_{n(k)}) &= \log\overline{F}(t) + \varepsilon(\gamma)\log\int_0^1 g_V^\gamma(u) f^{\gamma-1}(\overline{F}_t^{-1}(u) + t)du\\ &\geq \log\overline{F}(t) + \varepsilon(\gamma)\log\int_0^1 g_V^\gamma(u) f^{\gamma-1}(\overline{F}^{-1}(u))du = \log\overline{F}(t) + \mathcal{H}_\gamma(U_{n(k)}),\end{aligned}$$

for all $\gamma > 0$, and this completes the proof. □

When computing the lower bounds of Theorems 4.4 and 4.5 for $\gamma > 1$, it is possible to determine the maximum value among the lower bounds.

Example 4.2. Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables having a common Weibull distribution with the survival function $\overline{F}(x) = e^{-\sqrt{x}},\ x > 0$. It is not hard to see that

$$\mathcal{H}_\gamma(X;t) = \frac{\gamma}{1-\gamma}\sqrt{t} + \log(2) + \varepsilon(\gamma)\log\frac{\Gamma(2-0.5\gamma, \gamma\sqrt{t})}{\gamma^{1-0.5\gamma}},\ t > 0.$$

The objective is to establish lower and upper bounds for the Renyi entropy of $U^t_{8(4)}$. It is easy to verify that $v^\star = e^{-\frac{n-1}{k-1}} = 0.09697197$ and, hence, $g_{8(4)}(v^\star) = 4.465042$. Hence, based on the findings of Theorem 4.4, we can derive bounds for the Renyi entropy of $U^t_{8(4)}$ for $\gamma > 1$ ($0 < \gamma < 1$) in the following manner:

$$\mathcal{H}_\gamma(U^t_{8(4)}) \geq (\leq) \frac{\gamma}{1-\gamma}\left(\log g_{8(4)}(v^\star) + \sqrt{t}\right) + \log(2) + \varepsilon(\gamma)\log\frac{\Gamma(2-0.5\gamma, \gamma\sqrt{t})}{\gamma^{1-0.5\gamma}}, \tag{4.8}$$

for all $t > 0$. Furthermore, we can derive the lower bound, as expressed in (4.7), in the following manner:

$$\mathcal{H}_\gamma(U^t_{8(4)}) \geq \varepsilon(\gamma)\log\int_0^1 g^\gamma_{8(4)}(u)u^{\gamma-1}(-\log u)^{1-\gamma}du + \log 2 - \sqrt{t},\ t > 0, \tag{4.9}$$

for all $\gamma > 0$. Under the assumption of a Weibull distribution, we have computed the bounds for $\mathcal{H}_\gamma(U^t_{8(4)})$ using (4.8) (dashed line) and (4.9) (dotted line). Additionally, we obtained the exact value of $\mathcal{H}_\gamma(U^t_{8(4)})$ directly from (4.4). The results are depicted in Figure 8. Upon examination, we observe that the lower bound given by (4.9) (dotted line) outperforms the lower bound provided by (4.8) for $\gamma > 1$. The graphical representation in Figure 8 visually demonstrates the superiority of the lower bound obtained from (4.9) over (4.8) for $\gamma > 1$.

