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Research article

Further study on Hopf bifurcation and hybrid control strategy in BAM neural networks concerning time delay

Qingyi Cui[1], Changjin Xu[2,*], Wei Ou[1], Yicheng Pang[1], Zixin Liu[1], Jianwei Shen[3], Muhammad Farman[4,5] and Shabir Ahmad[6]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China
[2] Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550025, China
[3] School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, China
[4] Faculty of Arts and Sciences, Department of Mathematics, Near East University, 99138, Nicosia, Cyprus
[5] Department of Computer Science and Mathematics, Lebanese American University, 1107–2020, Beirut, Lebanon
[6] Department of Mathematics, University of Malakand, Chakdara, Dir Lower, Khyber Pakhtunkhwa, Pakistan

* **Correspondence:** Email: xcj403@126.com; Tel: +1-868-851-0704.

Abstract: Delayed dynamical system plays a vital role in describing the dynamical phenomenon of neural networks. In this article, we proposed a class of new BAM neural networks involving time delay. The traits of solution and bifurcation behavior of the established BAM neural networks involving time delay were probed into. First, the existence and uniqueness is discussed using a fixed point theorem. Second, the boundedness of solution of the formulated BAM neural networks involving time delay was analyzed by applying an appropriate function and inequality techniques. Third, the stability peculiarity and bifurcation behavior of the addressed delayed BAM neural networks were investigated. Fourth, Hopf bifurcation control theme of the formulated delayed BAM neural networks was explored by virtue of a hybrid controller. By adjusting the parameters of the controller, we could control the stability domain and Hopf bifurcation onset, which was in favor of balancing the states of different neurons in engineering. To verify the correctness of gained major outcomes, computer simulations were performed. The acquired outcomes of this article were new and own enormous theoretical meaning in designing and dominating neural networks.

Keywords: BAM neural networks; trait of solution; Hopf bifurcation; stability; time delay;

hybrid controller
Mathematics Subject Classification: 34C23, 34K18, 37GK15, 39A11, 92B20

1. Introduction

We all know that neural networks have displayed great application prospect in massive fields such as associative memory, combinatorial optimization, artificial intelligence, pattern recognition, signal processing and disease diagnosis [1–5]. In order to make fully use of neural networks to serve mankind, it is important for us to explore the dynamic properties of neural networks. In particular, time delay usually appears in neural networks due to the delay of signal propagation among different neurons. Thus, delayed neural networks are more reasonable to describe the interaction of real neurons. During the past several decades, many researchers pay great attention to the dynamics of delayed neural networks. Due to the existence of time delay, many dynamical behaviors of neural networks will change. In many cases, neural networks will lose their stability, produce periodic vibration, give rise to Hopf bifurcation, bring about chaotic phenomenon, and so on [6–11]. During the past decades, a number of outstanding achievements on neural networks with delays have been acquired. For instance, Abdurahman and Jiang [11] dealt with the general decay projective synchronization of memristor-based BAM neural networks with delays via nonlinear control technique. Kong et al. [12] explored the stability of almost periodic solutions to discontinuous BAM neural networks involving hybrid time-varying delays and D operator. Xu et al. [13] discussed the bifurcation issue of a simplified tri-neuron BAM networks with both time delays in frequency domain. Pratap et al. [14] made a detailed analysis on quasi-pinning synchronization and stabilization in a class of fractional-order BAM neural networks involving delays and discontinuous neuron activations. Maharajan et al. [15] set up a new global robust exponential stability condition for uncertain inertial-type BAM neural networks with discrete and distributed delays. For details, one can see [16–38].

Bidirectional associative memory (BAM) neural networks, which were first proposed by Kosto [11], are a class of special neural networks. They include neurons in two fields and are able to store bipolar vector pairs [12]. In 2008, Sprott [13] proposed the following 4D BAM neural networks:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - au_1(t), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - au_2(t), \\ \dot{u}_3(t) = \tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - au_4(t), \end{cases} \tag{1.1}$$

where $u_i (i = 1, 2, 3, 4)$ denote the state of the i neuron, a is a positive constant and the formula -au_i has a relation with frictional damping and guarantees the boundedness of the solutions. Sprott [13] had studied the chaotic phenomenon of the 4D BAM neural networks (1.1). Noticing that different self-feedback time delays in the first neuron and the second neuron often exist in system (1.1), Vaishwar and Yadav [14] modified system (1.1) as the following 4D BAM neural networks concerning both inconsistent time delays:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - au_1(t-\theta_1), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - au_2(t-\theta_2), \\ \dot{u}_3(t) = \tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - au_4(t), \end{cases} \tag{1.2}$$

where $\theta_1 > 0, \theta_2 > 0$ stand for two delays. Vaishwar and Yadav [14] investigated the stability behavior and Hopf bifurcation phenomenon of neural network model (1.2). Here, we would like to point out that all the neurons have self-feedback time delays. That is to say, the first neuron u_1, the second neuron u_2, the third neuron u_3, and the fourth neuron u_4 have different self-feedback time delays. For simplicity, we assume that self-feedback time delays for four neurons are same, then relying on the model (1.2), we can formulate the following 4D BAM neural networks concerning time delay:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - au_1(t-\theta), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - au_2(t-\theta), \\ \dot{u}_3(t) = \tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t-\theta), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - au_4(t-\theta), \end{cases} \tag{1.3}$$

where θ stands for a delay.

Compared with the model (1.2), there exist more self-feedback delays (for neurons u_3 and neuron u_4) in model (1.2). Usually, delay will lead to the vary of dynamical behavior of neural networks. However, Vaishwar and Yadav [14] did not consider this problem. This motivates us to explore this model (1.2). In addition, delay-driven Hopf bifurcation plays a significant role in describing the interaction law in nonlinear delayed dynamical systems [14–35]. In neural network area, delay-driven Hopf bifurcation can effectively describe the transformation relationship of the state of different neurons. Thus we think that it is an important thing to deal with the delay-Hopf bifurcation in many kinds of neural networks. Stimulated by the viewpoint above, we are about to handle the delay-Hopf bifurcation and the Hopf bifurcation control problem for system (1.3). In particular, we handle the following topics: (I) Prove the existence, uniqueness and non-negativeness of the solution to system (1.3). (II) Investigate the stability behavior and the emergence Hopf bifurcation of system (1.3). (III) Dominate Hopf bifurcation of system (1.3) by virtue of hybrid controller.

The main highlights of this manuscript are stated as follows: (I) Based on the previous studies, novel 4D BAM neural networks concerning time delay are formulated. (II) A new delay-independent criterion on the stability and the appearance of bifurcation of system (1.3) is acquired. (III) Taking advantage of hybrid controller, the stability domain and the time of the emergence of bifurcation of system (1.3) have been successfully adjusted. (IV) The influence of delay on the stability and the emergence of bifurcation of system (1.3) is explored. (V) The control tactics can help us explore the control issue of Hopf bifurcation of plenty of differential dynamical models in lots of fields.

We plan this study as follows. We explore the existence and uniqueness, non-negativeness of the solution to system (1.3) in Segment 2. The stability behavior and the emergence of bifurcation of system (1.3) are discussed in Segment 3. The control of Hopf bifurcation of system (1.3) via a proper hybrid controller including state feedback and parameter perturbation is dealt with in Segment 4. Segment 5 carries out computer simulations to support the correctness of the acquired outcomes. Segment 6 draws a simple conclusion.

2. Properties of solution

In this part, we consider the existence and uniqueness, boundedness of the solution to system (1.3) by making use of the fixed point theorem and inequality techniques.

Theorem 2.1. *Let* $\Omega = \{u_1, u_2, u_3, u_4) \in R^4 : \max\{|u_1|, |u_2|, |u_3|, |u_4|\} \le \mathcal{U}\}$*, where* $\mathcal{U}$ *represents a positive constant. For every* $(u_{10}, u_{20}, u_{30}, u_{40}) \in \Omega$*, system (1.3) under the initial value* $(u_{10}, u_{20}, u_{30}, u_{40})$ *owns a unique solution* $U = (u_1, u_2, u_3, u_4) \in \Omega$.

Proof. Let

$$h(U) = (h_1(U), h_2(U), h_3(U), h_4(U)), \tag{2.1}$$

where

$$\begin{cases} h_1(U) = \tanh[u_4(t) - u_2(t)] - au_1(t-\theta), \\ h_2(U) = \tanh[u_1(t) + u_4(t)] - au_2(t-\theta), \\ h_3(U) = \tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t-\theta), \\ h_4(U) = \tanh[u_3(t) - u_2(t)] - au_4(t-\theta). \end{cases} \tag{2.2}$$

For every $U, \tilde{U} \in \Omega$, one gains

$$\begin{aligned}
&||h(U) - h(\tilde{U})|| \\
&= |\tanh[u_4(t) - u_2(t)] - au_1(t-\theta) \\
&\quad - [\tanh[\tilde{u}_4(t) - \tilde{u}_2(t)] - a\tilde{u}_1(t-\theta)]| \\
&\quad + |\tanh[u_1(t) + u_4(t)] - au_2(t-\theta) \\
&\quad - [\tanh(\tilde{u}_1(t) + \tilde{u}_4(t)) - a\tilde{u}_2(t-\theta)]| \\
&\quad + |\tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t-\theta) \\
&\quad - [\tanh(\tilde{u}_1(t) + \tilde{u}_2(t) - \tilde{u}_4(t)) - a\tilde{u}_3(t-\theta)]| \\
&\quad + |\tanh[u_3(t) - u_2(t)] - au_4(t-\theta) \\
&\quad - [\tanh(\tilde{u}_3(t) - \tilde{u}_2(t)) - a\tilde{u}_4(t-\theta))]| \\
&\le (2+a)|u_1(t) - \tilde{u}_1(t)| + (3+a)|u_2(t) - \tilde{u}_2(t)| \\
&\quad + (1+a)|u_3(t) - \tilde{u}_3(t)| + (3+a)|u_4(t) - \tilde{u}_4(t)| \\
&\le (3+a)[|u_1(t) - \tilde{u}_1(t)| + |u_2(t) - \tilde{u}_2(t)| \\
&\quad + |u_3(t) - \tilde{u}_3(t)| + |u_4(t) - \tilde{u}_4(t)|] \\
&= (3+a)||U - \tilde{U}||.
\end{aligned} \tag{2.3}$$

Thus, $h(U)$ conforms to Lipschitz condition with respect to U (see [17]). According to fixed point theorem, we can lightly conclude that Theorem 2.1 is right.

