AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12511–12528.
DOI: 10.3934/math.2024612
Received: 02 February 2024
Revised: 08 March 2024
Accepted: 20 March 2024
Published: 01 April 2024

Research article

Interval-valued Pythagorean fuzzy entropy and its application to multi-criterion group decision-making

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Abstract: Entropy is an important tool of information measurement in the fuzzy set and its inference. The research on information measurement based on interval-valued Pythagorean fuzzy sets mostly involves the distance formula for interval-valued Pythagorean fuzzy numbers, but seldom involves the measurement of fuzziness. In view of this situation, we have aimed to propose new interval-valued Pythagorean fuzzy entropy and weighted exponential entropy schemes. Based on the interval-valued Pythagorean fuzzy weighted averaging operator, a strategy based on weighted exponential entropy is proposed to solve the multi-criteria group decision-making (MCGDM) problem in the interval-valued Pythagorean environment. Two examples illustrate that this paper provides a feasible new method to solve the MCGDM problem in an interval-valued Pythagorean fuzzy (IVPF) environment. Finally, by comparing with the existing methods, it is concluded that the entropy measure of IVPF schemes and the corresponding MCGDM method can select the optimal solution of the practical problem more precisely and accurately. Therefore, the comparative analysis shows that the proposed measurement method has the characteristics of flexibility and universality.

Keywords: interval-valued Pythagorean fuzzy set; entropy measure; weighted exponential entropy measure; decision-making
Mathematics Subject Classification: 03E72, 03G25, 08A72

1. Introduction

Numerous academics have studied the fuzzy set theory in-depth and extensively since Zadeh [1] first introduced it in 1965. With focus on fuzzy sets, Bulgarian academic Atanassov [2] introduced the idea of intuitive fuzzy sets in 1986. Subsequently, his related research [3–9] produced positive findings, indicating that the intuitive fuzzy set should take into account three information measures: membership, non-membership, and hesitancy. However, the intuitive fuzzy set can only describe the case in which the sum of membership and non-membership is greater than or equal to 0 and less than or equal to $1(u^2 + v^2 \leq 1)$, which introduces significant limitations to the decision making process and narrows its scope of application. For example, it is simple to observe that $0.3 + 0.9 > 1$ does not match the constraints provided by the intuitive fuzzy set when the number of members is 0.3 and the non-membership is 0.9. To this end, Yager [10, 11] defined the Pythagorean fuzzy complement operation in 2014 to propose the Pythagorean fuzzy set with the sum of squares of membership and non-membership less than or equal to $1(0 \leq u^2 + v^2 \leq 1)$, which allows decision makers to perform decision-making without modifying the intuitive fuzzy decision value. Compared to the intuitive fuzzy set, the Pythagorean fuzzy set is more flexible to the multi-attribute decision-making problem [12, 13]. At present, Pythagorean fuzzy sets have been widely applied and far-reaching in the fields of medical diagnosis [14–20], attribute decision making, and pattern recognition.

The generalization based on the original theory is the hot topic of fuzzy mathematical theory research. With the concept of an interval-valued Pythagorean fuzzy set(IVPFS), Peng and Yang [21] expanded the range of membership and non-membership from the number between closed intervals [0,1] to sub-intervals of closed intervals [0,1]. To date, many achievements have been made in the fuzzy set. For example, Peng and Yang [21] presented the IVPFS operator and its basic properties; Wei and Lu [22] newly defined the accuracy functions for two IVPFSs and their application in the decision-making process. A multi-attribute decision-making technique based on interval-valued Pythagorean fuzzy linguistic information has been presented by Du et al. [23].

In fuzzy set theory, entropy is a useful tool for measuring fuzzy information since it can measure it effectively. The information gets increasingly unclear the higher the entropy measure. The fuzzy entropy notion was initially introduced by Zadeh [24] in 1965. Interval-valued conceptual formulation of the fuzzy entropy measure was also added to the literature [25]. Sun and Li [26] proposed the IVPFS entropy measure and similarity and applied them to the field of pattern recognition. Peng et al. [27] proposed several new definitions of the Pythagorean fuzzy entropy measure and provided the relationship between entropy, distance measure, and similarity measure. Xu [28] applied two-stage multi-criteria decision making based on interval valued Pythagorean normal fuzzy information to the location of COVID-19 fighting hospital module. Scholars such as Mishra et al. [29] used a decision support system based on interval-valued Pythagorean fuzzy entropy to evaluate blockchain-based healthcare supply chains, applying theory to practice. The distance measure and similarity measure have been discussed and studied by many scholars, and they are all important measures in solving multi-attribute decision-making problems. Therefore, the metric method for IVPFS has a very broad research prospect.

The major topic of discussion is the IVPFS weighted exponential entropy metric. This work addresses decision-making difficulties by utilizing the weighted exponential entropy measure of IVPFSs. However, decision-making approaches using the interval-valued Pythagorean fuzzy entropy measure have yielded few studies that address uncertain socioeconomic environments. Driven by the growing practical significance of fuzzy decision-making techniques, we use the membership and non-membership of IVPFSs to introduce the exponential entropy measure. Additionally, the weighted interval-valued Pythagorean fuzzy exponential entropy metric is shown in analogous mode.

The parameters of the entropy measure of the IVPFS that was employed in the study are among the fundamental definitions and ideas pertaining to Pythagorean fuzzy sets and IVPFSs that are briefly reviewed in Section 2 of this paper. In Section 3, we define the corresponding weighted exponential entropy measure and introduce and establish the exponential entropy measure of the IVPFS. First, through two instances, we present the use of the interval-valued Pythagorean fuzzy exponential entropy measure in decision-making in Section 4. Second, we present a proposal for the multi-criteria group decision-making (MCGDM) technique based on IVPFSs by using the weighted exponential entropy measurement. We then solve two case studies to examine the advantageous aspects of the suggested MCGDM approach. The advantages of the IVPFS over the Pythagorean fuzzy set in practical problem application are further demonstrated in the same section through a comparative analysis of the proposed MCGDM method with the existing entropy and the corresponding group decision-making method.

2. Preliminaries

First, we introduce the basic criteria of the IVPFS uncertainty measure and the construction process for the interval-valued fuzzy entropy measure by using the theory of fuzzy sets. Second, we assess the drawbacks of the current Pythagorean fuzzy entropy measure and its use in multi-attribute decision-making.

Definition 1. *Yager [11] defines an IVPFS P; a finite universe of discourse X is expressed as*

$$P = \{\langle x, \tilde{\mu}_p(x), \tilde{\nu}_p(x)\rangle | x \in X\}$$

where $\tilde{\mu}_p(x) = [\tilde{\mu}_p^-(x), \tilde{\mu}_p^+(x)] \subseteq [0,1]$, $\tilde{\nu}_p(x) = [\tilde{\nu}_p^-(x), \tilde{\nu}_p^+(x)] \subseteq [0,1]$.

The numbers $\tilde{\mu}_p(x), \tilde{\nu}_p(x) \subseteq [0,1]$ *indicate the degree of membership and non-membership of x to P, respectively.*

For every IVPFS in X we will suppose that $\tilde{\pi}_p(x) = [\tilde{\pi}_p^-(x), \tilde{\pi}_p^+(x)] \subsetneqq [0,1]$, *i.e., the interval valued Pythagorean fuzzy index, degree of interval-valued Pythagorean fuzzy index, degree of hesitation of x in P. For a fuzzy set P in X,* $\tilde{\pi}_p^-(x) = \sqrt{1 - (\tilde{\mu}_p^+(x))^2 - (\tilde{\nu}_p^+(x))^2}, \tilde{\pi}_p^+(x) = \sqrt{1 - (\tilde{\mu}_p^-(x))^2 - (\tilde{\nu}_p^-(x))^2}$ *when* $\tilde{\mu}_p^-(x) \geqslant 0, \tilde{\nu}_p^-(x) \geqslant 0, (\tilde{\mu}_p^+(x))^2 + (\tilde{\nu}_p^+(x))^2 \leqslant 1, \forall x \in X$.

IVPFS(X) is the total IVPFS on X.

For ease of use, $<\tilde{\mu}_p(x), \tilde{\nu}_p(x)>$ *is called the interval-valued Pythagorean fuzzy number, and it is and abbreviated as* $<\tilde{\mu}_p, \tilde{\nu}_p>$.

