Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12550–12569.
DOI:10.3934/math.2024614
Received: 20 February 2024
Revised: 17 March 2024
Accepted: 22 March 2024
Published: 01 April 2024

Research article

The coefficient multipliers on H^2 and $\mathcal{D}^2$ with Hyers–Ulam stability

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Abstract: In this paper, we investigated the Hyers–Ulam stability of the coefficient multipliers on the Hardy space H^2 and the Dirichlet space $\mathcal{D}^2$. We also investigated the Hyers–Ulam stability of the coefficient multipliers between Dirichlet and Hardy spaces. We provided the necessary and sufficient conditions for the coefficient multipliers to have Hyers–Ulam stability on Hardy space H^2, on Dirichlet space $\mathcal{D}^2$, and between Dirichlet and Hardy spaces. We also showed that the best constant of Hyers–Ulam stability exists under different circumstances. Moreover, some illustrative examples were discussed.

Keywords: Hyers–Ulam stability; coefficient multipliers; Dirichlet spaces; Hardy spaces
Mathematics Subject Classification: 39B72, 39B82, 47B91

1. Introduction

In analytic function theory, it is an important question to describe the coefficient multipliers between various spaces of analytic functions. The coefficient multipliers allow to obtain information on the Taylor coefficients of analytic functions in certain function spaces or make it possible to tell whether a given function is in a particular space of functions by observing the Taylor coefficients. In the research of operator theory, the coefficient multipliers also play an important role. Through the actions and behaviors of the coefficient multipliers on function spaces, we can get the element information of the specific function spaces, reveal the structure of some function spaces, and find correlations among different function spaces. This facilitates the study of the properties of other operators related to coefficient multipliers. The operator equations of the coefficient multipliers are often discussed in relation to problems within operator theory. Usually, the exact solutions for operator equations are not easy to obtain. In view of this, it is necessary to investigate the approximate solutions, and we may ask whether these lie near the exact solutions. This relates to the question of Hyers–Ulam stability. Generally, we say that an operator equation has Hyers–Ulam stability if, for every solution of the perturbed equation, there is an exact solution that is close to it. In other words, if a specific operator equation is replaced by an operator inequality, when can one assert that solutions of the latter lie near the exact solution of the operator equation?

In view of the importance of this issue and the relevant research works that have been done, our research work was carried out.

Let A, B be normed spaces and consider a mapping $T : A \to B$. We say that T has Hyers–Ulam stability property (briefly, T is HU-stable) if there exists a constant $K > 0$ such that for any $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \varepsilon$, there exists an $f_0 \in A$ with $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$ (see [5,17]). The number K is called a Hyers–Ulam stability constant (briefly, HUS-constant) and the infimum of all HUS constants of T is denoted by K_T. Generally, K_T is not a HUS constant of T (see [4]).

The first important result, which we now call the Hyers–Ulam stability, is due to Hyers [6], who gave an answer to a question posed by Ulam [18] concerning group homomorphisms: let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given any $\varepsilon > 0$, is there a $\delta > 0$ such that, if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? Rassias [16] generalized the result of Hyers in 1978. Since then, the stability of many differential, integral, operator, functional equations have been extensively investigated.

Hyers–Ulam stability is widely used in many fields. For example, Hyers–Ulam stability analysis plays a significant role in the research of fractional order differential equations and systems. In 2023, Luo, Wang, Caraballo, and Zhu investigated Hyers–Ulam stability of Caputo-type fractional fuzzy stochastic differential equations with delay by the monotone iterative technique combined with the method of upper and lower solutions in [10]. In 2022, Luo, Abdeljawad, and Luo studied Ulam–Hyers stability results for a novel nonlinear nabla Caputo fractional variable-order difference system by applying Krasnoselskii's fixed point theorem in [8]. In 2020, Luo and Luo gave some existence and Hyers–Ulam stability results for a class of fractional-order delay differential equations with non-instantaneous impulses by Krasnoselskii's fixed point theorem and the generalized Gronwall's inequality in [9]. In 2022, Wang, Luo, and Zhu investigated Ulam–Hyers stability of Caputo-type fuzzy fractional differential equations with time-delays in [22]. In 2022, Eidinejad, Saadati, Allahviranloo, Kiani, Noeiaghdam, and Gamiz gave some results concerning the existence of a unique solution and the Hyers–Ulam–H–Fox stability of the conformable fractional differential equation by matrix-valued fuzzy controllers in [2].

In recent years, the Hyers–Ulam stability of operators on the different function spaces and operator algebras have been addressed by several researchers. In 2024, Keshavarz, Heydari, and Anderson investigated the Hyers–Ulam stability for mth differential operators on weighted Hardy spaces H^2_β in [7]. In [20,21], Wang and Xu discussed the Hyers–Ulam stability of differential operators on Hilbert spaces of entire functions and the reproducing kernel function spaces and gave several sufficient and necessary conditions for the differential operator to be stable on these spaces. In [14], the authors studied the stability of multipliers on Banach algebras. In [13], Miura, Hirasawa, and Takahasi investigated Hyers–Ulam stability of linear differential operators T_h ($h \in H(\mathbb{C})$) on the entire function space $H(\mathbb{C})$ and gave a sufficient and necessary condition for the T_h to be stable on $H(\mathbb{C})$. In [17], the authors gave a characterization for the weighed composition operators to have Hyers–Ulam stability on Banach space $C(X)$ and obtained a sufficient and necessary condition. In [15], Popa and Raşa investigated the stability of some classical operators from approximation theory.

Let A, B be two spaces of analytic functions on the unit disc $\mathbb{D}$ in the complex plane. Given a complex sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$, we define the operator T_λ as follows. For $f \in A$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, let $(T_\lambda f)(z) = \sum_{n=0}^{\infty} \lambda_n a_n z^n$. If $T_\lambda : A \to B$, then the sequence $\{\lambda_n\}$ is said to be a coefficient multiplier from A into B. This concept can be simply written as $\{\lambda_n\} \in (A, B)$ (see [1]).

For the completeness of the description, we briefly introduce some early work on the study of coefficient multipliers. Hardy and Littlewood [3] showed that the fractional integration is a multiplier of H^p into H^q under certain conditions. Duren [1] investigated the work of Hardy and Littlewood and then gave some simplified sufficient conditions such that the sequence $\{\lambda_n\}$ is a multiplier of H^p into H^q. Vukotic [19] investigated the coefficient multipliers of Bergman spaces and obtained some sufficient conditions and some necessary conditions for the coefficient multipliers. In particular, the coefficient multipliers from A^1 into A^2 were studied, and a necessary and sufficient condition was obtained. Wu and Yang [23] studied the multipliers between Dirichlet spaces and provided some interesting results. Since then, the coefficient multipliers of various function spaces have been extensively investigated, and some new results on the coefficient multipliers on Hardy, Bergman, Bloch, BMOA, Lipschitz, and Besov spaces have been obtained (see [11, 12] and the references therein).