Theorem 2.2. *If* $\theta = 0$*, then every solution to system (1.3) starting with* R^4_+ *is uniformly bounded.*

Label

$$U(t) = u_1(t) + u_2(t) + u_3(t) + u_4(t). \tag{2.4}$$

Then

$$\begin{aligned}
\dot{U}_1(t) &= \dot{u}_1(t) + \dot{u}_2(t) + \dot{u}_3(t) + \dot{u}_4(t) \\
&= \tanh[u_4(t) - u_2(t)] - au_1(t)
\end{aligned}$$

$$
\begin{aligned}
& +\tanh[u_1(t)+u_4(t)]-au_2(t) \\
& +\tanh[u_1(t)+u_2(t)-u_4(t)]-au_3(t) \\
& +\tanh[u_3(t)-u_2(t)]-au_4(t-\theta) \\
\leq\ & -a(u_1(t)+u_2(t)+u_3(t)+u_4(t))+4 \\
=\ & -aU(t)+4, \qquad (2.5)
\end{aligned}
$$

which results in

$$\dot{U}_1(t)+aU(t)\leq 4. \qquad (2.6)$$

Then, one acquires

$$U(t)\to\frac{4}{a},\ \text{as } t\to\infty. \qquad (2.7)$$

The proof of Theorem 2.2 comes to an end.

3. Study of bifurcation

Clearly, system (1.3) admits a unique zero equilibrium point: $E(0,0,0,0)$. The linear system of system (1.3) at $E(0,0,0,0)$ can be expressed as

$$
\begin{cases}
\dot{u}_1(t)=u_4(t)-u_2(t)-au_1(t-\theta),\\
\dot{u}_2(t)=u_1(t)+u_4(t)-au_2(t-\theta),\\
\dot{u}_3(t)=u_1(t)+u_2(t)-u_4(t)-au_3(t-\theta),\\
\dot{u}_4(t)=u_3(t)-u_2(t)-au_4(t-\theta).
\end{cases} \qquad (3.1)
$$

The characteristic equation of system (3.1) owns the following expression:

$$
\det\begin{bmatrix}
\lambda+ae^{-\lambda\theta} & 1 & 0 & -1\\
-1 & \lambda+ae^{-\lambda\theta} & 0 & -1\\
-1 & -1 & \lambda+ae^{-\lambda\theta} & 1\\
0 & 1 & -1 & \lambda+ae^{-\lambda\theta}
\end{bmatrix}=0, \qquad (3.2)
$$

which generates

$$
\begin{aligned}
&\lambda^4+3\lambda^2-\lambda+1+(a_1\lambda^3+a_2\lambda+a_3)e^{-\lambda\theta}\\
&+(a_4\lambda^2+a_5)e^{-2\lambda\theta}+a_6\lambda e^{-3\lambda\theta}+a_7e^{-4\lambda\theta}=0, \qquad (3.3)
\end{aligned}
$$

where

$$
\begin{cases}
a_1=4a,\\
a_2=6a,\\
a_3=-a,\\
a_4=6a^2,\\
a_5=3a^2\\
a_6=4a^3,\\
a_7=a^4.
\end{cases} \qquad (3.4)
$$

When $\theta = 0$, then Eq (3.3) becomes:

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{3.5}$$

where

$$\begin{cases} b_1 = a_1, \\ b_2 = 3 + a_4, \\ b_3 = a_2 + a_6 - 1, \\ b_4 = 1 + a_3 + a_5 + a_7. \end{cases} \tag{3.6}$$

If

$$(A_1) \begin{cases} \Delta_1 = b_1 > 0, \\ \Delta_2 = \det \begin{bmatrix} b_1 & 1 \\ b_3 & b_2 \end{bmatrix} > 0, \\ \Delta_3 = \begin{bmatrix} b_1 & 1 & 0 \\ b_3 & b_2 & b_1 \\ 0 & b_4 & b_3 \end{bmatrix} > 0, \\ \Delta_4 = b_4\Delta_3 > 0 \end{cases}$$

is fulfilled, then the four roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of Eq (3.5) have negative real parts. Thus the zero equilibrium point $E(0, 0, 0, 0)$ of system (1.3) with $\theta = 0$ is locally asymptotically stable.

By (3.3), we get

$$\begin{aligned} &(\lambda^4 + 3\lambda^2 - \lambda + 1)e^{-2\lambda\theta} + (a_1\lambda^3 + a_2\lambda + a_3)e^{\lambda\theta} \\ &+(a_4\lambda^2 + a_5) + a_6\lambda e^{-\lambda\theta} + a_7 e^{-2\lambda\theta} = 0. \end{aligned} \tag{3.7}$$

Suppose that $\lambda = i\phi$ is the root of Eq (3.7). Then Eq (3.7) becomes:

$$\begin{aligned} &[(i\phi)^4 + 3(i\phi)^2 - i\phi + 1]e^{2i\phi\theta} + [a_1(i\phi)^3 + a_2 i\phi + a_3]e^{i\phi\theta} \\ &+a_4(i\phi)^2 + a_5) + a_6(i\phi)e^{-i\phi\theta} + a_7 e^{-2i\phi\theta} = 0, \end{aligned} \tag{3.8}$$

which generates

$$\begin{aligned} &(\phi^4 - 3i\phi^2 - i\phi + 1)(\cos 2\phi\theta + i \sin 2\phi\theta) \\ &+(-a_1 i\phi^3 + a_2 i\phi + a_3)(\cos\phi\theta + i\sin\phi\theta) \\ &+(-a_4 i\phi^2 + a_5) + a_6 i\phi(\cos\phi\theta - i\sin\phi\theta) \\ &+a_7(\cos 2\phi\theta - i\sin 2\phi\theta) = 0. \end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{cases} (\phi^4 + 1 + a_7)\cos 2\phi\theta + (3\phi^2 + \phi)\sin 2\phi\theta \\ \qquad + a_3\cos\phi\theta + (a_6\phi - a_2 + a_1\phi^3)\sin\phi\theta + a_5 = 0, \\ (\phi^4 + 1 - a_7)\sin 2\phi\theta - (3\phi^2 + \phi)\cos 2\phi\theta \\ \qquad + (a_2 - a_1\phi^3)\cos\phi\theta + (a_3 - a_6\phi)\sin\phi\theta - a_4\phi^2 = 0. \end{cases} \tag{3.10}$$

Notice that $\sin\phi\theta = \pm\sqrt{1 - \cos^2\phi\theta}$, then it follows from the first equation of system (3.10) that

$$(\phi^4 + 1 + a_7)(2\cos^2\phi\theta - 1) + (3\phi^2 + \phi)\cos\phi\theta(\pm\sqrt{1 - \cos^2\phi\theta})$$

$$+a_3\cos\phi\theta + (a_6\phi - a_2 + a_1\phi^3)(\pm\sqrt{1-\cos^2\phi\theta}) + a_5 = 0. \tag{3.11}$$

By (3.11), one derives

$$\begin{aligned}&[(\phi^4+1+a_7)(2\cos^2\phi\theta - 1) + a_3\cos\phi\theta + a_5]^2\\ &= (3\phi^2+\phi)\cos\phi\theta + (a_6\phi - a_2 + a_1\phi^3)]^2(1-\cos^2\phi\theta),\end{aligned} \tag{3.12}$$

which leads to

$$\epsilon_1\cos^4\phi\theta + \epsilon_2\cos^3\phi\theta + \epsilon_3\cos^2\phi\theta + \epsilon_4\cos\phi\theta + \epsilon_5 = 0, \tag{3.13}$$

where

$$\begin{cases}\epsilon_1 = 4(\phi^4+1+a_7)^2 + (3\phi^2+\phi)^2,\\ \epsilon_2 = 2a_3(\phi^4+1+a_7) + 2(3\phi^2+\phi)(a_6\phi - a_2 + a_1\phi^3),\\ \epsilon_3 = a_3^2 - (3\phi^2+\phi)^2 + (a_6\phi - a_2 + a_1\phi^3)^2,,\\ \epsilon_4 = 2(a_5 - \phi^4 - 1 - a_7) - 2(3\phi^2+\phi)(a_6\phi - a_2 + a_1\phi^3),\\ \epsilon_5 = (a_5 - \phi^4 - 1 - a_7)^2 - (a_6\phi - a_2 + a_1\phi^3)^2.\end{cases} \tag{3.14}$$

By (3.13), we can get the expression of $\cos\phi\theta$. Here we assume that

$$\cos\phi\theta = f_1(\phi), \tag{3.15}$$

where $f_1(\phi)$ is a function with respect to ϕ. In a similar way, we can also get the expression of $\sin\phi\theta$. Here we assume that

$$\sin\phi\theta = f_2(\phi), \tag{3.16}$$

where $f_2(\phi)$ is a function with respect to ϕ. Using (3.15) and (3.16), we get

$$f_1^2(\phi) + f_2^2(\phi) = 1. \tag{3.17}$$

We can obtain the root of (3.17) by computer. Here we denote the root of (3.1) by ϕ_0. Using (3.15), one gains

$$\theta_0^{(h)} = \frac{1}{\phi_0}\left[\arccos f_1(\phi_0) + 2h\pi\right], \tag{3.18}$$

where $h = 0, 1, 2, \cdots$. Denote $\theta_0 = \min_{\{h=0,1,2,\cdots\}}\{\theta_0^{(h)}\}$ and assume that when $\theta = \theta_0$, (3.3) admits a pair of imaginary roots $\pm i\phi_0$.

