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Research article

A study on the existence results of boundary value problems of fractional relaxation integro-differential equations with impulsive and delay conditions in Banach spaces

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Abstract: The aim of this paper was to provide systematic approaches to study the existence of results for the system fractional relaxation integro-differential equations. Applied problems require definitions of fractional derivatives, allowing the utilization of physically interpretable boundary conditions. Impulsive conditions serve as basic conditions to study the dynamic processes that are subject to sudden changes in their state. In the process, we converted the given fractional differential equations into an equivalent integral equation. We constructed appropriate mappings and employed the Schaefer's fixed-point theorem and the Banach fixed-point theorem to show the existence of a unique solution. We presented an example to show the applicability of our results.

Keywords: Riemann-Liouville fractional derivative; fractional relaxation impulsive integro differential equations; Liouville-Caputo fractional derivative; existence; uniqueness; delay; fixed point
Mathematics Subject Classification: 26A33, 32C25, 34A12, 34K45

1. Introduction

This paper concerns the existence and uniqueness of a solution for the following impulsive fractional relaxation and integro-differential equations with delay conditions

$$\begin{cases} D^{\varepsilon}\, {}^{LC}D^{\varpi}y(\mathfrak{r}) + \lambda y(\mathfrak{r}) = \mathfrak{P}(\mathfrak{r}, y_{\mathfrak{r}}, I^{\vartheta}y(\mathfrak{r})),\ \mathfrak{r} \in [0, \mathsf{T}] = \mathfrak{I},\ \mathfrak{r} \neq \mathfrak{r}_{\mathfrak{k}},\ \lambda \in \mathfrak{R}, \\ \Delta y(\mathfrak{r}_{\mathfrak{k}}) = \tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^{-})),\ \mathfrak{k} = 1, 2, ..., \mathfrak{n}, \\ y(\mathfrak{r}) = \rho(\mathfrak{r}),\ \mathfrak{r} \in (-\infty, 0], \\ {}^{LC}D^{\varpi}y(0) = {}^{LC}D^{\varpi}y(\mathsf{T}) = 0,\ y(0) = \ell \int_0^{\mathsf{T}} y_s ds + \ell_1,\ \ell, \ell_1 \in \mathfrak{R}, \end{cases} \tag{1.1}$$

where ε and ϖ are the fractional derivative order of R-L fractional derivative D^{ε} and L-C fractional derivative ${}^{LC}D^{\varpi}$, $1 < \varepsilon < 2, \vartheta \in (0, 1)$ is order of R-L fractional integral I^{ϑ} and $\mathfrak{P} : \mathfrak{I} \times \aleph \times \mathfrak{R} \to \mathfrak{R}$ is a nonlinear continuous function. $\tilde{I}_{\mathfrak{k}} : \mathfrak{R} \to \mathfrak{R},$ the jump of y at $\mathfrak{r} = \mathfrak{r}_{\mathfrak{k}}$ is denoted by $\Delta y(\mathfrak{r}_{\mathfrak{k}}) = y(\mathfrak{r}_{\mathfrak{k}}^{+}) - y(\mathfrak{r}_{\mathfrak{k}}^{-})$, the right limit of $y(\mathfrak{r})$ at $\mathfrak{r} = \mathfrak{r}_{\mathfrak{k}}$ is $y(\mathfrak{r}_{\mathfrak{k}}^{+})$ and the left limit of $y(\mathfrak{r})$ at $\mathfrak{r} = \mathfrak{r}_{\mathfrak{k}}$ is $y(\mathfrak{r}_{\mathfrak{k}}^{-})$, $\mathfrak{k} = 1, 2, ..., \mathfrak{n}$. $\bar{C}_{\wr} := \bar{C}_{\wr}((-\infty, 0], \mathfrak{R})$ is the space of continuous functions.

We describe $y_{\mathfrak{r}}$ by

$$y_{\mathfrak{r}}(s) = y(\mathfrak{r} + s) \text{ where } \mathfrak{r} \in \mathfrak{I}, \text{ and } -\infty < s \leq 0.$$

Here, $y_{\mathfrak{r}}(.)$ portrays the state's history variance from time $\mathfrak{r} - \infty$ to $\mathfrak{r}$.

Differential equations with delay involve systems in which the future state not only depends on the current but also on the past state. Reforestation is a straightforward example found in nature. After being replanted, a cut forest will take at least 20 years to reach any kind of maturity; this could be much longer for some species of trees (redwoods, for instance). Therefore, it is obvious that time delays must be included in any mathematical model (refer [1, 2] for similar applications). In a study by Bouriah and colleagues [3], they briefly explored the presence and stability of fractional differential equations that incorporate both delay and impulse conditions. In another work [4], X. Ma et al. delved into the existence of nearly periodic solutions for fractional impulsive neutral stochastic differential equations with extended delay. These differential equations are currently a popular area of research and find wide application as mathematical models in real-world scenarios, as evidenced in the book [5]. Additionally, Wattanakejorn et al. [6] conducted research into the existence of solutions for relaxation differential equations that include impulsive delay boundary conditions.

