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Research article

Novel inequalities for subadditive functions via tempered fractional integrals and their numerical investigations

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Abstract: In this paper, we proposed some new integral inequalities for subadditive functions and the product of subadditive functions. Additionally, a novel integral identity was established and a number of inequalities of the Hermite-Hadamard type for subadditive functions pertinent to tempered fractional integrals were proved. Finally, to support the major results, we provided several examples of subadditive functions and corresponding graphs for the newly proposed inequalities.

Keywords: Hermite-Hadamard inequality; subadditivity; tempered fractional integral operators; Hölder's inequality; power-mean inequality; estimations
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction and preliminaries

The primary contribution to the concept of subadditive functions is due to the work of Hille and Phillips [1]. An extract from Rosenbaum's study on subadditive functions with several variables is also given [2]. The concepts of additivity, subadditivity, and superadditivity are used in many branches of mathematics as well as in mathematical inequalities, see [3–8].

Definition 1.1. *[9]* $F : I \subset R \to [0, \infty)$ *is said to be subadditive function on* I*, if*

$$F(\Theta_1 + \Theta_2) \leq F(\Theta_1) + F(\Theta_2) \tag{1.1}$$

holds for all $\Theta_1, \Theta_2 \in I$ *and* $\Theta_1 + \Theta_2 \in I$*. If the equality is achieved, then* F *is said to be additive, otherwise superadditive.*

Since many optimization issues require maximizing or minimizing a convex function while taking certain restrictions into account, convex functions are crucial in optimization. Additionally, they have appealing characteristics like a singular global minimum and nearly universal differentiability. Numerous fields, including optimization, game theory, economics, and computer science, use convexity theory. It offers strong tools for understanding and resolving optimization issues, and it has sparked the creation of several algorithms for quickly calculating answers to these issues.

Definition 1.2. *[10]* $F : I \subset R \to R$ *is said to be convex, if*

$$F(\delta\Theta_1 + (1-\delta)\Theta_2) \leq \delta F(\Theta_1) + (1-\delta)F(\Theta_2) \tag{1.2}$$

holds for all $\Theta_1, \Theta_2 \in I$ *with* $\delta \in [0, 1]$.

The given Hermite-Hadamard inequality (H-H) has a strong connection with convex functions.

Theorem 1.1. *[11, 12] If* $F : I \subset R \to R$ *is convex on* I*, then*

$$F\left(\frac{\Theta_1 + \Theta_2}{2}\right) \leq \frac{1}{\Theta_2 - \Theta_1} \int_{\Theta_1}^{\Theta_2} F(\delta) d\delta \leq \frac{F(\Theta_1) + F(\Theta_2)}{2} \tag{1.3}$$

holds for all $\Theta_1, \Theta_2 \in I$ *and* $\Theta_1 < \Theta_2$.

There are numerous well-known inequalities that may be obtained using the convexity feature, see [13–18].

Let us denote by $L[\Theta_1, \Theta_2]$ the set of all Lebesgue integrable functions on $[\Theta_1, \Theta_2]$ and by I° the interior set of I. The following H-H type for continuous subadditive functions were recently established by Sarikaya and Ali [19].

Theorem 1.2. *If* $F : I \subset R \to [0, \infty)$ *is a continuous subadditive function with* $\Theta_1, \Theta_2 \in I^\circ$ *and* $0 < \Theta_1 < \Theta_2$*, then*

$$\frac{1}{2}F(\Theta_1 + \Theta_2) \leq \frac{1}{\Theta_2 - \Theta_1} \int_{\Theta_1}^{\Theta_2} F(\delta) d\delta \leq \frac{1}{\Theta_1} \int_0^{\Theta_1} F(\delta) d\delta + \frac{1}{\Theta_2} \int_0^{\Theta_2} F(\delta) d\delta. \tag{1.4}$$

The fractional calculus focuses on integrals and derivatives of fractional orders. It extends the usual ideas of differentiation and integration to orders that are not integers. The order of differentiation or integration in fractional calculus can be any real number, including non-integer numbers. For instance, taking the square root of a derivative or integral corresponds to a half-order derivative or integral. Numerous disciplines, including physics, engineering, economics, and signal processing, use fractional calculus.

Definition 1.3. *[20, 21] Let* $\mathtt{F} \in L[\Theta_1, \Theta_2]$*, where* $0 \leq \Theta_1 < \Theta_2$*. For* $\alpha > 0$ *and* $\lambda \geq 0$*, the tempered fractional integral operators* $I^{\alpha,\lambda}_{\Theta_1^+}\mathtt{F}$ *and* $I^{\alpha,\lambda}_{\Theta_2^-}\mathtt{F}$ *are defined as*

$$I^{\alpha,\lambda}_{\Theta_1^+}\mathtt{F}(x) := \frac{1}{\Gamma(\alpha)} \int_{\Theta_1}^{x} (x-\delta)^{\alpha-1} e^{-\lambda(x-\delta)} \mathtt{F}(\delta) d\delta \quad (x > \Theta_1), \tag{1.5}$$

and

$$I^{\alpha,\lambda}_{\Theta_2^-}\mathtt{F}(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\Theta_2} (\delta-x)^{\alpha-1} e^{-\lambda(\delta-x)} \mathtt{F}(\delta) d\delta \quad (x < \Theta_2), \tag{1.6}$$

respectively.

Definition 1.4. *Let* $x, \lambda \geq 0$ *and* $\alpha > 0$*, then* λ*-incomplete gamma function is given by*

$$\gamma_\lambda(\alpha, x) := \int_0^x \delta^{\alpha-1} e^{-\lambda\delta} d\delta.$$

If $\lambda = 1$, then

$$\gamma(\alpha, x) := \int_0^x \delta^{\alpha-1} e^{-\delta} d\delta.$$

The following portions of this work are inspired by the aforementioned findings: We found some H-H inequalities for subadditive functions and their product using tempered fractional integrals in Section 2. We propose various fractional inequalities for subadditive functions relevant to tempered fractional integral operators with the help of a new lemma in Section 3. In Section 4, we provide a few numerical examples and graphs with numerical estimations to support our findings. In Section 5, some conclusions and new ideas for future research are discussed.

