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Research article

Geometry of tubular surfaces and their focal surfaces in Euclidean 3-space

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Abstract: In this study, we examined the focal surfaces of tubular surfaces in Euclidean 3-space E^3. We achieved some significant results for these surfaces in accordance with the modified orthogonal frame. Additionally, we proposed a few geometric invariants that illustrated the geometric characteristics of these surfaces, such as flat, minimal, Weingarten, and linear-Weingarten surfaces, using the traditional methods of differential geometry. Additionally, the asymptotic and geodesic curves of these surfaces have been researched. At last, we presented an example as an instance of use to validate our theoretical findings.

Keywords: tubular surfaces; focal surfaces; modified orthogonal frame
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1. Introduction

In the study of line congruence, focal surfaces are well recognized. A focal surface is one of these line congruences, which are surfaces that are created by changing one surface to another by lines. Line congruence has been presented in the field of visualization by Pottmann et al. in [1]. They can be utilized to envision the strain and intensity circulation on a plane, temperature, precipitation, and ozone over the earth's surface, and so forth. Prior to further processing, the quality of a surface is evaluated using focal surfaces; for further information, see, for instance, [2–5]. Numerous researches have been done on focal surfaces and curves; for examples, see [6–9]. Sasai [10] described the modified orthogonal frame of a space curve in Euclidean 3-space as a helpful tool for examining analytic curves with singular points when the Frenet frame is ineffective. The modified orthogonal frame has recently been the subject of various investigations [11–16].

The envelope of a moving sphere with variable radius is characterized as a canal surface, which is frequently used in solid and surface modeling. A canal surface is an envelope of a one-parameter set of spheres centered at the center curve $c(s)$ with radius $r(s)$. The spheres that are specified by $r(s)$ and $c(s)$ are combined to form a canal surface, which is obtained by the spine curve. These surfaces have a wide range of uses, including form reconstruction, robot movement planning, the creation of blending surfaces, and the easy sight of long and thin objects like pipes, ropes, poles, and live intestines. The term "tubular surface" refers to these canal surfaces if $r(s)$ is constant. Different frames have been used to study tubular surfaces; for more details, see [17–23]. The paper is organized as follows: The fundamental ideas and the modified orthogonal frame are presented in Section 2. In Section 3, we create tubular surfaces with a modified orthogonal frame and provide some findings from these surfaces. Section 4 provides the focal surfaces of tubular surfaces in accordance with the modified orthogonal frame in E^3. At last, an example that confirms our findings is presented in Section 5.

2. Basic concepts

Let $\alpha = \alpha(s)$ be a space curve with respect to the arc-length s in E^3 and $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be the tangent, principal, and binormal unit vectors at each point on $\alpha(s)$, respectively, then we have the Serret-Frenet equations:

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix}, \tag{2.1}$$

where κ and τ are, respectively, the curvature and torsion functions of α.

Since, the Serret-Frenet frame is inadequate for studying analytic space curves, of which curvatures have discrete zero points since the principal normal and binormal vectors may be discontinuous at zero points of the curvature, Sasai presented an orthogonal frame and obtained a formula, which corresponds to the Frenet-Serret equation [10].

Let $\alpha : I \longrightarrow E^3$ be an analytic curve. We suppose that the curvature $\kappa(s)$ of α is not identically zero. We express an orthogonal frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ as

$$\mathbf{T} = \frac{d\alpha}{ds}, \mathbf{N} = \frac{d\mathbf{T}}{ds}, \ \mathbf{B} = \mathbf{T} \times \mathbf{N}. \tag{2.2}$$

The relations between the frames $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ at nonzero points of κ are expressed as follows:

$$\begin{aligned} \mathbf{T} &= \mathbf{t}, \\ \mathbf{N} &= \kappa\mathbf{n}, \\ \mathbf{B} &= \kappa\mathbf{b}, \end{aligned} \tag{2.3}$$

and we have

$$\begin{aligned} &\langle \mathbf{T}, \mathbf{T} \rangle = 1, \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = \kappa^2, \\ &\langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0. \end{aligned} \tag{2.4}$$

From Eqs (2.1) and (2.3), a straightforward calculation leads to

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \dfrac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \dfrac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}, \tag{2.5}$$

and

$$\tau(s) = \frac{det(\alpha', \alpha'', \alpha''')}{\kappa^2},$$

is the torsion of α. Therefore, the frame denoted by Eq (2.5) is called the modified orthogonal frame.

Let $\Upsilon(s,\theta)$ be a surface in E^3 and $\mathbf{U}(s,\theta)$ be the unit normal vector field on $\Upsilon(s,\theta)$ defined by $\mathbf{U} = \frac{\Upsilon_s \times \Upsilon_\theta}{\|\Upsilon_s \times \Upsilon_\theta\|}$, where $\Upsilon_s = \frac{\partial \Upsilon}{\partial s}$ and $\Upsilon_\theta = \frac{\partial \Upsilon}{\partial \theta}$ are the tangent vectors of $\Upsilon(s,\theta)$. The metric (first fundamental form) I of $\Upsilon(s,\theta)$ is defined by

$$I = g_{11} ds^2 + 2 g_{12} ds d\theta + g_{22} d\theta^2,$$

where $g_{11} = \langle \Upsilon_s, \Upsilon_s \rangle$, $g_{12} = \langle \Upsilon_s, \Upsilon_\theta \rangle$, and $g_{22} = \langle \Upsilon_\theta, \Upsilon_\theta \rangle$.