Fractional calculus has been widely used in many fields of applied science and engineering. For example, it has been used in systems biology, physics, chemistry, and biochemistry. Fractional-order models can reflect the complex behaviors of various diseases more accurately and deeply than classical integer-order models. Fractional-order systems are better than integer-order systems because they contain the genetic characteristics of memory (see reference [7–9]). Substantial growth has been achieved in the concept of fractional derivatives and its applications in current history, as evidenced by the references [10–13]. In [14], the authors examined the study of a multi-term time-fractional delay differential system with monotonic conditions. Aissani and Benchohra [15] discussed fractional integro-differential equations with state-dependent delay. Kaliraj and colleagues [16] investigated the existence of results for a nonlocal neutral fractional differential equation using the concept of the Caputo derivative with impulsive conditions. In [17], the authors analyzed existence and stability results for impulsive fractional integro-differential equations with integral boundary conditions.

Nonetheless, differential equations with impulse conditions have attracted a lot of interest. For instance, impulsive effects are known to occur in many biological phenomena involving thresholds, bursting rhythm models in biology and medicine, optimal control models in economics, pharmacokinetics, and frequency-modulated systems. For example, in [18] the authors explored the efficacy of activated charcoal in detoxifying a body suffering from methanol poisoning by using impulsive conditions. In [19], Karthikeyan and others investigated the impulsive fractional integro-differential equations with boundary conditions. Zeng [20] examined the existence results for fractional impulsive delay feedback control systems with caputo fractional derivatives. The authors in [21] discussed the existence and uniqueness of a nonlocal fractional differential equation of Sobolev type with impulses. Liu et al. [22] discussed the existence of positive solutions for the ϕ-Hilfer fractional differential equation with random impulses and boundary value conditions. In [23], Shu et al. studied the mild solution of impulsive fractional evolution equations.

In [24], the authors studied the existence and uniqueness of positive solutions of the given non-linear fractional relaxation differential equation

$$\begin{cases} {}^{LC}D^{\alpha}\varkappa(t) + \lambda\varkappa(t) = f(t, \varkappa(t)), \quad 0 < t \leq 1, \\ \varkappa(0) = \varkappa_0 > 0, \end{cases}$$

where ${}^{LC}D^{\alpha}$ is the Liouville-Caputo fractional derivative, $\alpha \in (0, 1]$. By using the fixed-point theorems and the method of the lower and upper solutions, the existence and uniqueness of solutions have been examined.

A. Lachouri, A. Djoudi, and A. Ardjouni [25] discussed the existence and uniqueness of solutions for the below fractional relaxation integro-differential equations with boundary conditions

$$\begin{cases} D^{\beta}\,{}^{LC}D^{\alpha}\varkappa(t) + \lambda\varkappa(t) = f(t, \varkappa(t), I^{r}\varkappa(t)), \ \lambda \in \mathbb{R}, \quad 0 < t < T, \\ {}^{LC}D^{\alpha}\varkappa(0) = {}^{LC}\ D^{\alpha}\varkappa(T) = 0, \quad \varkappa(0) = a\int_0^T \varkappa_s ds + b, \quad a, b \in \mathbb{R}, \end{cases}$$

where ${}^{LC}D^{\alpha}$ and D^{β} are Liouville-Caputo (L-C) fractional derivative and the Riemann-Liouville (R-L) fractional derivative of orders α and β, respectively, $\alpha \in (0, 1)$, $\beta \in (1, 2)$, I^{r} is the Riemann-Liouville fractional integral of order $r \in (0, 1)$, and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function.

Motivated by the above works, we studied the existence and uniqueness of solutions for Eq 1.1. In this system, we included the impulse condition and infinite delay for integro-fractional differential equations with integral boundary conditions. Main results are proved by using Schaefer's and Banach fixed- point theorems.

The paper is structured as follows: Some fundamental terms and preliminary facts are given in Section 2. In Section 3, we discuss the existence and uniqueness of the main problem. An example is given in Section 4 to highlight the theoretical findings.

2. Preliminaries

This section presents some terminologies and results connected to fractional calculus.

Let $(\aleph, \|.\|_{\aleph})$ be the seminormed linear space of functions mapping from $(-\infty, 0]$ to $\mathfrak{R}$, let $\bar{C}_{\wr}(\mathfrak{I}) = \bar{C}_{\wr}(\mathfrak{I}, \mathfrak{R})$ denote the Banach space of continuous functions provided with the norm $\|y\|_{\infty} := \sup\{|y(\mathfrak{r})| : \mathfrak{r} \in [0, \mathsf{T}]\}$, where $\mathfrak{I} := [0, \mathsf{T}]$ and consider the space $\tilde{\mathcal{PC}}_{\wr}([0, \mathsf{T}], \mathfrak{R}) = \{y : [0, \mathsf{T}] \to \mathfrak{R} : y \in$