Motivated by the above work, we investigate the Hyers–Ulam stability of the coefficient multipliers T_λ on Hardy spaces H^2 and Dirichlet spaces $\mathcal{D}^2$. We also investigate the Hyers–Ulam stability of the coefficient multipliers between Dirichlet and Hardy spaces, and some illustrative examples are also discussed.

The organization of the paper is as follows: Section 2 is devoted to the fundamental definitions of the Hardy space and Dirichlet space and a review of some basic properties of these spaces. Additionally, we review some existing results concerning the coefficient multipliers T_λ on Hardy space H^2 and give a sufficient condition concerning the coefficient multipliers on Dirichlet space $\mathcal{D}^2$. In Section 3, a necessary and sufficient condition for the coefficient multipliers on Dirichlet space $\mathcal{D}^2$ to have Hyers–Ulam stability is obtained, and the best constant of Hyers–Ulam stability is also discussed. In Section 4, a necessary and sufficient condition for the coefficient multipliers on Hardy space H^2 to be Hyers–Ulam stable is obtained, and the best constant of Hyers–Ulam stability is also discussed. In Sections 5 and 6, we focus our attention on the investigation of the Hyers–Ulam stability of the coefficient multipliers between Dirichlet and Hardy spaces. We give some necessary and sufficient conditions for the coefficient multipliers to have Hyers–Ulam stability between Dirichlet and Hardy spaces, and the best constant of Hyers–Ulam stability is also discussed under different circumstances. In Section 7, we summarize the main results and ideas of this paper.

Throughout this paper, let $\mathbb{C}$ denote the complex plane and let $\mathbb{D}$ denote the unit open disc over complex plane.

2. The coefficient multipliers on Hardy space H^2 and Dirichlet space $\mathcal{D}^2$

In this section, we recall Hardy space H^2 and Dirichlet space $\mathcal{D}^2$ and the fundamental properties of them, we recall some sufficient conditions, for which a complex sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ can be the coefficient multiplier on Hardy space H^2, and we give a sufficient condition concerning the coefficient multipliers on Dirichlet space $\mathcal{D}^2$.

First, we recall Hardy space H^2 and Dirichlet space $\mathcal{D}^2$ and the fundamental properties of them.

Let $H(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$. For $0 < p < \infty$ and $0 < r < 1$, for a holomorphic function f defined on $\mathbb{D}$, we set

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}.$$

We define the Hardy space $H^p = H^p(\mathbb{D})$ as

$$H^p = \left\{f \in H(\mathbb{D}) : \sup_{0<r<1} M_p(f,r) < \infty\right\},$$

and for $f \in H^p$, we set

$$\|f\|_{H^p(\mathbb{D})} = \sup_{0<r<1} M_p(f,r).$$

Furthermore, we define H^∞ as the space of holomorphic functions that are bounded on the unit disc, endowed with the sup-norm.

The particular importance of H^2 is due to the fact that H^2 is a Hilbert space. If f is holomorphic on $\mathbb{D}$, then it admits power series expansion $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. It is easy to show that

$$\|f\|_{H^2(\mathbb{D})} = \sup_{0<r<1} M_2(f,r) = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}},$$

that is, $f \in H^2$ if and only if $\sum\limits_{n=0}^{\infty} |a_n|^2$ is finite. The norm $\|\cdot\|_{H^2(\mathbb{D})}$ can be simply written as $\|\cdot\|_{H^2}$ without causing confusion.

It is well known that for $0 < p < \infty$, each function in H^p can be approximated in norm by polynomials. Thus, H^p is also characterized as the closure of polynomials in the space L^p. It is also said that the polynomials are dense in H^p.

In what follows, $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ is the normalized area measure on the unit disc $\mathbb{D}$. The Dirichlet space $\mathcal{D}^2$ consists of all analytic functions f on $\mathbb{D}$ such that the Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

For convenience, we define the norm on the Dirichlet space $\mathcal{D}^2$ as follows

$$\|f\|_{\mathcal{D}^2} = \sqrt{\|f\|_{H^2}^2 + D(f)},\ f \in \mathcal{D}^2,$$

where the norm $\|f\|_{H^2}$ is the norm in which f is the vector in Hardy space H^2.

For $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n,\ g(z) = \sum\limits_{n=0}^{\infty} b_n z^n \in \mathcal{D}^2$, the inner product concerning the above norm is

$$\langle f, g\rangle = \sum_{n=0}^{\infty}(n+1)a_n\overline{b_n},$$

where $\overline{b_n}$ is the complex conjugation of the complex number b_n. For every $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^2$, by an easy calculation, we obtain

$$\|f\|_{\mathcal{D}^2} = \left\{ \sum_{n=0}^{\infty} (n+1)|a_n|^2 \right\}^{\frac{1}{2}}.$$

It is well known that the sequence

$$\left\{ \frac{z^n}{\sqrt{n+1}} \right\}_{n=0}^{\infty}$$

is an orthonormal basis of $\mathcal{D}^2$ and the polynomials are dense in $\mathcal{D}^2$.

In [1], Duren gave sufficient conditions for a complex sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ to be the coefficient multiplier on Hardy space H^2.

Theorem 2.1. ([1]) If $0 < p \leqslant 2 \leqslant q < \infty$, $\alpha = \frac{1}{p} - \frac{1}{q}$, and $\lambda_n = O(n^{-\alpha})$, then $\{\lambda_n\}$ is a multiplier of H^p into H^q. The same is true if $0 < p \leqslant 1$ and $q = \infty$, but not if $1 < p < q = \infty$. The number α is best: for each $a < \alpha$, there is a sequence $\{\lambda_n\}$ with $\lambda_n = O(n^{-a})$ that is not a multiplier of H^p into H^q.

By Theorem 2.1, it is easy to obtain the following result.

Corollary 2.2. If a sequence $\{\lambda_n\}$ satisfies $\lambda_n = O(1)$, then $\{\lambda_n\} \in (H^2, H^2)$.

Next, we give a sufficient condition concerning the coefficient multipliers on Dirichlet space $\mathcal{D}^2$.

Theorem 2.3. If a sequence $\{\lambda_n\}$ satisfies $\lambda_n = O(1)$, then $\{\lambda_n\} \in (\mathcal{D}^2, \mathcal{D}^2)$.