Now the following assumption is needed:

$$(A_2)\ \ \mathcal{U}_{1R}\mathcal{U}_{2R} + \mathcal{U}_{1I}\mathcal{U}_{2I} > 0,$$

where

$$\begin{cases}\mathcal{U}_{1R} = 1 + (3a_1\phi_0^2 + a_2)\cos\phi_0\theta_0 + 2a_4\phi_0\sin 2\phi_0\theta_0\\ \qquad + a_6\cos 3\phi_0\theta_0,\\ \mathcal{U}_{1I} = -4\phi_0^3 - 6\phi_0 - (3a_1\phi_0^2 + a_2)\sin\phi_0\theta_0\\ \qquad + 2a_4\phi_0\cos 2\phi_0\theta_0 - a_6\sin 3\phi_0\theta_0,\\ \mathcal{U}_{2R} = a_3\phi_0\sin\phi_0\theta_0 - \left(a_1\phi_0^3 - a_2\phi_0\right)\phi_0\cos\phi_0\theta_0\\ \qquad + \phi_0(a_5 - a_4\phi_0^2)\cos 2\phi_0\theta_0 - 3a_6\phi_0^2\cos 3\phi_0\theta_0\\ \qquad + 4a_7\phi_0\sin 4\phi_0\theta_0,\\ \mathcal{U}_{2I} = \left(-a_1\phi_0^3 + a_2\phi_0\right)\phi_0\sin\phi_0\theta_0 + a_2\phi_0\cos\phi_0\theta_0\\ \qquad - \phi_0(a_5 - a_4\phi_0^2)\sin 2\phi_0\theta_0 + 3a_6\phi_0^2\sin 3\phi_0\theta_0\\ \qquad + 4a_7\phi_0\cos 4\phi_0\theta_0.\end{cases} \tag{3.19}$$

Lemma 3.1. *Let $\lambda(\theta) = \varsigma_1(\theta) + i\varsigma_2(\theta)$ be the root of Eq (3.3) near $\theta = \theta_0$ such that $\varsigma_1(\theta_0) = 0, \varsigma_2(\theta_0) = \phi_0$, then Re* $\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_0,\phi=\phi_0} > 0$.

Proof. In view of Eq (3.3), one gains

$$\begin{aligned}&\left(4\lambda^3 - 6\lambda - 1\right)\frac{d\lambda}{d\theta} + (3a_1\lambda^2 + a_2)e^{-\lambda\theta}\frac{d\lambda}{d\theta}\\&-e^{-\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right)\left(a_1\lambda^3 + a_2\lambda + a_3\right) + 2a_4\lambda\frac{d\lambda}{d\theta}e^{-2\lambda\theta}\\&-e^{-2\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right)\left(a_4\lambda^2 + a_5\right) + a_6e^{-3\lambda\theta}\frac{d\lambda}{d\theta}\\&-3a_6\lambda e^{-3\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right) - 4a_7e^{-4\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right) = 0,\end{aligned} \tag{3.20}$$

which leads to

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = \frac{\mathcal{U}_1(\lambda)}{\mathcal{U}_2(\lambda)} - \frac{\theta}{\lambda}, \tag{3.21}$$

where

$$\begin{cases}\mathcal{U}_1(\lambda) = 4\lambda^3 - 6\lambda + 1 + (3a_1\lambda^2 + a_2)e^{-\lambda\theta} + 2a_4\lambda e^{-2\lambda\theta} + a_6e^{-3\lambda\theta},\\ \mathcal{U}_2(\lambda) = \lambda e^{-\lambda\theta}\left(a_1\lambda^3 + a_2\lambda + a_3\right) + 2\lambda(a_4\lambda^2 + a_5)e^{-2\lambda\theta}\\ \qquad\qquad + 3a_6\lambda^2e^{-3\lambda\theta} + 4a_7\lambda e^{-4\lambda\theta}.\end{cases} \tag{3.22}$$

Then

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\phi=\phi_0} = \mathrm{Re}\left[\frac{\mathcal{U}_1(\lambda)}{\mathcal{U}_2(\lambda)}\right]_{\theta=\theta_0,\phi=\phi_0} = \frac{\mathcal{U}_{1R}\mathcal{U}_{2R} + \mathcal{U}_{1I}\mathcal{U}_{2I}}{\mathcal{U}_{2R}^2 + \mathcal{U}_{2I}^2}. \tag{3.23}$$

Taking advantage of (A_2), one gets

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_0,\phi=\phi_0} > 0. \tag{3.24}$$

The proof ends.

Relying on the analysis above, we can lightly set up the following assertion.

Theorem 3.1. *If $(A_1), (A_2)$ hold, then the zero equilibrium point $E(0,0,0,0)$ of model (1.3) is locally asymptotically stable if $\theta \in [0, \theta_0)$ and model (1.3) produces a Hopf bifurcation around the zero equilibrium point $E(0,0,0,0)$ when $\theta = \theta_0$.*

Remark 3.1. *In many cases, the self-feedback time delay of every neuron in neural networks is different. If there exist two different delays in model (1.3), we can also explore the Hopf bifurcation of system (1.3). We leave it for future work.*

4. Control of bifurcation

In this part, we will deal with the Hopf bifurcation problem of system (1.5) via a suitable hybrid controller consisting of state feedback and parameter perturbation. Taking advantage of the idea in [19,20], we obtain the following controlled finance model:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - au_1(t-\theta), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - au_2(t-\theta), \\ \dot{u}_3(t) = \rho_1\{\tanh[u_1(t) + u_2(t) - u_4(t)] - au_3(t-\theta)\} + \rho_2 u_3(t-\theta), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - au_4(t-\theta), \end{cases} \tag{4.1}$$

where ρ_1, ρ_2 represent feedback gain parameters. Systems (4.1) and (1.3) have the same unique equilibrium point $E(0,0,0,0)$. The linear system of system (4.1) at $E(0,0,0,0)$ is given by

$$\begin{cases} \dot{u}_1(t) = u_4(t) - u_2(t) - au_1(t-\theta), \\ \dot{u}_2(t) = u_1(t) + u_4(t) - au_2(t-\theta), \\ \dot{u}_3(t) = \rho_1 u_1(t) + \rho_1 u_2(t) - \rho_1 u_4(t) - (\rho_1 a - \rho_2) u_3(t-\theta), \\ \dot{u}_4(t) = u_3(t) - u_2(t) - au_4(t-\theta). \end{cases} \tag{4.2}$$

The characteristic equation of system (3.1) owns the following expression:

$$\det \begin{bmatrix} \lambda + ae^{-\lambda\theta} & 1 & 0 & -1 \\ -1 & \lambda + ae^{-\lambda\theta} & 0 & -1 \\ -\rho_1 & -\rho_1 & \lambda + (\rho_1 a - \rho_2)e^{-\lambda\theta} & \rho_1 \\ 0 & 1 & -1 & \lambda + ae^{-\lambda\theta} \end{bmatrix} = 0, \tag{4.3}$$

which generates

$$\begin{aligned} &\lambda^4 + \alpha_1\lambda^2 + \alpha_2 + (\alpha_3\lambda^3 + \alpha_4\lambda)e^{-\lambda\theta} \\ &+(\alpha_5\lambda^2 + \alpha_6)e^{-2\lambda\theta} + \alpha_7\lambda e^{-3\lambda\theta} + \alpha_8 e^{-4\lambda\theta} = 0, \end{aligned} \tag{4.4}$$

where

$$\begin{cases} \alpha_1 = 2 + \rho_1, \\ \alpha_2 = \rho_1, \\ \alpha_3 = 3a + \rho_1 a + \rho_2, \\ \alpha_4 = 2(2a + \rho_1 a + \rho_2), \\ \alpha_5 = 3a(a + a\rho_1 + \rho_2), \\ \alpha_6 = a(2a\rho_1 + 2\rho_2 + a), \\ \alpha_7 = a^2(3a\rho_1 + 3\rho_2 + a), \\ \alpha_8 = a^3(a\rho_1 + \rho_2). \end{cases} \tag{4.5}$$

When $\theta = 0$, then Eq (4.4) becomes:

$$\lambda^4 + \beta_1\lambda^3 + \beta_2\lambda^2 + \beta_3\lambda + \beta_4 = 0, \tag{4.6}$$

where

$$\begin{cases} \beta_1 = \alpha_1, \\ \beta_2 = \alpha_1 + \alpha_5, \\ \beta_3 = \alpha_4 + \alpha_7, \\ \beta_4 = \alpha_2 + \alpha_6 + \alpha_8. \end{cases} \tag{4.7}$$

