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Research article

Uncertainty principle for vector-valued functions

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Abstract: The uncertainty principle for vector-valued functions of $L^2(\mathbb{R}^n, \mathbb{R}^m)$ with $n \geq 2$ are studied. We provide a stronger uncertainty principle than the existing one in literature when $m \geq 2$. The phase and the amplitude derivatives in the sense of the Fourier transform are considered when $m = 1$. Based on these definitions, a generalized uncertainty principle is given.

Keywords: uncertainty principle; vector-valued functions; Fourier derivative
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1. Introduction

The uncertainty principle is one of the most famous theories of quantum mechanics. Gabor's work [10] is considered to be the foundation for the study of uncertainty principle in the area of signal analysis. Since then, rich forms of the uncertainty principle have appeared in mathematical forms. In [1–4, 6, 7, 16, 17], the uncertainty principle for signal functions defined on the real line, on the circle, on the Euclidean space, and on the sphere, etc., were intensively studied. In [4], the so called phase and amplitude derivatives of signal function are defined in the sense of the Fourier transform, so it provides us a method for studying the uncertainty principle based on the Fourier transform of signal functions. In [5, 13], the theory of uncertainty principle is generalized from the complex domain to the hypercomplex domain using quaternion algebras, associated with the quaternion Fourier transform. The algorithm in [5] provides us a method to estimate the probability by the data and predict whether the missing signals can be recovered. In [8, 15], the uncertainty principle for doubly periodic signal functions was studied. A doubly periodic signal function is regarded as a function of $L^2(\mathbb{R}^2)$. The uncertainty principle for signal functions belonging to $L^2(\mathbb{R}^n)$ was studied in [11], and the result is the following theorem:

Theorem 1.1. *[11] Let $h \in L^2(\mathbb{R}^n)$ and let $\hat{h}$ be the Fourier transformation of h. Then for any $x_0, \xi_0 \in \mathbb{R}^n$*

$$(\int_{\mathbb{R}^n} |\xi - \xi_0|^2 |\hat{h}(\xi)|^2 d\xi)(\int_{\mathbb{R}^n} |x - x_0|^2 |h(x)|^2 dx) \geq \frac{n^2}{16\pi^2} ||h||_2^4. \tag{1.1}$$

The equality holds if and only if $h(x) = ce^{2\pi i x\cdot\xi_0} e^{-\alpha|x-x_0|^2/2}$, where $\alpha > 0$ and $c \in \mathbb{C}$.

Here, $\hat{h}$, the Fourier transform of h, is given by

$$\hat{h}(\omega) := \int_{\mathbb{R}^n} h(t) e^{-2\pi i \omega\cdot t} dt. \tag{1.2}$$

Let $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$, and let $f_j \in L^2(\mathbb{R}^n)$, $j = 1, \ldots, m$. Denote by $f = (f_1, \cdots, f_m)$ a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Thus, $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$, and the norm of f can be written as

$$||f||^2 = \int_{\mathbb{R}^n} |f(t)|^2 dt = \sum_{j=1}^{m} \int_{\mathbb{R}^n} |f_j(t)|^2 dt.$$

Since each entry f_j of f is in $L^2(\mathbb{R}^n)$, the Fourier transform of f_j is well defined. As a consequence, the Fourier transform of f is denoted by

$$\hat{f} := (\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_m),$$

where $\hat{f}_j$ are given by (1.2). In the following definition, we define several concepts which are frequently used in the rest of this paper.

Definition 1.2. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$. The mean of time t and of Fourier frequency ω is defined by*

$$\langle t \rangle := (\sum_{j=1}^{m} \int_{\mathbb{R}^n} t_1 |f_j(t)|^2 dt, \sum_{j=1}^{m} \int_{\mathbb{R}^n} t_2 |f_j(t)|^2 dt, \cdots, \sum_{j=1}^{m} \int_{\mathbb{R}^n} t_n |f_j(t)|^2 dt),$$

and by

$$\langle \omega \rangle := (\sum_{j=1}^{m} \int_{\mathbb{R}^n} \omega_1 |\hat{f}_j(\omega)|^2 d\omega, \sum_{j=1}^{m} \int_{\mathbb{R}^n} \omega_2 |\hat{f}_j(\omega)|^2 d\omega, \cdots, \sum_{j=1}^{m} \int_{\mathbb{R}^n} \omega_n |\hat{f}_j(\omega)|^2 d\omega).$$

Let $\langle t_k \rangle = \sum_{j=1}^{m} \int_{\mathbb{R}^n} t_k |f_j(t)|^2 dt$ and let $\langle \omega_k \rangle = \sum_{j=1}^{m} \int_{\mathbb{R}^n} \omega_k |\hat{f}_j(\omega)|^2 d\omega$ for $k = 1, 2, \cdots, n$. The variance of t and of ω is defined by

$$\sigma_t^2 := \sum_{k=1}^{n} \sum_{j=1}^{m} \int_{\mathbb{R}^n} |t_k - \langle t_k \rangle|^2 |f_j(t)|^2 dt,$$

and by

$$\sigma_\omega^2 := \sum_{k=1}^{n} \sum_{j=1}^{m} \int_{\mathbb{R}^n} |\omega_k - \langle \omega_k \rangle|^2 |\hat{f}_j(\omega)|^2 d\omega.$$

In Theorem 1.1, inequality (1.1) is the uncertainty principle for $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ with the special case when $n \geq 2$ and $m = 1$. In our notations, it can be represented as

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{n^2}{16\pi^2} \|f\|_2^4. \tag{1.3}$$

When $m = 1$, the definitions of the means and the variances about time and about frequency of the signal function $f(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R}^n)$ can be found in [14]. Meanwhile, the following theorem proposes a stronger form of the uncertainty principle for $f \in L^2(\mathbb{R}^n)$ when $n \geq 2$.

Theorem 1.3. *[14] Let $f(t) = \rho(t)e^{i\varphi(t)} \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$. Suppose that the gradients $\nabla\rho$, $\nabla\varphi$, and ∇f all exist and that $\frac{\partial f}{\partial t_k}$, $t_k f \in L^2(\mathbb{R}^n)$ for $k = 1, 2, \cdots, n$. Then*

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2}\Big[\sum_{k=1}^{n} \int_{\mathbb{R}^n} |(t_k - \langle t_k \rangle)(\frac{\partial \varphi(t)}{\partial t_k} - 2\pi\langle\omega_k\rangle)|\rho^2(t)dt\Big]^2. \tag{1.4}$$

If $\frac{\partial \varphi}{\partial t_k}$ are continuous and $\rho(t) \neq 0$ almost everywhere, then the equality of (1.4) holds if and only if f is in one of the following 2^n forms

$$f(t) = d_1 e^{-\lambda_1 |t - \langle t \rangle|^2/2} e^{\frac{i}{2}\lambda_2 \sum_{k=1}^{n} (-1)^{\ell_k} (t_k - \langle t_k \rangle)^2 + c + 2\pi i t \cdot \langle \omega \rangle}, \quad k = 1, 2, \cdots, n,$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\ell_k \in \mathbb{N}_+$, and d_1, λ_1 satisfy the equation $d_1^{\frac{2}{n}} \sqrt{\frac{\pi}{\lambda_1}} = 1$.