$\bar{C}_{\wr}((\mathfrak{r}_{\mathfrak{k}}, \mathfrak{r}_{\mathfrak{k}+1}], \mathfrak{R})$, $\mathfrak{k} = 0, ..., \mathfrak{n}$, and there exists $y(\mathfrak{r}_{\mathfrak{k}}^-)$ and $y(\mathfrak{r}_{\mathfrak{k}}^+)$, $\mathfrak{k} = 1, ..., \mathfrak{n}$, with $y(\mathfrak{r}_{\mathfrak{k}}) = y(\mathfrak{r}_{\mathfrak{k}}^-)\}$ equipped with the norm $\|y\|_{\tilde{\mathcal{PC}}_{\wr}} = \sup_{\mathfrak{r}\in[0,\mathsf{T}]} |y(\mathfrak{r})|$. Consider the space $\aleph_b = \{y : (-\infty, \mathsf{T}] \to \mathfrak{R} \backslash y \in \tilde{\mathcal{PC}}_{\wr}(\mathfrak{I}, \mathfrak{R}) \cap \aleph\}$. $\mathfrak{AC}(\mathfrak{I})$ is the space of absolutely continuous valued functions from $\mathfrak{I}$ to $\mathfrak{R}$, and set $\mathfrak{AC}^m(\mathfrak{I}) = \{y : \mathfrak{I} \to \mathfrak{R} : y, y', y'', \ldots, y^{m-1} \in \bar{C}_{\wr} \quad and \quad y^{m-1} \in \mathfrak{AC}(\mathfrak{I})\}$.

Consider $(\aleph, \|.\|_{\aleph})$ fulfilling the underlying axioms listed below,

(A1) If $y : (-\infty, \mathsf{T}] \to \mathfrak{R}$ and $y_0 \in \aleph$, then $K^*, H^*, M^* > 0$, are constants such that for any $\mathfrak{r} \in \mathfrak{I}$ the subsequent conditions retain:

- $y_{\mathfrak{r}}$ is in $\aleph$, and $y_{\mathfrak{r}}$ is continuous on $[0, \mathsf{T}) \backslash \{\mathfrak{r}_1, \mathfrak{r}_2, ..., \mathfrak{r}_m\}$,
- $\|y_{\mathfrak{r}}\|_{\aleph} \leq K^* \|y_1\|_{\aleph} + M^* \sup_{s\in[0,t]} |y(s)|$,
- $\|y(\mathfrak{r})\| \leq H^* \|y_{\mathfrak{r}}\|_{\aleph}$.

(A2) $y_{\mathfrak{r}}$ is a $\aleph-$ valued continuous function on $\mathfrak{I}$, for the function $y(.)$ in (A1),

(A3) $\aleph$'s space is complete.

Definition 2.1. *[26] $\mathfrak{h}_{\wr} : \mathfrak{I} \to \mathfrak{R}$ is function with fractional integral order ϖ, $\varpi > 0$ is specified by*

$$I^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = \frac{1}{\Gamma(\varpi)} \int_0^{\mathfrak{r}} (\mathfrak{r} - s)^{\varpi-1} \mathfrak{h}_{\wr}(s) ds,$$

given the integral exists.

Definition 2.2. *[26] $\mathfrak{h}_{\wr} : \mathfrak{I} \to \mathfrak{R}$ is with L-C fractional derivative order ϖ, $\varpi > 0$ is specified by*

$$^{LC}D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = D^{\varpi}\left[\mathfrak{h}_{\wr}(\mathfrak{r}) - \sum_{j=0}^{m-1} \frac{\mathfrak{h}_{\wr}^{(j)}(0)}{j!} \mathfrak{r}^j\right],$$

where

$$m = 1 + [\varpi] \quad for \quad \varpi \notin \mathbb{N}_0, \quad m = \varpi \quad for \quad \varpi \in \mathbb{N}_0, \tag{2.1}$$

and $D^{\varpi}_{0^+}$ is a R-L fractional derivative of order ϖ specified by

$$D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = D^m I^{m-\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = \frac{1}{\Gamma(n-\varpi)} \frac{d^m}{d\mathfrak{r}^m} \int_0^{\mathfrak{r}} (\mathfrak{r} - s)^{m-\varpi-1} \mathfrak{h}_{\wr}(s) ds.$$

The L-C fractional derivative $^{LC}D^{\varpi}_{0^+}$ exists for $y \in \mathfrak{AC}^m(\mathfrak{I})$. Here, it is denoted by

$$^{LC}D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = I^{m-\varpi}y^{(m)}(\mathfrak{r}) = \frac{1}{\Gamma(n-\varpi)} \int_0^{\mathfrak{r}} (\mathfrak{r} - s)^{m-\varpi-1} \mathfrak{h}_{\wr}^{(m)}(s) ds.$$

Note that we obtain, $^{LC}D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = \mathfrak{h}_{\wr}^{(m)}(\mathfrak{r})$ whenever $\varpi = m$.