When an IVPFS(P) satisfies that, $\forall x \in X, \tilde{\mu}_p^-(x) = \tilde{\mu}_p^+(x), \tilde{\nu}_p^-(x) = \tilde{\nu}_p^+(x)$, *then IVPFS(P) degenerates into a PFS(P), where PFS denotes a Pythagorean fuzzy set.*

Definition 2. *For all* $P_1, P_2 \in IVPFS(X)$, *their relations and operations are defined by Peng and Yang [21] as follows:*

(1) $P_1 \leqslant P_2$ *if and only if* $\tilde{\mu}_{p_1}^-(x) \leqslant \tilde{\mu}_{p_2}^-(x), \tilde{\mu}_{p_1}^+(x) \leqslant \tilde{\mu}_{p_2}^+(x), \tilde{\nu}_{p_1}^-(x) \geqslant \tilde{\nu}_{p_2}^-(x), \tilde{\nu}_{p_1}^+(x) \geqslant \tilde{\nu}_{p_2}^+(x)$;

(2) $P_1 = P_2$ *if and only if* $\tilde{\mu}_{p_1}^-(x) = \tilde{\mu}_{p_2}^-(x), \tilde{\mu}_{p_1}^+(x) = \tilde{\mu}_{p_2}^+(x), \tilde{\nu}_{p_1}^-(x) = \tilde{\nu}_{p_2}^-(x), \tilde{\nu}_{p_1}^+(x) = \tilde{\nu}_{p_2}^+(x)$;

(3)
$P_1 \bigcup P_2 = \{\langle x, [max(\tilde{\mu}_{p_1}^-(x), \tilde{\mu}_{p_2}^-(x)), max(\tilde{\mu}_{p_1}^+(x), \tilde{\mu}_{p_2}^+(x))], [min(\tilde{\nu}_{p_1}^-(x), \tilde{\nu}_{p_2}^-(x)), min(\tilde{\nu}_{p_1}^+(x), \tilde{\nu}_{p_2}^+(x))]\rangle | x \in X\}$;

(4)
$P_1 \bigcap P_2 = \{\langle x, [min(\tilde{\mu}_{p_1}^-(x), \tilde{\mu}_{p_2}^-(x)), min(\tilde{\mu}_{p_1}^+(x), \tilde{\mu}_{p_2}^+(x))], [max(\tilde{\nu}_{p_1}^-(x), \tilde{\nu}_{p_2}^-(x)), max(\tilde{\nu}_{p_1}^+(x), \tilde{\nu}_{p_2}^+(x))]\rangle | x \in X\}$;

(5) $P^c = \{\langle x, [\tilde{\nu}_p^-(x), \tilde{\nu}_p^+(x)], [\tilde{\mu}_p^-(x), \tilde{\mu}_p^+(x)]\rangle | x \in X\}$.

Definition 3. *For any $P \in IVPFS(X)$, a real function $F(P) : IVPFS(X) \to [0, 1]$ is called an entropy measure for IVPFSs, if F(P) satisfies the conditions of the following axioms:*

(F_1) $F(P) = 0$ *if P is a crisp set, i.e.,* $P = ([1, 1], [0, 0])$ *or* $P = ([0, 0], [1, 1])$ *for each* $x_i \in X$*;*

(F_2) $F(P) = 0$ *if and only if* $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)]$*, for each* $x_i \in X$*;*

(F_3) $F(P) = F(P^c)$*;*

(F_4) $F(P_1) \leqslant F(P_2)$ *if* $P_1 \subseteq P_2$ *when* $\tilde{\mu}_{p_2}^-(x_i) \leqslant \tilde{\nu}_{p_2}^-(x_i)$ *and* $\tilde{\mu}_{p_2}^+(x_i) \leqslant \tilde{\nu}_{p_2}^+(x_i)$ *for each* $x_i \in X$*, or* $P_2 \subseteq P_1$ *when* $\tilde{\mu}_{p_2}^-(x_i) \geqslant \tilde{\nu}_{p_2}^-(x_i)$ *and* $\tilde{\mu}_{p_2}^+(x_i) \geqslant \tilde{\nu}_{p_2}^+(x_i)$ *for each* $x_i \in X$*.*

Definition 4. *Let* $p = ([\tilde{\mu}^-, \tilde{\mu}^+], [\tilde{\nu}^-, \tilde{\nu}^+]), p_1 = ([\tilde{\mu}_{p_1}^-, \tilde{\mu}_{p_1}^+], [\tilde{\nu}_{p_1}^-, \tilde{\nu}_{p_1}^+])$*, and* $p_2 = ([\tilde{\mu}_{p_2}^-, \tilde{\mu}_{p_2}^+], [\tilde{\nu}_{p_2}^-, \tilde{\nu}_{p_2}^+])$ *be three interval-valued Pythagorean fuzzy numbers (IVPFNs), and Bengio et al. [30] gave these following operations:*

(1)$p_1 \oplus p_2 = \langle [\sqrt{(\tilde{\mu}_{p_1}^-)^2 + (\tilde{\mu}_{p_2}^-)^2 - (\tilde{\mu}_{p_1}^-)^2(\tilde{\mu}_{p_2}^-)^2}, \sqrt{(\tilde{\mu}_{p_1}^+)^2 + (\tilde{\mu}_{p_2}^+)^2 - (\tilde{\mu}_{p_1}^+)^2(\tilde{\mu}_{p_2}^+)^2}], [\tilde{\nu}_{p_1}^- \tilde{\nu}_{p_2}^-, \tilde{\nu}_{p_1}^+ \tilde{\nu}_{p_2}^+] \rangle$*;*

(2)$p_1 \otimes p_2 = \langle [\tilde{\mu}_{p_1}^- \tilde{\mu}_{p_2}^-, \tilde{\mu}_{p_1}^+ \tilde{\mu}_{p_2}^+], [\sqrt{(\tilde{\nu}_{p_1}^-)^2 + (\tilde{\nu}_{p_2}^-)^2 - (\tilde{\nu}_{p_1}^-)^2(\tilde{\nu}_{p_2}^-)^2}, \sqrt{(\tilde{\nu}_{p_1}^+)^2 + (\tilde{\nu}_{p_2}^+)^2 - (\tilde{\nu}_{p_1}^+)^2(\tilde{\nu}_{p_2}^+)^2}] \rangle$*;*

(3)$\lambda p = \langle [\sqrt{1 - (1 - (\tilde{\mu}^-)^2)^\lambda}, \sqrt{1 - (1 - (\tilde{\mu}^+)^2)^\lambda}], [(\nu^-)^\lambda, (\nu^+)^\lambda] \rangle$*;*

(4)$p^\lambda = \langle [(\mu^-)^\lambda, (\mu^+)^\lambda], [\sqrt{1 - (1 - (\tilde{\nu}^-)^2)^\lambda}, \sqrt{1 - (1 - (\tilde{\nu}^+)^2)^\lambda}] \rangle$.

Definition 5. *Using an interval-valued Pythagorean fuzzy weighted average (IPFWA) operator, the aggregated value of* $p_i = ([\tilde{\mu}_{p_i}^-, \tilde{\mu}_{p_i}^+], [\tilde{\nu}_{p_i}^-, \tilde{\nu}_{p_i}^+])$ $(i = 1, 2, ..., n)$*, given a collection of IVPFNs, is defined by Garg [31] as*

$$\begin{aligned} IPFWA(p_1, p_2, ..., p_n) &= \oplus_{i=1}^n \lambda_i \alpha_i \\ &= \langle [\sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^-)^2)^{\lambda_i}}, \sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^+)^2)^{\lambda_i}}], \\ &[\prod_{i=1}^n (1 - (\nu_i^-))^{\lambda_i}, [\prod_{i=1}^n (1 - (\nu_i^+))^{\lambda_i}] \rangle. \end{aligned} \tag{2.1}$$

3. Interval-valued Pythagorean fuzzy entropy and its application to multi-criterion group decision-making

This section begins with a review of Pal and Pal's [32] exponential entropy measure for the fuzzy set P, as shown below:

$$ {}^eH(P) = \frac{1}{n(\sqrt{e} - 1)} \sum_{i=1}^n [\mu_p(x_i) e^(1 - \mu_p(x_i)) + (1 - \mu_p(x_i)) e^{\mu_p(x_i)} - 1]. \tag{3.1}$$

Assuming for the moment that P is an interval-valued intuitionistic fuzzy set (IVIFS), the procedure described by Zhang et al. [25], It is reasonable to think of the element x_i average potential membership degree to IVIFS P as follow:

$$\begin{aligned} \tilde{\mu}_p(x_i) &= \frac{1}{2}[\frac{\tilde{\mu}_p^-(x_i) + \tilde{\mu}_p^+(x_i)}{2} + 1 - \frac{\tilde{\nu}_p^-(x_i) + \tilde{\nu}_p^+(x_i)}{2}] \\ &= \frac{\tilde{\mu}_p^-(x_i) + \tilde{\mu}_p^+(x_i) + 2 - \tilde{\nu}_p^-(x_i) - \tilde{\nu}_p^+(x_i)}{4}. \end{aligned} \tag{3.2}$$

However, the above method does not consider the hesitancy of IVIFSs when converting an IVIFS into a fuzzy set. Using the above method of converting a fuzzy set to an IVIFS, an IVPFS entropy measure similar in form to the exponential entropy measure of Pal and Pal is introduced below.