2. Main results

Let us take $\mathtt{Q} := [0, \infty)$, where $\mathtt{Q}^\circ := (0, \infty)$. Throughout this paper, we will use the above notations for our simplicity.

Theorem 2.1. *Let* $\mathtt{F} : \mathtt{Q} \to \mathtt{Q}$ *be a continuous subadditive function with* $\Theta_1, \Theta_2 \in \mathtt{Q}^\circ$ *and* $\Theta_1 < \Theta_2$*. Then for* $\lambda \geq 0$ *and* $\alpha > 0$*, we have*

$$\begin{aligned}
\frac{1}{2}\mathtt{F}(\Theta_1 + \Theta_2) \leq & \frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha, \Theta_2 - \Theta_1)} \left[I^{\alpha,\lambda}_{\Theta_1^+}\mathtt{F}(\Theta_2) + I^{\alpha,\lambda}_{\Theta_2^-}\mathtt{F}(\Theta_1) \right] \\
\leq & \frac{(\Theta_2 - \Theta_1)^\alpha}{2\gamma_\lambda(\alpha, \Theta_2 - \Theta_1)} \\
& \times \left\{ \frac{1}{\Theta_1^\alpha} \int_0^{\Theta_1} \delta^{\alpha-1} e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)\delta} \mathtt{F}(\delta) d\delta + \frac{1}{\Theta_1^\alpha} \int_0^{\Theta_1} (\Theta_1 - \delta)^{\alpha-1} e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)(\Theta_1-\delta)} \mathtt{F}(\delta) d\delta \right. \\
& \left. + \frac{1}{\Theta_2^\alpha} \int_0^{\Theta_2} \delta^{\alpha-1} e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)\delta} \mathtt{F}(\delta) d\delta + \frac{1}{\Theta_2^\alpha} \int_0^{\Theta_2} (\Theta_2 - \delta)^{\alpha-1} e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)(\Theta_2-\delta)} \mathtt{F}(\delta) d\delta \right\}.
\end{aligned} \tag{2.1}$$

Proof. Using the hypothesis of subadditive function $\mathtt{F}$ on $\mathtt{Q}$, we have

$$\mathtt{F}(\Theta_1 + \Theta_2) = \mathtt{F}(\delta\Theta_1 + (1-\delta)\Theta_2 + \delta\Theta_2 + (1-\delta)\Theta_1) \leq \mathtt{F}(\delta\Theta_1 + (1-\delta)\Theta_2) + \mathtt{F}((1-\delta)\Theta_1 + \delta\Theta_2). \tag{2.2}$$

Upon multiplication of (2.2) by $\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}$, and then integrating with respect to δ over $[0,1]$, we get

$$\begin{aligned}
&\frac{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^\alpha}F(\Theta_1+\Theta_2)\\
&\leq \int_0^1 \delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}F(\delta\Theta_1+(1-\delta)\Theta_2)d\delta + \int_0^1 \delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}F((1-\delta)\Theta_1+\delta\Theta_2)d\delta\\
&= \frac{1}{(\Theta_2-\Theta_1)^\alpha}\left[\int_{\Theta_1}^{\Theta_2}(\Theta_2-\delta)^{\alpha-1}e^{-\lambda(\Theta_2-\delta)}F(\delta)d\delta + \int_{\Theta_1}^{\Theta_2}(\delta-\Theta_1)^{\alpha-1}e^{-\lambda(\delta-\Theta_1)}F(\delta)d\delta\right]\\
&= \frac{\Gamma(\alpha)}{(\Theta_2-\Theta_1)^\alpha}\left[I^{\alpha,\lambda}_{\Theta_1^+}F(\Theta_2)+I^{\alpha,\lambda}_{\Theta_2^-}F(\Theta_1)\right].
\end{aligned}$$

Hence,

$$\frac{1}{2}F(\Theta_1+\Theta_2) \leq \frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[I^{\alpha,\lambda}_{\Theta_1^+}F(\Theta_2)+I^{\alpha,\lambda}_{\Theta_2^-}F(\Theta_1)\right].$$

This concludes the first side (left) of (2.1). Next, for the other side (right) of (2.1), since F is subadditive on Q, one has

$$F(\delta\Theta_1+(1-\delta)\Theta_2) \leq F(\delta\Theta_1)+F((1-\delta)\Theta_2) \tag{2.3}$$

and

$$F((1-\delta)\Theta_1+\delta\Theta_2) \leq F((1-\delta)\Theta_1)+F(\delta\Theta_2). \tag{2.4}$$

By adding (2.3) and (2.4), we get

$$F(\delta\Theta_1+(1-\delta)\Theta_2)+F((1-\delta)\Theta_1+\delta\Theta_2) \leq F(\delta\Theta_1)+F(\delta\Theta_2)+F((1-\delta)\Theta_1)+F((1-\delta)\Theta_2). \tag{2.5}$$

Multiplying both sides of (2.5) by $\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}$, and following the same procedure as above, we complete the proof. □

Remark 2.1. *Choosing $\lambda = 0$ and $\alpha = 1$ in Theorem 2.1, we obtain Theorem 1.2.*