In this paper, we propose a form of uncertainty principle for $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ with $n, m \geq 2$:

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2}\Big[\sum_{k=1}^{n} \sum_{j=1}^{m} \int_{\mathbb{R}^n} |(t_k - \langle t_k \rangle)(\frac{\partial \varphi_j}{\partial t_k} - 2\pi\langle\omega_k\rangle)|\rho_j^2(t)dt\Big]^2. \tag{1.5}$$

It is straightforward to verify that (1.5) reduces to (1.4) if $m = 1$. Hence, (1.5) can be regarded as an appropriate generalization of (1.4) into the case of vector-valued functions.

In Theorem 1.3, it is required that $\nabla\rho$, $\nabla\varphi$, and ∇f all exist. However, general signal functions do not have such good properties. We establish Fourier partial derivatives of f and prove a form of the uncertainty principle, which is

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2}\Big[\sum_{k=1}^{n} \int_{\mathbb{R}^n} |(t_k - \langle t_k \rangle)(D_k\varphi(t) - 2\pi\langle\omega_k\rangle)|\rho^2(t)dt\Big]^2, \tag{1.6}$$

where $D_k\varphi(t)$ are properly defined in our proof. Easy verification shows that (1.6) reduces to (1.4) if $\nabla\rho$, $\nabla\varphi$, and ∇f all exist. Therefore, (1.6) can be regard as a generalization of (1.4).

This paper is organized as follows. In Section 2, we prove a form of uncertainty principle for vector-valued signal functions $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ with conditions that the classical first order partial derivatives of f_j, ρ_j, and φ_j exist at all points, and that $\partial f_j/\partial t_k, t_k f_j \in L^2(\mathbb{R}^n)$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. This result generalizes the uncertainty principle obtained in [14]. In Section 3, we assume that $t_j f(t), \omega_j \hat{f}(\omega) \in L^2(\mathbb{R}^n)$, $j = 1, \ldots, n$ for signal functions $f \in L^2(\mathbb{R}^n)$. The Fourier phase and amplitude derivatives $\nabla_{\mathscr{F}}\varphi(t)$ and $\nabla_{\mathscr{F}}\rho(t)$ are properly defined. We prove a form of the uncertainty principle based on these Fourier transform derivatives, which also generalizes the result in [14].

2. Uncertainty principle for vector-valued functions

In this section, we study uncertainty principle for functions $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$. Each entry of $f = (f_1, f_2, \cdots, f_m)$ can be written as $f_j(t) = \rho_j(t)e^{i\varphi_j(t)}$, $j = 1, \ldots, m$. We assume that the classical first order partial derivatives of f_j, ρ_j, and φ_j exist at all points, and that $\partial f_j/\partial t_k, t_k f_j \in L^2(\mathbb{R}^n)$ for j=1, …, m and $k = 1, \ldots, n$. The main theorem of this section is Theorem 2.3. We will use the relation

$$\int_{\mathbb{R}^n} \sum_{k=1}^{n} t_k \rho_j \frac{\partial \rho_j}{\partial t_k} dt = -\frac{n}{2} \int_{\mathbb{R}^n} |\rho_j(t)|^2 dt \tag{2.1}$$

in the proof of the main theorem. Equality (2.1) holds because of

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k=1}^{n} t_k \rho_j \frac{\partial \rho_j}{\partial t_k} dt = \frac{1}{2} \sum_{k=1}^{n} \int_{\mathbb{R}^n} t_k \rho_j \frac{\partial \rho_j}{\partial t_k} + t_k \rho_j \frac{\partial \rho_j}{\partial t_k} dt \\
=&\frac{1}{2} \sum_{k=1}^{n} \Big(\int_{\mathbb{R}^n} -\frac{\partial (t_k \rho_j)}{\partial t_k} \rho_j dt + \int_{\mathbb{R}^n} t_k \rho_j \frac{\partial \rho_j}{\partial t_k} dt \Big) \\
=&-\frac{1}{2} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \Big[\frac{\partial (t_k \rho_j)}{\partial t_k} - t_k \frac{\partial \rho_j}{\partial t_k}\Big] \rho_j dt \\
=&-\frac{n}{2} \int_{\mathbb{R}^n} |\rho_j(t)|^2 dt.
\end{aligned}$$

Here, we have used the fact that if $f_j \in L^2(\mathbb{R}^n)$, then $\rho_j = |f_j| \to 0$ when $|t| \to \infty$.

Lemma 2.1. *Let $f = (f_1, f_2, \cdots, f_m) \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ and write $f_j(t) = \rho_j(t)e^{\varphi_j(t)}$, $j = 1, 2, \cdots, m$. Suppose that $\langle \omega \rangle = 0$ and that $\nabla \rho_j(t)$, $\nabla \varphi_j(t)$, and $\nabla f_j(t)$ all exist and $\frac{\partial f_j(t)}{\partial t_k} \in L^2(\mathbb{R}^n)$ for $j = 1, 2, \cdots, m$, $k = 1, 2, \cdots, n$. Then,*

$$\sigma_\omega^2 = \frac{1}{4\pi^2} \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \Big(\frac{\partial \rho_j(t)}{\partial t_k}\Big)^2 dt + \frac{1}{4\pi^2} \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \rho_j^2(t) \Big(\frac{\partial \varphi_j(t)}{\partial t_k}\Big)^2 dt.$$

Proof. By the definition of σ_ω^2 and by the assumption of $\langle \omega \rangle = 0$, it follows that

$$\begin{aligned}
\sigma_\omega^2 &= \int_{\mathbb{R}^n} |\omega|^2 |\hat{f}(\omega)|^2 d\omega \\
&= \sum_{j=1}^{m} \int_{\mathbb{R}^n} |\omega|^2 |\hat{f}_j(\omega)|^2 d\omega \\
&= \sum_{j=1}^{m} \frac{1}{4\pi^2} \int_{\mathbb{R}^n} |\widehat{\nabla f_j}(\omega)|^2 d\omega \\
&=\frac{1}{4\pi^2} \sum_{j=1}^{m} \int_{\mathbb{R}^n} |\nabla f_j(t)|^2 dt \\
&=\frac{1}{4\pi^2} \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \Big|\frac{\partial f_j(t)}{\partial t_k}\Big|^2 dt
\end{aligned}$$

$$=\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\Big(\frac{\partial\rho_j(t)}{\partial t_k}e^{i\varphi_j(t)}+i\rho_j(t)\frac{\partial\varphi_j(t)}{\partial t_k}e^{i\varphi_j(t)}\Big)^2dt$$
$$=\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\Big(\frac{\partial\rho_j(t)}{\partial t_k}\Big)^2+\rho_j^2(t)\Big(\frac{\partial\varphi_j(t)}{\partial t_k}\Big)^2dt$$
$$=\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\Big(\frac{\partial\rho_j(t)}{\partial t_k}\Big)^2dt+\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\rho_j^2(t)\Big(\frac{\partial\varphi_j(t)}{\partial t_k}\Big)^2dt.$$

This completes the proof. □

The following lemma proves a form of uncertainty for $f\in L^2(\mathbb{R}^n,\mathbb{R}^m)$ with the extra assumption that $\langle t\rangle=0$ and $\langle\omega\rangle=0$. The main result of Theorem 2.3 can be derived easily from the following lemma.