Lemma 2.1. *[26] Assume $\varpi > 0$ and m be the provided by (2.1). If $\mathfrak{h}_{\wr} \in \mathfrak{AC}^m(\mathfrak{I}, \mathfrak{R})$, then*

$$(I^{\varpi LC}D^{\varpi}\mathfrak{h}_{\wr})(\mathfrak{r}) = \mathfrak{h}_{\wr}(\mathfrak{r}) - \sum_{j=0}^{m-1} \frac{\mathfrak{h}_{\wr}^{(j)}(0)}{j!} \mathfrak{r}^j,$$

where $\mathfrak{h}_{\wr}^{(j)}$ is the normal derivative of $\mathfrak{h}_{\wr}$ of order j.

Lemma 2.2. *[26] For $\varpi > 0$ and m be provided by (2.1), then the general solution of the L-C fractional differential equation ${}^{LC}D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = 0$ is*

$$\mathfrak{h}_{\wr}(\mathfrak{r}) = b_0 + b_1\mathfrak{r} + b_2\mathfrak{r}^2 + ... + b_{m-1}\mathfrak{r}^{m-1},$$

where $b_k \in \mathfrak{R}$, $k = 0, 1, 2, ..., m-1$. Additionally, the general solution of the R-L fractional differential equation

$$D^{\varpi}\mathfrak{h}_{\wr}(\mathfrak{r}) = 0,$$

is

$$\mathfrak{h}_{\wr}(\mathfrak{r}) = b_1\mathfrak{r}^{\varpi-1} + b_2\mathfrak{r}^{\varpi-2} + b_3\mathfrak{r}^{\varpi-3} + ... + b_m\mathfrak{r}^{\varpi-m}, \quad b_k \in \mathfrak{R}, \quad k = 1, 2, ..., m.$$

Lemma 2.3. *[26] For any $0 \leq \varpi,\ \varepsilon < \infty$ and, then*

$$\frac{1}{\Gamma(\varpi)}\int_0^{\mathfrak{r}} (\mathfrak{r}-s)^{\varepsilon-1}s^{\varpi-1}ds = \frac{\Gamma(\varepsilon)}{\Gamma(\varpi+\varepsilon)}\mathfrak{r}^{\varpi+\varepsilon-1}.$$

Lemma 2.4. *[3] (Banach contraction mapping theorem) Let Θ be a non-empty subset of a Banach space $(E^*, \|\cdot\|)$, which is convex and closed, and $\phi^* : \Theta \to \Theta$ be an any contraction mapping, then has a unique fixed point.*

Lemma 2.5. *[3] (Schaefer's fixed-point theorem) Let E^* be a Banach space and $\phi^* : E^* \to E^*$ be a completely continuous operator. If the set $A = \{y \in E^* : y = \lambda\phi^* y,$ for some $\lambda \in (0, 1)\}$ is bounded. Then, the operator has a fixed point.*

Lemma 2.6. *[3] (Arzela-Ascoli theorem) Let $A \subset \tilde{\mathcal{PC}}_{\wr}(\mathfrak{I}, \mathfrak{R})$. A is relatively compact if: (i) A is uniformly bounded, i.e., there exists $M > 0$ such that*

$$|f(x)| < M for every f \in A and x \in (t_k, t_{k+1}), k = 1, ..., m.$$

(ii) A is equicontinuous on (t_k, t_{k+1}) i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each, $x, \bar{x} \in (t_k, t_{k+1})$, $|x - \bar{x}| \leq \delta$ implies $|f(x) - f(\bar{x})| \leq \epsilon$, for every $f \in A$.

Lemma 2.7. *(Lebesgue Dominated Convergence Theorem) Suppose g is Lebesgue integrable on E. The sequence f_n of measurable functions satisfies: (i). $|f_n| \leq g$ a.e. on E for $n \in N$ (ii). $f_n \to f$ a.e. on E. Then, $f \in L(E)$ and $\lim_{n\to\infty}\int_E f_n dx = \int_E f dx$.*

Lemma 2.8. *[25] For any $\mathfrak{h}_{\wr} \in \bar{C}_{\wr}(\mathfrak{I})$, then the problem*

$$D^{\varepsilon}\ {}^{LC}D^{\varpi}y(\mathfrak{r}) + \lambda y(\mathfrak{r}) = \mathfrak{h}_{\wr}(\mathfrak{r}),\ \mathfrak{r} \neq \mathfrak{r}_{\mathfrak{k}},\ \mathfrak{r} \in [0, \top],\ \lambda \in \mathfrak{R},$$
$${}^{LC}D^{\varpi}y(0) = {}^{LC}D^{\varpi}y(\top) = 0,\ y(0) = \ell\int_0^{\top} y_s ds + \ell_1,\ \ell, \ell_1 \in \mathfrak{R},$$

is identical to the integral equation

$$\begin{aligned} y(\mathfrak{r}) =& \frac{1}{\Gamma(\varpi+\varepsilon)}\left(\int_0^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_0^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ &- \frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\top^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}y_{\upsilon}d\upsilon\right) + \ell\int_0^{\top} y_{\upsilon}d\upsilon + \ell_1. \end{aligned}$$