Proof. Suppose that $f = \sum\limits_{n=0}^{\infty} a_n z^n \in \mathcal{D}^2$, then f is a analytic function on the unit disc $\mathbb{D}$ and satisfies that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} \leqslant 1$ and

$$\|f\|_{\mathcal{D}^2} = \left\{ \sum_{n=0}^{\infty} (n+1)|a_n|^2 \right\}^{\frac{1}{2}} < \infty.$$

Here, for the convenience of calculating the norm of the function in the space $\mathcal{D}^2$, each related function is expressed as a linear combination of orthonormal basis in the space. It is easy to get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \sqrt{n+1}\, a_n \frac{z^n}{\sqrt{n+1}}.$$

We have

$$T_\lambda f(z) = \sum_{n=0}^{\infty} \lambda_n a_n z^n = \sum_{n=0}^{\infty} \lambda_n \sqrt{n+1}\, a_n \frac{z^n}{\sqrt{n+1}}.$$

Suppose that $|\lambda_n| < C, C > 0$, let $g(z) = T_\lambda f(z) = \sum\limits_{n=0}^{\infty} \lambda_n a_n z^n$. We can obtain that

$$\lim_{n\to\infty} \sqrt[n]{|\lambda_n a_n|} \leqslant \lim_{n\to\infty} \sqrt[n]{C|a_n|} = \lim_{n\to\infty} \sqrt[n]{C} \cdot \lim_{n\to\infty} \sqrt[n]{|a_n|} \leqslant 1,$$

so the function g is also an analytic function on the unit disc $\mathbb{D}$. It is easy to get

$$\|g(z)\|_{\mathcal{D}^2}^2 = \|T_\lambda f(z)\|_{\mathcal{D}^2}^2 = \sum_{n=0}^{\infty} |\lambda_n|^2 |a_n|^2 (n+1) \leqslant C^2 \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty.$$

Thus, we get that $g \in \mathcal{D}^2$. Therefore, the sequence $\{\lambda_n\}$ is a multiplier from $\mathcal{D}^2$ to $\mathcal{D}^2$. The proof is complete. □

In order to study the Hyers–Ulam stability of the coefficient multipliers on Hardy space H^2 and Dirichlet space $\mathcal{D}^2$, throughout this paper, we suppose that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is bounded.

3. Hyers–Ulam stability of the coefficient multipliers T_λ on Dirichlet space $\mathcal{D}^2$

In this section, we investigate the Hyers–Ulam stability of the coefficient multipliers on Dirichlet space $\mathcal{D}^2$ and we give a necessary and sufficient condition for the coefficient multipliers to have Hyers–Ulam stability. We also show that the best constant of Hyers–Ulam stability exists.

Theorem 3.1. Suppose that $\lambda = \{\lambda_n\}$ satisfies $\lambda_n = O(1)$, then the following conditions are equivalent:
(a) the sequence $\left\{\frac{1}{\lambda_n}\right\}$ is bounded;
(b) the multiplier T_λ is Hyers–Ulam stable on Dirichlet space $\mathcal{D}^2$.

Proof. (b)⇒(a). Suppose that T_λ is stable with Hyers–Ulam stability constant K on Dirichlet space $\mathcal{D}^2$. For any $\varepsilon > 0$, $f, g \in \mathcal{D}^2$ and f, g satisfy $\| T_\lambda f - g \|_{\mathcal{D}^2} \leqslant \varepsilon$, then exists $f_0 \in \mathcal{D}^2$ and $K > 0$ such that $T_\lambda f_0 = g$ and $\|f_0 - f\|_{\mathcal{D}^2} < K\varepsilon$. If $f = \frac{z^n}{\lambda_n \sqrt{n+1}}$ for any nonnegative integer n, $g = 0, \varepsilon = 1$, we have $\| T_\lambda f - g \|_{\mathcal{D}^2} = \left\| \frac{z^n}{\sqrt{n+1}} \right\|_{\mathcal{D}^2} \leqslant 1 = \varepsilon$. Then, exists $f_0 \in \mathcal{D}^2$ such that $T_\lambda f_0 = 0$ and $\|f_0 - f\|_{\mathcal{D}^2} = \left\| f_0 - \frac{z^n}{\lambda_n \sqrt{n+1}} \right\|_{\mathcal{D}^2} < K \cdot 1 = K$. Thus, we have $\|f\|_{\mathcal{D}^2} - \|f_0\|_{\mathcal{D}^2} < K$. We obtain that $\|f\|_{\mathcal{D}^2} < \|f_0\|_{\mathcal{D}^2} + K$. Therefore, $\left|\frac{1}{\lambda_n}\right| \left\| \frac{z^n}{\sqrt{n+1}} \right\|_{\mathcal{D}^2} < \|f_0\|_{\mathcal{D}^2} + K$. We obtain that $\left|\frac{1}{\lambda_n}\right| < \|f_0\|_{\mathcal{D}^2} + K$.

(a)⇒(b). Suppose that the sequence $\left\{\frac{1}{\lambda_n}\right\}$ is bounded, and let $M = \sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\}$. Since the polynomials are dense in $\mathcal{D}^2$, we just need to show that T_λ is Hyers–Ulam stable on the polynomials dense subspace $\mathcal{P}$. Take any two polynomials $p(z) = \sum_{n=0}^{r} a_n z^n \in \mathcal{P}$ and $q(z) = \sum_{n=0}^{s} b_n z^n \in \mathcal{P}$, where r, s are all nonnegative integers.

When $r > s$, we have

$$
\begin{aligned}
T_\lambda p - q &= \sum_{n=0}^{r} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s} \lambda_n a_n z^n + \sum_{n=s+1}^{r} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) z^n + \sum_{n=s+1}^{r} \lambda_n a_n z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}} + \sum_{n=s+1}^{r} \lambda_n a_n \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}. \qquad (3.1)
\end{aligned}
$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (3.1), we obtain

$$\|T_\lambda p - q\|_{\mathcal{D}^2}$$

$$= \left[\sum_{n=0}^{s}(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=s+1}^{r}(n+1)|\lambda_n a_n|^2\right]^{\frac{1}{2}} < \varepsilon. \tag{3.2}$$

We take $p_0 \in \mathcal{P} \subset \mathcal{D}^2$ to be the function defined by

$$p_0(z) = \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n,$$

it is easy to show that

$$T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z).$$

Thus, we obtain

$$\begin{aligned}
p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n \\
&= \sum_{n=0}^{s} a_n z^n + \sum_{n=s+1}^{r} a_n z^n - \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n \\
&= \sum_{n=0}^{s}\left(a_n - \frac{b_n}{\lambda_n}\right)z^n + \sum_{n=s+1}^{r} a_n z^n \\
&= \sum_{n=0}^{s}\frac{\lambda_n a_n - b_n}{\lambda_n}z^n + \sum_{n=s+1}^{r}\frac{1}{\lambda_n}\lambda_n a_n z^n \\
&= \sum_{n=0}^{s}\frac{\lambda_n a_n - b_n}{\lambda_n}\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}} + \sum_{n=s+1}^{r}\frac{1}{\lambda_n}\cdot\lambda_n a_n\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}}.
\end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we get

$$\begin{aligned}
\|p - p_0\|_{\mathcal{D}^2} &= \left[\sum_{n=0}^{s}\left|\frac{1}{\lambda_n}\right|^2(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=s+1}^{r}\left|\frac{1}{\lambda_n}\right|^2(n+1)|\lambda_n a_n|^2\right]^{\frac{1}{2}} \\
&\leqslant \left[\sum_{n=0}^{s}M^2(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=s+1}^{r}M^2(n+1)|\lambda_n a_n|^2\right]^{\frac{1}{2}} \\
&= M\left[\sum_{n=0}^{s}(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=s+1}^{r}(n+1)|\lambda_n a_n|^2\right]^{\frac{1}{2}} \\
&< M\cdot\varepsilon.
\end{aligned}$$