Lemma 2.2. *Let $f=(f_1,f_2,\cdots,f_m)\in L^2(\mathbb{R}^n,\mathbb{R}^m)$ with $\|f\|_2=1$, and write $f_j(t)=\rho_j(t)e^{\varphi_j(t)}$, j=$1,2,\cdots,m$. Suppose that $\langle t\rangle=0$, $\langle\omega\rangle=0$, $\nabla\rho_j(t)$, $\nabla\varphi_j(t)$, and $\nabla f_j(t)$ all exist, and that $\frac{\partial f_j(t)}{\partial t_k}, t_kf_j(t)\in L^2(\mathbb{R}^n)$, $j=1,2,\cdots,m,\ k=1,2,\cdots,n$. Then*

$$\sigma_\omega^2\sigma_t^2\geq\frac{n^2}{16\pi^2}+\frac{1}{4\pi^2}\Big[\int_{\mathbb{R}^n}\sum_{k=1}^{n}\sum_{j=1}^{m}|t_k\frac{\partial\varphi_j}{\partial t_k}||f_j|^2dt\Big]^2.\tag{2.2}$$

If $\frac{\partial\varphi_j}{\partial t_k}$ are continuous and $\rho_j\neq0$ almost everywhere, then the equality of (2.2) *holds if and only if*

$$f_j(t)=d_je^{-\lambda_1|t|^2/2}e^{\frac{i}{2}\lambda_2\sum_{k=1}^{n}(-1)^{\ell_k}t_k^2}+C,\ \ j=1,\dots,m.\tag{2.3}$$

Here, $\lambda_{j_1},\lambda_{j_2}$, and $\{d_j\}_{j=1}^m$ are positive real numbers, while $\{\ell_k\}_{k=1}^n$ are positive integers.

Proof. By Lemma 2.1, it follows that

$$\sigma_t^2\sigma_\omega^2=\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\Big(\frac{\partial\rho_j}{\partial t_k}\Big)^2dt\int_{\mathbb{R}^n}\sum_{j=1}^{m}\sum_{k=1}^{n}|t_k|^2|f_j|^2dt$$
$$+\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\rho_j^2\Big(\frac{\partial\varphi_j}{\partial t_k}\Big)^2dt\int_{\mathbb{R}^n}\sum_{j=1}^{m}\sum_{k=1}^{n}|t_k|^2|f_j|^2dt.$$

So, in order to prove (2.2), we could prove two separate inequalities. The first inequality can be proved as follows:

$$\begin{aligned}\frac{1}{4\pi^2}\Big|\int_{\mathbb{R}^n}\sum_{k,j}\frac{\partial\rho_j}{\partial t_k}t_k\rho_jdt\Big|^2\leq&\frac{1}{4\pi^2}\Big(\int_{\mathbb{R}^n}\sum_{k,j}\Big|\frac{\partial\rho_j}{\partial t_k}\Big||t_k|\rho_jdt\Big)^2\\\leq&\frac{1}{4\pi^2}\int_{\mathbb{R}^n}\sum_{k,j}\Big(\frac{\partial\rho_j}{\partial t_k}\Big)^2dt\int_{\mathbb{R}^n}\sum_{k,j}|t_k|^2|f_j(t)|^2dt.\end{aligned}\tag{2.4}$$

Equality (2.1) shows that

$$\int_{\mathbb{R}^n}\sum_{k=1}^{n}t_k\rho_j\frac{\partial\rho_j}{\partial t_k}dt=-\frac{n}{2}\int_{\mathbb{R}^n}|\rho_j|^2dt.$$

Thus,

$$
\begin{aligned}
&\frac{1}{4\pi^2}\int_{\mathbb{R}^n}\sum_{k,j}\Big(\frac{\partial\rho_j}{\partial t_k}\Big)^2dt\int_{\mathbb{R}^n}\sum_{k,j}|t_k|^2|f_j|^2dt\\
\geq&\frac{1}{4\pi^2}|-\frac{n}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^n}|\rho_j|^2dt|^2\\
=&\frac{n^2}{16\pi^2}(\int_{\mathbb{R}^n}|f|^2dt)^2\\
=&\frac{n^2}{16\pi^2}.
\end{aligned}
\tag{2.5}
$$

The second inequality holds because of

$$
\begin{aligned}
&\frac{1}{4\pi^2}\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\rho_j^2\Big(\frac{\partial\varphi_j}{\partial t_k}\Big)^2dt\int_{\mathbb{R}^n}\sum_{j=1}^{m}\sum_{k=1}^{n}|t_k|^2|f_j|^2dt\\
\geq&\frac{1}{4\pi^2}(\int_{\mathbb{R}^n}\sum_{k,j}|\frac{\partial\varphi_j}{\partial t_k}||t_k|\rho_j^2dt)^2\\
=&\frac{1}{4\pi^2}(\int_{\mathbb{R}^n}\sum_{k,j}|\frac{\partial\varphi_j}{\partial t_k}t_k|\rho_j^2dt)^2.
\end{aligned}
\tag{2.6}
$$

By (2.5) and (2.6), inequality (2.2) follows.