Also, we can define the second fundamental form of $\Upsilon(s,\theta)$ as

$$II = h_{11} ds^2 + 2 h_{12} ds d\theta + h_{22} d\theta^2,$$

where $h_{11} = \langle \Upsilon_{ss}, \mathbf{U} \rangle$, $h_{12} = \langle \Upsilon_{s\theta}, \mathbf{U} \rangle$, $h_{22} = \langle \Upsilon_{\theta\theta}, \mathbf{U} \rangle$, and $\mathbf{U}$ is the unit normal vector of the surface. The Gaussian curvature K and the mean curvature H are, respectively, expressed as:

$$\begin{aligned} K &= \frac{h_{11} h_{22} - h_{12}^2}{g_{11} g_{22} - g_{12}^2}, \\ H &= \frac{h_{11} g_{22} - 2 g_{12} h_{12} + g_{11} h_{22}}{2(g_{11} g_{22} - g_{12}^2)}. \end{aligned} \tag{2.6}$$

3. Tubular surfaces with modified orthogonal frame

In this part, we obtain a tubular surface with modified orthogonal frame and give some important properties of this surface in E^3. The tubular surface with respect to the modified orthogonal frame has the parametrization:

$$\Upsilon(s,\theta) = c(s) + \frac{r}{\kappa(s)} (\cos\theta \mathbf{N}(s) + \sin\theta \mathbf{B}(s)), \tag{3.1}$$

where $c(s)$ is the center curve, $r = constant$, and $\kappa \neq 0$. The derivatives of $\Upsilon(s,\theta)$ are given by

$$\begin{aligned} \Upsilon_s &= (1 - r\kappa \cos\theta)\mathbf{T} + \frac{\tau r}{\kappa}(-\sin\theta \mathbf{N} + \cos\theta \mathbf{B}), \\ \Upsilon_\theta &= \frac{r}{\kappa}(-\sin\theta \mathbf{N} + \cos\theta \mathbf{B}), \\ \Upsilon_{ss} &= (-r\kappa' \cos\theta + r\kappa\tau \sin\theta)\mathbf{T} \\ &+ (1 - r\kappa \cos\theta - \frac{r}{\kappa}\tau' \sin\theta - \frac{r}{\kappa}\tau^2 \cos\theta)\mathbf{N} + (\frac{r}{\kappa}\tau' \cos\theta - \frac{r}{\kappa}\tau^2 \sin\theta)\mathbf{B}, \end{aligned}$$

$$\Upsilon_{s\theta} = (r\kappa \sin\theta)\mathbf{T} + \frac{\tau r}{\kappa}(-\cos\theta\mathbf{N} - \sin\theta\mathbf{B}),$$
$$\Upsilon_{\theta\theta} = \frac{r}{\kappa}(-\cos\theta\mathbf{N} - \sin\theta\mathbf{B}). \tag{3.2}$$

Therefore, we obtain

$$\begin{aligned} g_{11} &= (1 - r\kappa\cos\theta)^2 + r^2\tau^2, \\ g_{12} &= \tau r^2, \\ g_{22} &= r^2, \\ g &= g_{11}g_{22} - g_{12}^2 = r^2(1 - r\kappa\cos\theta)^2 \neq 0. \end{aligned} \tag{3.3}$$

The unit normal vector field $\mathbf{U}$ is given by

$$\mathbf{U}(s) = -\frac{1}{\kappa}\cos\theta\mathbf{N} - \frac{1}{\kappa}\sin\theta\mathbf{B}, \tag{3.4}$$

and we have

$$\begin{aligned} h_{11} &= -\kappa\cos\theta(1 - r\kappa\cos\theta) + \tau^2 r, \\ h_{12} &= \tau r, \\ h_{22} &= r. \end{aligned} \tag{3.5}$$

From Eq (2.6), we find

$$\begin{aligned} K &= -\frac{\kappa\cos\theta}{r(1 - r\kappa\cos\theta)}, \\ H &= \frac{1 - 2r\kappa\cos\theta}{2r(1 - r\kappa\cos\theta)}. \end{aligned} \tag{3.6}$$

From Eqs (3.1) and (3.6), we note that, if the Gaussian curvature K is zero, then the tubular surface is generated by a moving sphere with the radius $r = 1$, [18].