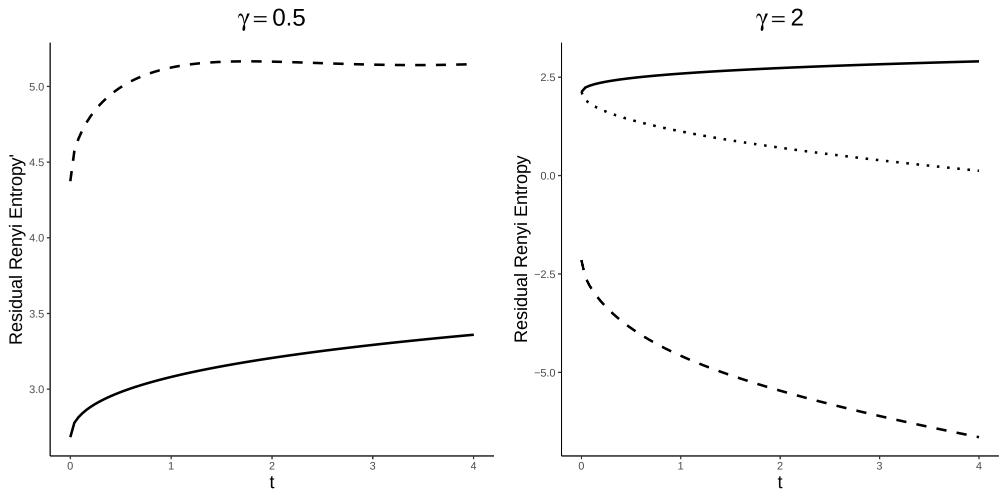

Figure 8. Exact bounds of $\mathcal{H}_\gamma(U^t_{n(k)})$ for the Weibull distribution with respect to time t.

In the context of considering the uncertainties associated with two k-records, we investigate the ordering of Renyi entropies of k-records under the condition that, in the first record, in both records there are greater unspecified thresholds of $t > 0$. In this regard, we present a theorem that compares the residual Renyi entropies of these two records.

Theorem 4.6. *Let $U^{X,t}_{n(k)}$ and $U^{Y,t}_{n(k)}$ denote two residual records coming from i.i.d. sequences $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ having the common CDFs F and G, respectively. When $X \leq_d Y$ and either X or Y possesses the IFR property, we can establish that $\mathcal{H}_\gamma(U^{X,t}n(k)) \leq \mathcal{H}_\gamma(U^{Y,t}_{n(k)})$ holds true for all $\gamma > 0$.*

Proof. By utilizing the relationship defined in Eq (4.4), it reduces to demonstrating that X_t is smaller than Y_t in the dispersive order, i.e., $X_t \leq_d Y_t$. Given the assumption that X is less than Y in the dispersive order and that either X or Y possesses the IFR property, by applying the proof provided in Theorem 5 of the reference [31], we can establish that $X_t \leq_d Y_t$. Hence, the theorem holds. □

Example 4.3. Let us consider residual k-records $U_{n(k)}^{X,t}$ and $U_{n(k)}^{Y,t}$ based on sequences of i.i.d. residual random variables, denoted as $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$, having the common survival functions $\overline{F}$ and $\overline{G}$, respectively. Consider the scenario where X pursues a Weibull distribution with shape parameter 3 and scale parameter 1 (i.e., $X \sim W(3,1)$), and Y pursues a Weibull distribution with shape parameter 2 and scale parameter 1 (i.e., $Y \sim W(2,1)$). It is evident that X is stochastically smaller than or equal to Y in the dispersive order, i.e., $X \leq_d Y$. Additionally, both X and Y are characterized by the IFR property. By applying Theorem 4.6, we can conclude that for any $\gamma > 0$, the Renyi entropy of $U_{n(k)}^{X,t}$ is less than or equal to the Renyi entropy of $U_{n(k)}^{Y,t}$, i.e., $\mathcal{H}_\gamma(U_{n(k)}^{X,t}) \leq \mathcal{H}_\gamma(U_{n(k)}^{Y,t})$.

5. Numerical computations

Numerical studies have always played a complementary role to the theoretical approach in the literature. In this section, we conduct a simulation study to estimate the Renyi entropy of the nth upper k-record based on the maximum likelihood estimator (MLE) of a random sample in the parametric case. First, we obtain the MLE of the Renyi entropy of the nth upper k-record based on a simulated random sample from the exponential distribution with mean $\frac{1}{\lambda}$. Consider a random sample $X_1, X_2, \ldots, X_n$ drawn from an exponential distribution with mean $\frac{1}{\lambda}$ with PDF

$$f(t) = \lambda e^{-\lambda t},\ t \geq 0, \lambda > 0.$$

By using (2.4), it is obvious that the Renyi entropy of the nth upper k-record can be expressed as follows:

$$\mathcal{H}_\gamma(U_{n(k)}) = \frac{1}{1-\gamma} \log\left[\frac{k^{n\gamma}}{\Gamma^\gamma(n)} \frac{\lambda^{\gamma-1}\Gamma(\gamma(n-1)+1)}{(k\gamma)^{\gamma(n-1)+1}}\right]. \tag{5.1}$$