If

$$(A_3)\begin{cases}\Lambda_1=\beta_1>0,\\ \Lambda_2=\det\begin{bmatrix}\beta_1 & 1\\ \beta_3 & \beta_2\end{bmatrix}>0,\\ \Lambda_3=\begin{bmatrix}\beta_1 & 1 & 0\\ \beta_3 & \beta_2 & \beta_1\\ 0 & \beta_4 & \beta_3\end{bmatrix}>0,\\ \Lambda_4=\beta_4\Lambda_3>0\end{cases}$$

is fulfilled, then the four roots $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ of Eq (4.4) have negative real parts. Thus the zero equilibrium point $E(0,0,0,0)$ of system (4.1) with $\theta=0$ is locally asymptotically stable.
By (4.4), we get

$$\begin{aligned}&(\lambda^4+\alpha_1\lambda^2+\alpha_2)e^{2\lambda\theta}+(\alpha_3\lambda^3+\alpha_4\lambda)e^{\lambda\theta}\\&+(\alpha_5\lambda^2+\alpha_6)+\alpha_7\lambda e^{-\lambda\theta}+\alpha_8e^{-2\lambda\theta}=0.\end{aligned}\tag{4.8}$$

Suppose that $\lambda=i\varphi$ is the root of Eq (4.8). Then Eq (4.8) becomes:

$$\begin{aligned}&[(i\varphi)^4+\alpha_1(i\varphi)^2+\alpha_2]e^{2i\varphi\theta}+[\alpha_3(i\varphi)^3+\alpha_4(i\varphi)]e^{i\varphi\theta}\\&+[\alpha_5(i\varphi)^2+\alpha_6]+\alpha_7i\varphi e^{-i\varphi\theta}+\alpha_8e^{-2i\varphi\theta}=0,\end{aligned}\tag{4.9}$$

which generates

$$\begin{aligned}&(\varphi^4-\alpha_1\varphi^2+\alpha_2)(\cos 2\varphi\theta+i\sin 2\varphi\theta)\\&+(-i\alpha_3\varphi^3+i\alpha_4\varphi)(\cos\varphi\theta+i\sin\varphi\theta)\\&+(-\alpha_5\varphi^2+\alpha_6)+i\alpha_7\varphi(\cos\varphi\theta-i\sin\varphi\theta)\\&+\alpha_8(\cos 2\varphi\theta-i\sin 2\varphi\theta)=0.\end{aligned}\tag{4.10}$$

It follows from (4.10) that

$$\begin{cases}(\varphi^4-\alpha_1\varphi^2+\alpha_2+\alpha_8)\cos 2\varphi\theta+[(\alpha_7-\alpha_4)\varphi+\alpha_3\varphi^3]\sin\phi\theta\\ \qquad+\alpha_6-\alpha_5\varphi^2=0,\\ (\varphi^4-\alpha_1\varphi^2+\alpha_2-\alpha_8)\sin 2\varphi\theta+(\alpha_4\varphi-\alpha_3\varphi^3)\cos\varphi\theta+\alpha_7\varphi=0.\end{cases}\tag{4.11}$$

In view of $\sin\varphi\theta=\pm\sqrt{1-\cos^2\varphi\theta}$, then it follows from the second equation of (4.11) that

$$2(\varphi^4-\alpha_1\varphi^2+\alpha_2-\alpha_8)(\pm\sqrt{1-\cos^2\varphi\theta})\cos\varphi\theta+(\alpha_4\varphi-\alpha_3\varphi^3)\cos\varphi\theta+\alpha_7\varphi=0,\tag{4.12}$$

which leads to

$$\epsilon_1\cos^4\varphi\theta+\epsilon_2\cos^2\varphi\theta+\epsilon_3\cos\varphi\theta+\epsilon_4=0,\tag{4.13}$$

where

$$\begin{cases}\epsilon_1=4(\varphi^4-\alpha_1\varphi+\alpha_2-\alpha_8)^2,\\ \epsilon_2=(\alpha_4\varphi-\alpha_3\varphi^3)^2-4(\varphi^4-\alpha_1\varphi+\alpha_2-\alpha_8)^2,\\ \epsilon_3=2\alpha_7\varphi(\alpha_4\varphi-\alpha_3\varphi^3),\\ \epsilon_4=\alpha_7^2\varphi^2.\end{cases}\tag{4.14}$$

By virtue of computer software, we can solve the value of $\cos\varphi\theta$. Here we assume that

$$\cos\varphi\theta = h_1(\varphi). \tag{4.15}$$

Then we can solve the value of $\sin\varphi\theta$. Here we assume that

$$\sin\varphi\theta = h_2(\varphi). \tag{4.16}$$

Using (4.15) and (4.16), we have

$$h_1^2(\varphi) + h_2^2(\varphi) = 1. \tag{4.17}$$

By virtue of (4.17), we can easily obtain the value of φ (say φ_0) via computer software. According to (4.15), we get

$$\theta_j = \frac{1}{\varphi_0}[\arccos f_1(\varphi_0) + 2j\pi],\ j = 0, 1, 2, \cdots. \tag{4.18}$$

Denote

$$\theta_* = \min_{\{j=0,1,2,\cdots\}}\{\theta_j\}. \tag{4.19}$$

Now we know that when $\theta = \theta_*$, (4.4) has a pair of imaginary roots $\pm i\varphi_0$.

Now we give the following hypothesis:

$$(A_4)\ \ \mathcal{V}_{1R}\mathcal{V}_{2R} + \mathcal{V}_{1I}\mathcal{V}_{2I} > 0,$$

where

$$\begin{cases} \mathcal{V}_{1R} = (\alpha_4 - 2\alpha_3\varphi_0^2)\cos\varphi_0\theta_* + 2\alpha_5\varphi_0\sin\varphi_0\theta_*, \\ \mathcal{V}_{1I} = 2\alpha_1\varsigma_0 - 4\varphi_0^3 - (\alpha_4 - 2\alpha_3\varphi_0^2)\sin\varphi_0\theta_* + 2\alpha_5\varphi_0\cos\varphi_0\theta_*, \\ \mathcal{V}_{2R} = (\alpha_3\varphi_0^3 - \alpha_4\varphi_0)\varphi_0\cos\varphi_0\theta_* + 2\varphi_0(\alpha_6 - \alpha_5\varphi_0^2)\sin 2\varphi_0\theta_* \\ \qquad - 3\alpha_7\varphi_0^2\cos 3\varphi_0\theta_* + 4\alpha_8\varphi_0\sin 4\varphi_0\theta_*, \\ \mathcal{V}_{2I} = (\alpha_4\varphi_0 - \alpha_3\varphi_0^3)\varphi_0\sin\varphi_0\theta_* + 2\varphi_0(\alpha_6 - \alpha_5\varphi_0^2)\cos 2\varphi_0\theta_* \\ \qquad - 3\alpha_7\varphi_0^2\sin 3\varphi_0\theta_* + 4\alpha_8\varphi_0\cos 4\varphi_0\theta_*. \end{cases} \tag{4.20}$$

Lemma 4.1. *Let $\lambda(\theta) = \xi_1(\theta) + i\xi_2(\theta)$ be the root of Eq (4.4) around $\theta = \theta_*$ satisfying $\xi_1(\theta_*) = 0, \xi_2(\theta_*) = \varphi_0$, then Re $\left(\frac{d\lambda}{d\theta}\right)\Big|_{\theta=\theta_*,\varphi=\varphi_0} > 0$.*

Proof. By Eq (4.4), one gets

$$\begin{aligned} &\left(3\lambda^2 + 2\alpha_1\lambda\right)\frac{d\lambda}{d\theta} + (2\alpha_3\lambda^2 + \alpha_4)e^{-\lambda\theta}\frac{d\lambda}{d\theta} \\ &-e^{-\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right)\left(\alpha_3\lambda^3 + \alpha_4\lambda\right) + 2\alpha_5\lambda e^{-2\lambda\theta}\frac{d\lambda}{d\theta} \\ &-2e^{-2\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right)(\alpha_5\lambda^2 + \alpha_6) + \alpha_7 e^{-3\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right) \\ &-3e^{-3\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right)\alpha_7\lambda - 4\alpha_8 e^{-4\lambda\theta}\left(\frac{d\lambda}{d\theta}\theta + \lambda\right) = 0, \end{aligned} \tag{4.21}$$

which implies

$$\left(\frac{d\lambda}{d\sigma}\right)^{-1} = \frac{\mathcal{V}_1(\lambda)}{\mathcal{V}_2(\lambda)} - \frac{\theta}{\lambda}, \tag{4.22}$$

where

$$\begin{cases} \mathcal{V}_1(\lambda) = 4\lambda^3 + 2\alpha_1\lambda + (2\alpha_3\lambda^2 + \alpha_4)e^{-\lambda\theta} + 2\alpha_5\lambda e^{-2\lambda\sigma}, \\ \mathcal{V}_2(\lambda) = \lambda e^{-\lambda\theta}\left(\alpha_3\lambda^3 + \alpha_4\lambda\right) + 2\lambda e^{-2\lambda\theta}\left(\alpha_5\lambda^2 + \alpha_6\right) \\ \qquad + 3\lambda^2\alpha_7 e^{-3\lambda\sigma} + 4\alpha_8\lambda e^{-4\lambda\sigma}. \end{cases} \tag{4.23}$$

Hence

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\theta=\theta_*,\varphi=\varphi_0} = \mathrm{Re}\left[\frac{\mathcal{V}_1(\lambda)}{\mathcal{V}_2(\lambda)}\right]_{\theta=\theta_*,\varphi=\varphi_0} = \frac{\mathcal{V}_{1R}\mathcal{V}_{2R} + \mathcal{V}_{1I}\mathcal{V}_{2I}}{\mathcal{V}_{2R}^2 + \mathcal{V}_{2I}^2}. \tag{4.24}$$

Taking advantage of (A_4), one gets

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\sigma}\right)^{-1}\right]_{\theta=\theta_*,\varphi=\varphi_0} > 0, \tag{4.25}$$

which ends the proof.