3.1. Interval-valued Pythagorean fuzzy exponential entropy

3.1.1. Basic theory

Definition 6. *Let $P \in IVPFS(X), X = x_1, x_2, ..., x_n$; then, the exponential entropy of the IVPFS is defined as*

$$E_H(P) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \{(\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4}) e^{(1-\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4})} + (1 - \frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4}) e^{(\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4} - 1)}\}. \tag{3.3}$$

We demonstrate the validity and logic of the exponential entropy measure (3.3) of the suggested IVPFS in the ensuing theorem.

Theorem 1. *The $E_H(P)$ defined in (3.3) is the exponential entropy measure of an interval-valued Pythagorean fuzzy set (IVPFS).*

Proof. All we have to do is demonstrate that the IVPFS satisfies the (F_1)–(F_4) in Definition 3 in order to establish its entropy measure.

(F_1): Let P be a crisp set; then, we have either $P = ([1,1],[0,0])$ or $P = ([0,0],[1,1])$ i.e. $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [1,1]$ and $[\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)] = [0,0]$ or $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [0,0]$ and $[\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)] = [1,1]$ for each $x_i \in X$; we can get $E_H(P) = 0$.

Conversely, let

$$\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4} = \zeta_p(x_i). \tag{3.4}$$

Then,

$$E_H(P) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} [\zeta_p(x_i) e^{(1-\zeta_p(x_i))} + (1-\zeta_p(x_i)) e^{\zeta_p(x_i)} - 1] \tag{3.5}$$

is the same as that derived by Pal and Pal [32], as the exponential entropy measure (3.1) becomes zero if $\zeta_p(x_i) = 0$ or 1 for each $x_i \in X$.

$$i.e. \frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4} = 0 \tag{3.6}$$

$$or \frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4} = 1 \tag{3.7}$$

for each $x_i \in X$. Now, (3.6) and (3.7) hold if either $P = ([1,1],[0,0])$ or $P = ([0,0],[1,1])$ i.e. P is a crisp set.

(F_2): If $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)]$ for each $x_i \in X$, obviously by (3.3), we to get $E_H(P) = 1$.

In turn, assuming that $E_H(P) = 1$, the need to prove that $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)]$.

From (3.5), we get

$$E_H(P) = \frac{1}{n}\sum_{i=1}^{n} g(\zeta_p(x_i))$$

where

$$g(\zeta_p(x_i)) = \frac{\zeta_p(x_i)e^{1-\zeta_p(x_i)} + (1-\zeta_p(x_i))e^{\zeta_p(x_i)} - 1}{\sqrt{e}-1} \tag{3.8}$$

for each $x_i \in X$.

Now,

$$\begin{aligned} E_H(P) = 1 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} g(\zeta_p(x_i)) = 1 \\ \Rightarrow g(\zeta_p(x_i) = 1 \end{aligned} \tag{3.9}$$

for each $x_i \in X$.

Equation (3.8) is differentiated with respect to (w.r.t.) $\zeta_p(x_i)$ and is equal to zero, we get

$$\begin{aligned} \frac{\partial g(\zeta_p(x_i))}{\partial(\zeta_p(x_i))} &= \frac{e^{(1-\zeta_p(x_i))} - \zeta_p(x_i)e^{(1-\zeta_p(x_i))} - e^{\zeta_p(x_i)} + (1-\zeta_p(x_i))e^{\zeta_p(x_i)}}{\sqrt{e}-1} = 0 \\ &\Rightarrow \frac{\partial g(\zeta_p(x_i))}{\partial(\zeta_p(x_i))} = \frac{(1-\zeta_p(x_i))e^{(1-\zeta_p(x_i))} - \zeta_p(x_i)e^{\zeta_p(x_i))}}{\sqrt{e}-1} = 0 \\ &\Rightarrow (1-\zeta_p(x_i))e^{(1-\zeta_p(x_i))} = \zeta_p(x_i)e^{\zeta_p(x_i)} \\ &\Rightarrow 1-\zeta_p(x_i) = \zeta_p(x_i) \\ &\Rightarrow \zeta_p(x_i) = \frac{1}{2}, \end{aligned} \tag{3.10}$$

for each $x_i \in X$.

$$\begin{aligned} \frac{\partial^2 g(\zeta_p(x_i))}{\partial^2(\zeta_p(x_i))} &= \frac{-e^{(1-\zeta_p(x_i))} - (1-\zeta_p(x_i))e^{(1-\zeta_p(x_i))} - e^{\zeta_p(x_i)} + (1-\zeta_p(x_i))e^{\zeta_p(x_i)}}{\sqrt{e}-1} \\ &= \frac{(\zeta_p(x_i)-2)e^{(1-\zeta_p(x_i))} - (1+\zeta_p(x_i))e^{\zeta_p(x_i)}}{\sqrt{e}-1} < 0, \end{aligned} \tag{3.11}$$

for each $x_i \in X$.

So $g(\zeta_p(x_i))$ is maximum at $\zeta_p(x_i) = \frac{1}{2}$. From above, we see that it is a convex function. Therefore, the (3.8) can yield the maximum value at the time, i.e., $[\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)] = [\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)]$.

(F_3): Given that $F^c = \{\langle x_i, [\tilde{\nu}_p^-(x_i), \tilde{\nu}_p^+(x_i)], [\tilde{\mu}_p^-(x_i), \tilde{\mu}_p^+(x_i)]\rangle | x_i \in X\}$ and (3.3), it is obvious to get that $F(P) = F(P^c)$.

(F_4): If we set $\alpha = (\tilde{\mu}_p^-(x_i))^2 + (\tilde{\mu}_p^+(x_i))^2, \beta = (\tilde{\nu}_p^-(x_i))^2 + (\tilde{\nu}_p^+(x_i))^2$, then

$$E_H(P) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n} f(\alpha,\beta).$$

Then, we have

$$f(\alpha,\beta) = \frac{\alpha+2-\beta}{4}e^{(1-\frac{\alpha+2-\beta}{4})} + (1-\frac{\alpha+2-\beta}{4})e^{\frac{\alpha+2-\beta}{4}} - 1, \tag{3.12}$$

where $\alpha, \beta \in [0,1]$.

Subtracting α and β from (3.12) to obtain the partial derivative, we get

$$\frac{\partial f}{\partial \alpha} = \frac{1}{4}[\frac{\beta+2-\alpha}{4}e^{(1-\frac{\alpha+2-\beta}{4})} - \frac{\alpha+2-\beta}{4}e^{\frac{\alpha+2-\beta}{4}}], \tag{3.13}$$

$$\frac{\partial f}{\partial \beta} = \frac{1}{4}[\frac{\alpha+2-\beta}{4}e^{(1-\frac{\beta+2-\alpha}{4})} - \frac{\beta+2-\alpha}{4}e^{\frac{\beta+2-\alpha}{4}}]. \tag{3.14}$$

Setting $\frac{\partial f}{\partial \alpha} = 0$ and $\frac{\partial f}{\partial \beta} = 0$ to find the critical points, we get

$$\alpha = \beta. \tag{3.15}$$

According to (3.13) and (3.15), we get that $\frac{\partial f}{\partial \alpha} \geqslant 0$ when $\alpha \leqslant \beta$ and $\frac{\partial f}{\partial \alpha} \leqslant 0$ when $\alpha \geqslant \beta$ for any $\alpha, \beta \in [0,1]$. Therefore,$f(\alpha, \beta)$ is increasing w.r.t. α for $\alpha \leqslant \beta$ and decreasing when $\alpha \geqslant \beta$.