Next, we discuss the conditions such that the equality of (2.2) holds. The second inequality of (2.4) is going to be an equality if and only if there exists $\lambda_1\in\mathbb{R}$ with $\lambda_1>0$ such that

$$|\frac{\partial\rho_j}{\partial t_k}|=\lambda_1|t_k|\rho_j,\ k=1,\ldots,n,\ j=1,\ldots,m,$$

for all $t\in\mathbb{R}^n$. The first inequality of (2.4) is going to be an equality if and only if either

$$\frac{\partial\rho_j}{\partial t_k}=\lambda_1t_k\rho_j(t),\ k=1,\ldots,n,\ j=1,\ldots,m$$

or

$$\frac{\partial\rho_j}{\partial t_k}=-\lambda_1t_k\rho_j(t),\ k=1,\ldots,n,\ j=1,\ldots,m$$

is true. If the first one is true, it follows that

$$\rho_j(t)=d_je^{\lambda_1|t|^2/2},\ j=1,\ldots,m.$$

Obviously, the function $\rho_j(t)=d_je^{\lambda_1|t|^2/2}$ is not in $L^2(\mathbb{R}^n)$. Therefore, we must have

$$\frac{\partial\rho_j}{\partial t_k}=-\lambda_1t_k\rho_j(t),\ k=1,\ldots,n,\ j=1,\ldots,m$$

and then

$$\rho_j(t) = d_j e^{-\lambda_1 |t|^2/2}, \ j = 1, \ldots, m.$$

Here, d_j are positive real numbers.

The equality of (2.6) holds if and only if there exists $\lambda_2 \in \mathbb{R}$ with $\lambda_2 > 0$ such that

$$\rho_j |\frac{\partial \varphi_j}{\partial t_k}| = \lambda_2 |t_k| \rho_j, \ k = 1, \ldots, n, \ j = 1, \ldots, m,$$

for all $t \in \mathbb{R}^n$. Since $\rho_j(t) \neq 0$ almost everywhere and since $\frac{\partial \varphi_j}{\partial t_k}$ are continuous for $k = 1, \ldots, n$, $j = 1, \ldots, m$, we have

$$|\frac{\partial \varphi_j}{\partial t_k}| = \lambda_2 |t_k|, \ k = 1, \ldots, n, \ j = 1, \ldots, m.$$

When $k = 1$, we have

$$\frac{\partial \varphi_j}{\partial t_1} = \pm \lambda_2 t_1.$$

Thus,

$$\varphi_j(t) = \pm \frac{1}{2} \lambda_2 t_1^2 + C_1. \tag{2.7}$$

When $k = 2$, we have

$$\frac{\partial \varphi_j}{\partial t_2} = \pm \lambda_2 t_2. \tag{2.8}$$

Plugging (2.7) into (2.8) implies that

$$C_1 = \pm \frac{1}{2} \lambda_2 t_1^2 + C_2,$$

and then

$$\varphi_j(t) = \pm \frac{1}{2} \lambda_2 t_1^2 \pm \frac{1}{2} \lambda_2 t_2^2 + C_2.$$

Continue this process, when $k = n$, we have

$$\varphi_j(t) = (-1)^{\ell_1} \frac{1}{2} \lambda_2 t_1^2 + (-1)^{\ell_2} \frac{1}{2} \lambda_2 t_2^2 + \cdots + (-1)^{\ell_n} \frac{1}{2} \lambda_2 t_n^2 + C,$$

where $\ell_1, \ldots, \ell_n$ are positive integers. Combining the formulas of ρ_j we have obtained, then

$$f_j(t) = \rho_j(t) e^{i\varphi_j(t)} = d_j e^{-\lambda_1 |t|^2/2} e^{\frac{i}{2} \lambda_2 \sum_{k=1}^n (-1)^{\ell_k} t_k^2} + C, \ j = 1, \ldots, m.$$

Therefore, the equality of (2.2) holds if and only if every $f_j(t)$ is in one of the forms of (2.3). This completes the proof. □

By Lemma 2.2, we can prove a form of uncertainty principle for $f \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ without the assumption that $\langle t\rangle = 0$ and $\langle\omega\rangle = 0$, which is the main result of this section. In the proof of the following theorem, the notations ρ^g, φ^g, $\langle t\rangle_g$, $\langle\omega\rangle_g$, σ_t^g, and σ_ω^g represent the corresponding notation with respect to the function g.

Theorem 2.3. *Let $f = (f_1, f_2, \cdots, f_m) \in L^2(\mathbb{R}^n, \mathbb{R}^m)$ with $\|f\|_2 = 1$, and write $f_j(t) = \rho_j(t)e^{\varphi_j(t)}$, j=$1, 2, \cdots, m$. Suppose that $\nabla\rho_j(t)$, $\nabla\varphi_j(t)$, and $\nabla f_j(t)$ all exist and $\frac{\partial f_j(t)}{\partial t_k}, t_k f_j(t) \in L^2(\mathbb{R}^n)$, j=$1, 2, \cdots, m$, $k = 1, 2, \cdots, n$. Then*

$$\sigma_t^2\sigma_\omega^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2}[\sum_{k=1}^{n}\sum_{j=1}^{m}\int_{\mathbb{R}^n}|(t_k - \langle t_k\rangle)(\frac{\partial\varphi_j}{\partial t_k} - 2\pi\langle\omega_k\rangle)|\rho_j^2(t)dt]^2. \tag{2.9}$$

If $\frac{\partial\varphi_j}{\partial t_k}$ are continuous and $\rho_j \neq 0$ almost everywhere, then the equality of (2.9) holds if and only if

$$f_j(t) = e^{2\pi i t\cdot\langle\omega\rangle}d_j e^{-\lambda_1|t-\langle t\rangle|^2/2}e^{\frac{i}{2}\lambda_2\sum_{k=1}^{n}(-1)^{\ell_k}(t_k-\langle t_k\rangle)^2} + C,\ \ j = 1, \ldots, m.$$

Here, $\lambda_{j_1}, \lambda_{j_2}$, and $\{d_j\}_{j=1}^m$ are positive real numbers, while $\{\ell_k\}_{k=1}^n$ are positive integers.

Proof. Now the quantities $\langle t\rangle$ and $\langle\omega\rangle$ are not 0. Let

$$g_j(t) = e^{-2\pi i(t+\langle t\rangle)\cdot\langle\omega\rangle}f_j(t + \langle t\rangle) = \rho_j^g(t)e^{i\varphi_j^g(t)},$$

for $j = 1, 2, \cdots, m$. Then $g = (g_1, \ldots, g_m) \in L^2(\mathbb{R}^n, \mathbb{R}^m)$. The mean of time t of the signal g is

$$\begin{aligned}\langle t_k\rangle_g &= \sum_{j=1}^{m}\int_{\mathbb{R}^n} t_k|g_j(t)|^2dt\\ &= \sum_{j=1}^{m}\int_{\mathbb{R}^n} t_k|f_j(t + \langle t\rangle)|^2dt\\ &= \sum_{j=1}^{m}\int_{\mathbb{R}^n}(t_k - \langle t_k\rangle)|f_j(t)|^2dt\\ &= \sum_{j=1}^{m}\int_{\mathbb{R}^n} t_k|f_j(t)|^2dt - \langle t_k\rangle\sum_{j=1}^{m}\int_{\mathbb{R}^n}|f_j(t)|^2dt\\ &=0.\end{aligned}$$