When $r = s$, we have

$$T_\lambda p - q = \sum_{n=0}^{r}\lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{s} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}.
\end{aligned}
\tag{3.4}
$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (3.4) we get

$$
\|T_\lambda p - q\|_{\mathcal{D}^2} = \left[\sum_{n=0}^{s} (n+1) |\lambda_n a_n - b_n|^2 \right]^{\frac{1}{2}} < \varepsilon.
\tag{3.5}
$$

Take $p_0 \in \mathcal{P} \subset \mathcal{D}^2$ to be the function defined by

$$
p_0(z) = \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n,
$$

it is easy to show that

$$
T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z).
$$

Thus, we have

$$
\begin{aligned}
p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n \\
&= \sum_{n=0}^{s} \left(a_n - \frac{b_n}{\lambda_n} \right) z^n \\
&= \sum_{n=0}^{s} \frac{\lambda_n a_n - b_n}{\lambda_n} z^n \\
&= \sum_{n=0}^{s} \frac{\lambda_n a_n - b_n}{\lambda_n} \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}.
\end{aligned}
\tag{3.6}
$$

By (3.5) and (3.6), we obtain

$$
\begin{aligned}
\|p - p_0\|_{\mathcal{D}^2} &= \left[\sum_{n=0}^{s} \left| \frac{1}{\lambda_n} \right|^2 (n+1) |\lambda_n a_n - b_n|^2 \right]^{\frac{1}{2}} \\
&\leqslant M \left[\sum_{n=0}^{s} (n+1) |\lambda_n a_n - b_n|^2 \right]^{\frac{1}{2}}
\end{aligned}
$$

$$< \quad M \cdot \varepsilon.$$

When $r < s$, we have

$$\begin{aligned} T_\lambda p - q &= \sum_{n=0}^{r} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\ &= \sum_{n=0}^{r} \lambda_n a_n z^n - \sum_{n=0}^{r} b_n z^n - \sum_{n=r+1}^{s} b_n z^n \\ &= \sum_{n=0}^{r} (\lambda_n a_n - b_n) z^n - \sum_{n=r+1}^{s} b_n z^n \\ &= \sum_{n=0}^{r} (\lambda_n a_n - b_n) \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}} - \sum_{n=r+1}^{s} b_n \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}. \end{aligned} \tag{3.7}$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (3.7) we get

$$\|T_\lambda p - q\|_{\mathcal{D}^2} = \left[\sum_{n=0}^{r} (n+1) |\lambda_n a_n - b_n|^2 + \sum_{n=r+1}^{s} (n+1) |b_n|^2 \right]^{\frac{1}{2}} < \varepsilon. \tag{3.8}$$

Take $p_0 \in \mathcal{P} \subset \mathcal{D}^2$ to be the function defined by

$$p_0(z) = \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n,$$

it is easy to show that

$$T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z).$$

Thus, we get

$$\begin{aligned} p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n \\ &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{r} \frac{b_n}{\lambda_n} z^n - \sum_{n=r+1}^{s} \frac{b_n}{\lambda_n} z^n \\ &= \sum_{n=0}^{r} \left(a_n - \frac{b_n}{\lambda_n} \right) z^n - \sum_{n=r+1}^{s} \frac{b_n}{\lambda_n} z^n \\ &= \sum_{n=0}^{r} \frac{\lambda_n a_n - b_n}{\lambda_n} z^n - \sum_{n=r+1}^{s} \frac{1}{\lambda_n} b_n z^n \\ &= \sum_{n=0}^{r} \frac{(\lambda_n a_n - b_n)}{\lambda_n} \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}} - \sum_{n=r+1}^{s} \frac{b_n}{\lambda_n} \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$\begin{aligned}
&\|p - p_0\|_{\mathcal{D}^2} \\
= &\left[\sum_{n=0}^{r}\left|\frac{1}{\lambda_n}\right|^2 (n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=r+1}^{s}\left|\frac{1}{\lambda_n}\right|^2 (n+1)|b_n|^2\right]^{\frac{1}{2}} \\
\leqslant &\left[\sum_{n=0}^{r} M^2(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=r+1}^{s} M^2(n+1)|b_n|^2\right]^{\frac{1}{2}} \\
= &M\left[\sum_{n=0}^{r}(n+1)|\lambda_n a_n - b_n|^2 + \sum_{n=r+1}^{s}(n+1)|b_n|^2\right]^{\frac{1}{2}} \\
< &M \cdot \varepsilon.
\end{aligned}$$

Therefore, the coefficient multiplier T_λ is Hyers–Ulam stable on $\mathcal{D}^2$. The proof is complete. □

Next, we will show that the best constant of Hyers–Ulam stability of the coefficient multiplier T_λ exists.

Theorem 3.2. Suppose that $\lambda = \{\lambda_n\}$, and the coefficient multiplier T_λ is Hyers–Ulam stable on $\mathcal{D}^2$, then $K_{T_\lambda} = \sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\}$ and K_{T_λ} is a HUS constant of T_λ.

Proof. Since the coefficient multiplier T_λ is Hyers–Ulam stable on $\mathcal{D}^2$, then for any $\varepsilon > 0$, $f, g \in \mathcal{D}^2$ and f, g satisfy $\|T_\lambda f - g\|_{\mathcal{D}^2} \leqslant \varepsilon$, then exists $f_0 \in \mathcal{D}^2$ and $K > 0$ such that $T_\lambda f_0 = g$ and $\|f_0 - f\|_{\mathcal{D}^2} < K\varepsilon$. For $f = \frac{1}{\lambda_n}\frac{z^n}{\sqrt{n+1}}$, where n is any nonnegative integer, $g = 0$, $\varepsilon = 1$, we have $\|T_\lambda f - g\|_{\mathcal{D}^2} = \left\|\frac{z^n}{\sqrt{n+1}}\right\|_{\mathcal{D}^2} = 1 \leqslant \varepsilon$, then exists $K > 0, f_0 = 0 \in \mathcal{D}^2$ such that $T_\lambda f_0 = 0 = g$ and $\|f_0 - f\|_{\mathcal{D}^2} = \left\|0 - \frac{1}{\lambda_n}\frac{z^n}{\sqrt{n+1}}\right\|_{\mathcal{D}^2} = \left|\frac{1}{\lambda_n}\right|\left\|\frac{z^n}{\sqrt{n+1}}\right\|_{\mathcal{D}^2} = \left|\frac{1}{\lambda_n}\right| < K\varepsilon = K \cdot 1 = K$. Thus, we have $\left|\frac{1}{\lambda_n}\right| < K$, and we obtain that $\sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\} \leqslant K$. From the proof of Theorem 3.1, we know that $M = \sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\}$ is a HUS constant. Therefore, $K_{T_\lambda} = \sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\}$. We obtain that K_{T_λ} is a HUS constant of T_λ. □

Next, several examples are given to illustrate the results of the above two theorems.

Example 3.3. A trivial example is the coefficient multiplier T_λ, where the sequence $\lambda = \{\lambda_n\}, \lambda_n \equiv c \in \mathbb{C}, c \neq 0$. By Theorems 3.1 and 3.2, it is easy to show that T_λ is Hyers–Ulam stable on $\mathcal{D}^2$, and the best HUS constant of T_λ is $\frac{1}{|c|}$.