Relaying on the discussion above, we can lightly gain the following outcome.

Theorem 4.1. *If $(A_3), (A_4)$ hold, then the equilibrium point $E(0,0,0,0)$ of model (4.1) is locally asymptotically stable if $\theta \in [0, \theta_*)$ and a Hopf bifurcation of model (4.1) arises around the equilibrium point $E(0,0,0,0)$ when $\theta = \theta_*$.*

Remark 4.1. *In 2008, Sprott [13] investigated the chaotic behavior of the 4D neural network model (1.1); In 2022, Vaishwar and Yadav [14] dealt with the stability and Hopf-bifurcation of the 4D neural network model (1.2). Both works of [13] and [14] are only concerned with the integer-order neural networks. In this study, we deal with the stability, Hopf bifufcation and its control issue of the 4D fractional-order neural network model (1.3) concerning delay. All the exploration ways are different from those in [13] and [14]. Relying on this viewpoint, we argue that the exploration ways and the acquired outcomes are important supplement to the works of [13] and [14].*

Remark 4.2. *In this paper, we have dealt with the Hopf bifurcation control via hybrid controller. Of course, we can also deal with the Hopf bifurcation control via other hybrid controllers. We will explore this topic in near future.*

5. Illustrated examples

Example 5.1. Consider the following 4D BAM neural networks concerning time delay:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - 0.25u_1(t-\theta), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - 0.25u_2(t-\theta), \\ \dot{u}_3(t) = \tanh[u_1(t) + u_2(t) - u_4(t)] - 0.25u_3(t-\theta), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - 0.25u_4(t-\theta). \end{cases} \tag{5.1}$$

It is not difficult to gain that system (5.1) has a unique zero equilibrium point $E(0,0,0,0)$. One is lightly able to verify that both conditions (A_1) and (A_2) of Theorem 3.1 hold true. Making use of computer software, one gains $\phi_0 = 2.0083, \theta_0 \approx 0.8$. To examine the accuracy of the gained outcomes of Theorem 3.1, we give both different time delay values. Let $\theta = 0.76$ and $\theta = 1.1$. For $\theta = 0.76 < \theta_0 \approx 0.8$, the computer simulation outcomes are provided in Figure 1. In terms of Figure 1, one can easily understand that $u_1 \to 0, u_2 \to 0, u_3 \to 0, u_4 \to 0$ when $t \to +\infty$. In other words, the zero equilibrium point $E(0,0,0,0)$ of system (5.1) maintains locally asymptotically stable situation. For $\theta = 1.1 > \theta_0 \approx 0.8$, the computer simulation outcomes are provided in Figure 2. In terms of Figure 2, one can easily understand that u_1, u_2, u_3, u_4 are going to maintain periodic motion around the value 0, respectively. In other words, a cluster of limit cycles (i.e., Hopf bifurcations) happen around the zero equilibrium point $E(0,0,0,0)$. In addition, the bifurcation plots, which clearly display the bifurcation value of system (5.1), are provided in Figures 3–6. In terms of Figures 3–6, one is lightly able to understand that the bifurcation point of system (5.1) is $\theta_0 \approx 0.8$.

Example 5.2. Consider the following controlled 4D BAM neural networks concerning time delay:

$$\begin{cases} \dot{u}_1(t) = \tanh[u_4(t) - u_2(t)] - 0.25u_1(t-\theta), \\ \dot{u}_2(t) = \tanh[u_1(t) + u_4(t)] - 0.25u_2(t-\theta), \\ \dot{u}_3(t) = \rho_1\{\tanh[u_1(t) + u_2(t) - u_4(t)] - 0.25u_3(t-\theta)\} + \rho_2 u_3(t-\theta), \\ \dot{u}_4(t) = \tanh[u_3(t) - u_2(t)] - 0.25u_4(t-\theta). \end{cases} \tag{5.2}$$

It is not difficult to gain that system (5.2) has a unique zero equilibrium point $E(0,0,0,0)$. One is lightly able to verify that both conditions (A_3) and (A_4) of Theorem 4.1 hold true. Making use of computer software, one gains $\varphi_0 = 5.6712, \theta_* \approx 0.62$. To examine the accuracy of the gained outcomes of Theorem 4.1, we give both different time delay values. Let $\theta = 0.6$ and $\theta = 0.7$. For $\theta = 0.6 < \theta_0 \approx 0.62$, the computer simulation outcomes are provided in Figure 7. In terms of Figure 7, one can easily understand that $u_1 \to 0, u_2 \to 0, u_3 \to 0, u_4 \to 0$ when $t \to +\infty$. In other words, the zero equilibrium point $E(0,0,0,0)$ of system (5.2) maintains locally asymptotically stable situation. For $\theta = 0.7 > \theta_* \approx 0.62$, the computer simulation outcomes are provided in Figure 8. In terms of Figure 8, one can easily understand that u_1, u_2, u_3, u_4 are going to maintain periodic motion around the value 0, respectively. In other words, a cluster of limit cycles (i.e., Hopf bifurcations) happen around the zero equilibrium point $E(0,0,0,0)$. In addition, the bifurcation plots, which clearly display the bifurcation value of system (5.2), are provided in Figures 9–12. In terms of Figures 9–12, one is lightly able to understand that the bifurcation point of system (5.2) is $\theta_* \approx 0.62$.

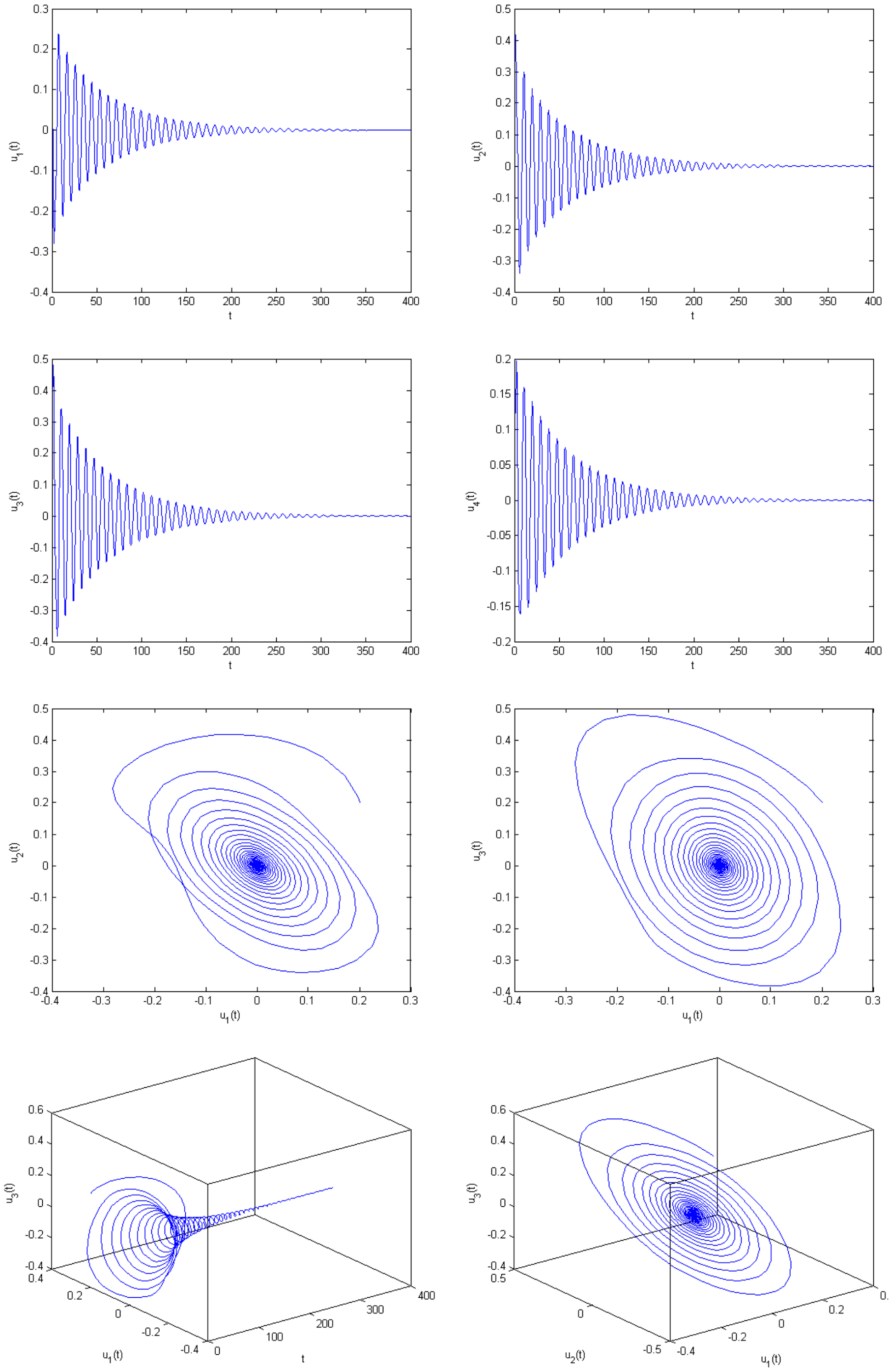

Figure 1. Computer simulation figures of system (5.1) concerning the delay $\theta = 0.76 < \theta_0 \approx 0.8$. The zero equilibrium point $E(0, 0, 0, 0)$ maintains locally asymptotically stable situation.