The MLE for λ is obtained as

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\overline{X}}.$$

To evaluate the performance of the proposed MLE for $\mathcal{H}_\gamma(U_{n(k)})$ on simulated exponential data, we calculate the average bias and mean square error (MSE) of the estimator. Since the MLE is an invariant estimator, we can estimate $\mathcal{H}\gamma(Un(k))$ for an exponential distribution using the MLE of λ, which is expressed as follows:

$$\begin{aligned}\widehat{\mathcal{H}}_\gamma(U_{n(k)}) &= \frac{1}{1-\gamma} \log\left[\frac{k^{n\gamma}}{\Gamma^\gamma(n)} \frac{\widehat{\lambda}^{\gamma-1}\Gamma(\gamma(n-1)+1)}{(k\gamma)^{\gamma(n-1)+1}}\right] \\ &= \frac{1}{1-\gamma} \log\left[\frac{k^{n\gamma}}{\Gamma^\gamma(n)} \frac{\Gamma(\gamma(n-1)+1)}{\overline{X}^{\gamma-1}(k\gamma)^{\gamma(n-1)+1}}\right],\end{aligned} \tag{5.2}$$

for $k = 1, 2, \cdots, n$. The bias and MSE of the estimates are computed for different sample sizes ($n = 20, 30, 40, 50$) and various values of the parameters λ, γ, and k. The estimates are obtained from 5000 repetitions. The results are presented in Tables 1–3. It is evident from the tables that the bias and MSE of the estimator for the Renyi entropy of the *n*th upper *k*-record decrease as the sample size increases. This indicates that as the sample size becomes larger, the accuracy of the estimator improves.

Table 1. The bias and MSE of the estimate of Renyi entropy of the *n*th upper *k*-record value for different choices of γ when $\lambda = 0.5$.

		$\gamma = 0.25$		$\gamma = 0.5$		$\gamma = 1.5$		$\gamma = 2$	
n	k	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	1	-0.723572	0.568931	-0.722112	0.563712	-0.719403	0.572620	-0.714141	0.573233
	5	-0.716497	0.568104	-0.714180	0.571727	-0.718204	0.570663	-0.719447	0.563243
	8	-0.716839	0.567735	-0.711634	0.572746	-0.714919	0.569857	-0.717714	0.561210
	10	-0.720055	0.569197	-0.719609	0.565455	-0.724326	0.569313	-0.718558	0.566871
	15	-0.716368	0.563779	-0.720652	0.581269	-0.714743	0.570726	-0.716109	0.567842
25	1	-0.710577	0.554173	-0.705725	0.560503	-0.712234	0.551846	-0.713835	0.553181
	5	-0.711691	0.556429	-0.709240	0.547388	-0.713106	0.546327	-0.711553	0.552503
	8	-0.711625	0.555559	-0.712987	0.552518	-0.712949	0.545672	-0.713238	0.554599
	10	-0.706001	0.552868	-0.712386	0.547289	-0.713963	0.552246	-0.710721	0.550837
	15	-0.710208	0.552865	-0.713606	0.548887	-0.717116	0.554143	-0.717924	0.547950
30	1	-0.705337	0.536905	-0.712371	0.541493	-0.710517	0.533954	-0.706567	0.533200
	5	-0.706220	0.534238	-0.712157	0.536727	-0.713667	0.542586	-0.708138	0.538059
	8	-0.706371	0.537250	-0.708027	0.538675	-0.712718	0.538023	-0.708954	0.538286
	10	-0.709739	0.532792	-0.708000	0.539679	-0.705055	0.536008	-0.710137	0.538086
	15	-0.712938	0.534418	-0.709319	0.535311	-0.709323	0.535483	-0.709814	0.536602
50	1	-0.699265	0.519117	-0.703930	0.516939	-0.703269	0.514384	-0.706338	0.516335
	5	-0.703137	0.513672	-0.704965	0.512922	-0.701257	0.518997	-0.699364	0.516436
	8	-0.702592	0.518221	-0.705283	0.515041	-0.704264	0.514328	-0.703589	0.518070
	10	-0.702760	0.518590	-0.704987	0.514618	-0.704003	0.514521	-0.703010	0.508464
	15	-0.702423	0.509657	-0.701563	0.515991	-0.703708	0.514922	-0.702866	0.513654

Table 2. The bias and MSE of the estimate of Renyi entropy of the *n*th upper *k*-record value for different choices of γ when $\lambda = 1$.