Example 3.4. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{n+5}{n+8}i^{2n}$, i is the imaginary unit and n is nonnegative integer. It is evident that the sequence satisfies $\lambda_n = O(1)$; by Theorem 2.3, we have $\{\lambda_n\} \in (\mathcal{D}^2, \mathcal{D}^2)$. By $\frac{1}{\lambda_n} = \left(1 + \frac{3}{n+5}\right)\frac{1}{i^{2n}}$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is bounded. By Theorems 3.1 and 3.2, it is easy to show that T_λ is Hyers–Ulam stable on $\mathcal{D}^2$, and the best HUS constant of T_λ is $\frac{8}{5}$.

Example 3.5. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{\sqrt{3}(n+4)}{n+5} + \frac{\sqrt{8}(n+4)}{n+5}i$, and n is nonnegative integer. We obtain that the sequence satisfies $|\lambda_n| \leqslant \sqrt{11}$. By Theorem 2.3, we have $\{\lambda_n\} \in (\mathcal{D}^2, \mathcal{D}^2)$. By

$\frac{1}{\lambda_n} = \frac{\sqrt{3}(n+5)}{11(n+4)} - \frac{\sqrt{8}(n+5)}{11(n+4)}i$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is bounded and $\left|\frac{1}{\lambda_n}\right| \leqslant \frac{5\sqrt{11}}{44}$. By Theorem 3.1 and 3.2, the multiplier T_λ is Hyers–Ulam stable on $\mathcal{D}^2$, and the best HUS constant of T_λ is $\frac{5\sqrt{11}}{44}$.

Example 3.6. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{10i}{(n+5)^3}$, and n is nonnegative integer. It is evident that the sequence satisfies $|\lambda_n| = \frac{10}{(n+5)^3} \leqslant \frac{2}{25}$; by Theorem 2.3, we have $\{\lambda_n\} \in (\mathcal{D}^2, \mathcal{D}^2)$. By $\frac{1}{\lambda_n} = \frac{(n+5)^3}{10i}$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is unbounded. By Theorem 3.1, the multiplier T_λ is not Hyers–Ulam stable on $\mathcal{D}^2$.

4. Hyers–Ulam stability of the coefficient multipliers T_λ on Hardy space H^2

In this section, we investigate the Hyers–Ulam stability of the coefficient multipliers on Hardy space H^2 and we give a necessary and sufficient condition for the coefficient multipliers to have Hyers–Ulam stability. We also show that the best constant of Hyers–Ulam stability exists by using the same method of Section 3. We omit the proofs of the following main results.

Theorem 4.1. Suppose that $\lambda = \{\lambda_n\}$ satisfies $\lambda_n = O(1)$, then the following conditions are equivalent:
(a) the multiplier T_λ is Hyers–Ulam stable on Hardy space H^2;
(b) the sequence $\left\{\frac{1}{\lambda_n}\right\}$ is bounded.

Theorem 4.2. Suppose that $\lambda = \{\lambda_n\}$ and the coefficient multiplier T_λ is Hyers–Ulam stable on H^2, then $K_{T_\lambda} = \sup\left\{\left|\frac{1}{\lambda_n}\right| : n \geqslant 0\right\}$ and K_{T_λ} is a HUS constant of T_λ.

Next, several examples are given to illustrate the results of the above two theorems.

Example 4.3. A trivial example is the coefficient multiplier T_λ, where the sequence $\lambda = \{\lambda_n\}, \lambda_n \equiv c \in \mathbb{C}, c \neq 0$. By Theorem 4.1 and 4.2, it is easy to show that T_λ is Hyers–Ulam stable on H^2, and the best HUS constant of T_λ is $\frac{1}{|c|}$.

Example 4.4. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = 5 + i^n$, i is the imaginary unit and n is nonnegative integer. It is evident that the sequence satisfies $\lambda_n = O(1)$; by Corollary 2.2, we have $\{\lambda_n\} \in (H^2, H^2)$. By $\frac{1}{\lambda_n} = \frac{1}{5+i^n}$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is bounded. By Theorem 4.1 and 4.2, T_λ is Hyers–Ulam stable on H^2, and the best HUS constant of T_λ is $\frac{1}{4}$.

Example 4.5. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{\sqrt{n+3}}{\sqrt{n+4}} + \frac{\sqrt{n+3}}{\sqrt{n+4}}i$, and n is nonnegative integer. We get that the sequence satisfies $|\lambda_n| \leqslant \sqrt{2}$. By Corollary 2.2, we have $\{\lambda_n\} \in (H^2, H^2)$. By $\frac{1}{\lambda_n} = \frac{\sqrt{n+4}}{2\sqrt{n+3}} - \frac{\sqrt{n+4}}{2\sqrt{n+3}}i$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is bounded and $\left|\frac{1}{\lambda_n}\right| \leqslant \sqrt{\frac{2}{3}}$. By Theorem 4.1 and 4.2, the coefficient multiplier T_λ is Hyers–Ulam stable on H^2, and the best HUS constant of T_λ is $\sqrt{\frac{2}{3}}$.

Example 4.6. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{7+8i}{(n+2)^4}$, and n is nonnegative integer. We get that the sequence satisfies $|\lambda_n| = \frac{\sqrt{113}}{(n+2)^4} \leqslant \frac{\sqrt{113}}{16}$. By Corollary 2.2, we have $\{\lambda_n\} \in (H^2, H^2)$. By $\frac{1}{\lambda_n} = \frac{(n+2)^4}{7+8i}$, we obtain that $\left\{\frac{1}{\lambda_n}\right\}$ is unbounded. By Theorem 4.1, the coefficient multiplier T_λ is not Hyers–Ulam stable on H^2.

5. The coefficient multipliers from Hardy space to Dirichlet space with Hyers–Ulam stability

In this section, we focus our attention on the investigation of the Hyers–Ulam stability of the coefficient multipliers from H^2 to $\mathcal{D}^2$.

Theorem 5.1. If a sequence $\{\lambda_n\}$ satisfies $\lambda_n \sqrt{n+1} = O(1)$, then $\{\lambda_n\} \in (H^2, \mathcal{D}^2)$.

Proof. Suppose that $f \in H^2$ and $f = \sum\limits_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|_{H^2} = \left\{ \sum_{n=0}^{\infty} |a_n|^2 \right\}^{\frac{1}{2}} < \infty.$$

Suppose that the sequence $\left\{\lambda_n \sqrt{n+1}\right\}$ satisfies $|\lambda_n \sqrt{n+1}| \leqslant C$, where $C > 0$. By

$$(T_\lambda f)(z) = \sum_{n=0}^{\infty} \lambda_n a_n z^n,$$

we obtain

$$\sum_{n=0}^{\infty} (n+1)|\lambda_n a_n|^2 = \sum_{n=0}^{\infty} |\lambda_n|^2 (n+1)|a_n|^2 \leqslant C^2 \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Thus $T_\lambda f \in \mathcal{D}^2$. So $\{\lambda_n\} \in (H^2, \mathcal{D}^2)$. □

Next, some examples are given to illustrate the result of the theorem.