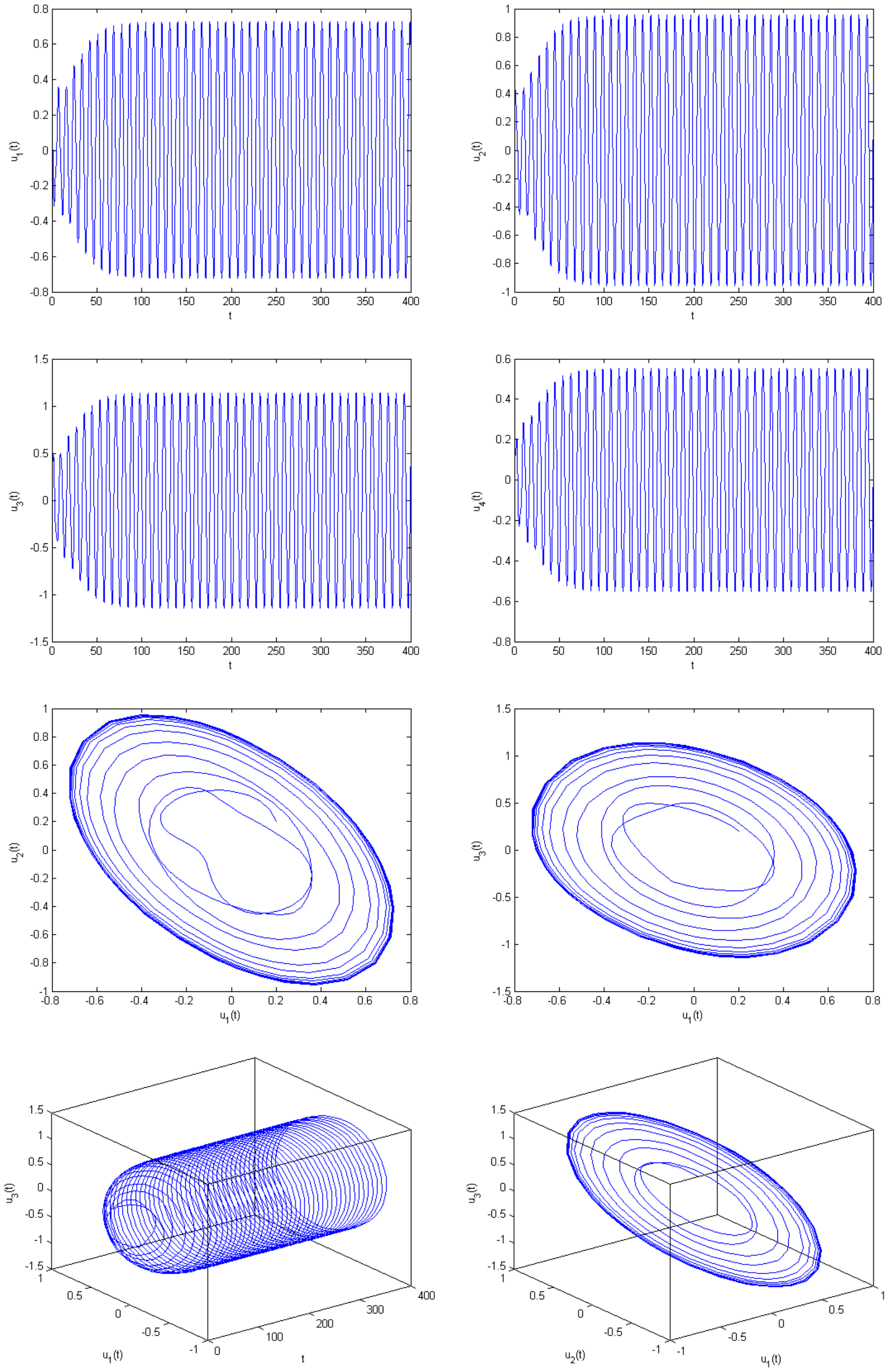

Figure 2. Computer simulation figures of system (5.1) concerning the delay $\theta = 1.1 > \theta_0 \approx 0.8$. A cluster of limit cycles (i.e., Hopf bifurcations) happen near the zero equilibrium point $E(0, 0, 0, 0)$.

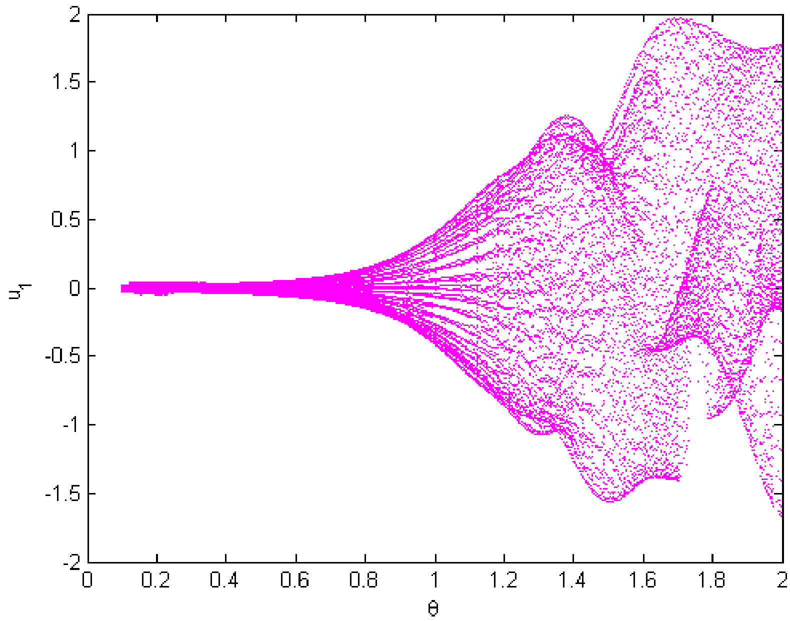

Figure 3. Bifurcation figure of system (5.1): x-axis denotes the time t and y-axis denotes the state variable u_1. The bifurcation point $\theta_0 \approx 0.8$.

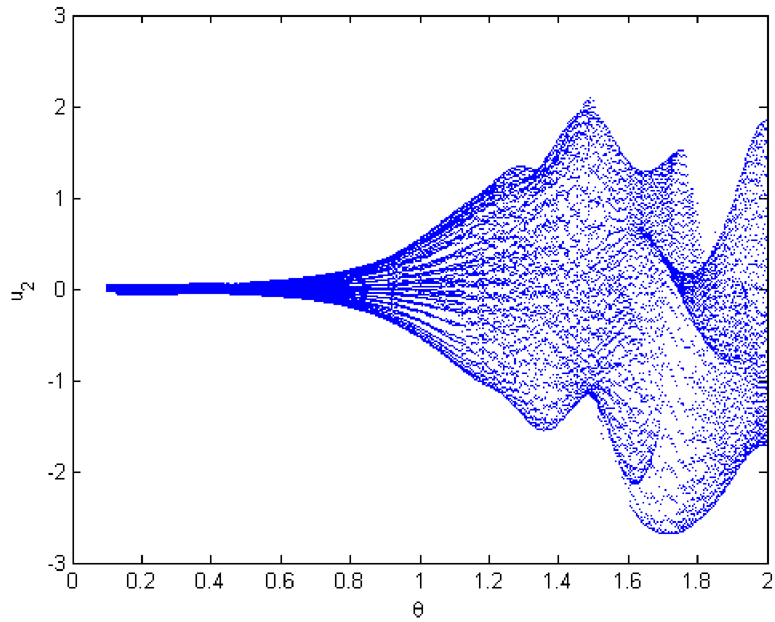

Figure 4. Bifurcation figure of system (5.1): x-axis denotes the time t and y-axis denotes the state variable u_2. The bifurcation point $\theta_0 \approx 0.8$.

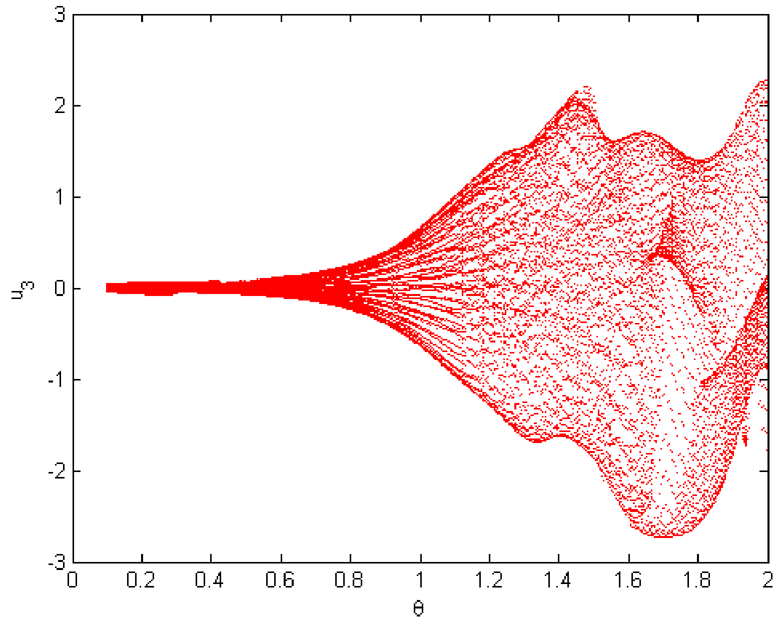

Figure 5. Bifurcation figure of system (5.1): x-axis denotes the time t and y-axis denotes the state variable u_3. The bifurcation point $\theta_0 \approx 0.8$.