Similarly, we get that $\frac{\partial f}{\partial \beta} \leqslant 0$ when $\alpha \leqslant \beta$ and $\frac{\partial f}{\partial \beta} \geqslant 0$ when $\alpha \geqslant \beta$. Therefore,$f(\alpha, \beta)$ is decreasing w.r.t. β for $\alpha \leqslant \beta$ and increasing when $\alpha \geqslant \beta$.

Now, if $P_1 \subseteq P_2$ with $\tilde{\mu}^-_{p_2}(x_i) \leqslant \tilde{\nu}^-_{p_2}(x_i)$ and $\tilde{\mu}^+_{p_2}(x_i) \leqslant \tilde{\nu}^+_{p_2}(x_i)$ for each $x_i \in X$, then we have that $\tilde{\mu}^-_{p_1}(x_i) \leqslant \tilde{\mu}^-_{p_2}(x_i) \leqslant \tilde{\nu}^-_{p_2}(x_i) \leqslant \tilde{\nu}^-_{p_1}(x_i)$ and $\tilde{\mu}^+_{p_1}(x_i) \leqslant \tilde{\mu}^+_{p_2}(x_i) \leqslant \tilde{\nu}^+_{p_2}(x_i) \leqslant \tilde{\nu}^+_{p_1}(x_i)$. It implies that $\tilde{\mu}^-_{p_1}(x_i) \leqslant \tilde{\nu}^-_{p_1}(x_i)$ and $\tilde{\mu}^+_{p_1}(x_i) \leqslant \tilde{\nu}^+_{p_1}(x_i)$ for each $x_i \in X$. Therefore, from the monotonic nature of $f(\alpha, \beta)$ and (3.3), we obtain that $E_H(P_1) \leqslant E_H(P_2)$.

Likewise, when $P_1 \supseteq P_2$ with $\tilde{\mu}^-_{p_2}(x_i) \geqslant \tilde{\nu}^-_{p_2}(x_i)$ and $\tilde{\mu}^+_{p_2}(x_i) \geqslant \tilde{\nu}^+_{p_2}(x_i)$ for each $x_i \in X$, one can also prove that $E_H(P_1) \leqslant E_H(P_2)$. □

3.1.2. Case application

The application and consistency of the suggested exponential and weighted exponential entropy measurements of the IVPFS are shown in this section.

Example 1. *Consider the decision-making case presented by Wang et al. [33]. Assume that a corporate entity's treasurer is assessing the four possibilities $G = \{G_1, G_2, G_3, G_4\}$. The company requires the plans that are evaluated by senior financial executives to have the following characteristics: risk (a_1), development (a_2), social and political issues (a_3), and environmental impact (a_4). Assume that a treasurer frequently assesses the available options for each feature by using the IVPFN. The assumed decision matrix is given by*

$$R = \begin{bmatrix} \langle[0.42,0.48],[0.4,0.5]\rangle & \langle[0.6,0.7],[0.05,0.25]\rangle & \langle[0.4,0.5],[0.2,0.5]\rangle & \langle[0.55,0.75],[0.15,0.25]\rangle \\ \langle[0.4,0.5],[0.4,0.5]\rangle & \langle[0.5,0.8],[0.1,0.2]\rangle & \langle[0.3,0.6],[0.3,0.4]\rangle & \langle[0.6,0.7],[0.1,0.3]\rangle \\ \langle[0.3,0.5],[0.4,0.5]\rangle & \langle[0.1,0.3],[0.2,0.4]\rangle & \langle[0.7,0.8],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.1,0.2]\rangle \\ \langle[0.2,0.4],[0.4,0.5]\rangle & \langle[0.6,0.7],[0.2,0.3]\rangle & \langle[0.5,0.6],[0.2,0.3]\rangle & \langle[0.7,0.8],[0.1,0.2]\rangle \end{bmatrix}.$$

With the help of the suggested entropy measure (3.1), we can quickly compute the following:

$${}^eH(G_1) = 0.9213, {}^eH(G_2) = 0.9215, {}^eH(G_3) = 0.9001, {}^eH(G_4) = 0.8821,$$

i.e.

$${}^eH(G_4) <^e H(G_3) <^e H(G_1) <^e H(G_2),$$

which coincides with the ranking result obtained in [31]. This can illustrate the rationality of exponential fuzzy entropy.

3.2. Interval-valued Pythagorean fuzzy weighted exponential entropy

3.2.1. Basic theory

By taking the weight of each element $x_i \in X$, a weighted exponential entropy measure of an IVPFS P is proposed as follows:

$$E_H^{\omega}(P) = \frac{1}{n(\sqrt{e}-1)} \times \sum_{i=1}^{n} \omega_i \{ (\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4}) e^{(1 - \frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4})}$$
$$+ (1 - \frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4}) e^{(\frac{\tilde{\mu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 + 2 - \tilde{\nu}_p^-(x_i)^2 - \tilde{\nu}_p^+(x_i)^2}{4})} - 1 \} \tag{3.16}$$

where $\omega_i \in [0,1], i = 1, 2, ..., n$, and $\sum_{i=1}^{n} \omega_i = 1$.

If we consider that $\omega_i = \frac{1}{n}, i = 1, 2, ..., n$, then $E_H^{\omega}(P) = E_H(P)$.

It can easily be shown that the weighted exponential entropy $E_H^{\omega}(P)$ of an IVPFS also satisfies the conditions of the entropy properties (F_1)–(F_4) in Definition 3.

In this part, the axiomatic definitions of IVPFSs exponential entropy and weighted exponential entropy are introduced and proved. Then, the application of IVPFSs exponential entropy is given via an example, and its rationality is verified.

3.2.2. Case application

Example 2. *Consider the decision-making situation given in [34]. The project manager of an asset management company plans to evaluate the investment projects, and there are four investment projects $I = \{I_1, I_2, I_3, I_4\}$, namely, the automobile company (I_1), food company (I_2), computer company (I_3) and equipment company (I_4). The three factors $a = \{a_1, a_2, a_3\}$ should be considered in the evaluation, respectively indicating the potential electronic risks (a_1), development vitality (a_2), social influence and development prospects (a_3). Suppose that the decision matrix is interval-valued in the Pythagorean fuzzy environment, as follows:*

$$R = \begin{bmatrix} \langle [0.4, 0.5], [0.3, 0.4] \rangle & \langle [0.4, 0.6], [0.2, 0.4] \rangle & \langle [0.1, 0.3], [0.5, 0.6] \rangle \\ \langle [0.6, 0.7], [0.2, 0.3] \rangle & \langle [0.6, 0.7], [0.2, 0.3] \rangle & \langle [0.4, 0.7], [0.1, 0.2] \rangle \\ \langle [0.3, 0.6], [0.3, 0.4] \rangle & \langle [0.5, 0.6], [0.3, 0.4] \rangle & \langle [0.5, 0.6], [0.1, 0.3] \rangle \\ \langle [0.7, 0.8], [0.1, 0.2] \rangle & \langle [0.6, 0.7], [0.1, 0.3] \rangle & \langle [0.3, 0.4], [0.1, 0.2] \rangle \end{bmatrix}.$$

Each element in the matrix represents the evaluation value for the investment project under the corresponding attribute, represented by the Pythagorean fuzzy number. The standard weight is $\omega = (0.35, 0.25, 0.4)^T$.

By using the proposed entropy measure (3.16), we can easily compute the following:

$$^eH(I_1) = 0.3223, ^e H(I_2) = 0.2970, ^e H(I_3) = 0.3214, ^e H(I_3) = 0.2879.$$

i.e.

$$^eH(I_4) <^e H(I_2) <^e H(I_3) <^e H(I_1).$$

According to the principle of minimum uncertainty, the project I_1 should be invested in principle, which is the same conclusion as [34], so the above definition of the exponential fuzzy entropy measure (3.16) can be justified.

3.3. Group decision-making method based on interval-valued Pythagorean weighted exponential fuzzy entropy

Now the proposed interval-valued Pythagorean weighted exponential fuzzy entropy will be applied to the MCGDM problem. Before this, a MCGDM method using interval-valued Pythagorean weighted exponential fuzzy entropy is introduced, and a case study is presented to illustrate its application in practice.

This section provides a group decision-making approach with IVPFSs as the preferred alternative. When the weight part is known, the weighted exponential entropy of IVPFSs is used to solve the MCGDM problem in the interval-valued Pythagorean fuzzy environment with known experts and unknown standard weights.