Example 5.2. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{i}{\sqrt{n+1}}$, n is nonnegative integer. It is evident that the sequence $\{\lambda_n\}$ satisfies $\lambda_n \sqrt{n+1} = O(1)$; by Theorem 5.1, we have $\{\lambda_n\} \in (H^2, \mathcal{D}^2)$.

Example 5.3. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{6+7i}{n+1}$, i is the imaginary unit and n is nonnegative integer. We get that the sequence $\left\{\lambda_n \sqrt{n+1}\right\}$ is bounded. By Theorem 5.1, we have $\{\lambda_n\} \in (H^2, \mathcal{D}^2)$.

Next, we prove the main results of this section.

Theorem 5.4. Suppose that $\lambda = \{\lambda_n\}$ satisfies $\lambda_n \sqrt{n+1} = O(1)$, then the coefficient multiplier T_λ : $H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable if and only if the sequence $\left\{\frac{1}{\lambda_n \sqrt{n+1}}\right\}$ is bounded.

Proof. Sufficiency. Suppose that the sequence $\left\{\frac{1}{\lambda_n \sqrt{n+1}}\right\}$ is bounded, and let $M = \sup \left\{ \left|\frac{1}{\lambda_n \sqrt{n+1}}\right| : n \geqslant 0 \right\}$. Since the polynomials are dense in H^2 and $\mathcal{D}^2$, we just need to show that T_λ is Hyers–Ulam stable on the polynomials dense subspace $\mathcal{P}$ of H^2 and $\mathcal{D}^2$. Take any two polynomials $p(z) = \sum\limits_{n=0}^{r} a_n z^n \in \mathcal{P} \subset H^2$ and $q(z) = \sum\limits_{n=0}^{s} b_n z^n \in \mathcal{P} \subset \mathcal{D}^2$, where r, s are all nonnegative integers.

When $r = s$, we have

$$T_\lambda p - q = \sum_{n=0}^{r} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{s} \lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) z^n \\
&= \sum_{n=0}^{s} (\lambda_n a_n - b_n) \cdot \sqrt{n+1} \cdot \frac{z^n}{\sqrt{n+1}}. \qquad (5.1)
\end{aligned}
$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (5.1) we get

$$
\|T_\lambda p - q\|_{\mathcal{D}^2} = \left[\sum_{n=0}^{s} |\lambda_n a_n - b_n|^2 (n+1) \right]^{\frac{1}{2}} < \varepsilon. \qquad (5.2)
$$

Take $p_0 \in \mathcal{P} \subset H^2$ to be the function defined by

$$
p_0(z) = \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n,
$$

it is easy to show that

$$
T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z),
$$

where $q \in \mathcal{P} \subset \mathcal{D}^2$. Thus, we have

$$
\begin{aligned}
p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n \\
&= \sum_{n=0}^{s} \left(a_n - \frac{b_n}{\lambda_n} \right) z^n \\
&= \sum_{n=0}^{s} \frac{\lambda_n a_n - b_n}{\lambda_n} z^n. \qquad (5.3)
\end{aligned}
$$

By (5.2) and (5.3), we obtain

$$
\begin{aligned}
\|p - p_0\|_{H^2} &= \left[\sum_{n=0}^{s} \left| \frac{\lambda_n a_n - b_n}{\lambda_n} \right|^2 \right]^{\frac{1}{2}} \\
&= \left[\sum_{n=0}^{s} \left| \frac{1}{\lambda_n \sqrt{n+1}} \right|^2 |\lambda_n a_n - b_n|^2 (n+1) \right]^{\frac{1}{2}} \\
&\leqslant \left[\sum_{n=0}^{s} M^2 |\lambda_n a_n - b_n|^2 (n+1) \right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= M\left[\sum_{n=0}^{s}|\lambda_n a_n - b_n|^2(n+1)\right]^{\frac{1}{2}} \\
&< M\cdot\varepsilon.
\end{aligned}
$$

When $r < s$, we have

$$
\begin{aligned}
T_\lambda p - q &= \sum_{n=0}^{r}\lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{r}\lambda_n a_n z^n - \sum_{n=0}^{r} b_n z^n - \sum_{n=r+1}^{s} b_n z^n \\
&= \sum_{n=0}^{r}(\lambda_n a_n - b_n) z^n - \sum_{n=r+1}^{s} b_n z^n \\
&= \sum_{n=0}^{r}(\lambda_n a_n - b_n)\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}} - \sum_{n=r+1}^{s} b_n\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}}. \qquad (5.4)
\end{aligned}
$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (5.4) we get

$$
\|T_\lambda p - q\|_{\mathcal{D}^2} = \left[\sum_{n=0}^{r}|\lambda_n a_n - b_n|^2(n+1) + \sum_{n=r+1}^{s}|b_n|^2(n+1)\right]^{\frac{1}{2}} < \varepsilon. \qquad (5.5)
$$

Take $p_0 \in \mathcal{P} \subset H^2$ to be the function defined by

$$
p_0(z) = \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n,
$$

it is easy to obtain that

$$
T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z),
$$

where $q \in \mathcal{P} \subset \mathcal{D}^2$. Thus, we get

$$
\begin{aligned}
p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n \\
&= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{r}\frac{b_n}{\lambda_n}z^n - \sum_{n=r+1}^{s}\frac{b_n}{\lambda_n}z^n \\
&= \sum_{n=0}^{r}\left(a_n - \frac{b_n}{\lambda_n}\right)z^n - \sum_{n=r+1}^{s}\frac{b_n}{\lambda_n}z^n \\
&= \sum_{n=0}^{r}\frac{\lambda_n a_n - b_n}{\lambda_n}z^n - \sum_{n=r+1}^{s}\frac{1}{\lambda_n}b_n z^n. \qquad (5.6)
\end{aligned}
$$

By (5.5) and (5.6), we obtain

$$\begin{aligned}
& \|p - p_0\|_{H^2} \\
= & \left[\sum_{n=0}^{r}\left|\frac{\lambda_n a_n - b_n}{\lambda_n}\right|^2 + \sum_{n=r+1}^{s}\left|\frac{b_n}{\lambda_n}\right|^2\right]^{\frac{1}{2}} \\
= & \left[\sum_{n=0}^{r}\left|\frac{1}{\lambda_n\sqrt{n+1}}\right|^2 |\lambda_n a_n - b_n|^2(n+1) + \sum_{n=r+1}^{s}\left|\frac{1}{\lambda_n\sqrt{n+1}}\right|^2 |b_n|^2(n+1)\right]^{\frac{1}{2}} \\
\leqslant & \left[\sum_{n=0}^{r} M^2|\lambda_n a_n - b_n|^2(n+1) + \sum_{n=r+1}^{s} M^2|b_n|^2(n+1)\right]^{\frac{1}{2}} \\
= & M\left[\sum_{n=0}^{r}|\lambda_n a_n - b_n|^2(n+1) + \sum_{n=r+1}^{s}|b_n|^2(n+1)\right]^{\frac{1}{2}} \\
< & M\cdot\varepsilon.
\end{aligned}$$