Also, the curvatures of the tubular surface $\Upsilon(s, \theta)$ satisfy the relation:

$$H = \frac{1}{2}(Kr + \frac{1}{r}), \tag{3.7}$$

and the shape operator of $\Upsilon(s, \theta)$ is given by

$$\begin{aligned} S &= \frac{1}{g}\begin{bmatrix} h_{11}g_{22} - h_{12}g_{12} & h_{12}g_{22} - h_{22}g_{12} \\ -h_{11}g_{12} + h_{12}g_{11} & -h_{12}g_{12} + h_{22}g_{11} \end{bmatrix} \\ &= \frac{1}{r^2(1 - r\kappa\cos\theta)^2}\begin{bmatrix} r^2(-\kappa\cos\theta(1 - r\kappa\cos\theta)) & 0 \\ \tau r(1 - r\kappa\cos\theta) & r(1 - r\kappa\cos\theta)^2 \end{bmatrix}. \end{aligned} \tag{3.8}$$

It follows that the principal curvatures of $\Upsilon(s, \theta)$ are obtained as:

$$\begin{aligned} k_1 &= \frac{1}{r}, \\ k_2 &= -\frac{\kappa\cos\theta}{1 - r\kappa\cos\theta} = Kr. \end{aligned} \tag{3.9}$$

Proposition 3.1. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3, then $\Upsilon(s,\theta)$ is not a flat surface.*

Proof. The proof can be obtained by straightforward calculations from Eq (3.6).

Proposition 3.2. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3, then $\Upsilon(s,\theta)$ is minimal if and only if*

$$r = \frac{1}{2\kappa\cos\theta}.$$

Proof. The result is obtained directly from Eq (3.6).

Theorem 3.1. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame, then*
(i) the s-curves of $\Upsilon(s,\theta)$ are asymptotic curves if and only if

$$r = \frac{\kappa\cos\theta}{\kappa^2\cos\theta^2 + \tau^2},$$

(ii) the θ-curves of $\Upsilon(s,\theta)$ cannot be asymptotic curves.

Proof. From the definition of asymptotic curves, we obtain

$$\begin{aligned}\langle \Upsilon_{ss}, \mathbf{U}\rangle =&0,\\ \langle \Upsilon_{\theta\theta}, \mathbf{U}\rangle =&0.\end{aligned}$$

i. From Eq (3.5), we can get

$$\begin{aligned} h_{11} &= -\kappa\cos\theta(1 - r\kappa\cos\theta) + \tau^2 r = 0,\\ r &= \frac{\kappa\cos\theta}{\kappa^2\cos\theta^2 + \tau^2}.\end{aligned}$$

ii. $\Upsilon(s,\theta)$ is regular if $h_{22} \neq 0$. Thus, the s-curves of $\Upsilon(s,\theta)$ cannot be asymptotic curves.

Theorem 3.2. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame, then*
i. s-curves of $\Upsilon(s,\theta)$ are geodesic if and only if,

$$r\kappa^2\cos^2\theta - 2\kappa\cos\theta + \tau^2 r = c,$$

where c is a constant.
ii. θ-curves of $\Upsilon(s,\theta)$ are geodesic.

Proof. From the definition of geodesic curves, we find that, $\Upsilon_{ss} \times \mathbf{U} = 0$ and $\Upsilon_{\theta\theta} \times \mathbf{U} = 0$.
i. According to Eqs (3.2) and (3.4), we obtain

$$\begin{aligned}\Upsilon_{ss} \times \mathbf{U} =&(\kappa\sin\theta(r\kappa\cos\theta - 1) + r\tau')\mathbf{T} + (\frac{r}{\kappa}\sin\theta(\tau\kappa\sin\theta - \kappa'\cos\theta))\mathbf{N}\\ &+(\frac{r}{\kappa}\cos\theta(-\tau\kappa\sin\theta + \kappa'\cos\theta))\mathbf{B}.\end{aligned}$$

Since $\mathbf{T}, \mathbf{N}$, and B are linearly independent, then $\Upsilon_{ss} \times \mathbf{U} = 0$ if and only if

$$\kappa\sin\theta(r\kappa\cos\theta - 1) + r\tau' = 0,$$

$$\frac{r}{\kappa}\sin\theta(\tau\kappa\sin\theta-\kappa'\cos\theta)=0,$$
$$\frac{-r}{\kappa}\cos\theta(\tau\kappa\sin\theta-\kappa'\cos\theta)=0.$$

When the last two equations are taken into consideration together and the necessary operations are done, we get

$$r\kappa\kappa'\cos^2\theta-\kappa'\cos\theta+r\tau\tau'=0,$$
$$r\kappa^2\cos^2\theta-2\kappa\cos\theta+\tau^2r=c,$$

where c is constant.

ii. Also, from Eqs (3.2) and (3.4), we have $\boldsymbol{\Upsilon}_{\theta\theta}\times\mathbf{U}=0$. Thus, θ- parameter curves are geodesic curves.

Definition 3.1. *The pair $(X,Y), X\neq Y$ of the curvatures K,H of a tubular surface $\boldsymbol{\Upsilon}(s,\theta)$ is said to be a (X,Y)-Weingarten surface if $\Phi(X,Y)=0$, where the Jacobi function Φ is defined as $X_sY_\theta-Y_sX_\theta=0$ [24].*

Definition 3.2. *The pair $(X,Y), X\neq Y$ of the curvatures K,H of the tubular surface $\boldsymbol{\Upsilon}(s,\theta)$ is said to be a (X,Y)-linear Weingarten surface if $\boldsymbol{\Upsilon}(s,\theta)$ satisfies the following relation:*

$$aX+bY=c,$$

where $(a,b,c)\epsilon R$ and $(a,b,c)\neq(0,0,0)$ [25].