Proof. Taking the integrator operator I^{ε} to the above first equation and from Lemma 2.2, we get

$$^{LC}D^{\varpi}y(\mathfrak{r}) = I^{\varepsilon}\mathfrak{h}_{\wr}(\mathfrak{r}) - \lambda I^{\varepsilon}y((r)) + a_1\mathfrak{r}^{\varepsilon-1} + a_2\mathfrak{r}^{\varepsilon-2}. \tag{2.2}$$

According to conditions $^{LC}D^{\varpi}y(0) =^{LC} D^{\varpi}y(\top) = 0$*, it yields*

$$a_1 = \frac{1}{\top^{\varepsilon-1}}(\lambda I^{\varepsilon}y(\top) - I^{\varepsilon}\mathfrak{h}_{\wr}(\top)),\ a_2 = 0.$$

Replacing a_1 and a_2 by their values in (3), we get

$$^{LC}D^{\varpi}y(\mathfrak{r}) = I^{\varepsilon}\mathfrak{h}_{\wr}(\mathfrak{r}) - \lambda I^{\varepsilon}y(\mathfrak{r}) + \frac{\mathfrak{r}^{\varepsilon-1}}{\top^{\varepsilon-1}}(\lambda I^{\varepsilon}y(\top) - I^{\varepsilon}\mathfrak{h}_{\wr}(\top)). \tag{2.3}$$

Taking the integrator operator I^{ϖ} again to the above equation and using Lemmas 2.2 and 2.3, we obtain

$$y(\mathfrak{r}) = I^{\varpi+\varepsilon}\mathfrak{h}_{\wr}(\mathfrak{r}) - \lambda I^{\varpi+\varepsilon}y(\mathfrak{r}) + \frac{\Gamma(\varepsilon)\mathfrak{r}^{\varepsilon+\varpi-1}}{\top^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}(I^{\varepsilon}\mathfrak{h}_{\wr}(\top) - \lambda I^{\varepsilon}y(\top)) + a_3, \tag{2.4}$$

using the integral condition, we find

$$a_3 = \ell\int_0^{\top} y_s ds + \ell_1.$$

Substituting the value of a_3, we obtain the integral equation.

Lemma 2.9. *[25] For any $\mathfrak{h}_{\wr} \in \bar{C}_{\wr}(\mathfrak{I})$, then the problem*

$$\begin{cases} D^{\varepsilon\ LC}D^{\varpi}y(\mathfrak{r}) + \lambda y(\mathfrak{r}) = \mathfrak{h}_{\wr}(\mathfrak{r}),\ \mathfrak{r} \neq \mathfrak{r}_{\mathfrak{k}},\ \mathfrak{r} \in [0, \top],\ \lambda \in \mathfrak{R}, \\ \Delta y(\mathfrak{r}_{\mathfrak{k}}) = \tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-)),\ \mathfrak{k} = 1, 2, ..., \mathfrak{n}, \\ y(\mathfrak{r}) = \rho(\mathfrak{r}),\ -\infty < \mathfrak{r} \leq 0, \\ ^{LC}D^{\varpi}y(0) =^{LC} D^{\varpi}y(\top) = 0,\ y(0) = \ell\int_0^{\top} y_s ds + \ell_1,\ \ell, \ell_1 \in \mathfrak{R}, \end{cases} \tag{2.5}$$

is identical to the integral equation

$$\begin{cases} \rho(\mathfrak{r});\ \mathfrak{r} \in (-\infty, 0] \\ \dfrac{1}{\Gamma(\varpi+\varepsilon)}\left(\displaystyle\int_0^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_0^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ -\dfrac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\top^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\displaystyle\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ +\ell\displaystyle\int_0^{\top} y_{\upsilon}d\upsilon + \ell_1\ if\ \mathfrak{r} \in [0, \mathfrak{r}_1] \\ \vdots \\ \dfrac{1}{\Gamma(\varpi+\varepsilon)}\displaystyle\sum_{i=1}^{\mathfrak{k}}\left(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i} (\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i} (\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ +\dfrac{1}{\Gamma(\varpi+\varepsilon)}\left(\displaystyle\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ -\dfrac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\top^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\displaystyle\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}\mathfrak{h}_{\wr}(\upsilon)d\upsilon - \lambda\int_0^{\top} (\top-\upsilon)^{\varepsilon-1}y_{\upsilon}d\upsilon\right) \\ +\ell\displaystyle\int_0^{\top} y_{\upsilon}d\upsilon + \ell_1 + \sum_{\mathfrak{k}=1}^{m}\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))\ if\ \mathfrak{r} \in (\mathfrak{r}_{\mathfrak{k}}, \mathfrak{r}_{\mathfrak{k}+1}]. \end{cases}$$

3. Main results

We require the following hypothesis:

(H1) Take the constants $k_1 > 0,\ k_2 \in (0,1)$ such that

$$|\mathfrak{P}(\mathfrak{r}, y, y_1) - \mathfrak{P}(\mathfrak{r}, y^*, y_1^*)| \le k_1 \|y - y^*\|_{\aleph} + k_2 |y_1 - y_1^*|,$$

for any $\mathfrak{r} \in \mathfrak{I}$, for each $y, y^* \in \aleph$ $i = 1, 2,$ and $y_1, y_1^* \in \mathfrak{R}$.