Theorem 2.2. *Let $\Phi,\Psi : Q \to Q$ be two continuous subadditive functions with $\Theta_1,\Theta_2 \in Q^\circ$ and $\Theta_1 < \Theta_2$. Then for $\lambda \geq 0$ and $\alpha > 0$, we have*

$$\begin{aligned}
&\frac{1}{2}\Phi(\Theta_1+\Theta_2)\Psi(\Theta_1+\Theta_2)\\
\leq&\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[I^{\alpha,\lambda}_{\Theta_1^+}\Phi(\Theta_2)\Psi(\Theta_2)+I^{\alpha,\lambda}_{\Theta_2^-}\Phi(\Theta_1)\Psi(\Theta_1)\right]+\frac{(\Theta_2-\Theta_1)^\alpha}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\\
&\times\left\{\int_0^1\left[\delta^{\alpha-1}+(1-\delta)^{\alpha-1}\right]e^{-\lambda(\Theta_2-\Theta_1)\delta}\left[\Phi(\delta\Theta_1)\Psi(\delta\Theta_2)+\Phi(\delta\Theta_2)\Psi(\delta\Theta_1)\right]d\delta\right.\\
&\quad+\frac{1}{\Theta_1^\alpha}\int_0^{\Theta_1}\delta^{\alpha-1}e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)\delta}\left[\Phi(\delta)\Psi(\Theta_1-\delta)+\Phi(\Theta_1-\delta)\Psi(\delta)\right]d\delta\\
&\quad\left.+\frac{1}{\Theta_2^\alpha}\int_0^{\Theta_2}\delta^{\alpha-1}e^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)\delta}\left[\Phi(\delta)\Psi(\Theta_2-\delta)+\Phi(\Theta_2-\delta)\Psi(\delta)\right]d\delta\right\}
\end{aligned} \tag{2.6}$$

and

$$\frac{\Gamma(\alpha)}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,\lambda}_{\Theta_1^+}\Phi(\Theta_2)\Psi(\Theta_2)+\mathrm{I}^{\alpha,\lambda}_{\Theta_2^-}\Phi(\Theta_1)\Psi(\Theta_1)\right]$$
$$\leq \frac{(\Theta_2-\Theta_1)^\alpha}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}$$
$$\times\left\{\int_0^1\left[\delta^{\alpha-1}\mathrm{e}^{-\lambda(\Theta_2-\Theta_1)\delta}+(1-\delta)^{\alpha-1}\mathrm{e}^{-\lambda(\Theta_2-\Theta_1)(1-\delta)}\right][\Phi(\delta\Theta_1)\Psi((1-\delta)\Theta_2)+\Phi(\delta\Theta_2)\Psi((1-\delta)\Theta_1)]\,\mathrm{d}\delta\right.$$
$$+\frac{1}{\Theta_1^\alpha}\int_0^{\Theta_1}\left[\delta^{\alpha-1}\mathrm{e}^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)\delta}+(\Theta_1-\delta)^{\alpha-1}\mathrm{e}^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)(\Theta_1-\delta)}\right]\Phi(\delta)\Psi(\delta)\mathrm{d}\delta$$
$$\left.+\frac{1}{\Theta_2^\alpha}\int_0^{\Theta_2}\left[\delta^{\alpha-1}\mathrm{e}^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)\delta}+(\Theta_2-\delta)^{\alpha-1}\mathrm{e}^{-\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)(\Theta_2-\delta)}\right]\Phi(\delta)\Psi(\delta)\mathrm{d}\delta\right\}. \tag{2.7}$$

Proof. Using (2.2) and the hypothesis of subadditive functions Φ, and Ψ on $\mathtt{Q}$, one has

$$\Phi(\Theta_1+\Theta_2)\leq\Phi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_2+(1-\delta)\Theta_1) \tag{2.8}$$

and

$$\Psi(\Theta_1+\Theta_2)\leq\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Psi(\delta\Theta_2+(1-\delta)\Theta_1). \tag{2.9}$$

Multiplying (2.8) and (2.9), we get

$$\begin{aligned}
&\Phi(\Theta_1+\Theta_2)\Psi(\Theta_1+\Theta_2)\\
&\leq[\Phi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_2+(1-\delta)\Theta_1)]\,[\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Psi(\delta\Theta_2+(1-\delta)\Theta_1)]\\
&=\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_2+(1-\delta)\Theta_1)\\
&\quad+\Phi(\delta\Theta_2+(1-\delta)\Theta_1)\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_2+(1-\delta)\Theta_1)\Psi(\delta\Theta_2+(1-\delta)\Theta_1)\\
&\leq\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_2+(1-\delta)\Theta_1)\Psi(\delta\Theta_2+(1-\delta)\Theta_1)\\
&\quad+[\Phi(\delta\Theta_1)+\Phi((1-\delta)\Theta_2)]\,[\Psi(\delta\Theta_2)+\Psi((1-\delta)\Theta_1)]\\
&\quad+[\Phi(\delta\Theta_2)+\Phi((1-\delta)\Theta_1)]\,[\Psi(\delta\Theta_1)+\Psi((1-\delta)\Theta_2)]\\
&=\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi(\delta\Theta_2+(1-\delta)\Theta_1)\Psi(\delta\Theta_2+(1-\delta)\Theta_1)\\
&\quad+\Phi(\delta\Theta_1)\Psi(\delta\Theta_2)+\Phi(\delta\Theta_1)\Psi((1-\delta)\Theta_1)+\Phi((1-\delta)\Theta_2)\Psi(\delta\Theta_2)\\
&\quad+\Phi((1-\delta)\Theta_2)\Psi((1-\delta)\Theta_1)+\Phi(\delta\Theta_2)\Psi(\delta\Theta_1)+\Phi(\delta\Theta_2)\Psi((1-\delta)\Theta_2)\\
&\quad+\Phi((1-\delta)\Theta_1)\Psi(\delta\Theta_1)+\Phi((1-\delta)\Theta_1)\Psi((1-\delta)\Theta_2).
\end{aligned} \tag{2.10}$$

Multiplying both sides of (2.10) by $\delta^{\alpha-1}\mathrm{e}^{-\lambda(\Theta_2-\Theta_1)\delta}$ and integrating with respect to δ over $[0,1]$, we obtain (2.6).