		$\gamma = 0.25$		$\gamma = 0.5$		$\gamma = 1.5$		$\gamma = 2$	
n	k	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	1	-0.020759	0.053079	-0.023149	0.050987	-0.027957	0.051829	-0.030385	0.055046
	5	-0.028763	0.051505	-0.026178	0.050486	-0.020245	0.051568	-0.023435	0.052691
	8	-0.032085	0.051423	-0.033675	0.050485	-0.027125	0.051564	-0.027220	0.052493
	10	-0.025197	0.052157	-0.029183	0.052112	-0.022448	0.052150	-0.025951	0.052020
	15	-0.026822	0.052507	-0.018934	0.052625	-0.023091	0.051922	-0.023160	0.052353
25	1	-0.018420	0.042192	-0.018407	0.041429	-0.023150	0.041449	-0.022388	0.041553
	5	-0.021719	0.041065	-0.017738	0.041917	-0.022827	0.040748	-0.017727	0.040347
	8	-0.021331	0.042071	-0.021282	0.039726	-0.020575	0.041062	-0.018686	0.041289
	10	-0.018933	0.040485	-0.023537	0.041685	-0.025199	0.041462	-0.018041	0.040687
	15	-0.018747	0.041590	-0.021664	0.039868	-0.025146	0.040042	-0.022504	0.040269
30	1	-0.017938	0.033794	-0.016626	0.034327	-0.016713	0.033391	-0.017305	0.034871
	5	-0.011626	0.034420	-0.016033	0.034685	-0.019911	0.034814	-0.017391	0.035195
	8	-0.012365	0.033793	-0.017986	0.033753	-0.019336	0.033837	-0.021089	0.033500
	10	-0.020154	0.032541	-0.015602	0.033992	-0.019249	0.034011	-0.017132	0.034425
	15	-0.016845	0.033864	-0.016096	0.034972	-0.015493	0.034529	-0.017059	0.034555
50	1	-0.006822	0.020029	-0.009460	0.020310	-0.011465	0.021160	-0.012787	0.019426
	5	-0.008034	0.020381	-0.007225	0.020042	-0.009197	0.020205	-0.010454	0.020537
	8	-0.009479	0.020006	-0.010698	0.021015	-0.012118	0.020140	-0.009141	0.020919
	10	-0.009439	0.019920	-0.009974	0.020340	-0.009807	0.021056	-0.012961	0.020056
	15	-0.010529	0.020523	-0.013843	0.019692	-0.009743	0.020506	-0.009763	0.020715

Table 3. The bias and MSE of the estimate of Renyi entropy of the nth upper k-record value for different choices of γ when $\lambda = 2$.