(H2) Consider the constants $K_1 > 0$ and $0 < K_2 < 1$ such that

$$|\mathfrak{P}(\mathfrak{r}, y, y^*)| \le K_1 \|y\|_{\aleph} + K_2 |y^*|,$$

for each $\mathfrak{r} \in \mathfrak{I}$, for any $y \in \aleph$ and $y^* \in \mathfrak{R}$.

(H3) There exists $\wp > 0$ such that

$$|\tilde{I}_k(y) - \tilde{I}_k(y^*)| \le \wp \|y - y^*\|_{\aleph},\ \forall\ y, y^* \in \mathfrak{R} \text{ with } k = 1, 2, ..., \mathfrak{n}.$$

Theorem 3.1. *Assume (H1) and (H3) holds. If* $\nabla < 1$, *then Eq* (1.1) *has a solution that is unique on* $(-\infty, \mathsf{T}]$.

Where

$$\nabla = \left(\frac{(\mathfrak{n}+1)}{\Gamma(\varpi + \varepsilon + 1)} + \frac{1}{\varepsilon \Gamma(\varpi + \varepsilon)} \right) \mathsf{T}^{\varpi + \varepsilon} \left[k_1 + k_2 H^* \frac{\mathsf{T}^{\theta}}{\Gamma(\theta + 1)} + |\lambda| \right] + \mathfrak{n} \wp.$$

Proof. Indicate the operator $\Pi : \aleph_b \to \aleph_b$ *as*

$$(\Pi y)(\mathfrak{r}) = \begin{cases} \rho(\mathfrak{r});\ \mathfrak{r} \in (-\infty, 0] \\ \dfrac{1}{\Gamma(\varpi+\varepsilon)} \displaystyle\sum_{i=1}^{\mathfrak{k}} \left(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i} (\mathfrak{r}_i - \upsilon)^{\varpi+\varepsilon-1} \mathfrak{P}(\upsilon, y_\upsilon, I^\theta y(\upsilon)) d\upsilon - \lambda \int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i} (\mathfrak{r}_i - \upsilon)^{\varpi+\varepsilon-1} y_\upsilon d\upsilon \right) \\ + \dfrac{1}{\Gamma(\varpi+\varepsilon)} \left(\displaystyle\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}} (\mathfrak{r} - \upsilon)^{\varpi+\varepsilon-1} \mathfrak{P}(\upsilon, y_\upsilon, I^\theta y(\upsilon)) d\upsilon - \lambda \int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}} (\mathfrak{r} - \upsilon)^{\varpi+\varepsilon-1} y_\upsilon d\upsilon \right) \\ - \dfrac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)} \left(\displaystyle\int_0^{\mathsf{T}} (\mathsf{T} - \upsilon)^{\varepsilon-1} \mathfrak{P}(\upsilon, y_\upsilon, I^\theta y(\upsilon))) d\upsilon - \lambda \int_0^{\mathsf{T}} (\mathsf{T} - \upsilon)^{\varepsilon-1} y_\upsilon d\upsilon \right) \\ + \ell \displaystyle\int_0^{\mathsf{T}} y_\upsilon d\upsilon + \ell_1 + \sum_{\mathfrak{k}=1}^{m} \tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-)) \text{ if } \mathfrak{r} \in (\mathfrak{r}_{\mathfrak{k}}, \mathfrak{r}_{\mathfrak{k}+1}]. \end{cases}$$

Let $\tilde{z}(.) : (-\infty, \mathsf{T}] \to \mathfrak{R}$ *be a function indicated by*

$$\tilde{z}(\mathfrak{r}) = \begin{cases} \rho(\mathfrak{r});\ \ \mathfrak{r} \in (-\infty, 0], \\ \ell \displaystyle\int_0^{\mathsf{T}} y_\upsilon d\upsilon + \ell_1;\ \ \mathfrak{r} \in \mathfrak{I}. \end{cases} \tag{3.1}$$

Then $\tilde{z}_0 = \rho$, $\forall\ u \in C(\mathfrak{I})$, *with* $u(0) = 0$, *define the function* $\tilde{u}$ *as*

$$\tilde{u} = \begin{cases} 0;\ \ -\infty < \mathfrak{r} \le 0, \\ u(\mathfrak{r});\ \ \mathfrak{r} \in \mathfrak{I}. \end{cases} \tag{3.2}$$