By subadditivity of Φ and Ψ on $\mathtt{Q}$, we have

$$\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\leq\Phi(\delta\Theta_1)+\Phi((1-\delta)\Theta_2) \tag{2.11}$$

and

$$\Psi(\delta\Theta_1+(1-\delta)\Theta_2)\leq\Psi(\delta\Theta_1)+\Psi((1-\delta)\Theta_2). \tag{2.12}$$

Multiplying inequalities (2.11) and (2.12), we get

$$\begin{aligned}&\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_1+(1-\delta)\Theta_2) \\ \leq&\Phi(\delta\Theta_1)\Psi(\delta\Theta_1)+\Phi(\delta\Theta_1)\Psi((1-\delta)\Theta_2)+\Phi((1-\delta)\Theta_2)\Psi(\delta\Theta_1)+\Phi((1-\delta)\Theta_2)\Psi((1-\delta)\Theta_2).\end{aligned} \tag{2.13}$$

Similarly,

$$\begin{aligned}&\Phi((1-\delta)\Theta_1+\delta\Theta_2)\Psi((1-\delta)\Theta_1+\delta\Theta_2) \\ \leq&\Phi((1-\delta)\Theta_1)\Psi((1-\delta)\Theta_1)+\Phi((1-\delta)\Theta_1)\Psi(\delta\Theta_2)+\Phi(\delta\Theta_2)\Psi((1-\delta)\Theta_1)+\Phi(\delta\Theta_2)\Psi(\delta\Theta_2).\end{aligned} \tag{2.14}$$

Adding (2.13) and (2.14), we obtain

$$\begin{aligned}&\Phi(\delta\Theta_1+(1-\delta)\Theta_2)\Psi(\delta\Theta_1+(1-\delta)\Theta_2)+\Phi((1-\delta)\Theta_1+\delta\Theta_2)\Psi((1-\delta)\Theta_1+\delta\Theta_2) \\ \leq&\Phi(\delta\Theta_1)\Psi(\delta\Theta_1)+\Phi(\delta\Theta_1)\Psi((1-\delta)\Theta_2)+\Phi((1-\delta)\Theta_2)\Psi(\delta\Theta_1)+\Phi((1-\delta)\Theta_2)\Psi((1-\delta)\Theta_2) \\ &+\Phi((1-\delta)\Theta_1)\Psi((1-\delta)\Theta_1)+\Phi((1-\delta)\Theta_1)\Psi(\delta\Theta_2)+\Phi(\delta\Theta_2)\Psi((1-\delta)\Theta_1)+\Phi(\delta\Theta_2)\Psi(\delta\Theta_2).\end{aligned} \tag{2.15}$$

Multiplying both sides of (2.15) by $\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}$, and following the same procedure as above, we have (2.7). □

Remark 2.2. *Choosing $\lambda = 0$ and $\alpha = 1$ in Theorem 2.2, we obtain ([19], Theorem 4).*

Theorem 2.3. *Let $\mathtt{F} : \mathtt{Q} \to \mathtt{Q}$ be a continuous subadditive function with $\Theta_1, \Theta_2 \in \mathtt{Q}^\circ$ and $\Theta_1 < \Theta_2$. Then for $\lambda \geq 0$ and $\alpha > 0$, we have*

$$\begin{aligned}&\frac{1}{2}\mathtt{F}(\Theta_1+\Theta_2) \\ \leq&\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^+}\mathtt{F}(\Theta_2)+\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^-}\mathtt{F}(\Theta_1)\right] \\ \leq&\frac{2^{\alpha-1}(\Theta_2-\Theta_1)^\alpha}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\times\left\{\frac{1}{\Theta_1^\alpha}\int_0^{\frac{\Theta_1}{2}}\delta^{\alpha-1}e^{-2\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)\delta}\mathtt{F}(\delta)d\delta+\frac{1}{\Theta_1^\alpha}\int_{\frac{\Theta_1}{2}}^{\Theta_1}(\Theta_1-\delta)^{\alpha-1}e^{-2\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_1}\right)(\Theta_1-\delta)}\mathtt{F}(\delta)d\delta\right. \\ &\left.+\frac{1}{\Theta_2^\alpha}\int_0^{\frac{\Theta_2}{2}}\delta^{\alpha-1}e^{-2\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)\delta}\mathtt{F}(\delta)d\delta+\frac{1}{\Theta_2^\alpha}\int_{\frac{\Theta_2}{2}}^{\Theta_2}(\Theta_2-\delta)^{\alpha-1}e^{-2\lambda\left(\frac{\Theta_2-\Theta_1}{\Theta_2}\right)(\Theta_2-\delta)}\mathtt{F}(\delta)d\delta\right\}.\end{aligned} \tag{2.16}$$

Proof. From subadditivity of $\mathtt{F}$, we have

$$\mathtt{F}(\Theta_1+\Theta_2)\leq\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1+\frac{\delta}{2}\Theta_2\right)+\mathtt{F}\left(\frac{\delta}{2}\Theta_1+\frac{2-\delta}{2}\Theta_2\right). \tag{2.17}$$

Multiplying both sides of (2.17) by $\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}$, and then integrating over $[0,1]$, we get

$$\begin{aligned}&\frac{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^\alpha}\mathtt{F}(\Theta_1+\Theta_2) \\ \leq&\int_0^1\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}\mathtt{F}\left(\frac{\delta}{2}\Theta_1+\frac{2-\delta}{2}\Theta_2\right)d\delta+\int_0^1\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1+\frac{\delta}{2}\Theta_2\right)d\delta\end{aligned}$$

$$=\left(\frac{2}{\Theta_2-\Theta_1}\right)^{\alpha}\Gamma(\alpha)\left[\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^+}\mathrm{F}(\Theta_2)+\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^-}\mathrm{F}(\Theta_1)\right].$$

Hence,

$$\frac{1}{2}\mathrm{F}(\Theta_1+\Theta_2)\leq\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^+}\mathrm{F}(\Theta_2)+\mathrm{I}^{\alpha,2\lambda}_{\left(\frac{\Theta_1+\Theta_2}{2}\right)^-}\mathrm{F}(\Theta_1)\right],$$

which proves the left part of (2.16). Consequently, to prove the right part, we have

$$\mathtt{F}\left(\frac{\delta}{2}\Theta_1+\frac{2-\delta}{2}\Theta_2\right)\leq\mathtt{F}\left(\frac{\delta}{2}\Theta_1\right)+\mathtt{F}\left(\frac{2-\delta}{2}\Theta_2\right)\tag{2.18}$$

and

$$\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1+\frac{\delta}{2}\Theta_2\right)\leq\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1\right)+\mathtt{F}\left(\frac{\delta}{2}\Theta_2\right).\tag{2.19}$$