Assuming that set $A = \{A_1, A_2, ..., A_n\}$ is n alternatives, k experts need to rank n schemes according to m criteria. The set of experts is denoted as $Y = \{Y_1, Y_2, ..., Y_k\}$, and $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_k\}^T$ is the weight vector $\sum\limits_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \in [0, 1]$ $(i \in N)$, corresponding to all experts. The criterion set is denoted as $Z = \{Z_1, Z_2, ..., Z_m\}$, $\omega = \{\omega_1, \omega_2, ..., \omega_k\}^T$ is the weight vector corresponding to all criteria, and the weight ω_j of the criterion Z_j satisfies $\sum\limits_{j=1}^{m} \omega_j = 1$ and $\omega_j \in [0, 1]$ $(j \in N)$.

To solve this problem effectively, we propose a group decision-making method based on IVPFS's weighted exponential entropy measure $E_H^{\omega}(P)$, as follows:

Step 1. Find the Pythagorean interval-valued fuzzy decision-making matrix (IVPFDM). $R^{(i)} = (r_{ij}^{(i)})_{n\times m}$ indicates the score that the expert gives the applicant.

Step 2. During the group decision-making process, the available scores played by the experts are collected, and according to these available scores, an IVPFDM composed of $z_{lj} = ([\mu_{lj}^-, \mu_{lj}^+], [\nu_{lj}^-, \nu_{lj}^+])$ is expressed. To this end, we use the interval-valued Pythagorean power weighted average operator

$$\begin{aligned} z_{lj} &= ([\mu_{lj}^-, \mu_{lj}^+], [\nu_{lj}^-, \nu_{lj}^+]) = IVPFWA_{\lambda}(r_{ij}^{(1)}, r_{ij}^{(2)}, ..., r_{ij}^{(k)}) = \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus ... \oplus \lambda_k r_{ij}^{(k)} \\ &= ([\sqrt{1 - \sum_{i=1}^{k}(1 - (\mu_{lj}^-)^2)^{\lambda_i}}, \sqrt{1 - \sum_{i=1}^{k}(1 - (\mu_{lj}^+)^2)^{\lambda_i}}], [\sum_{i=1}^{k}(\nu_{lj}^-)^{\lambda_i}, \sum_{i=1}^{k}(\nu_{lj}^+)^{\lambda_i}]). \end{aligned} \tag{3.17}$$

Step 3. The weight vector for calculating the criterion.

Here, in the interval-valued Pythagorean fuzzy, we use the (3.3) to determine the standard weights where the standard values are completely unknown:

$$\omega_j = \frac{1 - E_H^j}{m - \sum_{j=1}^{m} E_H^j}, j = 1, 2, ..., m. \tag{3.18}$$

Step 4. For each alternative A_i, the weighted interval-valued information measure $E_H^{\omega}(A)$ was calculated by using (3.16).

Step 5. Ordering of substitutes:

According to the value obtained in Step 4, the scheme is ranked in ascending order, and the scheme with the lowest $E_H^{\omega}(A)$ value is the best scheme.

Using the scenarios mentioned by Dugenci [35] and Park et al. [36], we want to confirm the validity of the suggested MCGDM approach for weighted exponential fuzzy entropy.

Example 3. *Choosing the best employees [35]*

Six applicants passed the initial screening and went through additional testing in order to be considered for a position as advertising experts with a service company. A review team comprising three specialists was constituted by the company to assess each of the six candidates and carry out interviews as part of the selection process. In order to make the selection, experts proposed five criteria: communication skills (C_1)*, international language fluency* (C_2) *, performance stability* (C_3)*, early experience* (C_4)*, and confidence* (C_5)*.*

Use the MCGDM method of the interval-valued Pythagorean weighted exponential fuzzy entropy measure, to select the best candidate, as follows:

Step 1. Determine the IVPFDM.

Three experts rated the six candidates based on five evaluation criteria, as shown in Tables 1–3.

Table 1. Expert 1's IVPFDM $R^{(1)}$ for the personnel selection problem.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$
A_2	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.3]\rangle$	$\langle[0.8, 0.9], [0.0, 0.1]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$
A_3	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$
A_4	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.5, 0.8], [0.1, 0.2]\rangle$	$\langle[0.6, 0.8], [0.0, 0.1]\rangle$	$\langle[0.4, 0.5], [0.3, 0.4]\rangle$
A_5	$\langle[0.4, 0.6], [0.1, 0.3]\rangle$	$\langle[0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$
A_6	$\langle[0.5, 0.8], [0.1, 0.2]\rangle$	$\langle[0.4, 0.6], [0.2, 0.3]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.4, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$

Table 2. Expert 2's IVPFDM $R^{(2)}$ for the personnel selection problem.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$
A_2	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$
A_3	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.6, 0.8], [0.0, 0.1]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$
A_4	$\langle[0.4, 0.5], [0.3, 0.4]\rangle$	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$
A_5	$\langle[0.5, 0.6], [0.3, 0.4]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.4, 0.6], [0.2, 0.3]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$
A_6	$\langle[0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$

Table 3. Expert 3's IVPFDM $R^{(3)}$ for the personnel selection problem.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.4, 0.6], [0.3, 0.4]\rangle$
A_2	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$
A_3	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.4, 0.6], [0.2, 0.3]\rangle$
A_4	$\langle[0.5, 0.6], [0.3, 0.4]\rangle$	$\langle[0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.0, 0.2]\rangle$	$\langle[0.7, 0.8], [0.0, 0.1]\rangle$	$\langle[0.4, 0.7], [0.1, 0.2]\rangle$
A_5	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.7, 0.9], [0.0, 0.1]\rangle$	$\langle[0.5, 0.7], [0.0, 0.1]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$
A_6	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$

Step 2. On using (3.17), select the score of the candidate $A_l, l = 1, 2, ..., n$ selected by different experts, expressed by z_{lj}, and construct the interval-valued Pythagorean fuzzy decision matrix of the expert weight vector, $\lambda = (0.3, 0.5, 0.2)'$. Table 4 represents the interval-valued Pythagorean fuzzy matrix of the candidate selection questions.

Table 4. Values z_{lj} in the collective IVPFDM.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle[0.5542, 0.6547], [0.2000, 0.3000]\rangle$	$\langle[0.5181, 0.6188], [0.2000, 0.3178]\rangle$	$\langle[0.4147, 0.5150], [0.2828, 0.4625]\rangle$	$\langle[0.7000, 0.8000], [0.1000, 0.2000]\rangle$	$\langle[0.4337, 0.5940], [0.1762, 0.3249]\rangle$
A_2	$\langle[0.6000, 0.7241], [0.1741, 0.2766]\rangle$	$\langle[0.7000, 0.8000], [0.1000, 0.2000]\rangle$	$\langle[0.5229, 0.7000], [0.1149, 0.2449]\rangle$	$\langle[0.7353, 0.8855], [0.0000, 0.1149]\rangle$	$\langle[0.5940, 0.6967], [0.1625, 0.2656]\rangle$
A_3	$\langle[0.5627, 0.6664], [0.1625, 0.2814]\rangle$	$\langle[0.5988, 0.7564], [0.1231, 0.2259]\rangle$	$\langle[0.5813, 0.7287], [0.0000, 0.2408]\rangle$	$\langle[0.6177, 0.7427], [0.1149, 0.2169]\rangle$	$\langle[0.5181, 0.7206], [0.1149, 0.2169]\rangle$
A_4	$\langle[0.4542, 0.5940], [0.2158, 0.3249]\rangle$	$\langle[0.6405, 0.8299], [0.0000, 0.1732]\rangle$	$\langle[0.5229, 0.7353], [0.0000, 0.2000]\rangle$	$\langle[0.6232, 0.7560], [0.0000, 0.1414]\rangle$	$\langle[0.4542, 0.5988], [0.1966, 0.3016]\rangle$
A_5	$\langle[0.4734, 0.6232], [0.1732, 0.3194]\rangle$	$\langle[0.6331, 0.7748], [0.1231, 0.2259]\rangle$	$\langle[0.5471, 0.7287], [0.0000, 0.2132]\rangle$	$\langle[0.6188, 0.7836], [0.0000, 0.1741]\rangle$	$\langle[0.5229, 0.7000], [0.1000, 0.2000]\rangle$
A_6	$\langle[0.5229, 0.7560], [0.1414, 0.2449]\rangle$	$\langle[0.4734, 0.6232], [0.1741, 0.2766]\rangle$	$\langle[0.4170, 0.5542], [0.2828, 0.3873]\rangle$	$\langle[0.4734, 0.6967], [0.1625, 0.2656]\rangle$	$\langle[0.5542, 0.6829], [0.1625, 0.2656]\rangle$

Step 3. On using (3.18), the following different standard weight vectors are obtained $\omega = (0.1355, 0.2544, 0.1386, 0.3350, 0.1365)'$.