If $y(.)$ fulfills the integral equation

$$\begin{aligned}
y(\mathfrak{r}) =&\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\left(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{P}(\upsilon,y_\upsilon,I^\theta y(\upsilon))d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\left(\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{P}(\upsilon,y_\upsilon,I^\theta y(\upsilon))d\upsilon-\lambda\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&-\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\mathfrak{P}(\upsilon,y_\upsilon,I^\theta y(\upsilon))d\upsilon-\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\ell\int_0^{\mathsf{T}}y_\upsilon d\upsilon+\ell_1+\sum_{\mathfrak{k}=1}^{m}\tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-)).
\end{aligned}\tag{3.3}$$

We can decompose $y(.)$ as $y(\mathfrak{r})=\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r})$; for $\mathfrak{r}\in\mathfrak{I}$, which shows that $y_\mathfrak{r}=\tilde{u}_\mathfrak{r}+\tilde{z}_\mathfrak{r}$ for all $\mathfrak{r}\in\mathfrak{I}$, and $u(.)$ fulfills

$$\begin{aligned}
u(\mathfrak{r}) =&\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\left(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\left(\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&-\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{n}}\tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-)),
\end{aligned}\tag{3.4}$$

where

$$\mathfrak{h}_\wr(\mathfrak{r})=\mathfrak{P}(\mathfrak{r},\tilde{u}_\mathfrak{r}+\tilde{z}_\mathfrak{r},I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r})),\ y_\mathfrak{r}=\tilde{u}_\mathfrak{r}+\tilde{z}_\mathfrak{r}\ \textit{and}\ \tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-))=\tilde{I}_\mathfrak{k}(\tilde{u}(\mathfrak{r}_\mathfrak{k}^-)+\tilde{z}(\mathfrak{r}_\mathfrak{k}^-)).$$

Let Λ_0 be the Banach space

$$\Lambda_0=\{u\in C(\mathfrak{I});\ u_0=0\}.$$

The norm $\|.\|_\mathsf{T}$ in Λ_0 is denoted by

$$\|u\|_T=\|u_0\|_\aleph+\sup_{\mathfrak{r}\in\mathfrak{I}}|u(\mathfrak{r})|=\sup_{\mathfrak{r}\in\mathfrak{I}}|u(\mathfrak{r})|;\ u\in\Lambda_0.$$

Denote the operator $\Upsilon^*:\Lambda_0\to\Lambda_0$ by

$$\begin{aligned}
(\Upsilon^* u)(\mathfrak{r}) =&\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\left(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\left(\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\right)\\
&-\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}y_\upsilon d\upsilon\right)\\
&+\sum_{\mathfrak{k}=1}^{m}\tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-)).
\end{aligned}\tag{3.5}$$

As a result, the operators Π *and* Υ^* *have a fixed point, that are equivalent. Now, we shall show that* $\Upsilon^*:\Lambda_0\to\Lambda_0$ *is a contraction map.*

Take $u,u^{'}\in\Lambda_0$*, then* $\forall\ \mathfrak{r}\in\mathfrak{I}$,

$$\begin{aligned}|\Upsilon^*(u)(\mathfrak{r})-\Upsilon^*(u^{'})(\mathfrak{r})|\leq&\Big|\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\Big(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\Big)\\&-\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\Big(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}g(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\bar{y}_\upsilon d\upsilon\Big)\Big|\\&+\Big|\frac{1}{\Gamma(\varpi+\varepsilon)}\Big(\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\Big)\\&-\frac{1}{\Gamma(\varpi+\varepsilon)}\Big(\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}g(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\bar{y}_\upsilon d\upsilon\Big)\Big|\\&+\Big|\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\Big(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}y_\upsilon d\upsilon\Big)\\&-\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\Big(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}g(\upsilon)d\upsilon-\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\bar{y}_\upsilon d\upsilon\Big)\Big|\\&+\Big|\sum_{\mathfrak{k}=1}^{m}\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))-\sum_{\mathfrak{k}=1}^{m}\tilde{I}_{\mathfrak{k}}(\bar{y}(\mathfrak{r}_{\mathfrak{k}}^-))\Big|=G_1+G_2+G_3+G_4,\end{aligned}$$

where $\mathfrak{h}_\wr, g\in\bar{C}_\wr(\mathfrak{I})$ *like that*

$$\mathfrak{h}_\wr(\mathfrak{r})=\mathfrak{P}(\mathfrak{r},\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r})))\text{ and }g(\mathfrak{r})=\mathfrak{P}(\mathfrak{r},\tilde{u^{'}}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u^{'}}(\mathfrak{r})+\tilde{z}(\mathfrak{r}))).$$