By adding (2.18) and (2.19), we obtain

$$\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1+\frac{\delta}{2}\Theta_2\right)+\mathtt{F}\left(\frac{\delta}{2}\Theta_1+\frac{2-\delta}{2}\Theta_2\right)\leq\mathtt{F}\left(\frac{\delta}{2}\Theta_1\right)+\mathtt{F}\left(\frac{\delta}{2}\Theta_2\right)+\mathtt{F}\left(\frac{2-\delta}{2}\Theta_1\right)+\mathtt{F}\left(\frac{2-\delta}{2}\Theta_2\right).\tag{2.20}$$

Multiplying both sides of (2.20) by $\delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}$, and following the same procedures as done in earlier theorems, we have the right part. □

3. More midpoint type results

Lemma 3.1. *Assume that* $\mathtt{F}:\mathtt{Q}\to\mathtt{Q}$ *is a differentiable continuous function for* $\Theta_1,\Theta_2\in\mathtt{Q}^\circ$ *and* $\Theta_1<\Theta_2$. *Then for* $\lambda\geq 0$ *and* $\alpha>0$, *we have*

$$\begin{aligned}&\frac{\mathrm{F}(\Theta_1)+\mathrm{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\\&=\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\times\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left[\mathrm{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)-\mathrm{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right]d\delta.\end{aligned}\tag{3.1}$$

Proof. Let us denote, respectively,

$$\mathrm{I}_1:=\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\mathrm{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)d\delta$$

and

$$\mathrm{I}_2:=\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\mathrm{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)d\delta.$$

Then, we have

$$\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left[\mathrm{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)-\mathrm{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right]d\delta=\mathrm{I}_1-\mathrm{I}_2.\tag{3.2}$$

Using integration by parts, we get

$$\begin{aligned}
I_1 =& \frac{2}{\Theta_2-\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)F\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)\Big|_0^1 \\
&-\frac{2}{\Theta_2-\Theta_1}\int_0^1 \delta^{\alpha-1}e^{-\lambda(\Theta_2-\Theta_1)\delta}F\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)d\delta \\
=& \frac{2}{\Theta_2-\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,1)F(\Theta_2)-\left(\frac{2}{\Theta_2-\Theta_1}\right)^{\alpha+1}\times\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}\left(\delta-\frac{\Theta_1+\Theta_2}{2}\right)^{\alpha-1}e^{-2\lambda\left(\delta-\frac{\Theta_1+\Theta_2}{2}\right)}F(\delta)d\delta \\
=& \frac{2}{\Theta_2-\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,1)F(\Theta_2)-\left(\frac{2}{\Theta_2-\Theta_1}\right)^{\alpha+1}\Gamma(\alpha)I^{\alpha,2\lambda}_{\Theta_2^-}F\left(\frac{\Theta_1+\Theta_2}{2}\right).
\end{aligned}$$

Similarly,

$$I_2=-\frac{2}{\Theta_2-\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,1)F(\Theta_1)+\left(\frac{2}{\Theta_2-\Theta_1}\right)^{\alpha+1}\Gamma(\alpha)I^{\alpha,2\lambda}_{\Theta_1^+}F\left(\frac{\Theta_1+\Theta_2}{2}\right).$$

From Definition 1.4, we get

$$\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,1)=\frac{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^\alpha}.$$

Multiplying both sides of (3.2) by the factor $\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}$ and using above relation, we have the desired result (3.1). □

Theorem 3.1. *Suppose that* $F:Q\to Q$ *is a differentiable continuous function with* $\Theta_1,\Theta_2\in Q^\circ$ *and* $\Theta_1<\Theta_2$. *If* $|F'|^q$ *for* $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *is a subadditive function, then for* $\lambda\geq 0$ *and* $\alpha>0$, *we have*

$$\begin{aligned}
&\left|\frac{F(\Theta_1)+F(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[I^{\alpha,2\lambda}_{\Theta_1^+}F\left(\frac{\Theta_1+\Theta_2}{2}\right)+I^{\alpha,2\lambda}_{\Theta_2^-}F\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right| \\
\leq& \frac{2^{\frac{1-2q}{q}}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}C^{\frac{1}{p}}(\gamma,p) \\
&\times\left\{\left[\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}|F'(\delta)|^q d\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}|F'(\delta)|^q d\delta\right]^{\frac{1}{q}}+\left[\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}|F'(\delta)|^q d\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}|F'(\delta)|^q d\delta\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{3.3}$$

where

$$C(\gamma,p):=\int_0^1\left[\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\right]^p d\delta.$$

Proof. Under the assumption of Lemma 3.1, subadditivity of $|F'|^q$ on Q and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{F(\Theta_1)+F(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[I^{\alpha,2\lambda}_{\Theta_1^+}F\left(\frac{\Theta_1+\Theta_2}{2}\right)+I^{\alpha,2\lambda}_{\Theta_2^-}F\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right| \\
\leq& \frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\int_0^1|\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)|\left[\left|F'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)\right|+\left|F'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right|\right]d\delta
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left(\int_0^1 [\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)]^{\mathrm{p}}\,\mathrm{d}\delta\right)^{\frac{1}{\mathrm{p}}}\\
&\quad\times\left\{\left(\int_0^1\left|\mathrm{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)\right|^{\mathrm{q}}\mathrm{d}\delta\right)^{\frac{1}{\mathrm{q}}}+\left(\int_0^1\left|\mathrm{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right|^{\mathrm{q}}\mathrm{d}\delta\right)^{\frac{1}{\mathrm{q}}}\right\}\\
&\leq \frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{\frac{1}{\mathrm{p}}}(\gamma,\mathrm{p})\\
&\quad\times\left\{\left[\int_0^1\left(\left|\mathrm{F}'\left(\left(\frac{1+\delta}{2}\right)\Theta_2\right)\right|^{\mathrm{q}}+\left|\mathrm{F}'\left(\left(\frac{1-\delta}{2}\right)\Theta_1\right)\right|^{\mathrm{q}}\right)\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}\right.\\
&\quad\left.+\left[\int_0^1\left(\left|\mathrm{F}'\left(\left(\frac{1-\delta}{2}\right)\Theta_2\right)\right|^{\mathrm{q}}+\left|\mathrm{F}'\left(\left(\frac{1+\delta}{2}\right)\Theta_1\right)\right|^{\mathrm{q}}\right)\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}\right\}\\
&= \frac{2^{\frac{1-2\mathrm{q}}{\mathrm{q}}}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{\frac{1}{\mathrm{p}}}(\gamma,\mathrm{p})\\
&\quad\times\left\{\left[\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}+\left[\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}\right\}.
\end{aligned}
$$