Step 4. On using (3.16), and the above standard weight vectors ω, we calculate $E_H^{\omega}(A_l)$ for each of the alternatives A_l. The obtained values are given in Table 5.

Step 5. Rank the applicants.

Based on the $E_H^{\omega}(A_l)$ values and ranking presented in Table 5, we conclude that A_2 is the best candidate, consistent with the results in [34], which justified the MCGDM method of the interval-valued Pythagorean weighted exponential fuzzy entropy measure.

Table 5. WIVIM values computed for the personnel selection issue.

	$E_H^{\omega}(A_l)$	Rank
A_1	0.1744	5
A_2	0.1469	1
A_3	0.1683	4
A_4	0.1653	3
A_5	0.1654	2
A_6	0.1837	6

Example 4. *The best air conditioner to choose [36]*

In this case, the group decision-making issue involves a city development committee trying to establish a municipal library. The library should select the most suitable air conditioning equipment based on its specific needs. The air conditioning supplier offers four recommendations $B_l, (l = 1, 2, 3, 4)$ among the following five qualities: performance (q_1), maintainability (q_2), flexibility

(q_3), cost (q_4), and safety (q_5). Assuming that $q = \{q_1, q_2, q_3, q_4, q_5\}$ is a set of quality criteria, four existing committee experts $e = \{e_1.e_2, e_3, e_4\}$ are invited to evaluate the quality of the air conditioning, corresponding to a weight vector of $\lambda = (0.3, 0.2, 0.3, 0.2)^T$. About the quality standard $q = \{q_1, q_2, q_3, q_4, q_5\}$, the decision matrix $R^{(i)} = (r_{ij}^{(i)})_{n\times m}$ in the Pythagorean fuzzy environment. Each element in the matrix indicates the evaluation value for the air conditioner under the corresponding criterion, which is expressed by the Pythagorean fuzzy number.

Step 1. Determine the IVPFDM.
$R^{(i)} = (r_{ij}^{(i)})_{n\times m}$ shows the experts' selection scores, as shown in Tables 6–9.

Table 6. IVPFDM $R^{(1)}$ by expert 1 for the best air-conditioner selection problem.

	B_1	B_2	B_3	B_4
q_1	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$	$\langle[0.4, 0.5], [0.3, 0.5]\rangle$	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$
q_2	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.1, 0.3], [0.2, 0.4]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.7, 0.8]\rangle$
q_3	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.3, 0.4], [0.4, 0.5]\rangle$	$\langle[0.5, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.5, 0.8]\rangle$
q_4	$\langle[0.5, 0.7], [0.1, 0.2]\rangle$	$\langle[0.2, 0.4], [0.5, 0.6]\rangle$	$\langle[0.4, 0.6], [0.2, 0.3]\rangle$	$\langle[0.2, 0.3], [0.4, 0.6]\rangle$
q_5	$\langle[0.1, 0.4], [0.3, 0.5]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.2, 0.3], [0.5, 0.6]\rangle$

Table 7. IVPFDM $R^{(2)}$ by expert 2 for the best air-conditioner selection problem.

	B_1	B_2	B_3	B_4
q_1	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.4, 0.6], [0.3, 0.4]\rangle$	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$
q_2	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$	$\langle[0.1, 0.3], [0.3, 0.7]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.6, 0.8]\rangle$
q_3	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$	$\langle[0.3, 0.4], [0.4, 0.5]\rangle$	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.7, 0.8]\rangle$
q_4	$\langle[0.5, 0.6], [0.1, 0.3]\rangle$	$\langle[0.2, 0.3], [0.6, 0.7]\rangle$	$\langle[0.4, 0.6], [0.3, 0.4]\rangle$	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$
q_5	$\langle[0.1, 0.3], [0.3, 0.5]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.2, 0.4]\rangle$	$\langle[0.2, 0.4], [0.5, 0.6]\rangle$

Table 8. IVPFDM $R^{(3)}$ by expert 3 for the best air-conditioner selection problem.

	B_1	B_2	B_3	B_4
q_1	$\langle[0.4, 0.7], [0.1, 0.2]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4]\rangle$	$\langle[0.2, 0.4], [0.3, 0.4]\rangle$	$\langle[0.3, 0.4], [0.2, 0.4]\rangle$
q_2	$\langle[0.3, 0.5], [0.3, 0.4]\rangle$	$\langle[0.2, 0.4], [0.4, 0.5]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.6, 0.8]\rangle$
q_3	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$	$\langle[0.4, 0.5], [0.3, 0.4]\rangle$	$\langle[0.5, 0.7], [0.1, 0.3]\rangle$	$\langle[0.1, 0.3], [0.5, 0.7]\rangle$
q_4	$\langle[0.5, 0.6], [0.1, 0.3]\rangle$	$\langle[0.1, 0.2], [0.7, 0.8]\rangle$	$\langle[0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.2, 0.3], [0.5, 0.7]\rangle$
q_5	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.6, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.6, 0.8]\rangle$

Table 9. IVPFDM $R^{(4)}$ by expert 4 for the best air-conditioner selection problem.

	B_1	B_2	B_3	B_4
q_1	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.4, 0.5], [0.4, 0.5]\rangle$	$\langle[0.4, 0.5], [0.3, 0.4]\rangle$	$\langle[0.3, 0.4], [0.4, 0.5]\rangle$
q_2	$\langle[0.3, 0.4], [0.3, 0.4]\rangle$	$\langle[0.1, 0.2], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.3]\rangle$	$\langle[0.1, 0.3], [0.6, 0.7]\rangle$
q_3	$\langle[0.7, 0.8], [0.1, 0.2]\rangle$	$\langle[0.3, 0.4], [0.5, 0.6]\rangle$	$\langle[0.5, 0.8], [0.1, 0.2]\rangle$	$\langle[0.1, 0.2], [0.5, 0.8]\rangle$
q_4	$\langle[0.5, 0.6], [0.1, 0.3]\rangle$	$\langle[0.2, 0.3], [0.4, 0.6]\rangle$	$\langle[0.4, 0.5], [0.2, 0.3]\rangle$	$\langle[0.2, 0.3], [0.4, 0.5]\rangle$
q_5	$\langle[0.1, 0.2], [0.5, 0.7]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2]\rangle$	$\langle[0.5, 0.6], [0.3, 0.4]\rangle$	$\langle[0.3, 0.4], [0.5, 0.6]\rangle$

Step 2. Table 10 shows a collective IVPFDM with values indicated by z_{lj} based on the expert weight vector $\lambda = (0.3, 0.2, 0.3, 0.2)^T$ and the scores of the choices $B_l, (l = 1, 2, 3, 4)$ made by various experts for personnel selection by using (3.17).

Table 10. Collective IVPFDM with the values z_{lj}.

	B_1	B_2	B_3	B_4
q_1	$\langle[0.4807, 0.6405], [0.1625, 0.2814]\rangle$	$\langle[0.3545, 0.4734], [0.3249, 0.4939]\rangle$	$\langle[0.3445, 0.4983], [0.3000, 0.4277]\rangle$	$\langle[0.3000, 0.4337], [0.3249, 0.4850]\rangle$
q_2	$\langle[0.3000, 0.4639], [0.3464, 0.4638]\rangle$	$\langle[0.1382, 0.3194], [0.2670, 0.4516]\rangle$	$\langle[0.6341, 0.7836], [0.1000, 0.2169]\rangle$	$\langle[0.1000, 0.2241], [0.6284, 0.7789]\rangle$
q_3	$\langle[0.6232, 0.7241], [0.1231, 0.2259]\rangle$	$\langle[0.3341, 0.4337], [0.3837, 0.4850]\rangle$	$\langle[0.5529, 0.7748], [0.1000, 0.2259]\rangle$	$\langle[0.1000, 0.2351], [0.5348, 0.7686]\rangle$
q_4	$\langle[0.5000, 0.6341], [0.1000, 0.2656]\rangle$	$\langle[0.1763, 0.3118], [0.5486, 0.6746]\rangle$	$\langle[0.4337, 0.6118], [0.2169, 0.3178]\rangle$	$\langle[0.2241, 0.3232], [0.4277, 0.6059]\rangle$
q_5	$\langle[0.1863, 0.3907], [0.3622, 0.5348]\rangle$	$\langle[0.6341, 0.7560], [0.1231, 0.2259]\rangle$	$\langle[0.5337, 0.6793], [0.1762, 0.2980]\rangle$	$\langle[0.2035, 0.3231], [0.5281, 0.6541]\rangle$

Step 3. Equation (3.18) was used to produce the following criterion weight vector $\omega = (0.0437, 0.2672, 0.3059, 0.1523, 0.2308)^T$.