Now, we define the partial derivatives of the curvatures of $\boldsymbol{\Upsilon}(s,\theta)$:

$$\begin{aligned}K_s&=-\frac{\kappa'\cos\theta}{r(1-r\kappa\cos\theta)^2},\\K_\theta&=\frac{\kappa\sin\theta}{r(1-r\kappa\cos\theta)^2},\\H_s&=-\frac{r}{2}\Big(\frac{\kappa'\cos\theta}{r(1-r\kappa\cos\theta)^2}\Big),\\H_\theta&=\frac{r}{2}\Big(\frac{\kappa\sin\theta}{r(1-r\kappa\cos\theta)^2}\Big).\end{aligned}\tag{3.10}$$

Proposition 3.3. *Let $\boldsymbol{\Upsilon}(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame, then $\boldsymbol{\Upsilon}(s,\theta)$ is a Weingarten surface.*

Proof. (K,H)-Weingarten surface $\boldsymbol{\Upsilon}(s,\theta)$ satisfies Jacobi equation:

$$H_sK_\theta-H_\theta K_s=0,$$

$$H_sK_\theta=H_\theta K_s.$$

Thus, the conclusion follows from Eq (3.10).

Proposition 3.4. *Let $\boldsymbol{\Upsilon}(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame. If (K,H) is a linear Weingarten surface, then for $c=1$, the relations $a=-r^2$ and $b=2r$, hold.*

Proof. The (K, H)-linear Weingarten surface satisfies:

$$aK + bH = 1,$$

where $a, b \in R$ and $(a, b) \neq 0$. From Eq (3.7), we get

$$2r - b = (-\frac{\kappa \cos\theta}{r(1 - r\kappa \cos\theta)})(br + 2a),$$

and

$$2\kappa \cos\theta(-r^2 + rb + a) + 2r - b = 0,$$

so we find,

$$\begin{aligned} &2r - b = 0, \\ &2\kappa \cos\theta(-r^2 + rb + a) = 0, \end{aligned}$$

since $b = 2r$; therefore, $a = -r^2$ is obtained.

4. Focal surfaces of a tubular surface

Let $\boldsymbol{\Upsilon}(s, \theta)$ be a surface in E^3 parameterized by

$$\mathbf{C}(s, \theta, z) = \boldsymbol{\Upsilon}(s, \theta) + z\mathbf{E}(s, \theta), \tag{4.1}$$

where $\mathbf{E}(s, \theta)$ is the set of all unit vectors and z is a marked distance. For each (s, θ), Eq (4.1) indicates a line congruence and called a generatrix. Additionally, there exist two special points (real, imaginary, or unit) on the generatrix of $\mathbf{C}$. These points are called focal points, which are the osculating points with generatrix. Hence, focal surfaces are defined as a geometric locus of focal points. If $\mathbf{E}(s, \theta) = \mathbf{U}(s, \theta)$, then $\mathbf{C} = \mathbf{C}_u$ is a normal congruence. The parametric equation of focal surfaces of $\mathbf{C}_u$ is given as

$$\boldsymbol{\Upsilon}_i^*(s, \theta) = \boldsymbol{\Upsilon}(s, \theta) + \frac{1}{k_i}\mathbf{U}(s, \theta); i = 1, 2, \tag{4.2}$$

where k_1 and k_2 are the principle curvature functions of surfaces $\boldsymbol{\Upsilon}(s, \theta)$ [1].

In this section, we get focal surfaces of a tubular surface with the modified orthogonal frame in E^3. Also, we investigate the properties obtained for the tubular surface within the focal surfaces. Since $k_1 = \frac{1}{r}$, the focal surface is the curve $c(s)$. Hence, we obtain the focal surface $\boldsymbol{\Upsilon}^*(s, \theta)$ of $\boldsymbol{\Upsilon}(s, \theta)$ with the function $k_2 = -\frac{\kappa \cos\theta}{1 - r\kappa \cos\theta}$, as follows:

$$\boldsymbol{\Upsilon}^*(s, \theta) = c(s) + \frac{1}{\kappa \cos\theta}(\frac{1}{\kappa} \cos\theta\mathbf{N} + \frac{1}{\kappa} \sin\theta\, \mathbf{B}), \tag{4.3}$$

where $\kappa \neq 0$.

The derivatives of $\boldsymbol{\Upsilon}^*(s, \theta)$ are

$$\boldsymbol{\Upsilon}_s^* = \frac{1}{\kappa^2}(-\frac{\kappa'}{\kappa} - \tau \tan\theta)\mathbf{N} + \frac{1}{\kappa^2}(\tau - \frac{\kappa'}{\kappa} \tan\theta)\mathbf{B},$$

$$
\begin{aligned}
\Upsilon^*_\theta &= \frac{1}{\kappa^2}\sec^2\theta\mathbf{B},\\
\Upsilon^*_{ss} &= (\frac{\kappa'}{\kappa}+\tau\tan\theta)\mathbf{T}+(\frac{2\kappa'^2}{\kappa^4}+\frac{2\kappa'\tau\tan\theta}{\kappa^3}-\frac{\kappa''}{\kappa^3}-\frac{\tau'}{\kappa^2}\tan\theta-\frac{\tau^2}{\kappa^2})\mathbf{N}\\
&\quad+(-\frac{2\kappa'\tau}{\kappa^3}+\frac{2\kappa'^2\tan\theta}{\kappa^4}+\frac{\tau'}{\kappa^2}-\frac{\tau^2}{\kappa^2}\tan\theta-\frac{\kappa''}{\kappa^3}\tan\theta)\mathbf{B},\\
\Upsilon^*_{\theta\theta} &= \frac{2}{\kappa^2}\sec^2\theta\tan\theta\,\mathbf{B}.
\end{aligned}\tag{4.4}
$$