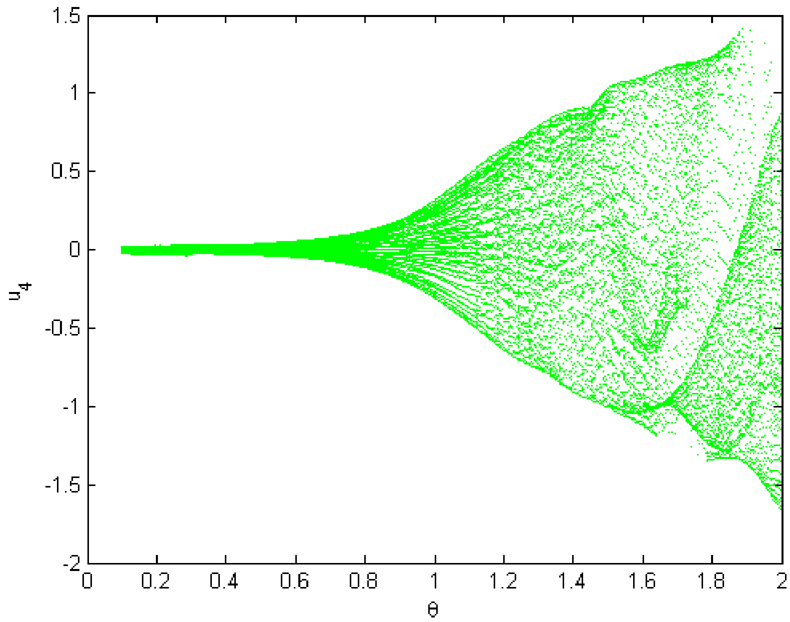

Figure 6. Bifurcation figure of system (5.1): x-axis denotes the time t and y-axis denotes the state variable u_4. The bifurcation point $\theta_0 \approx 0.8$.

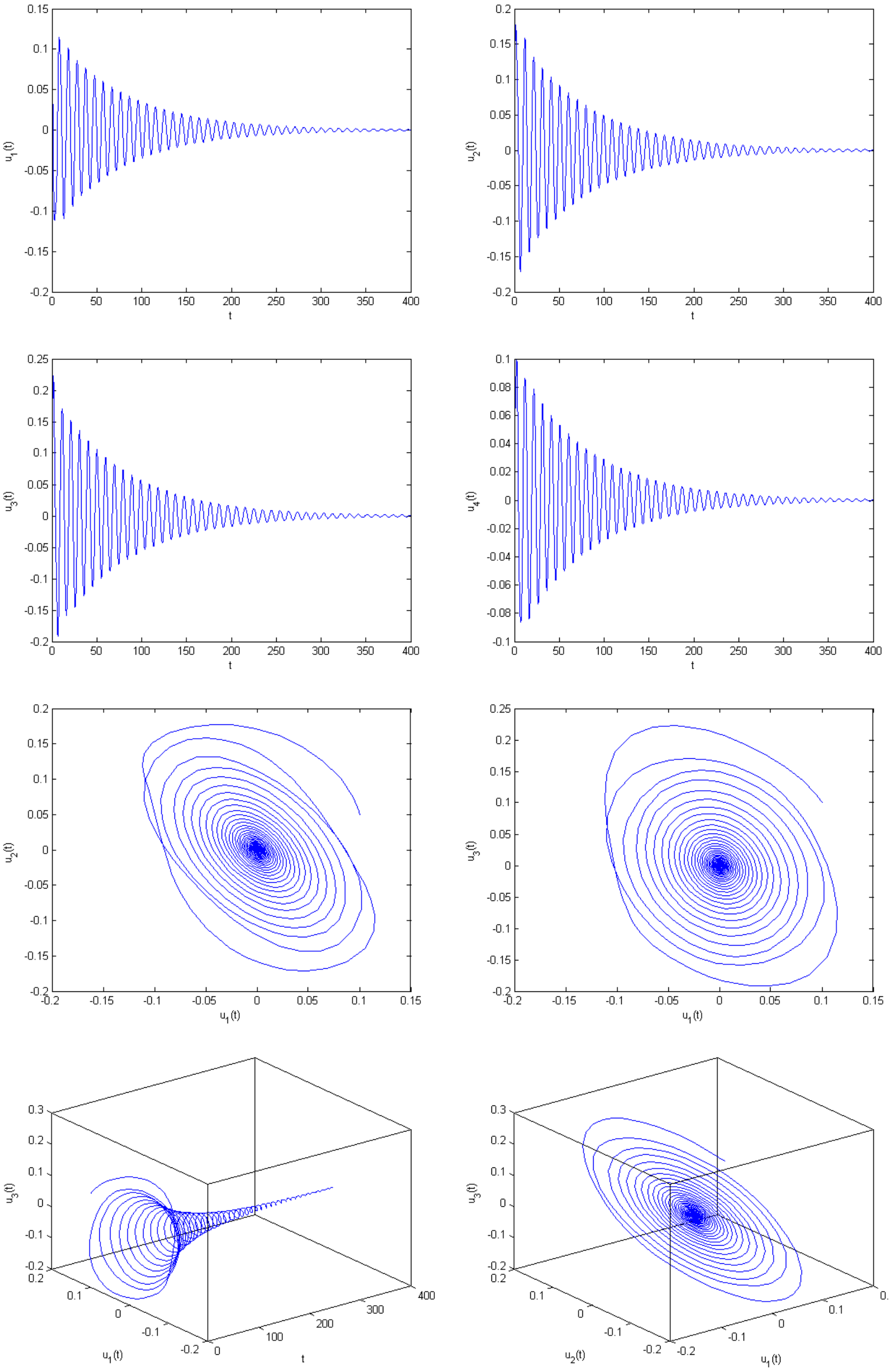

Figure 7. Computer simulation figures of system (5.2) concerning the delay $\theta = 0.6 < \theta_* \approx 0.62$. The zero equilibrium point $E(0, 0, 0, 0)$ maintains locally asymptotically stable situation.

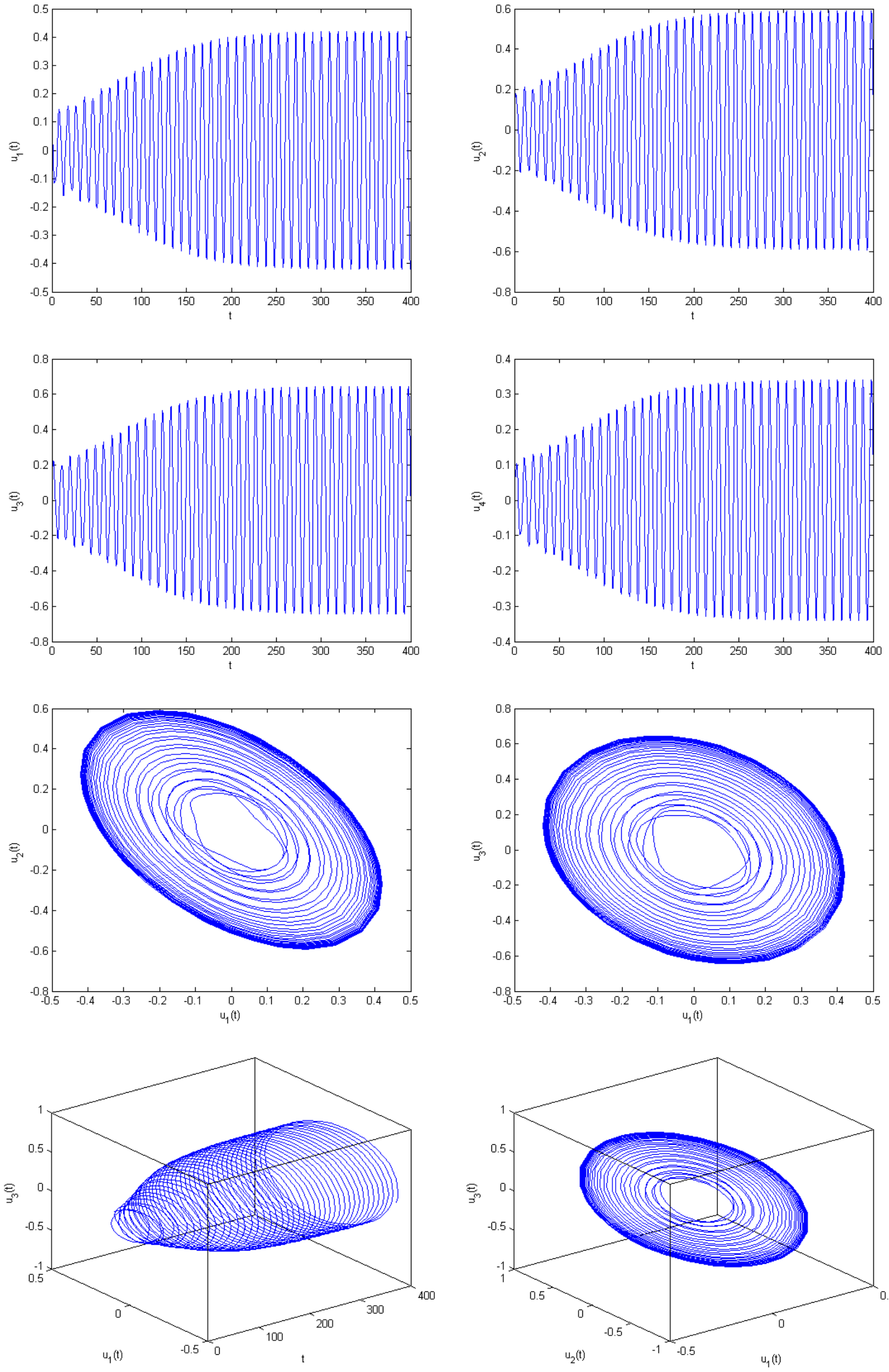

Figure 8. Computer simulation figures of system (5.2) concerning the delay $\theta = 0.7 > \theta_* \approx 0.62$. A cluster of limit cycles (i.e., Hopf bifurcations) happen near the zero equilibrium point $E(0, 0, 0, 0)$.