The proof of Theorem 3.1 is completed. □

Corollary 3.1. *Choosing* $|\mathrm{F}'|\leq \mathrm{K}$ *in Theorem 3.1, we get*

$$
\begin{aligned}
&\left|\frac{\mathrm{F}(\Theta_1)+\mathrm{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right|\\
&\leq\frac{2^{\frac{1-\mathrm{q}}{\mathrm{q}}}\mathrm{K}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{\frac{1}{\mathrm{p}}}(\gamma,\mathrm{p}).
\end{aligned}\tag{3.4}
$$

Theorem 3.2. *Assume that* $\mathtt{F}:\mathtt{Q}\to\mathtt{Q}$ *is a differentiable continuous function with* $\Theta_1,\Theta_2\in\mathtt{Q}^\circ$ *and* $\Theta_1<\Theta_2$*. If* $|\mathrm{F}'|^{\mathrm{q}}$ *for* $\mathrm{q}\geq 1$ *is a subadditive function, then for* $\lambda\geq 0$ *and* $\alpha>0$*, we have*

$$
\begin{aligned}
&\left|\frac{\mathrm{F}(\Theta_1)+\mathrm{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathrm{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right|\\
&\leq\frac{2^{\frac{1-2\mathrm{q}}{\mathrm{q}}}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{1-\frac{1}{\mathrm{q}}}(\gamma)\\
&\quad\times\left\{\left[\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_1}\right)|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_2}-1\right)|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}\right.\\
&\quad\left.+\left[\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_1}-1\right)|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_2}\right)|\mathrm{F}'(\delta)|^{\mathrm{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathrm{q}}}\right\},
\end{aligned}\tag{3.5}
$$

where

$$
\mathrm{C}(\gamma):=\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\mathrm{d}\delta=\frac{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^\alpha}-\frac{\gamma_\lambda(\alpha+1,\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^{\alpha+1}}.
$$

Proof. Under the assumption of Lemma 3.1, subadditivity of $|\mathtt{F}'|^{\mathtt{q}}$ on $\mathtt{Q}$ and power-mean inequality, we have

$$
\begin{aligned}
&\left|\frac{\mathtt{F}(\Theta_1)+\mathtt{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right|\\
\le&\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\int_0^1|\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)|\left[\left|\mathtt{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)\right|+\left|\mathtt{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right|\right]\mathrm{d}\delta\\
\le&\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left(\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\mathrm{d}\delta\right)^{1-\frac{1}{\mathtt{q}}}\\
&\times\left\{\left(\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left|\mathtt{F}'\left(\Theta_1+\frac{1+\delta}{2}(\Theta_2-\Theta_1)\right)\right|^{\mathtt{q}}\mathrm{d}\delta\right)^{\frac{1}{\mathtt{q}}}\right.\\
&\left.+\left(\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left|\mathtt{F}'\left(\Theta_1+\frac{1-\delta}{2}(\Theta_2-\Theta_1)\right)\right|^{\mathtt{q}}\mathrm{d}\delta\right)^{\frac{1}{\mathtt{q}}}\right\}\\
\le&\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{4\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{1-\frac{1}{\mathtt{q}}}(\gamma)\\
&\times\left\{\left[\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left(\left|\mathtt{F}'\left(\left(\frac{1+\delta}{2}\right)\Theta_2\right)\right|^{\mathtt{q}}+\left|\mathtt{F}'\left(\left(\frac{1-\delta}{2}\right)\Theta_1\right)\right|^{\mathtt{q}}\right)\mathrm{d}\delta\right]^{\frac{1}{\mathtt{q}}}\right.\\
&\left.+\left[\int_0^1\gamma_{\lambda(\Theta_2-\Theta_1)}(\alpha,\delta)\left(\left|\mathtt{F}'\left(\left(\frac{1-\delta}{2}\right)\Theta_2\right)\right|^{\mathtt{q}}+\left|\mathtt{F}'\left(\left(\frac{1+\delta}{2}\right)\Theta_1\right)\right|^{\mathtt{q}}\right)\mathrm{d}\delta\right]^{\frac{1}{\mathtt{q}}}\right\}\\
=&\frac{2^{\frac{1-2\mathtt{q}}{\mathtt{q}}}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{1-\frac{1}{\mathtt{q}}}(\gamma)\\
&\times\left\{\left[\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_1}\right)|\mathtt{F}'(\delta)|^{\mathtt{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_2}-1\right)|\mathtt{F}'(\delta)|^{\mathtt{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathtt{q}}}\right.\\
&\left.+\left[\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_1}-1\right)|\mathtt{F}'(\delta)|^{\mathtt{q}}\mathrm{d}\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_2}\right)|\mathtt{F}'(\delta)|^{\mathtt{q}}\mathrm{d}\delta\right]^{\frac{1}{\mathtt{q}}}\right\}.
\end{aligned}
$$