Also, it is straightforward to obtain that

$$\hat{g}_j(\omega) = e^{2\pi i\omega\cdot\langle t\rangle}\hat{f}_j(\omega + \langle\omega\rangle),$$

and then

$$\langle\omega_k\rangle_g = 0.$$

Therefore, the vector-valued function g satisfies the conditions of Lemma 2.2. Then we have

$$(\sigma_t^g)^2(\sigma_\omega^g)^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2}[\int_{\mathbb{R}^n}\sum_{k=1}^{n}\sum_{j=1}^{m}|t_k\frac{\partial\varphi_j^g}{\partial t_k}||g_j|^2dt]^2. \tag{2.10}$$

Because of

$$\int_{\mathbb{R}^n} |t_k|^2 |g_j(t)|^2 dt = \int_{\mathbb{R}^n} |t_k - \langle t_k \rangle|^2 |f_j(t)|^2 dt$$

and

$$\int_{\mathbb{R}^n} |\omega_k|^2 |\hat{g}_j(\omega)|^2 d\omega = \int_{\mathbb{R}^n} |\omega_k - \langle \omega_k \rangle|^2 |\hat{f}_j(\omega)|^2 d\omega,$$

we have $(\sigma_t^g)^2 = \sigma_t^2$ and $(\sigma_\omega^g)^2 = \sigma_\omega^2$. Also, since

$$\|g\|_2 = \|f\|_2 = 1$$

and

$$\int_{\mathbb{R}^n} \sum_{k=1}^n |t_k \frac{\partial \varphi_j^g(t)}{\partial t_k} |(\rho_j^g)^2(t) dt = \int_{\mathbb{R}^n} \sum_{k=1}^n |(t_k - \langle t_k \rangle)(\frac{\partial \varphi_j}{\partial t_k} - 2\pi \langle \omega_k \rangle)|\rho_j^2(t) dt,$$

we obtain (2.9), which is

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{n^2}{16\pi^2} + \frac{1}{4\pi^2} [\sum_{k=1}^n \sum_{j=1}^m \int_{\mathbb{R}^n} |(t_k - \langle t_k \rangle)(\frac{\partial \varphi_j}{\partial t_k} - 2\pi \langle \omega_k \rangle)|\rho_j^2(t) dt]^2.$$

The equality of (2.9) holds if and only if the equality of (2.10) holds. By Lemma 2.2, the equality of (2.10) holds if and only if

$$g_j(t) = d_j e^{-\lambda_1 |t|^2/2} e^{\frac{i}{2}\lambda_2 \sum_{k=1}^n (-1)^{\ell_k} t_k^2} + C.$$

By the relationship between f and g, i.e.,

$$f_j(t) = e^{2\pi i t \cdot \langle \omega \rangle} g_j(t - \langle t \rangle),$$

the equality of (2.9) holds if and only if

$$f_j(t) = e^{2\pi i t \cdot \langle \omega \rangle} d_j e^{-\lambda_1 |t - \langle t \rangle|^2/2} e^{\frac{i}{2}\lambda_2 \sum_{k=1}^n (-1)^{\ell_k} (t_k - \langle t_k \rangle)^2} + C.$$

This completes the proof. □

3. Fourier gradient and uncertainty principle

In this section, we go back to the situation of $m = 1$. The so called Fourier phase and amplitude derivatives of $f \in L^2(\mathbb{R}^n)$ are defined. Because, in general, the signal functions may not have ideal smoothness conditions, such as that $\nabla\rho$, $\nabla\varphi$, and ∇f all exist, which are assumed in Theorem 1.3. Lemma 3.1 guarantees that the Fourier transform derivative of $f \in L^2(\mathbb{R})$ is valid once $tf(t),\ \omega\hat{f}(\omega) \in L^2(\mathbb{R})$. This lemma is also fundamental for Definition 3.2.

Lemma 3.1. *[4] Assume that $f(t)$, $tf(t)$, and $\omega\hat{f}(\omega) \in L^2(\mathbb{R})$. Then $\hat{f} \in L^1(\mathbb{R})$, and $f(t)$ is almost everywhere equal to a function in $C_0(\mathbb{R})$. Moreover, there exists the Fourier transform derivative $(Df)(t) \in L^2(\mathbb{R})$ of f such that $(Df)^\wedge(\omega) = i\omega\hat{f}(\omega) \in L^2(\mathbb{R})$ and*

$$\lim_{a\to 0}\int_{-\infty}^{+\infty} |a^{-1}(f(t+a) - f(t)) - (Df)(t)|^2 dt = 0.$$

Therefore,

$$\liminf_{a\to 0} |a^{-1}(f(t+a) - f(t)) - (Df)(t)| = 0$$

holds almost everywhere on $\mathbb{R}$. If, in particular, f has classical derivatives f' almost everywhere on $\mathbb{R}$, then $(Df)(t) = f'$ almost everywhere on $\mathbb{R}$.

It is worth noting that the definition of the Fourier transform in [4] is slightly different from our definition. They define the Fourier transform of f to be

$$\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

However, we define the Fourier transform by (1.2). Under our definition, $\widehat{f'}(\omega) = 2\pi i\omega\hat{f}(\omega)$, and then the definition of Fourier derivative should be slightly changed. It should be a function $(Df)(t) \in L^2(\mathbb{R})$ such that $(Df)^\wedge(\omega) = 2\pi i\omega\hat{f}(\omega)$. Now we can introduce our definition of the Fourier partial derivative.

Definition 3.2. *Let $f \in L^2(\mathbb{R}^n)$. If $t_jf(t)$, $\omega_j\hat{f}(\omega) \in L^2(\mathbb{R}^n)$, $j = 1, \ldots, n$, and denote $g_j(\omega) := 2\pi i\omega_j\hat{f}(\omega)$, $j = 1, \ldots, n$. Then the Fourier transform partial derivative of f with respect to t_j is defined by*

$$D_jf(t) := \mathscr{F}^{-1}(g_j)(t).$$

Here, $\mathscr{F}^{-1}$ is the inverse Fourier transform operator.

Call $\nabla_{\mathscr{F}} := (D_1, D_2, \cdots, D_n)$ the Fourier gradient operator, and we have

$$\nabla_{\mathscr{F}} f = (D_1f, D_2f, \cdots, D_nf).$$

Definition 3.3. *Let $f(t) \in L^2(\mathbb{R}^n)$. Suppose that $\omega_j\hat{f}(\omega) \in L^2(\mathbb{R}^n)$, $j = 1, \ldots, n$. Rewrite $f(t) = \rho(t)e^{i\varphi(t)}$. The Fourier transform phase and amplitude derivatives are defined to be*

$$(D_j\rho)(t) := \begin{cases} \rho(t)\,\mathrm{Re}\,\frac{(D_jf)(t)}{f(t)}, & \text{if } f(t) \neq 0, \\ 0, & \text{if } f(t) = 0, \end{cases}$$

and

$$(D_j\varphi)(t) := \begin{cases} \mathrm{Im}\,\frac{(D_jf)(t)}{f(t)}, & \text{if } f(t) \neq 0, \\ 0, & \text{if } f(t) = 0. \end{cases}$$

Here, $j = 1, \ldots, n$.