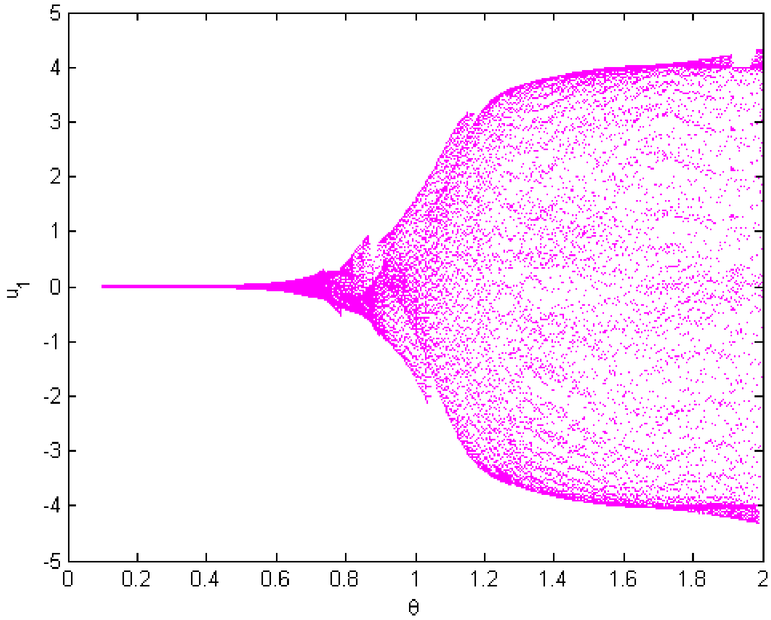

Figure 9. Bifurcation figure of system (5.2): x-axis denotes the time t and y-axis denotes the state variable u_1. The bifurcation point $\theta_* \approx 0.62$.

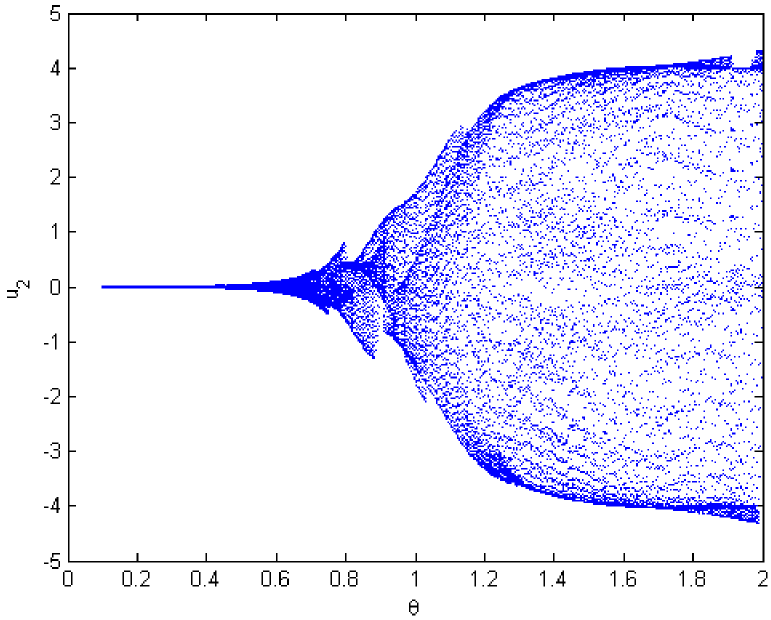

Figure 10. Bifurcation figure of system (5.2): x-axis denotes the time t and y-axis denotes the state variable u_2. The bifurcation point $\theta_* \approx 0.62$.

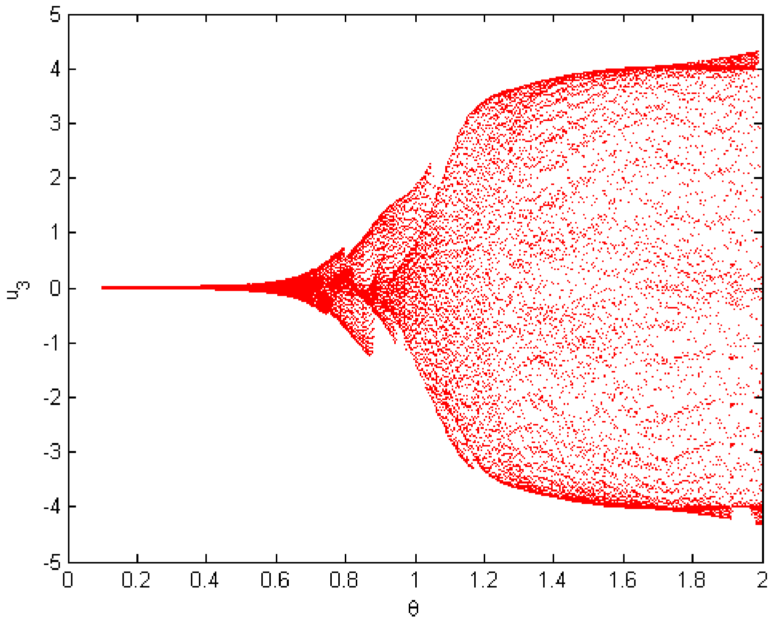

Figure 11. Bifurcation figure of system (5.2): x-axis denotes the time t and y-axis denotes the state variable u_3. The bifurcation point $\theta_* \approx 0.62$.

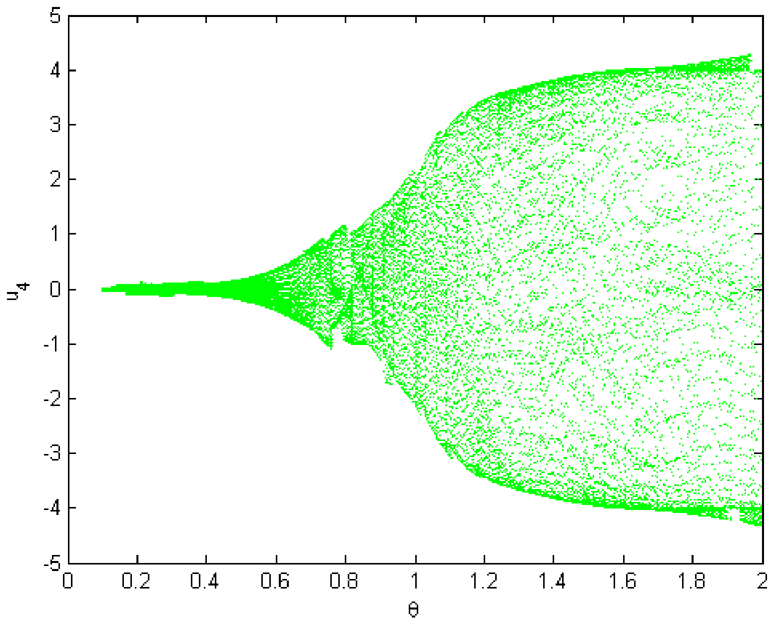

Figure 12. Bifurcation figure of system (5.2): x-axis denotes the time t and y-axis denotes the state variable u_4. The bifurcation point $\theta_* \approx 0.62$.

Remark 5.1. *From the computer simulation outcomes of Examples 5.1 and 5.2, we know that the bifurcation value of system (5.1) is* $\theta_0 \approx 0.8$ *and the bifurcation value of system (5.2) is* $\theta_* \approx 0.62$*. Thus we understand that the stability domain of system (5.1) is narrowed and the time of onset of Hopf bifurcation of system (5.1) is advanced via a suitable hybrid controller consisting of state feedback and parameter perturbation.*

6. Conclusions

Delayed-driven Hopf bifurcation is a vital dynamical behavior in delayed dynamical models. In particular, delayed-driven Hopf bifurcation in neural networks has attracted great interest from scientific community. In this study, novel 4D BAM neural networks involving time delay have been formulated. The existence and uniqueness, boundedness of solution of the addressed BAM neural networks concerning single delay have been explored. Regarding the delay as Hopf bifurcation parameter, a new delay-independent condition on the stability and the emergence of delay-driven Hopf bifurcation is acquired. The importance of delay in stabilizing networks and dominating the time of onset of delay-driven Hopf bifurcation is reveled. In order to adjust the stability domain and the time of delay-driven Hopf bifurcation of addressed BAM neural networks concerning single delay, we apply a proper hybrid controller including state feedback and parameter perturbation to explore the stability domain and the time of delay-driven Hopf bifurcation issue. The study shows that the designed hybrid controller is an effective controller. The acquired outcomes of this study have momentous theoretical significance in controlling and contriving neural networks. Also the obtained results have important application in artificial intelligence in various industrial applications, especially in terms of convolutional fusion framework for collaborative fault identification of rotating machinery, and Physics-Informed Residual Network (PIResNet) for rolling element bearing fault diagnostics [39]. What is more, the exploration idea of this study is able to investigate the Hopf bifurcation and its control issue in lots of differential dynamical systems. In this paper, we have dealt with the Hopf bifurcation and its control of integer-order neural networks via hybrid controller. Of course, we can also deal with the Hopf bifurcation and its control of fractional-order neural networks via hybrid controller [40–47]. We will explore this topic in near future.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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