The proof of Theorem 3.2 is completed. □

Corollary 3.2. *Taking* $\mathtt{q}=1$ *in Theorem 3.2, we have*

$$
\begin{aligned}
&\left|\frac{\mathtt{F}(\Theta_1)+\mathtt{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right|\\
\le&\frac{(\Theta_2-\Theta_1)^{\alpha+1}}{2\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\\
&\times\left\{\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_1}\right)|\mathtt{F}'(\delta)|\mathrm{d}\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_2}\right)|\mathtt{F}'(\delta)|\mathrm{d}\delta\right.\\
&\left.+\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_1}-1\right)|\mathtt{F}'(\delta)|\mathrm{d}\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_2}-1\right)|\mathtt{F}'(\delta)|\mathrm{d}\delta\right\}.
\end{aligned}
\tag{3.6}
$$

Corollary 3.3. *Choosing* $|\mathtt{F}'| \leq \mathrm{K}$ *in Theorem 3.2, we get*

$$\left|\frac{\mathtt{F}(\Theta_1)+\mathtt{F}(\Theta_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\left[\mathrm{I}^{\alpha,2\lambda}_{\Theta_1^+}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)+\mathrm{I}^{\alpha,2\lambda}_{\Theta_2^-}\mathtt{F}\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]\right|$$
$$\leq\frac{2^{\frac{1-2q}{q}}\mathrm{K}(\Theta_2-\Theta_1)^{\alpha+1}}{\gamma_\lambda(\alpha,\Theta_2-\Theta_1)}\mathrm{C}^{1-\frac{1}{q}}(\gamma)$$
$$\times\left\{\left[\frac{1}{\Theta_1}\int_0^{\frac{\Theta_1}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_1}\right)\mathrm{d}\delta+\frac{1}{\Theta_2}\int_{\frac{\Theta_2}{2}}^{\Theta_2}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_2}-1\right)\mathrm{d}\delta\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{\Theta_1}\int_{\frac{\Theta_1}{2}}^{\Theta_1}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,\frac{2\delta}{\Theta_1}-1\right)\mathrm{d}\delta+\frac{1}{\Theta_2}\int_0^{\frac{\Theta_2}{2}}\gamma_{\lambda(\Theta_2-\Theta_1)}\left(\alpha,1-\frac{2\delta}{\Theta_2}\right)\mathrm{d}\delta\right]^{\frac{1}{q}}\right\}. \tag{3.7}$$

Remark 3.1. *For suitable choices of α and λ in Theorems 3.1 and 3.2, one will able to get interesting integral inequalities.*

4. Numerical examples

It is shown that the functions $\mathtt{F}(\delta) = e^{-\delta}$ and $\sqrt{\delta}$ for all $\delta > 0$ are subadditive. It can be observed that these graphs illustrate and confirm the correctness of our obtained inequalities.

Example 4.1. If we take subadditive function $\mathtt{F}(\delta) = \sqrt{\delta}$ in Theorem 2.1 for all $\delta > 0$, $0 < \alpha < 1$ and $\lambda = 1$, then we observe the following numerical verification (see Table 1).

Table 1. Estimations of inequalities in Theorem 2.1 for $\mathtt{F}(\delta) = \sqrt{\delta}$, $\Theta_1 = 1$ and $\Theta_2 = 2$.

α	Left side	Middle side	Right side
0.10	0.866025	1.20951	1.31914
0.20	0.866025	1.21150	1.39740
0.30	0.866025	1.21314	1.45415
0.40	0.866025	1.21450	1.49635
0.50	0.866025	1.21562	1.52827
0.60	0.866025	1.21653	1.55268
0.70	0.866025	1.21728	1.57148
0.80	0.866025	1.21788	1.58599
0.90	0.866025	1.21837	1.59717

Example 4.2. If we take subadditive function $\mathtt{F}(\delta) = e^{-\delta}$ in Theorems 2.1 and 2.3 for all $\delta > 0$, $0 < \alpha < 1$ and $\lambda = 1$, then we observe the following numerical verifications (see Tables 2 and 3) and corresponding graphs (see Figures 1 and 2).

Table 2. Estimations of inequalities in Theorem 2.1 for $F(\delta) = e^{-\delta}$, $\Theta_1 = 1$ and $\Theta_2 = 2$.

α	Left side	Middle side	Right side
0.10	0.0248935	0.247754	1.21353
0.20	0.0248935	0.244558	1.18203
0.30	0.0248935	0.241912	1.15601
0.40	0.0248935	0.239727	1.13457
0.50	0.0248935	0.237928	1.11694
0.60	0.0248935	0.236451	1.1025
0.70	0.0248935	0.235245	1.09073
0.80	0.0248935	0.234267	1.08119
0.90	0.0248935	0.233479	1.07353

Table 3. Estimations of inequalities in Theorem 2.3 for $F(\delta) = e^{-\delta}, \Theta_1 = 1$ and $\Theta_2 = 2$.

α	Left side	Middle side	Right side
0.10	0.0248935	0.248907	1.22482
0.20	0.0248935	0.246526	1.20127
0.30	0.0248935	0.244423	1.18051
0.40	0.0248935	0.242560	1.16217
0.50	0.0248935	0.240906	1.14592
0.60	0.0248935	0.239435	1.13149
0.70	0.0248935	0.238122	1.11863
0.80	0.0248935	0.236947	1.10716
0.90	0.0248935	0.235894	1.09689

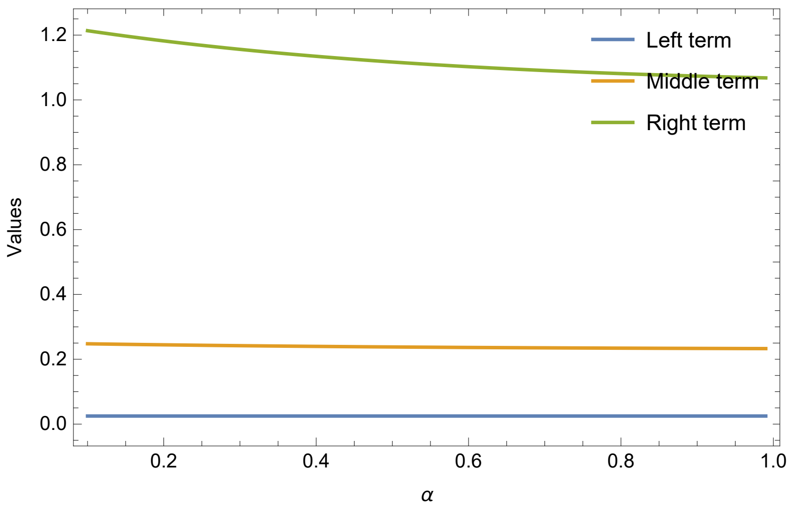

Figure 1. Graphical representation of Theorem 2.1 with $F(\delta) = e^{-\delta}$, $\Theta_1 = 1$, $\Theta_2 = 2$, $0 < \alpha < 1$.