From Eq (4.4), we get

$$
\begin{aligned}
g^*_{11} &= \frac{1}{\kappa^2}\sec^2\theta(\tau^2+\frac{\kappa'^2}{\kappa^2}),\\
g^*_{12} &= \frac{1}{\kappa^2}\sec^2\theta\left(\tau-\frac{\kappa'}{\kappa}\tan\theta\right),\\
g^*_{22} &= \frac{1}{\kappa^2}\sec^4\theta,\\
g^* &= g^*_{11}g^*_{22}-g^{*2}_{12}=\frac{1}{\kappa^4}\sec^4\theta(\tau\tan\theta+\frac{\kappa'}{\kappa})^2\neq 0,
\end{aligned}\tag{4.5}
$$

and

$$
\begin{aligned}
\Upsilon^*_s\times\Upsilon^*_\theta &= -\left(\frac{1}{\kappa^4}\sec^2\theta(\frac{\kappa'}{\kappa}+\tau\tan\theta)\right)\mathbf{T},\\
\|\Upsilon^*_s\times\Upsilon^*_\theta\| &= \frac{1}{\kappa^4}\sec^2\theta(\frac{\kappa'}{\kappa}+\tau\tan\theta),\\
\mathbf{U}^*(s,\theta) &= -\mathbf{T}.
\end{aligned}\tag{4.6}
$$

From Eqs (4.4) and (4.6), we obtain

$$
\begin{aligned}
h^*_{11} &= (-\frac{\kappa'}{\kappa}-\tau\tan\theta),\\
h^*_{12} &= 0,\\
h^*_{22} &= 0.
\end{aligned}\tag{4.7}
$$

Also, from Eqs (4.5) and (4.7), we get

$$
\begin{aligned}
K^* &= 0,\\
H^* &= -\frac{\kappa^3}{2\,(\kappa\tau\tan\theta+\kappa')}.
\end{aligned}\tag{4.8}
$$

Further, the shape operator of the surface $\Upsilon^*(s,\theta)$ is expressed as

$$
S^* = \frac{1}{\gamma}\begin{bmatrix} -\frac{1}{\kappa^2}\sec^4\theta(\frac{\kappa'}{\kappa}+\tau\tan\theta) & 0\\ \frac{1}{\kappa^2}\sec^2\theta(\tau-\frac{\kappa'}{\kappa}\tan\theta)(\frac{\kappa'}{\kappa}+\tau\tan\theta) & 0\end{bmatrix},\tag{4.9}
$$

where, $\gamma=\frac{1}{\kappa^4}\sec^4\theta(\tau\tan\theta+\frac{\kappa'}{\kappa})^2$.

Proposition 4.1. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame obtained with the parametrization given by Eq (3.1) and let $\Upsilon^*(s,\theta)$ be its focal surface with the parametrization given by Eq (4.3), then the focal surface $\Upsilon^*(s,\theta)$ is a flat surface.*

Proof. The proof can be obtained by using Eq (4.8) and straightforward calculations.

Proposition 4.2. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame obtained with the parametrization given by Eq (3.1) and let $\Upsilon^*(s,\theta)$ be its focal surface with the parametrization given by Eq (4.3), then the focal surface $\Upsilon^*(s,\theta)$ is not minimal surface.*

Proof. From Eq (4.8), we get

$$-\frac{\kappa^3}{2\left(\kappa\tau\tan\theta+\kappa'\right)}\neq 0.$$

So, $\Upsilon^*(s,\theta)$ is not minimal.

Theorem 4.1. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame obtained with the parametrization given by Eq (3.1) and let $\Upsilon^*(s,\theta)$ be its focal surface with the parametrization given by Eq (4.3), then*
(i) s-curves of $\Upsilon^*(s,\theta)$ cannot be asymptotic curves.
(ii) θ-curves of $\Upsilon^*(s,\theta)$ are asymptotic curves.

Proof. (i) From Eq (4.7), we get

$$\begin{aligned} h^*_{11} &= \langle \Upsilon^*_{ss}, \mathbf{U}^* \rangle, \\ &= \left(-\frac{\kappa'}{\kappa}-\tau\tan\theta\right)\neq 0. \end{aligned}$$

Since $\Upsilon^*(s,\theta)$ is regular; $h^*_{11}\neq 0$. Thus, s-parameter curves of the focal surface $\Upsilon^*(s,\theta)$ cannot be asymptotic curves.
(ii) Also, from Eq (4.7), we get

$$h^*_{22}=\langle \Upsilon^*_{\theta\theta}, \mathbf{U}^*\rangle = 0.$$

Since $h^*_{22}=0$, then θ-curves of $\Upsilon^*(s,\theta)$ are asymptotic curves.