When $r > s$, we have

$$\begin{aligned}
T_\lambda p - q &= \sum_{n=0}^{r}\lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s}\lambda_n a_n z^n + \sum_{n=s+1}^{r}\lambda_n a_n z^n - \sum_{n=0}^{s} b_n z^n \\
&= \sum_{n=0}^{s}(\lambda_n a_n - b_n)z^n + \sum_{n=s+1}^{r}\lambda_n a_n z^n \\
&= \sum_{n=0}^{s}(\lambda_n a_n - b_n)\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}} + \sum_{n=s+1}^{r}\lambda_n a_n\cdot\sqrt{n+1}\cdot\frac{z^n}{\sqrt{n+1}}. \qquad (5.7)
\end{aligned}$$

For any $\varepsilon > 0$, if p, q satisfy that $\|T_\lambda p - q\|_{\mathcal{D}^2} < \varepsilon$, by (5.7), we obtain

$$\begin{aligned}
& \|T_\lambda p - q\|_{\mathcal{D}^2} \\
= & \left[\sum_{n=0}^{s}|\lambda_n a_n - b_n|^2(n+1) + \sum_{n=s+1}^{r}|\lambda_n a_n|^2(n+1)\right]^{\frac{1}{2}} < \varepsilon. \qquad (5.8)
\end{aligned}$$

Take $p_0 \in \mathcal{P} \subset H^2$ to be the function defined by

$$p_0(z) = \sum_{n=0}^{s}\frac{b_n}{\lambda_n}z^n,$$

it is easy to get that

$$T_\lambda p_0(z) = \sum_{n=0}^{s} b_n z^n = q(z),$$

where $q \in \mathcal{P} \subset \mathcal{D}^2$. Thus, we obtain

$$\begin{aligned} p - p_0 &= \sum_{n=0}^{r} a_n z^n - \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n \\ &= \sum_{n=0}^{s} a_n z^n + \sum_{n=s+1}^{r} a_n z^n - \sum_{n=0}^{s} \frac{b_n}{\lambda_n} z^n \\ &= \sum_{n=0}^{s} \left(a_n - \frac{b_n}{\lambda_n}\right) z^n + \sum_{n=s+1}^{r} a_n z^n \\ &= \sum_{n=0}^{s} \frac{\lambda_n a_n - b_n}{\lambda_n} z^n + \sum_{n=s+1}^{r} \frac{1}{\lambda_n} \lambda_n a_n z^n. \end{aligned} \tag{5.9}$$

By (5.8) and (5.9), we get

$$\begin{aligned} \|p - p_0\|_{H^2} &= \left[\sum_{n=0}^{s} \left|\frac{\lambda_n a_n - b_n}{\lambda_n}\right|^2 + \sum_{n=s+1}^{r} \left|\frac{\lambda_n a_n}{\lambda_n}\right|^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{n=0}^{s} \left|\frac{1}{\lambda_n \sqrt{n+1}}\right|^2 |\lambda_n a_n - b_n|^2 (n+1) + \sum_{n=s+1}^{r} \left|\frac{1}{\lambda_n \sqrt{n+1}}\right|^2 |\lambda_n a_n|^2 (n+1)\right]^{\frac{1}{2}} \\ &\leqslant \left[\sum_{n=0}^{s} M^2 |\lambda_n a_n - b_n|^2 (n+1) + \sum_{n=s+1}^{r} M^2 |\lambda_n a_n|^2 (n+1)\right]^{\frac{1}{2}} \\ &= M\left[\sum_{n=0}^{s} |\lambda_n a_n - b_n|^2 (n+1) + \sum_{n=s+1}^{r} |\lambda_n a_n|^2 (n+1)\right]^{\frac{1}{2}} \\ &< M \cdot \varepsilon. \end{aligned}$$

Therefore, the coefficient multiplier $T_\lambda : H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable.

Necessity. Suppose that $T_\lambda : H^2 \to \mathcal{D}^2$ is stable with Hyers–Ulam stability constant K. For any $\varepsilon > 0$, $f \in H^2$, $g \in \mathcal{D}^2$ and f, g satisfy $\| T_\lambda f - g \|_{\mathcal{D}^2} \leqslant \varepsilon$, then exists $f_0 \in H^2$ and $K > 0$ such that $T_\lambda f_0 = g$ and $\|f_0 - f\|_{H^2} < K\varepsilon$. If $f = \frac{z^n}{\lambda_n \sqrt{n+1}}$ for any nonnegative integer n, $g = 0, \varepsilon = 1$, we have $\| T_\lambda f - g \|_{\mathcal{D}^2} = \left\|\frac{z^n}{\sqrt{n+1}}\right\|_{\mathcal{D}^2} \leqslant 1 = \varepsilon$. Then exists $f_0 \in H^2$ such that $T_\lambda f_0 = 0$ and $\|f_0 - f\|_{H^2} = \left\|f_0 - \frac{z^n}{\lambda_n \sqrt{n+1}}\right\|_{H^2} < K \cdot 1 = K$. Thus, we have $\|f\|_{H^2} - \|f_0\|_{H^2} < K$. We obtain that $\|f\|_{H^2} < \|f_0\|_{H^2} + K$. Therefore, $\left|\frac{1}{\lambda_n \sqrt{n+1}}\right| \|z^n\|_{H^2} < \|f_0\|_{H^2} + K$. We obtain that $\left|\frac{1}{\lambda_n \sqrt{n+1}}\right| < \|f_0\|_{H^2} + K$. Since n is any nonnegative integer, we get that the sequence $\left\{\frac{1}{\lambda_n \sqrt{n+1}}\right\}$ is bounded. The proof is complete. □

Next, we show that the best constant of Hyers–Ulam stability of the coefficient multiplier $T_\lambda : H^2 \to \mathcal{D}^2$ exists.

Theorem 5.5. Suppose that $\lambda = \{\lambda_n\}$, the coefficient multiplier $T_\lambda : H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable, then $K_{T_\lambda} = \sup\left\{\left|\frac{1}{\lambda_n\sqrt{n+1}}\right| : n \geqslant 0\right\}$ and K_{T_λ} is a HUS constant of T_λ.