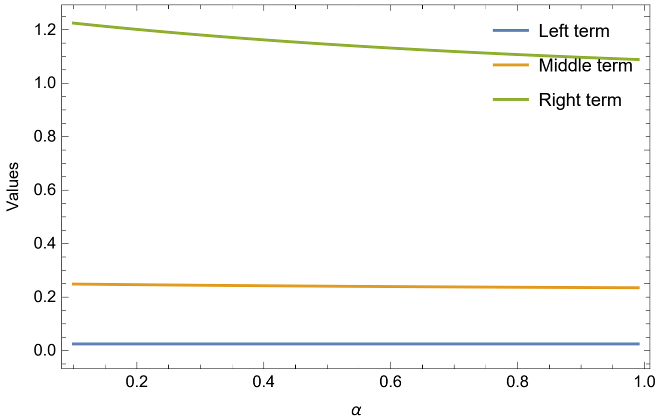

Figure 2. Graphical representation of Theorem 2.3 for $\mathtt{F}(\delta) = e^{-\delta}$, $\forall\, \delta > 0$ and $0 < \alpha < 1$.

Example 4.3. Taking $\Phi(\delta) = \Psi(\delta) = e^{-\delta}$ in Theorem 2.2 for all $\delta > 0$, $0 < \alpha < 1$ and $\lambda = 1$, then we observe the following numerical verifications (see Tables 4 and 5) and corresponding graphs (see Figures 3 and 4).

Table 4. Estimations of the first inequality of Theorem 2.2 for $\Phi(\delta) = \Psi(\delta) = e^{-\delta}$, $\Theta_1 = 1$ and $\Theta_2 = 2$.

α	Left side	Right side
0.10	0.00123937	5.73786
0.20	0.00123937	5.1814
0.30	0.00123937	4.7129
0.40	0.00123937	4.31749
0.50	0.00123937	3.98298
0.60	0.00123937	3.69935
0.70	0.00123937	3.45838
0.80	0.00123937	3.25324
0.90	0.00123937	3.07834

Table 5. Estimations of the second inequality of Theorem 2.2 for $\Phi(\delta) = \Psi(\delta) = e^{-\delta}$, $\Theta_1 = 1$ and $\Theta_2 = 2$.

α	Left side	Right side
0.10	0.0730897	1.48926
0.20	0.0700023	1.41475
0.30	0.0674546	1.35365
0.40	0.0653563	1.30362
0.50	0.0636327	1.26275
0.60	0.0622219	1.22945
0.70	0.0610724	1.20245
0.80	0.0601415	1.18068
0.90	0.0593939	1.16326

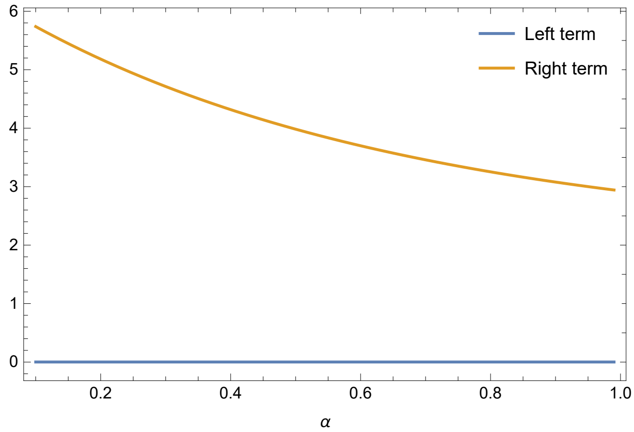

Figure 3. Graphs of first inequality of Theorem 2.2 for $\Phi(\delta) = \Psi(\delta) = e^{-\delta}$, $\forall\, \delta > 0$ and $0 < \alpha < 1$.

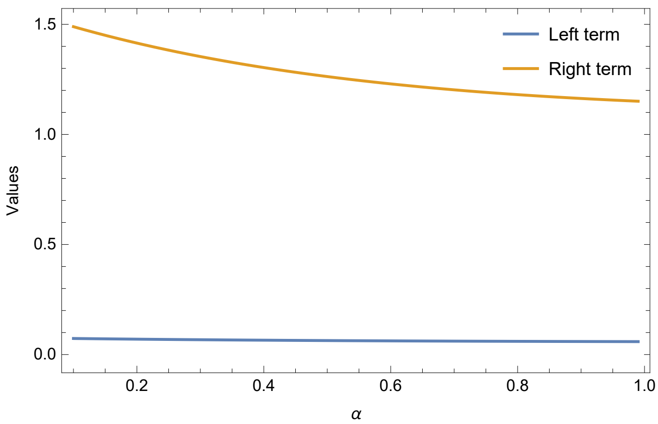

Figure 4. Graphs of second inequality of Theorem 2.2 for $\Phi(\delta) = \Psi(\delta) = e^{-\delta}$, $\forall\, \delta > 0$ and $0 < \alpha < 1$.

5. Conclusions

Several new fractional H-H types for subadditive functions and their products are developed in this paper. As an ancillary finding, we also obtain inequalities for subadditive functions using tempered fractional integrals and a new identity. To confirm the veracity of our findings, we give several examples of subadditive functions, their graphical representations, and numerical calculations. For those interested in this topic and working on it, our findings utilizing the tempered fractional integral operators provide a number of new opportunities and make it possible for them to create additional approximations for many other varieties of operators.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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