The following lemma proves that $f(t)$ is identical with an absolutely continuous function almost everywhere. It is crucial in one proof, which concerns the uncertainty principle studied by the Fourier transform of [4].

Lemma 3.4. *[4] Assume that* $1 \le p_1 \le 2,\ 1 \le p_2 \le 2,\ f(t) \in L^{p_1}(\mathbb{R})$, *and* $h(\omega) = i\omega\hat{f}(\omega) \in L^{p_2}(\mathbb{R})$. *Let*

$$g(t) = \int_a^t (Df)(u)du + f(a),$$

where a *is a Lebesgue point of* f. *Then* $f(t)$ *is identical almost everywhere with the absolutely continuous function* $g(t)$, *and*

$$(Df)(t) = g'(t) \text{ for almost all } t \in \mathbb{R}.$$

The following lemma generalizes Lemma 3.4 to a higher dimensional case.

Lemma 3.5. *Assume that* $f(t) \in L^2(\mathbb{R}^n)$, *and* $h_j(\omega) = i\omega_j\hat{f}(\omega) \in L^2(\mathbb{R}^n),\ j = 1, \dots, n$. *Let*

$$g(t) = \sum_{j=1}^{n} \int_{a_j}^{t_j} (D_j f)(a_1, \dots, u_j, \dots, a_n)du_j + f(a),$$

where a *is a Lebesgue point of* f. *Then* $f(t)$ *is identical almost everywhere with* $g(t)$, *and*

$$(D_j f)(t) = \frac{\partial g}{\partial t_j}(t) \text{ for almost all } t \in \mathbb{R}^n \text{ and } j = 1, \dots, n.$$

Moreover, g *is absolutely continuous in each argument.*

Proof. Let $a = (a_1, \dots, a_n)$ be a Lebesgue point of f. It can be observed that

$$g(a_1, \dots, t_j, \dots, a_n) = \int_{a_j}^{t_j} (D_j f)(a_1, \dots, u_j, \dots, a_n)du_j + f(a).$$

By Lemma 3.4, $f(a_1, \dots, t_j, \dots, a_n)$ is identical almost everywhere with the absolutely continuous function $g(a_1, \dots, t_j, \dots, a_n)$, and

$$(D_j f)(a_1, \dots, t_j, \dots, a_n) = \frac{\partial g}{\partial t_j}(a_1, \dots, t_j, \dots, a_n) \text{ for almost all } t_j \in \mathbb{R}.$$

Then we have $g(a) = f(a)$ and

$$(D_j f)(a) = \frac{\partial g}{\partial t_j}(a), \quad j = 1, \dots, n.$$

Since the points of $\mathbb{R}^n$ are almost everywhere Lebesgue points of f, we conclude that $g(t) = f(t)$ almost everywhere; meanwhile,

$$(D_j f)(t) = \frac{\partial g}{\partial t_j}(t) \text{ for almost all } t \in \mathbb{R}^n \text{ and } j = 1, \dots, n.$$

This completes the proof. □

In the following lemma, the variance of ω, which is σ_ω^2, of a signal function is represented by its Fourier phase and amplitude derivatives.

Lemma 3.6. *Assume that $f(t) \in L^2(\mathbb{R}^n)$, and $h_j(\omega) = i\omega_j \hat{f}(\omega) \in L^2(\mathbb{R}^n)$, $j = 1, \dots, n$. Then*

$$\sigma_\omega^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^n} |\nabla_{\mathscr{F}} \rho|^2(t) + \frac{1}{4\pi^2} \int_{\mathbb{R}^n} |\nabla_{\mathscr{F}} \varphi(t) - 2\pi \langle \omega \rangle|^2 \rho^2(t) dt.$$

Proof. Since $f(t) \in L^2(\mathbb{R}^n)$ and $\omega_j \hat{f}(\omega) \in L^2(\mathbb{R}^n)$, $j = 1, \dots, n$, σ_ω^2 is well defined. By the definition of σ_ω^2, we know that

$$\sigma_\omega^2 = \sum_{k=1}^{n} \int_{\mathbb{R}^n} |\omega_k - \langle \omega_k \rangle|^2 |\hat{f}(\omega)|^2 d\omega.$$

For each fixed k, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\omega_k - \langle \omega_k \rangle|^2 |\hat{f}(\omega)|^2 d\omega \\ =& \int_{\mathbb{R}^n} (\omega_k - \langle \omega_k \rangle) \hat{f}(\omega) \overline{(\omega_k - \langle \omega_k \rangle) \hat{f}(\omega)} d\omega \\ =& \int_{\mathbb{R}^n} [\frac{-i}{2\pi} (D_k f)(t) - \langle \omega_k \rangle f(t)] \overline{[\frac{-i}{2\pi} (D_k f)(t) - \langle \omega_k \rangle f(t)]} dt. \end{aligned}$$

The last equality follows from the Plancherel theorem, which states that $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$, see [12, p.156] for details. Also,

$$\begin{aligned} &\int_{\mathbb{R}^n} [\frac{-i}{2\pi} (D_k f) - \langle \omega_k \rangle f] \overline{[\frac{-i}{2\pi} (D_k f) - \langle \omega_k \rangle f]} dt \\ =& \frac{1}{4\pi^2} \int_{\mathbb{R}^n} (D_k f) \overline{(D_k f)} dt + \frac{i}{2\pi} \int_{\mathbb{R}^n} \langle \omega_k \rangle (D_k f) \bar{f} dt - \frac{i}{2\pi} \int_{\mathbb{R}^n} \langle \omega_k \rangle f \overline{(D_k f)} dt + \int_{\mathbb{R}^n} \langle \omega_k \rangle^2 |f|^2 dt \\ =& \frac{1}{4\pi^2} \int_{\mathbb{R}^n \setminus E} \left|\frac{D_k f}{f}\right|^2 |f|^2 dt - \frac{\langle \omega_k \rangle}{\pi} \int_{\mathbb{R}^n} \mathrm{Im}[(D_k f) \bar{f}] dt + \int_{\mathbb{R}^n} \langle \omega_k \rangle^2 |f|^2 dt, \end{aligned}$$

where $E := \{t \in \mathbb{R}^n : f(t) = 0\}$. Then we have

$$\int_{\mathbb{R}^n} |\omega_k - \langle \omega_k \rangle|^2 |\hat{f}(\omega)|^2 d\omega = \frac{1}{4\pi^2} \int_{\mathbb{R}^n \setminus E} \left|\frac{D_k f}{f}\right|^2 |f|^2 dt - \frac{\langle \omega_k \rangle}{\pi} \int_{\mathbb{R}^n} \mathrm{Im}[(D_k f) \bar{f}] dt + \int_{\mathbb{R}^n} \langle \omega_k \rangle^2 |f|^2 dt.$$