Step 4. Now, using (3.17) and the weights of the ω criteria, we compute $E_H^{\omega}(B_l)$ for each alternative $B_l, (l = 1, 2, 3, 4)$. Table 11 displays the calculated values.

Step 5. Sort the calculations.

Based on the calculation results and ranking results in Table 11, we find that B_3 is the best choice, which is the same as the results obtained by Park et al. [36], which can again justify the MCGDM method of the interval-valued Pythagorean weighted exponential fuzzy entropy measure.

Table 11. WIVIM values for the best air conditioner selection problem.

	$E_H^{\omega}(B_l)$	Rank
B_1	0.1862	4
B_2	0.1847	3
B_3	0.1725	1
B_4	0.1743	2

4. Comparative analysis

This subsection presents a comparison between the Sun and Li [27] existing approach and the proposed MCGDM method.

Entropy measure is an important tool to explain ambiguity. Pythagorean fuzzy sets only considers a single membership value and non-membership value. In order to compare the proposed fuzzy set entropy with the existing fuzzy entropy measure proposed by Sun and Li [27], this section will describe the use of the personnel selection problem in Example 3, according to the four classical conversion methods, i.e., (1): the minimum of the interval; (2): the maximum of the interval; (3): the average and the geometric mean of the maximum value; (4): Maximum value geometric mean method. The

IVPFDM R into four Pythagorean fuzzy decision-making matrices $R_1.R_2, R_3, R_4$, and the converted results are as follows:

$$R_1 = \begin{bmatrix} (0.5528, 0.2) & (0.5150, 0.2) & (0.4024, 0.2828) & (0.7, 0.1) & (0, 4319, 0.1762) \\ (0.6, 0.1741) & (0.7, 0.1) & (0.5218, 0.1149) & (0.7344, 0.0) & (0.5898, 0.1625) \\ (0.5551, 0.1625) & (0.5962, 0.0) & (0.5778, 0.0) & (0.6127, 0.1149) & (0.5150, 0.1149) \\ (0.4523, 0.2158) & (0.6378, 0.0) & (0.5218, 0.0) & (0.6224, 0.0) & (0.4523, 0.1966) \\ (0.4719, 0.1732) & (0.6296, 0.1231) & (0.5375, 0.0) & (0.6163, 0.0) & (0.5218, 0.1) \\ (0.5218, 0.1414) & (0.4719, 0.1741) & (0.4084, 0.2828) & (0.4719, 0.1625) & (0.5528, 0.1625) \end{bmatrix},$$

$$R_2 = \begin{bmatrix} (0.6536, 0.3) & (0.6163, 0.3178) & (0.5505, 0.4625) & (0.8, 0.2) & (0.5898, 0.3249) \\ (0.7234, 0.2766) & (0.8, 0.2) & (0.7, 0.2449) & (0.8851, 0.1149) & (0.6933, 0.2656) \\ (0.6603, 0.2814) & (0.7538, 0.1597) & (0.7219, 0.2408) & (0.7405, 0.2169) & (0.7186, 0.2169) \\ (0.5898, 0.3249) & (0.8268, 0.1732) & (0.7344, 0.2) & (0.7551, 0.1414) & (0.5962, 0.3016) \\ (0.6224, 0.3194) & (0.7741, 0.2259) & (0.7219, 0.2132) & (0.7831, 0.1741) & (0.7, 0.2) \\ (0.7551, 0.2449) & (0.6224, 0.2766) & (0.5528, 0.3873) & (0.6933, 0.2656) & (0.6822, 0.2656) \end{bmatrix},$$

$$R_3 = \begin{bmatrix} (0.6032, 0.25) & (0.56565, 0.2589) & (0.47645, 0.37265) & (0.75, 0.15) & (0.51085, 0.25055) \\ (0.6077, 0.22195) & (0.675, 0.07985) & (0.64985, 0.1204) & (0.6766, 0.1659) & (0.6168, 0.1659) \\ (0.52105, 0.27035) & (0.7323, 0.0866) & (0.6281, 0.1) & (0.68875, 0.0707) & (0.52425, 0.2491) \\ (0.54715, 0.2463) & (0.70185, 0.1745) & (0.6297, 0.1066) & (0.6997, 0.08705) & (0.6109, 0.15) \\ (0.5, 0.19315) & (0.54715, 0.22535) & (0.4806, 0.33505) & (0.5826, 0.21405) & (0.6175, 0.21405) \end{bmatrix},$$

$$R_4 = \begin{bmatrix} (0.6, 0.2449) & (0.5634, 0.2521) & (0.451, 0.3617) & (0.7483, 0.1414) & (0.5047, 0.2393) \\ (0.6588, 0.2194) & (0.7483, 0.1414) & (0.6044, 0.1677) & (0.8062, 0.339) & (0.6395, 0.2077) \\ (0.6054, 0.2138) & (0.6704, 0.4434) & (0.6458, 0.4907) & (0.6736, 0.1578) & (0.6083, 0.1578) \\ (0.5165, 0.2648) & (0.7262, 0.0) & (0.6229, 0.0) & (0.6855, 0.0) & (0.5193, 0.2448) \\ (0.542, 0.2352) & (0.6981, 0.1668) & (0.6229, 0.0) & (0.6947, 0.0) & (0.6044, 0.1414) \\ (0.6277, 0.186) & (0.5419, 0.2195) & (0.4751, 0.3309) & (0.572, 0.2077) & (0.6141, 0.2077) \end{bmatrix}.$$

Moreover, we use the entropy measures provided by Sun and Li [27] and Peng and Yang [21] to compute the performance of the created MCGDM technique as follows:

Sun and Li [27] interval-valued Pythagorean fuzzy entropy measure :

$$E_{Sun}(P) = \frac{1}{n} \sum_{i=1}^{n} \frac{\tilde{\mu}_p^-(x_i)^2 \bigwedge \tilde{\nu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 \bigwedge \tilde{\nu}_p^+(x_i)^2}{\tilde{\mu}_p^-(x_i)^2 \bigvee \tilde{\nu}_p^-(x_i)^2 + \tilde{\mu}_p^+(x_i)^2 \bigvee \tilde{\nu}_p^+(x_i)^2},$$

Peng et al. [21] Pythagorean fuzzy entropy measure :

$$E_{Peng}(P) = \frac{1}{|X|} \sum_{x_j \in X} \frac{(\mu_p(x_j))^2 \bigwedge (\nu_p(x_j))^2}{(\mu_p(x_j))^2 \bigvee (\nu_p(x_j))^2}.$$

In order to illustrate the advantages of the MCGDM method of interval Pythagorean weighted exponential fuzzy entropy $E_{Sun}(P)$, chose to use the interval-valued Pythagorean fuzzy entropy scheme proposed by Sun and Li [27] and the Pythagorean fuzzy entropy measure $E_{Peng}(P)$ proposed by Peng et al. [21]. The entropy of $A_i, (i = 1, 2, 3, 4, 5, 6)$ in the four Pythagorean fuzzy decision-making matrices R_1, R_2, R_3, R_4 were calculated separately, and the results are shown in Table 12.

Table 12. Results of the comparative study.

Method	Ranking order of alternatives	Best alternative
$E_{Sun}(P)$	$R: A_2 < A_3 < A_5 < A_4 < A_6 < A_1$	A_2
	$R_1: A_3 < A_5 < A_2 < A_4 < A_6 < A_1$	A_3
$E_{Peng}(P)$	$R_2: A_2 < A_3 < A_5 < A_4 < A_6 < A_1$	A_2
	$R_3: A_3 < A_2 < A_5 < A_4 < A_6 < A_1$	A_3
	$R_4: A_3 < A_5 < A_2 < A_4 < A_6 < A_1$	A_3
$E_H(P)$	$R: A_2 < A_3 < A_5 < A_4 < A_6 < A_1$	A_2

It can be ascertained from Table 12 that the sorting results for the Pythagorean fuzzy decision-making matrix R_2 and the method proposed by Sun and Li show that the proposed method comprehensively considered most of the decision-making information, whereas the decision matrices R_1, R_3, R_4 ignored the information in the conversion process, so the Pythagorean fuzzy entropy measure can be applied to select the optimal solution for many practical problems. It shows that the IVPFS is more flexible and extensive than the traditional Pythagorean fuzzy set. It can also be easily seen that the ranking scheme obtained by the proposed MCGDM method is similar to the sorting order of the existing sorting methods, and that the best selection scheme is consistent with the results calculated via the proposed decision-making method and entropy.