Theorem 4.2. *Let $\Upsilon(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame obtained with the parametrization given by Eq (3.1) and let $\Upsilon^*(s,\theta)$ be its focal surface with the parametrization given by Eq (4.3), then*
(i) s-parameter curves of $\Upsilon^*(s,\theta)$ are geodesic curves if and only if,

$$\frac{2\kappa}{\kappa'}=\frac{\tau}{\tau'}.$$

(ii) θ-parameter curves of $\Upsilon^*(s,\theta)$ are not geodesic curves.

Proof. (i) From Eqs (4.4) and (4.6), we have

$$\begin{aligned} \Upsilon^*_{ss}\times\mathbf{U}^* &= -\left(-\frac{2\kappa'^2\tau}{\kappa^3}+\frac{2\kappa'^2\tan\theta}{\kappa^4}+\frac{\tau'}{\kappa^2}-\frac{\tau^2\tan\theta}{\kappa^2}-\frac{\kappa''\tan\theta}{\kappa^3}\right)\mathbf{N} \\ &\quad +\left(\frac{2\kappa'}{\kappa^4}+\frac{2\kappa'\tau\tan\theta}{\kappa^3}-\frac{\tau'\tan\theta}{\kappa^2}-\frac{\kappa''}{\kappa^3}-\frac{\tau^2}{\kappa^2}\right)\mathbf{B}. \end{aligned}$$

Therefore, $\boldsymbol{\Upsilon}^*_{ss} \times \mathbf{U}^* = 0$ if and only if

$$\frac{2\kappa}{\kappa'} = \frac{\tau}{\tau'}.$$

(ii) From Eqs (4.4) and (4.6), we get

$$\boldsymbol{\Upsilon}^*_{\theta\theta} \times \mathbf{U}^* = \frac{-2\sec^2\theta\tan\theta}{\kappa^2}\mathbf{N}.$$

Since $\boldsymbol{\Upsilon}^*_{\theta\theta} \times \mathbf{U}^* \neq 0$, then θ-parameter curves are not geodesic curves of $\boldsymbol{\Upsilon}^*(s,\theta)$.

Hence, the proof is completed.

The partial derivatives of the curvatures of $\boldsymbol{\Upsilon}^*(s,\theta)$ are

$$\begin{aligned} K^*_s &= 0,\ K^*_\theta = 0, \\ H^*_s &= -\frac{3\kappa^2\kappa'}{2\left(\kappa\tau\tan\theta + \kappa'\right)} + \frac{\kappa^3\left(\tau\kappa'\tan\theta + \kappa\tau'\tan\theta + \kappa''\right)}{2\left(\kappa\tau\tan\theta + \kappa'\right)^2}, \\ H^*_\theta &= \frac{\kappa^4\tau\sec^2\theta}{2\left(\kappa\tau\tan\theta + \kappa'\right)^2}. \end{aligned} \tag{4.10}$$

Thus, we get the result:

Proposition 4.3. *Let $\boldsymbol{\Upsilon}(s,\theta)$ be a tubular surface in E^3 with a modified orthogonal frame obtained with the parametrization given by Eq (3.1) and let $\boldsymbol{\Upsilon}^*(s,\theta)$ be its focal surface with the parametrization given by Eq (4.3), then the focal surface $\boldsymbol{\Upsilon}^*(s,\theta)$ is a Weingarten surface but not a linear-Weingarten surface.*

Proof. The conclusion can be obtained easily from Eqs (4.8) and (4.10).

Definition 4.1. *[8] A surface $\boldsymbol{\Upsilon}(s,\theta)$ in E^3 with principal curvatures $k_1 \neq k_2$ has a generalized focal surface $\widetilde{\boldsymbol{\Upsilon}}(s,\theta)$ given by*

$$\widetilde{\boldsymbol{\Upsilon}}(s,\theta) = \boldsymbol{\Upsilon}(s,\theta) + f(k_1,k_2)\mathbf{U}(s,\theta),$$

where $f(k_1,k_2)$ is related to its principal curvatures.

If

$$f(k_1,k_2) = \frac{k_1^2 + k_2^2}{k_1 + k_2},$$

then $\widetilde{\boldsymbol{\Upsilon}}(s,\theta)$ is expressed as

$$\widetilde{\boldsymbol{\Upsilon}}(s,\theta) = \boldsymbol{\Upsilon}(s,\theta) + (\frac{k_1^2 + k_2^2}{k_1 + k_2})\mathbf{U}(s,\theta).$$

Therefore, by employing this definition and using Eq (3.9), we can obtain a generalized focal surface $\widetilde{\boldsymbol{\Upsilon}}(s,\theta)$ of $\boldsymbol{\Upsilon}(s,\theta)$ as

$$\widetilde{\boldsymbol{\Upsilon}}(s,\theta) = c(s) + \frac{1}{\kappa}(r - \frac{K^2r^4 + 1}{Kr^3 + r})(\cos\theta\mathbf{N} + \sin\theta\mathbf{B}).$$