Because of

$$\begin{aligned} &\frac{1}{4\pi^2} \int_{\mathbb{R}^n \setminus E} \left|\frac{D_k f}{f}\right|^2 |f|^2 dt \\ =& \frac{1}{4\pi^2} \int_{\mathbb{R}^n \setminus E} \mathrm{Re}^2 [\frac{(D_k f)(t)}{f(t)}] |f(t)|^2 dt + \frac{1}{4\pi^2} \int_{\mathbb{R}^n \setminus E} \mathrm{Im}^2 [\frac{(D_k f)(t)}{f(t)}] |f(t)|^2 dt \\ =& \frac{1}{4\pi^2} \int_{\mathbb{R}^n} (D_k \rho)^2 (t) dt + \frac{1}{4\pi^2} \int_{\mathbb{R}^n} (D_k \varphi)^2 (t) |f(t)|^2 dt \end{aligned}$$

and

$$\frac{\langle\omega_k\rangle}{\pi}\int_{\mathbb{R}^n}\mathrm{Im}[(D_kf)\bar{f}]dt=\frac{\langle\omega_k\rangle}{\pi}\int_{\mathbb{R}^n}(D_k\varphi)(t)|f(t)|^2dt.$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^n}|\omega_k-\langle\omega_k\rangle|^2|\hat{f}(\omega)|^2d\omega\\=&\frac{1}{4\pi^2}\int_{\mathbb{R}^n}(D_k\rho)^2dt+\frac{1}{4\pi^2}\int_{\mathbb{R}^n}(D_k\varphi)^2|f|^2dt-\frac{\langle\omega_k\rangle}{\pi}\int_{\mathbb{R}^n}(D_k\varphi)|f|^2dt+\int_{\mathbb{R}^n}\langle\omega_k\rangle^2|f|^2dt\\=&\frac{1}{4\pi^2}\int_{\mathbb{R}^n}(D_k\rho)^2dt+\frac{1}{4\pi^2}\int_{\mathbb{R}^n}[(D_k\varphi)-2\pi\langle\omega_k\rangle]^2|f|^2dt.\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}\sigma_\omega^2=&\sum_{k=1}^n\int_{\mathbb{R}^n}|\omega_k-\langle\omega_k\rangle|^2|\hat{f}(\omega)|^2d\omega\\=&\frac{1}{4\pi^2}\sum_{k=1}^n\int_{\mathbb{R}^n}(D_k\rho)^2dt+\frac{1}{4\pi^2}\sum_{k=1}^n\int_{\mathbb{R}^n}[(D_k\varphi)-2\pi\langle\omega_k\rangle]^2|f|^2dt\\=&\frac{1}{4\pi^2}\int_{\mathbb{R}^n}|\nabla_{\mathscr{F}}\rho|^2(t)+\frac{1}{4\pi^2}\int_{\mathbb{R}^n}\rho^2(t)|\nabla_{\mathscr{F}}\varphi(t)-2\pi\langle\omega\rangle|^2dt.\end{aligned}$$

Then, we have finished the proof. □

Theorem 3.7. *Let $f\in L^2(\mathbb{R}^n)$ with $\|f\|_2=1$. Suppose that $t_jf(t),\ \omega_j\hat{f}(\omega)\in L^2(\mathbb{R}^n)$, $j=1,\ldots,n$. Write $f(t)=\rho(t)e^{i\varphi(t)}$, then*

$$\sigma_t^2\sigma_\omega^2\geq\frac{n^2}{16\pi^2}+\frac{1}{4\pi^2}\Big[\sum_{k=1}^n\int_{\mathbb{R}^n}|(t_k-\langle t_k\rangle)(D_k\varphi(t)-2\pi\langle\omega_k\rangle)|\rho^2(t)dt\Big]^2.\tag{3.1}$$

Under the extra assumptions that $f(t)=\rho(t)e^{i\varphi(t)}$ has the classical partial derivatives $\frac{\partial f}{\partial t_j}$, $\frac{\partial\varphi}{\partial t_j}$, $\frac{\partial\rho}{\partial t_j}$ for $j=1,\ldots,n$, where $\frac{\partial\varphi}{\partial t_j}$ are continuous and ρ is non-zero almost everywhere, then the equality of (3.1) *is attained if and only if $f(t)$ has one of the following 2^n forms:*

$$f(t)=d_1e^{-\lambda_1|t-\langle t\rangle|^2/2}e^{\frac{i}{2}\lambda_2\sum_{k=1}^n(-1)^{\ell_k}(t_k-\langle t_k\rangle)^2+c+2\pi it\cdot\langle\omega\rangle},k=1,2,\cdots,n,$$

where $\lambda_1>0$, $\lambda_2>0$, $\ell_k\in\mathbb{N}_+$, and d_1, λ_1 satisfy equation $d_1^{\frac{2}{n}}\sqrt{\frac{\pi}{\lambda_1}}=1$.

Proof. By Definition 1.2, the variance of time is

$$\sigma_t^2=\sum_{k=1}^n\int_{\mathbb{R}^n}|t_k-\langle t_k\rangle|^2|f(t)|^2dt=\int_{\mathbb{R}^n}|t-\langle t\rangle|^2\rho(t)^2dt.$$

Here, in our considering section, $m=1$. Lemma 3.6 provides that

$$\sigma_\omega^2=\frac{1}{4\pi^2}\int_{\mathbb{R}^n}|\nabla_{\mathscr{F}}\rho|^2(t)+\frac{1}{4\pi^2}\int_{\mathbb{R}^n}\rho^2(t)|\nabla_{\mathscr{F}}\varphi(t)-2\pi\langle\omega\rangle|^2dt.$$

In order to prove inequality (3.1), it suffices to prove two separate inequalities. The first one is

$$(\int_{\mathbb{R}^n} |t-\langle t\rangle|^2\rho(t)^2dt)(\frac{1}{4\pi^2}\int_{\mathbb{R}^n} |\nabla_{\mathscr{F}}\rho|^2(t)dt) \geq \frac{n^2}{16\pi^2}, \tag{3.2}$$

and the second one is

$$\begin{aligned}&(\int_{\mathbb{R}^n} |t-\langle t\rangle|^2\rho(t)^2dt)(\frac{1}{4\pi^2}\int_{\mathbb{R}^n} \rho^2(t)|\nabla_{\mathscr{F}}\varphi(t)-2\pi\langle\omega\rangle|^2dt)\\ \geq&\frac{1}{4\pi^2}[\sum_{k=1}^{n}\int_{\mathbb{R}^n} |(t_k-\langle t_k\rangle)(D_k\varphi(t)-2\pi\langle\omega_k\rangle)|\rho^2(t)dt]^2.\end{aligned} \tag{3.3}$$