		$\gamma = 0.25$		$\gamma = 0.5$		$\gamma = 1.5$		$\gamma = 2$	
n	k	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	1	0.666239	0.499101	0.666763	0.486859	0.663885	0.492163	0.668948	0.492054
	5	0.667630	0.496912	0.669362	0.501246	0.664270	0.497375	0.667801	0.498755
	8	0.666578	0.504952	0.669458	0.501003	0.668022	0.489390	0.671396	0.496988
	10	0.670596	0.500284	0.667084	0.495751	0.667897	0.492739	0.666356	0.507184
	15	0.668470	0.491366	0.673631	0.501388	0.664114	0.499369	0.668836	0.493331
25	1	0.672902	0.490622	0.676473	0.495529	0.670497	0.499717	0.672626	0.493282
	5	0.674959	0.492944	0.671622	0.495291	0.669380	0.491898	0.672235	0.498776
	8	0.669700	0.495498	0.674938	0.494999	0.675294	0.495822	0.674558	0.497930
	10	0.675557	0.499685	0.672397	0.500511	0.671875	0.497052	0.671476	0.498202
	15	0.673004	0.496579	0.670080	0.488081	0.675153	0.493800	0.676860	0.495217
30	1	0.676921	0.489142	0.676960	0.490480	0.675866	0.490181	0.677345	0.496267
	5	0.679184	0.496520	0.677942	0.490316	0.670262	0.497349	0.675464	0.485984
	8	0.676251	0.492185	0.673295	0.492892	0.679993	0.487142	0.670006	0.491588
	10	0.678427	0.487291	0.677004	0.493599	0.678527	0.493825	0.676112	0.485600
	15	0.677903	0.491908	0.677469	0.487813	0.670491	0.492821	0.676096	0.494919
50	1	0.681410	0.485012	0.681908	0.489576	0.682303	0.485752	0.682743	0.489870
	5	0.684103	0.489767	0.683600	0.487172	0.684824	0.482836	0.682178	0.485626
	8	0.681539	0.487943	0.686444	0.490705	0.685825	0.489912	0.684794	0.486353
	10	0.682694	0.487194	0.683952	0.486003	0.683512	0.489597	0.686123	0.485053
	15	0.681053	0.487599	0.680047	0.488736	0.682141	0.484299	0.687791	0.485262

Hereafter, we present the performance of the given estimator on both actual and simulated data, assuming a standard exponential distribution.

Example 5.1. To examine the spread of the COVID-19 pandemic, Kasilingam *et al.* [32] utilized an exponential model. Specifically, they focused on identifying countries that exhibited early signs of containment measures up until March 26, 2020. The dataset used in their analysis consists of the percentage of infected cases in 42 countries, which are listed below:
Dataset: 1.56, 8.51, 2.17, 0.37, 1.09, 9.84, 4.95, 3.18, 11.37, 2.81, 6.22, 1.87, 0.00, 0.00, 9.05, 2.44, 1.38, 4.17, 3.74, 1.37, 2.33, 7.80, 2.10, 0.47, 2.54, 4.92, 0.09, 0.18, 1.72, 1.02, 0.62, 2.34, 0.50, 2.37, 3.65, 0.59, 5.76, 2.14, 0.88, 0.95, 4.17, 2.25.

In their paper, Mohammed *et al.* [33] demonstrated that the provided data exhibits a good fit, with the exponential distribution having a mean of one. To investigate the relationship between the parameters γ, k, the theoretical $\mathcal{H}_\gamma(U_{42(k)})$, and its empirical estimator for the real-life data, Table 4 presents both the theoretical and empirical estimator. It is observed that as the parameter λ increases, both the theoretical and empirical estimator decrease. Conversely, an increase in k leads to a decrease in both measures.

Table 4. Estimation of $\mathcal{H}_\gamma(U_{42(k)})$ based on standard exponential distribution for genuine COVID-19 infection data.

γ	0.25		0.5		1.5		2	
k	$\mathcal{H}_\gamma(U_{42(k)})$	$\widehat{\mathcal{H}}_\gamma(U_{42(k)})$	$\mathcal{H}_\gamma(U_{42(k)})$	$\widehat{\mathcal{H}}_\gamma(U_{42(k)})$	$\mathcal{H}_\gamma(U_{42(k)})$	$\widehat{\mathcal{H}}_\gamma(U_{42(k)})$	$\mathcal{H}_\gamma(U_{42(k)})$	$\widehat{\mathcal{H}}_\gamma(U_{42(k)})$
1	3.710080	4.804557	3.474969	4.569445	3.184577	4.279054	3.125347	4.219824
5	2.100642	3.195119	1.865531	2.960008	1.575139	2.669616	1.515909	2.610386
8	1.630638	2.725115	1.395527	2.490004	1.105136	2.199612	1.045905	2.140382
10	1.407495	2.501972	1.172384	2.266860	0.881992	1.976469	0.822762	1.917239
15	1.002030	2.096507	0.766919	1.861395	0.476527	1.571004	0.417297	1.511773