5. Computational example

Let us demonstrate the above considerations in a computational example. So, assume the center curve α is given by

$$\alpha(s) = \left(\cos\left(\frac{\sqrt{5}}{3}s\right), \sin\left(\frac{\sqrt{5}}{3}s\right), \frac{2s}{3}\right),$$

then, its modified orthogonal frame is calculated as follows:

$$\begin{aligned}
\mathbf{T} &= \left(-\frac{\sqrt{5}}{3}\sin\left(\frac{\sqrt{5}}{3}s\right), \frac{\sqrt{5}}{3}cos\left(\frac{\sqrt{5}}{3}s\right), \frac{2}{3}\right),\\
\mathbf{N} &= \left(-\frac{5}{9}\cos\left(\frac{\sqrt{5}}{3}s\right), -\frac{5}{9}sin\left(\frac{\sqrt{5}}{3}s\right), 0\right),\\
\mathbf{B} &= \left(\frac{10}{27}\sin\left(\frac{\sqrt{5}}{3}s\right), -\frac{10}{27}cos\left(\frac{\sqrt{5}}{3}s\right), \frac{5\sqrt{5}}{27}\right).
\end{aligned}$$

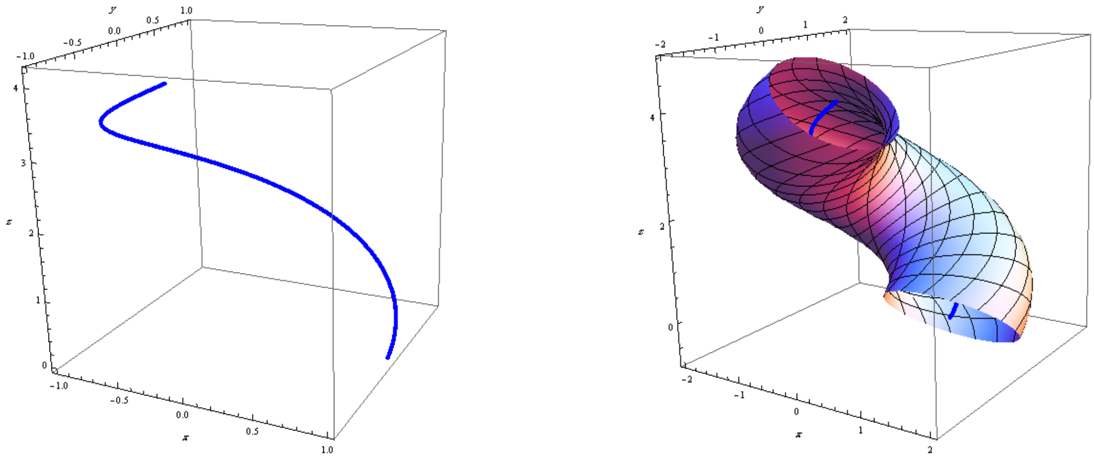
When the radius function $r(s) = 1$, we obtain the tubular surface:

$$\Upsilon(s,\theta) = \left(\begin{array}{c}
\cos\left(\frac{\sqrt{5}}{3}s\right) - \cos\theta\cos\left(\frac{\sqrt{5}}{3}s\right) + \frac{2}{3}\sin\theta\sin\left(\frac{\sqrt{5}}{3}s\right),\\
\sin\left(\frac{\sqrt{5}}{3}s\right) - \cos\theta\sin\left(\frac{\sqrt{5}}{3}s\right) - \frac{2}{3}\sin\theta\cos\left(\frac{\sqrt{5}}{3}s\right),\\
\frac{2s}{3} + \frac{\sqrt{5}}{3}\sin\theta).
\end{array}\right)$$

In addition, we get the focal surface of $\Upsilon(s,\theta)$ as

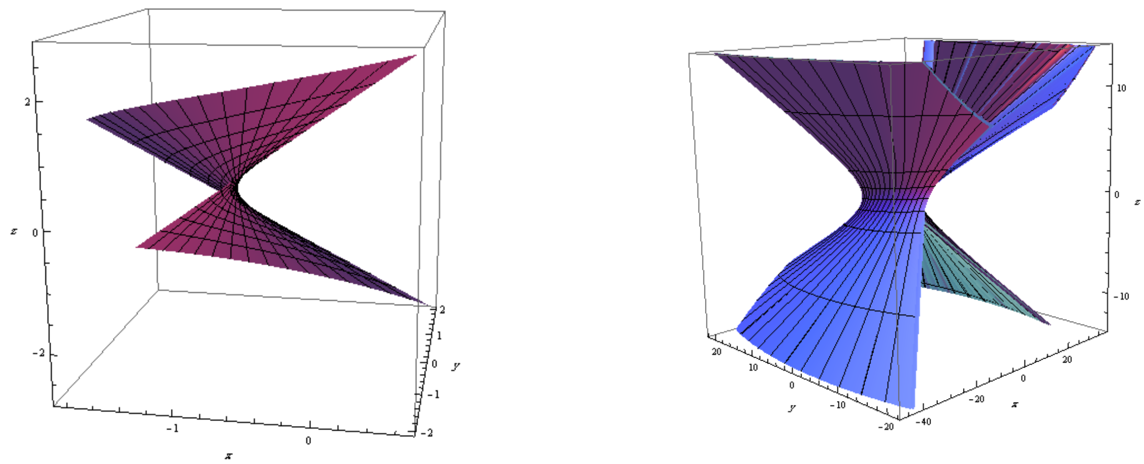
$$\Upsilon^{*}(s,\theta) = \left(\begin{array}{c}
\cos\left(\frac{\sqrt{5}}{3}s\right) - \frac{9}{5}\cos\left(\frac{\sqrt{5}}{3}s\right) + \frac{6}{5}\tan\theta\sin\left(\frac{\sqrt{5}}{3}s\right),\\
\sin\left(\frac{\sqrt{5}}{3}s\right) - \frac{9}{5}\sin\left(\frac{\sqrt{5}}{3}s\right) - \frac{6}{5}\tan\theta\cos\left(\frac{\sqrt{5}}{3}s\right),\\
\frac{2s}{3} + \frac{3\sqrt{5}}{3}\tan\theta).
\end{array}\right)$$