It is obvious that (3.2) is equivalent to (3.4)

$$(\int_{\mathbb{R}^n} |t-\langle t\rangle|^2\rho(t)^2dt)(\int_{\mathbb{R}^n} |\nabla_{\mathscr{F}}\rho|^2(t)dt) \geq \frac{n^2}{4}, \tag{3.4}$$

and that (3.3) is equivalent to (3.5)

$$\begin{aligned}&(\int_{\mathbb{R}^n} |t-\langle t\rangle|^2\rho(t)^2dt)(\int_{\mathbb{R}^n} \rho^2(t)|\nabla_{\mathscr{F}}\varphi(t)-2\pi\langle\omega\rangle|^2dt)\\ \geq&[\sum_{k=1}^{n}\int_{\mathbb{R}^n} |t_k-\langle t_k\rangle||D_k\varphi(t)-2\pi\langle\omega_k\rangle|\rho^2(t)dt]^2.\end{aligned} \tag{3.5}$$

Now, we prove (3.4). By Lemma 3.5, we may assume that $g(t)$ is a function that is equal to $f(t)$ almost everywhere and is absolutely continuous in each argument. Let M_n and N_n be two particular sequences of numbers tending to infinity as $n \to \infty$. In the following computation, let (t', t_k) represents the tuple $(t_1, ..., t_k, ..., t_n)$. Then we have

$$\begin{aligned}\frac{n^2}{4} =&[\frac{n}{2}\int_{\mathbb{R}^n} |f(t)|^2dt]^2\\ =&[\frac{n}{2}\int_{\mathbb{R}^n} |g(t)|^2dt]^2\\ =&[\frac{n}{2}\int_{\mathbb{R}^{n-1}} \lim_{m\to\infty}\int_{-N_m}^{M_m} |g(t', t_k)|^2dt_kdt']^2\\ =&\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}} \lim_{m\to\infty}\int_{-N_m}^{M_m} |g(t', t_k)|^2dt_kdt'\}^2\\ =&\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}} \{\lim_{m\to\infty}[(t_k-\langle t_k\rangle)|g(t', t_k)|^2|_{-N_m}^{M_m}] - \lim_{m\to\infty}\int_{-N_m}^{M_m}(t_k-\langle t_k\rangle)[\frac{\partial g}{\partial t_k}(t)\bar{g}(t)+g(t)\overline{\frac{\partial g}{\partial t_k}}(t)]dt_k\}dt'\}^2\\ =&\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\infty}(t_k-\langle t_k\rangle)[\frac{\partial g}{\partial t_k}(t)\bar{g}(t)+g(t)\overline{\frac{\partial g}{\partial t_k}}(t)]dt_kdt'\}^2\\ =&\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^{n}}(t_k-\langle t_k\rangle)[\frac{\partial g}{\partial t_k}(t)\bar{g}(t)+g(t)\overline{\frac{\partial g}{\partial t_k}}(t)]dt\}^2\end{aligned}$$

$$
\begin{aligned}
&=\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^n}(t_k-\langle t_k\rangle)[(D_kf)(t)\bar{f}(t)+f(t)\overline{(D_kf)(t)}]dt\}^2\\
&=\{\frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^n\setminus E}(t_k-\langle t_k\rangle)|f(t)|^2[\frac{(D_kf)(t)}{f(t)}+\frac{\overline{(D_kf)(t)}}{\bar{f}(t)}]dt\}^2\\
&=\{\sum_{k=1}^{n}\int_{\mathbb{R}^n\setminus E}(t_k-\langle t_k\rangle)|f(t)||f(t)|\operatorname{Re}\frac{(D_kf)(t)}{f(t)}dt\}^2\\
&=\{\sum_{k=1}^{n}\int_{\mathbb{R}^n}(t_k-\langle t_k\rangle)|f(t)|(D_k\rho)(t)dt\}^2\\
&=\{\int_{\mathbb{R}^n}(t-\langle t\rangle)|f(t)|\cdot(\nabla_{\mathscr{F}}\rho)(t)dt\}^2\\
&\le\int_{\mathbb{R}^n}|(t-\langle t\rangle)\rho(t)|^2dt\int_{\mathbb{R}^n}|\nabla_{\mathscr{F}}\rho|^2(t)dt,
\end{aligned}
$$

where $E=\{t\in\mathbb{R}^n : f(t)=0\}$.

By Hölder's inequality of vector-valued functions [9], it implies that

$$
\begin{aligned}
&[\sum_{k=1}^{n}\int_{\mathbb{R}^n}|t_k-\langle t_k\rangle||D_k\varphi(t)-2\pi\langle\omega_k\rangle|\rho^2(t)dt]^2\\
\le&[\int_{\mathbb{R}^n}(\sum_{k=1}^{n}|t_k-\langle t_k\rangle|^2)^{\frac{1}{2}}(\sum_{k=1}^{n}|D_k\varphi(t)-2\pi\langle\omega_k\rangle)|^2)^{\frac{1}{2}}\rho^2(t)dt]^2\\
\le&\int_{\mathbb{R}^n}\sum_{k=1}^{n}|t_k-\langle t_k\rangle|^2\rho^2(t)dt\int_{\mathbb{R}^n}\sum_{k=1}^{n}|D_k\varphi(t)-2\pi\langle\omega_k\rangle)|^2\rho^2(t)dt\\
=&\int_{\mathbb{R}^n}|t-\langle t\rangle|^2\rho(t)^2dt\int_{\mathbb{R}^n}\rho^2(t)|\nabla_{\mathscr{F}}\varphi(t)-2\pi\langle\omega\rangle|^2dt.
\end{aligned}
$$

Thus we proved (3.5). Therefore, inequality (3.1) holds.

It is not hard to verify that the 2^n types of functions in the statement of the theorem make (3.1) equalities. When we consider the necessity of the 2^n types, we assumed that the classical partial derivatives $\frac{\partial f}{\partial t_j}$, $\frac{\partial\varphi}{\partial t_j}$, $\frac{\partial\rho}{\partial t_j}$ for $j=1,\dots,n$ all exist, $\frac{\partial\varphi}{\partial t_j}$ are continuous, and ρ is almost everywhere non-zero. In this case,

$$\nabla_{\mathscr{F}}\varphi(t)=\nabla\varphi(t) \text{ and } \nabla_{\mathscr{F}}\rho=\nabla\rho.$$

The same proof as in that of Theorem 1.3 is valid. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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