6. Conclusions

In this study, we have emphasized the importance of using k-records to quantify uncertainty through the use of Renyi entropy. We performed a comparison between the Renyi entropy of k-records and the Renyi entropy of the original random variables. For the first time, we investigated the implications and properties of Renyi and RRE in the context of k-records. This study contributes to the growing literature on the information properties of recorded values. We first discuss the effects of Renyi entropy ordering under the usual stochastic and dispersive ordering and establish these relationships under certain conditions. This analysis provides insights into the ordering of record values based on Renyi entropy. Next, we focus on the monotonicity properties of the RRE of k-record values considering the aging properties of component lifetimes. This investigation helps us to understand how the RRE behaves when the component lifetimes change. We also derived an expression for the Renyi entropy of the residual nth upper k-records, especially when the first record exceeds a certain threshold. We then investigated several properties of this formula that shed light on the behavior and properties of Renyi entropy associated with k-record values. In general, these results provide valuable insights into the information properties of recorded values and extend the understanding of Renyi and RRE in the context of k-record values. In the last section, we performed a parametric estimation of the Renyi entropy of nth upper k-records. We found that the proposed estimator depends on the parameter λ and the sample size, both having a similar influence. We recall that the results presented in this paper have far-reaching implications for other information measures based on functional survival functional entropy. These measures include cumulative residual entropy, cumulative entropy, generalized cumulative residual entropy, and generalized cumulative entropy, among others. The exploration and extension of these concepts will be a focus of our future research efforts.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank three anonymous reviewers for their constructive comments and suggestions.

This work was supported by Researchers Supporting Project number (RSP2024R464), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. N. Ebrahimi, F. Pellerey, New partial ordering of survival functions based on the notion of uncertainty, *J. Appl. Probab.,* **32** (1995), 202–211. https://doi.org/10.2307/3214930
2. C. E. Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.,* **27** (1948), 379–423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x
3. R. D. Gupta, A. K. Nanda, α-and β-entropies and relative entropies of distributions, *J. Stat. Theor. Appl.,* **1** (2002), 177–190.
4. M. Asadi, N. Ebrahimi, E. S. Soofi, Dynamic generalized information measures, *Stat. Probab. Lett.,* **71** (2005), 85–98. https://doi.org/10.1016/j.spl.2004.10.033
5. A. K. Nanda, P. Paul, Some results on generalized residual entropy, *Inform. Sciences,* **176** (2006), 27–47. https://doi.org/10.1016/j.ins.2004.10.008
6. M. Mesfioui, M. Kayid, M. Shrahili, Renyi entropy of the residual lifetime of a reliability system at the system level, *Axioms,* **12** (2023), 320. https://doi.org/10.3390/axioms12040320
7. K. Chandler, The distribution and frequency of record values, *J. Roy. Stat. Soc. B*, **14** (1952), 220–228. https://doi.org/10.1111/j.2517-6161.1952.tb00115.x
8. N. Glick, Breaking records and breaking boards, *Am. Math. Mon.,* **85** (1978), 2–26. https://doi.org/10.1080/00029890.1978.11994501
9. M. Ahsanullah, *Record values-theory and applications*, University Press of America, 2004.
10. B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, *Records,* John Wiley and Sons, 2011.
11. U. Kamps, A concept of generalized order statistics, *J. Stat. Plan. Infer.,* **48** (1995), 1–23. https://doi.org/10.1016/0378-3758(94)00147-N
12. W. Dziubdziela, B. Kopociński, Limiting properties of the k-th record values, *Appl. Math.,* **2** (1976), 187–190. Available from: `http://eudml.org/doc/263206.`
13. S. Zarezadeh, M. Asadi, Results on residual Rényi entropy of order statistics and record values, *Inform. Sciences,* **180** (2010), 4195–4206. https://doi.org/10.1016/j.ins.2010.06.019
14. A. Habibi, N. R. Arghami, J. Ahmadi, Statistical evidence in experiments and in record values, *Commun. Stat.-Theor. M.,* **35** (2006), 1971–1983. https://doi.org/10.1080/03610920600762780
15. M. Abbasnezhad, N. R. Arghami, Potential statistical evidence in experiments and Renyi information, *J. Iran. Stat. Soc.,* **5** (2006), 39–52. Available from: `https://sid.ir/paper/567629/en.`
16. S. Baratpour, J. Ahmadi, N. R. Arghami, Entropy properties of record statistics, *Stat. Pap.,* **48** (2007), 197–213. https://doi.org/10.1007/s00362-006-0326-7
17. J. Jose, E. A. Sathar, Rényi entropy of k-records: Properties and applications, *REVSTAT-Stat. J.,* **20** (2022), 481–500. https://doi.org/10.57805/revstat.v20i4.385
18. P. S. Asha, M. Chacko, Residual Renyi entropy of k-record values, *Commun. Stat.-Theor. M.,* **45** (2016), 4874–4885. https://doi.org/10.1080/03610926.2014.932806