The center curve $\alpha(s)$ and the corresponding tubular surface $\Upsilon(s,\theta)$ are shown in Figures 1a and 1b, respectively.



(a) **(b)**

Figure 1. (a) The center curve $\alpha(s)$; (b) the tubular surface $\Upsilon(s,\theta)$.



(a) **(b)**

Figure 2. (a) The focal surface $\Upsilon^{*}(s,\theta)$ with $r(s) = 1$; (b) the generalized focal surface $\widetilde{\Upsilon}(s,\theta)$.

Also, the generalized focal surface of $\Upsilon(s,\theta)$ is obtained as follows:

$$\widetilde{\Upsilon}(s,\theta)=\begin{pmatrix} \cos\left(\frac{\sqrt{5}}{3}s\right)-\rho\cos\left(\frac{\sqrt{5}}{3}s\right)\cos\theta+\frac{2}{3}\rho\sin\theta\sin\left(\frac{\sqrt{5}}{3}s\right), \\ \sin\left(\frac{\sqrt{5}}{3}s\right)-\rho\sin\left(\frac{\sqrt{5}}{3}s\right)\cos\theta-\rho\frac{2}{3}\sin\theta\cos\left(\frac{\sqrt{5}}{3}s\right), \\ \frac{2s}{3}+\rho\frac{\sqrt{5}}{3}\sin\theta. \end{pmatrix};$$

$$\rho=\left(1-\frac{\left(\frac{\frac{-5}{9}\cos\theta}{1-\frac{5}{9}\cos\theta}\right)^{2}+1}{\left(\frac{\frac{-5}{9}\cos\theta}{1-\frac{5}{9}\cos\theta}\right)+1}\right).$$

The focal surface $\mathbf{\Upsilon}^*(s,\theta)$ is displayed in Figure 2a and the generalized focal surface $\widetilde{\mathbf{\Upsilon}}(s,\theta)$ is displayed in Figure 2b.

Finally, in the light of aforementioned calculations, we can conclude the above results as follows:

- While $\mathbf{\Upsilon}(s,\theta)$ is a not flat surface, the focal surface $\mathbf{\Upsilon}^*(s,\theta)$ is flat.
- $\mathbf{\Upsilon}(s,\theta)$ is a minimal surface, but the focal surface $\mathbf{\Upsilon}^*(s,\theta)$ is not minimal.
- The s-parameter curves of $\mathbf{\Upsilon}(s,\theta)$ are asymptotic, while the s-parameter curves of the $\mathbf{\Upsilon}^*(s,\theta)$ are non-asymptotic.
- The θ-parameter curves of the tubular surface $\mathbf{\Upsilon}(s,\theta)$ are not asymptotic, but the θ-parameter curves of the focal surface $\mathbf{\Upsilon}^*(s,\theta)$ are asymptotic.
- The s and θ parameters of the tubular surface $\mathbf{\Upsilon}(s,\theta)$ are geodesic curves, but the s parameter curve of $\mathbf{\Upsilon}^*(s,\theta)$ is a geodesic whereas the θ parameter curve is not a geodesic curve.
- $\mathbf{\Upsilon}(s,\theta)$ and focal surface $\mathbf{\Upsilon}^*(s,\theta)$ are Weingarten surfaces.
- $\mathbf{\Upsilon}(s,\theta)$ can be a linear Weingarten surface while the focal surface $\mathbf{\Upsilon}^*(s,\theta)$ is not a linear Weingarten surface.

6. Conclusions

In this paper, tubular surfaces and their focal surfaces have been studied in E^3. Some characteristics of the tubular surfaces have been presented such as minimal, Weingarten, linear-Weingarten, and flat. Afterward, focal surfaces of tubular surfaces have been obtained with modified orthogonal frame. Similar properties have been investigated for focal surfaces, that is, it has been shown that the focal surfaces are flat, Weingarten, and linear Weingarten, but they are not minimal surfaces. In addition, asymptotic and geodesic curves of the tubular and focal surfaces have been investigated. Finally, to confirm our results, a computational example is given and plotted.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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