5. Concluding remarks

(1) the interval-valued Pythagorean fuzzy entropy measure has the following benefits: The problem can be solved by using the interval-valued Pythagorean fuzzy entropy measure; (2) IVPFSs can represent different types of multi-attribute decision-making information, with good flexibility and practicability; (3) the interval-valued Pythagorean fuzzy entropy measure is an important measurement tool to solve multi-attribute decision-making problems.

As a generalization of Pythagorean fuzzy sets, IVPFSs are frequently utilized in cluster analysis, pattern recognition, and multi-attribute decision making. The idea of an interval-valued intuitionistic fuzzy entropy definition has been used in this research to construct an interval-valued Pythagorean fuzzy entropy measure following the establishment of the axiomatic criterion of this type of measure. Based on the example of an interval-valued intuitionistic fuzzy entropy measure, the proposed entropy has been used to solve the MCGDM problem in the interval-valued Pythagorean fuzzy environment. Lastly, the suggested method's reasonable stability and applicability to the solution selection problem have been demonstrated through a comparative study of the results in the existing literature, and this has been supplemented with the example results to support the conclusions drawn from the examples. As a result, the approach may effectively handle real-world decision-making problems and act as a substitute for the current approaches. We plan to extend the application of this strategy to more complex multi-objective optimization problems, pattern recognition, clustering, and economic decision-making problems.

Use of AI tools declaration

The author declares that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interests.

References

1. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96.
3. K. T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Heidelberg: Physica-Verlag, 1999.
4. Z. S. Xu, Intuitionistic fuzzy aggregation operators, *IEEE T. Fuzzy Syst.*, **15** (2007), 1179–1187. https://doi.org/10.1109/TFUZZ.2006.890678
5. Z. S. Xu, *Intuitionistic Fuzzy Information Aggregation*, Beijing: Science Press, 2008.
6. K. T. Atanassov, *On Intuitionistic Fuzzy Sets Theory*, Berlin: Springer, 2012.
7. Y. J. Lei, *Intuitionistic Fuzzy Set Theory and Its Application*, Beijing: Science Press, 2014.
8. X. N. Li, H. J. Yi, Intuitionistic fuzzy matroids, *J. Intell. Fuzzy Syst.*, **33** (2017), 3653–3663.
9. E. B. Jamkhaneh, H. Garg, Some new operations over the generalized intuitionistic fuzzy sets and their application to decision-making process, *Granul. Comput.*, **3** (2018), 111–122. https://doi.org/10.1007/s41066-017-0059-0
10. R. R. Yager, Pythagorean fuzzy subsets, In: *IEEE Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA /NAFIPS)*, 2013, 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375
11. R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE T. Fuzzy Syst.*, **22** (2014), 958–965. https://doi.org/10.1109/TFUZZ.2013.2278989
12. W. F. Liu, X. He, Pythagorean hesitant fuzzy set, in Chinese, *Fuzzy Syst. Math.*, **30** (2016), 108–113.
13. X. D. Peng, Y. Yang, J. P. Song, Pythagorean fuzzy soft set and its application, *Comput. Eng.*, **41** (2015), 225–229.
14. X. L. Zhang, Z. S. Xu, Extension of TOPSIS to multi-criteria decision making with Pythagorean fuzzy sets, *Int. J. Intell. Syst.*, **29** (2014), 1061–1078. https://doi.org/10.1002/int.21676
15. X. He, Y. X. Du, W. F. Liu, Pythagorean fuzzy power average operators, in Chinese, *Fuzzy Syst. Math.*, **30** (2016), 117–126.
16. X. L. Zhang, A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision, *Int. J. Intell. Syst.*, **31** (2016), 593–611. https://doi.org/10.1002/int.21796
17. G. W. Wei, Pythagorean fuzzy interaction aggregation operators and their applications to multiple attribute decision making, *J. Intell. Fuzzy Syst.*, **33** (2017), 2119–2132.

18. W. F. Liu, Y. X. Du, J. Chang, Pythagorean fuzzy interaction aggregation operators and applications in decision making, *Control Decis.*, **32** (2017), 1034–1040.

19. G. W. Wei, M. Lu, Pythagorean fuzzy power aggregation operators in multiple attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 169–186. https://doi.org/10.1002/int.21946

20. H. Ding, Y. L. Li, Multiple attribute group decision making method based on Pythagorean fuzzy power weighted average operator, *Comput. Eng. Appl.*, **54** (2018), 1–6.

21. X. D. Peng, Y. Yang, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operator, *Int. J. Intell. Syst.*, **31** (2016), 444–487. https://doi.org/10.1002/int.21790

22. G. W. Wei, M. Lu, Pythagorean fuzzy maclaurin symmetric mean operators in multiple attribute decision making, *Int. J. Intell. Syst.*, **33** (2016), 1043–1070. https://doi.org/10.1002/int.21911

23. Y. Q. Du, F. J. Hou, R. W. Zafar, Q. Yu, Y. Zhai, A novel method for multi-attribute decision making with interval-valued Pythagorean fuzzy linguistic information, *Int. J. Intell. Syst.*, **32** (2017), 1085–1112. https://doi.org/10.1002/int.21881

24. A. Zadeh, Fuzzy sets and systems, In: *Proceedings of the Symposium on Systems Theory. New York Polytechnic Institute of Brooklyn*, 1965, 29–37.

25. Q. S. Zhang, S. Y. Jiang , B. G. Jia, S. H. Luo, Some information measures for interval-valued intuitionistic fuzzy sets, *Int. J. Inform. Sci.*, **180** (2010), 5130–5145. https://doi.org/10.1016/j.ins.2010.08.038

26. Q. Q. Sun, X. N. Li, Information measures of interval valued Pythagorean fuzzy sets and their applications, in Chinese, *J. Shandong Uni. (Natural Science)*, **54** (2019), 43–53.

27. X. D. Peng, H. Y. Yuan, Y. Yang, Pythagorean fuzzy information measures and their applications, *Int. J. Intell. Syst.*, **32** (2017), 991–1029. https://doi.org/10.1002/int.21880

28. Y. Xu, A two-stage multi-criteria decision-making method with interval-valued q-Rung Orthopair fuzzy technology for selecting bike-sharing recycling supplier, *Eng. Appl. Artific. Intell.*, **119** (2023), 105827. https://doi.org/10.1016/j.engappai.2023.105827

29. A. R. Mishra, P. Rani, A. F. Alrasheedi, R. Dwivedi, Evaluating the blockchain-based healthcare supply chain using interval-valued Pythagorean fuzzy entropy-based decision support system, *Eng. Appl. Artific. Intell.*, **126** (2023), 107112. https://doi.org/10.1016/j.engappai.2023.107112

30. Y. Bengio, A. Courville, P. Vincent, Representation learning: a review and new perspectives, *IEEE T. Pattern Anal. Mach. Intell.*, **35** (2013), 1798–1828. https://doi.org/10.1109/TPAMI.2013.50

31. H. Garg, A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision making problem, *Int. J. Intell. Syst.*, **31** (2016), 529–540.

32. N. R. Pal, S. K. Pal, Entropy, a new definition and its applications, *IEEE T. Syst. Man Cybernet*, **21** (1991), 1260–1270. https://doi.org/10.1109/21.120079

33. Z. Wang, K. W. Li, W. Wang, An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights, *Inform. Sci.*, **179** (2016), 3026–3040. https://doi.org/10.1016/j.ins.2009.05.001

34. J. Ye, Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives, *Expert Syst. Appl.*, **38** (2011), 6179–6183. https://doi.org/10.1016/j.eswa.2010.11.052

35. M. Dugenci, A new distance measure for interval valued intuitionistic fuzzy sets and its application to group decision making problems with incomplete weights information, *Appl. Soft Comput.*, **41** (2016), 120–134. https://doi.org/10.1016/j.asoc.2015.12.026

36. J. H. Park, Y. Park, Y. Kwun, X. Tan, Extension of the TOPSIS method for decision making problems under interval-valued intuitionistic fuzzy environment, *Appl. Math. Model.*, **35** (2011), 2544–2556. https://doi.org/10.1016/j.apm.2010.11.025