19. P. S. Asha, M. Chacko, Residual Verma entropy of k-record values: Verma entropy of k-record values, *J. Kerala Stat. Assoc.,* **28** (2017), 28–45. Available from: `https://ojs.ksa.org.in/index.php/JKSA/article/view/22.`
20. M. Shrahili, M. Kayid, Residual tsallis entropy and record values: Some new insights, *Symmetry,* **15** (2023), 2040. https://doi.org/10.3390/sym15112040
21. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, Springer Science and Business Media, 2007. https://doi.org/10.1007/978-0-387-34675-5
22. B. E. Khaledi, R. Shojaei, On stochastic orderings between residual record values, *Stat. Probab. Lett.,* **77** (2007), 1467–1472. https://doi.org/10.1016/j.spl.2007.03.033
23. M. Abbasnejad, N. R. Arghami, Renyi entropy properties of order statistics, *Commun. Stat.-Theor. M.,* **40** (2010), 40–52. https://doi.org/10.1080/03610920903353683
24. I. Bagai, S. C. Kochar, On tail-ordering and comparison of failure rates, *Commun. Stat.-Theor. M.,* **15** (1986), 1377–1388. https://doi.org/10.1080/03610928608829189
25. M. Z. Raqab, W. A. Amin, A note on reliability properties of k-record statistics, *Metrika,* **46** (1997), 245–251. https://doi.org/10.1007/BF02717177
26. S. C. Kochar, Some partial ordering results on record values, *Commun. Stat.-Theor. M.,* **19** (1990), 299–306. https://doi.org/10.1080/03610929008830201
27. B. E. Khaledi, Some new results on stochastic orderings between generalized order statistics, *J. Iran. Stat. Soc.,* **4** (2022), 35–49. Available from: `https://sid.ir/paper/117852/en.`
28. R. C. Gupta, S. N. U. A. Kirmani, Closure and monotonicity properties of nonhomogeneous Poisson processes and record values, *Probab. Eng. Inform. Sci.,* **2** (1988), 475–484. https://doi.org/10.1017/S0269964800000188
29. U. Kamps, Reliability properties of record values from non-identically distributed random variables, *Commun. Stat.-Theor. M.,* **23** (1994), 2101–2112. https://doi.org/10.1080/03610929408831373
30. M. Z. Raqab, M. Asadi, On the mean residual life of records, *J. Stat. Plan. Infer.,* **138** (2008), 3660–3666. https://doi.org/10.1016/j.jspi.2007.11.014
31. N. Ebrahimi, S. N. U. A. Kirmani, Some results on ordering of survival functions through uncertainty, *Stat. Probab. Lett.,* **29** (1996), 167–176. https://doi.org/10.1016/0167-7152(95)00170-0
32. D. Kasilingam, S. P. S. Prabhakaran, D. K. Rajendran, V. Rajagopal, T. S. Kumar, A. Soundararaj, Exploring the growth of COVID-19 cases using exponential modelling across 42 countries and predicting signs of early containment using machine learning, *Transbound. Emerg. Dis.,* **68** (2021), 1001–1018. https://doi.org/10.1111/tbed.13764
33. M. S. Mohamed, A. Almohaimeed, M. M. Abd El-Raouf, Some new findings on the survival Rényi entropy and application of COVID-19 data, *Results Phys.,* **31** (2021), 104966. https://doi.org/10.1016/j.rinp.2021.104966

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