Proof. Since the coefficient multiplier $T_\lambda : H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable, then for any $\varepsilon > 0$, $f \in H^2$, $g \in \mathcal{D}^2$ and f, g satisfy $\| T_\lambda f - g \|_{\mathcal{D}^2} \leqslant \varepsilon$, then exists $f_0 \in H^2$ and $K > 0$ such that $T_\lambda f_0 = g$ and $\|f_0 - f\|_{H^2} < K\varepsilon$. For $f = \frac{z^n}{\lambda_n\sqrt{n+1}}$, $g = 0, \varepsilon = 1$, where n is any nonnegative integer, we have $\| T_\lambda f - g \|_{\mathcal{D}^2} = \left\|\frac{z^n}{\sqrt{n+1}}\right\|_{\mathcal{D}^2} = 1 \leqslant \varepsilon$, then exists $K > 0, f_0 = 0 \in H^2$ such that $T_\lambda f_0 = 0 = g$ and $\|f_0 - f\|_{H^2} = \left\|0 - \frac{z^n}{\lambda_n\sqrt{n+1}}\right\|_{H^2} = \left|\frac{1}{\lambda_n\sqrt{n+1}}\right| \|z^n\|_{H^2} = \left|\frac{1}{\lambda_n\sqrt{n+1}}\right| < K\varepsilon = K \cdot 1 = K$. Thus, we have $\left|\frac{1}{\lambda_n\sqrt{n+1}}\right| < K$ and we obtain that $\sup\left\{\left|\frac{1}{\lambda_n\sqrt{n+1}}\right| : n \geqslant 0\right\} \leqslant K$. From the proof of Theorem 5.4, we know that $M = \sup\left\{\left|\frac{1}{\lambda_n\sqrt{n+1}}\right| : n \geqslant 0\right\}$ is a HUS constant. Therefore, $K_{T_\lambda} = \sup\left\{\left|\frac{1}{\lambda_n\sqrt{n+1}}\right| : n \geqslant 0\right\}$. We obtain that K_{T_λ} is a HUS constant of T_λ. □

Next, several examples are given to illustrate the results of the above two theorems.

Example 5.6. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{2i}{\sqrt{n+1}}$, n is nonnegative integer. We obtain that the sequence $\left\{\frac{1}{\lambda_n\sqrt{n+1}}\right\}$ is bounded. By Theorem 5.4, we obtain that $T_\lambda : H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable. By Theorem 5.5, the best HUS constant of T_λ is $\frac{1}{2}$.

Example 5.7. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{2}{\sqrt{n+1}} + \frac{3}{\sqrt{n+1}}i$, n is nonnegative integer. It is evident that the sequence $\left\{\frac{1}{\lambda_n\sqrt{n+1}}\right\}$ is bounded; by Theorem 5.4, we obtain that $T_\lambda : H^2 \to \mathcal{D}^2$ is Hyers–Ulam stable. By Theorem 5.5, the best HUS constant of T_λ is $\frac{1}{\sqrt{13}}$.

6. The coefficient multipliers from Dirichlet space to Hardy space with Hyers–Ulam stability

In this section, by a duality argument, we have the following main results concerning the coefficient multipliers from Dirichlet space $\mathcal{D}^2$ to Hardy space H^2. We omit the proofs of them. Several examples are given to illustrate the results of the following main results.

Theorem 6.1. If a sequence $\{\lambda_n\}$ satisfies $\frac{\lambda_n}{\sqrt{n+1}} = O(1)$, then $\{\lambda_n\} \in (\mathcal{D}^2, H^2)$.

Theorem 6.2. Suppose that $\lambda = \{\lambda_n\}$ satisfies $\frac{\lambda_n}{\sqrt{n+1}} = O(1)$, then the coefficient multiplier $T_\lambda : \mathcal{D}^2 \to H^2$ is Hyers–Ulam stable if and only if the sequence $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is bounded.

Theorem 6.3. Suppose that $\lambda = \{\lambda_n\}$, the coefficient multiplier $T_\lambda : \mathcal{D}^2 \to H^2$ is Hyers–Ulam stable, then $K_{T_\lambda} = \sup\left\{\left|\frac{\sqrt{n+1}}{\lambda_n}\right| : n \geqslant 0\right\}$ and K_{T_λ} is a HUS constant of T_λ.

Next, several examples are given to illustrate the results of the above theorems.

Example 6.4. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \frac{2i}{\sqrt{n+1}}$, n is nonnegative integer. We obtain that the sequence $\left\{\frac{\lambda_n}{\sqrt{n+1}}\right\}$ is bounded. By Theorem 6.1, we have $\left\{\frac{2i}{\sqrt{n+1}}\right\}_{n=0}^{\infty} \in (\mathcal{D}^2, H^2)$. Since $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is unbounded, by Theorem 6.2, we obtain that $T_\lambda : \mathcal{D}^2 \to H^2$ is not Hyers–Ulam stable.

Example 6.5. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \sqrt{n+5}$, n is nonnegative integer. It is evident that the sequence $\left\{\frac{\lambda_n}{\sqrt{n+1}}\right\}$ is bounded; by Theorem 6.1, we have $\{\sqrt{n+5}\}_{n=0}^{\infty} \in (\mathcal{D}^2, H^2)$. Since $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is bounded, by Theorem 6.2 we obtain that $T_\lambda : \mathcal{D}^2 \to H^2$ is Hyers–Ulam stable. By Theorem 6.3, the best HUS constant of T_λ is $\frac{\sqrt{5}}{5}$.

Example 6.6. Take the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = 5\sqrt{n+1} + 12i\sqrt{n+1}$, n is nonnegative integer. We get that the sequence $\left\{\frac{\lambda_n}{\sqrt{n+1}}\right\}$ is bounded. By Theorem 6.1, we have $\{\lambda_n\} \in (\mathcal{D}^2, H^2)$. Since $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is bounded, by Theorem 6.2 we obtain that $T_\lambda : \mathcal{D}^2 \to H^2$ is Hyers–Ulam stable. By Theorem 6.3, the best HUS constant of T_λ is $\frac{1}{13}$.

Example 6.7. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \sqrt[5]{n+1}$, n is nonnegative integer. It is evident that the sequence $\left\{\frac{\lambda_n}{\sqrt{n+1}}\right\}$ is bounded; by Theorem 6.1, we have $\left\{\sqrt[5]{n+1}\right\}_{n=0}^{\infty} \in (\mathcal{D}^2, H^2)$. Since $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is unbounded, by Theorem 6.2 we obtain that $T_\lambda : \mathcal{D}^2 \to H^2$ is not Hyers–Ulam stable.

Example 6.8. We consider the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \sqrt[5]{n+1} + i\sqrt[7]{n+1}$, n is nonnegative integer. It is evident that the sequence $\left\{\frac{\lambda_n}{\sqrt{n+1}}\right\}$ is bounded; by Theorem 6.1, we have $\{\lambda_n\} \in (\mathcal{D}^2, H^2)$. Since $\left\{\frac{\sqrt{n+1}}{\lambda_n}\right\}$ is unbounded, by Theorem 6.2 we obtain that $T_\lambda : \mathcal{D}^2 \to H^2$ is not Hyers–Ulam stable.

7. Conclusions

In our work, we investigate the Hyers–Ulam stability of the coefficient multipliers on Hardy space H^2 and Dirichlet space $\mathcal{D}^2$. We also investigate the Hyers–Ulam stability of the coefficient multipliers between Dirichlet and Hardy spaces. These results show that the Hyers–Ulam stability of the coefficient multipliers T_λ on Hilbert spaces of analytic functions depends on the boundedness of a particular sequence associated with the sequence $\lambda = \{\lambda_n\}$. When T_λ is Hyers–Ulam stable, these results also show that the best constant of Hyers–Ulam stability of the coefficient multiplier T_λ exists under different circumstances.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the referees for many valuable comments and suggestions that helped to improve the presentation and quality of this article. This work is supported by the Fundamental Science Research Projects of Shanxi Province (202203021211110), China.

Conflict of interest

The author declares no conflict of interest.

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