From (H_1)*, we get*

$$\begin{aligned}|\mathfrak{h}_\wr(\mathfrak{r})-g(\mathfrak{r})|&=|\mathfrak{P}(\mathfrak{r},\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r})))-\mathfrak{P}(\mathfrak{r},\tilde{u^{'}}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u^{'}}(\mathfrak{r})+\tilde{z}(\mathfrak{r})))|\\&\leq\Big[k_1+k_2H^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\Big]||\tilde{u}_{\mathfrak{r}}-\tilde{u^{'}}_{\mathfrak{r}}||_{\aleph}\end{aligned}$$

$$||y_{\mathfrak{r}}-\bar{y}_{\mathfrak{r}}||=||\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}}-\tilde{u^{'}}_{\mathfrak{r}}-\tilde{z}_{\mathfrak{r}}||=||\tilde{u}_{\mathfrak{r}}-\tilde{u^{'}}_{\mathfrak{r}}||$$

and

$$\begin{aligned}|\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))-\tilde{I}_{\mathfrak{k}}(\bar{y}(\mathfrak{r}_{\mathfrak{k}}^-))|&=|\tilde{I}_{\mathfrak{k}}(\tilde{u}(\mathfrak{r}_{\mathfrak{k}}^-)+\tilde{z}((\mathfrak{r}_{\mathfrak{k}}^-)))-\tilde{I}_{\mathfrak{k}}(\tilde{u^{'}}(\mathfrak{r}_{\mathfrak{k}}^-)+\tilde{z}((\mathfrak{r}_{\mathfrak{k}}^-)))|\\&\leq m\wp||\tilde{u}_{\mathfrak{r}}-\tilde{u^{'}}_{\mathfrak{r}}||.\end{aligned}$$

Here,

$$\begin{aligned}G_1=&\Big|\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\Big(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon\Big)\\&-\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\Big(\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}g(\upsilon)d\upsilon-\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\bar{y}_\upsilon d\upsilon\Big)\Big|\end{aligned}$$

$$\leq \frac{1}{\Gamma(\varpi+\varepsilon)} \sum_{0<\mathfrak{r}_\mathfrak{k}<\mathfrak{r}} \int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_\mathfrak{k}} (\mathfrak{r}_\mathfrak{k}-\upsilon)^{\varpi+\varepsilon-1} |\mathfrak{h}_\wr(\upsilon)-g(\upsilon)| d\upsilon + \frac{1}{\Gamma(\varpi+\varepsilon)} |\lambda| \sum_{0<\mathfrak{r}_\mathfrak{k}<\mathfrak{r}} \int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_\mathfrak{k}} (\mathfrak{r}_\mathfrak{k}-\upsilon)^{\varpi+\varepsilon-1} |y_\upsilon-\bar{y}_\upsilon| d\upsilon$$
$$\leq \frac{\mathsf{T}^{\varpi+\varepsilon} m}{\Gamma(\varpi+\varepsilon+1)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{|\lambda| m}{\Gamma(\varpi+\varepsilon+1)} \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph.$$

$$\begin{aligned} G_2 =& \Big| \frac{1}{\Gamma(\varpi+\varepsilon)} \Big(\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} \mathfrak{h}_\wr(\upsilon) d\upsilon - \lambda \int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} y_\upsilon d\upsilon \Big) \\ & - \frac{1}{\Gamma(\varpi+\varepsilon)} \Big(\int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} g(\upsilon) d\upsilon - \lambda \int_{\mathfrak{r}_\mathfrak{k}}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} \bar{y}_\upsilon d\upsilon \Big) \Big| \\ & \leq + \frac{1}{\Gamma(\varpi+\varepsilon)} \int_{\mathfrak{r}_m}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} |\mathfrak{h}_\wr(\upsilon)-g(\upsilon)| d\upsilon + \frac{1}{\Gamma(\varpi+\varepsilon)} |\lambda| \int_{\mathfrak{r}_m}^{\mathfrak{r}} (\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1} |y_\upsilon-\bar{y}_\upsilon| d\upsilon \\ & + \frac{\mathsf{T}^{\varpi+\varepsilon}}{\Gamma(\varpi+\varepsilon+1)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{|\lambda|}{\Gamma(\varpi+\varepsilon+1)} \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph. \end{aligned}$$

$$\begin{aligned} G_3 =& \Big| \frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)} \Big(\int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} \mathfrak{h}_\wr(\upsilon) d\upsilon - \lambda \int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} y_\upsilon d\upsilon \Big) \\ & - \frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)} \Big(\int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} g(\upsilon) d\upsilon - \lambda \int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} \bar{y}_\upsilon d\upsilon \Big) \Big| \\ \leq & \frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)} \int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} |\mathfrak{h}_\wr(\upsilon)-g(\upsilon)| d\upsilon + \frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)} |\lambda| \int_0^{\mathsf{T}} (\mathsf{T}-\upsilon)^{\varepsilon-1} |y_\upsilon-\bar{y}_\upsilon| d\upsilon \\ \leq & \frac{\mathsf{T}^{\varepsilon+\varpi}}{\varepsilon\Gamma(\varepsilon+\varpi)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{\mathsf{T}^{\varepsilon+\varpi}}{\varepsilon\Gamma(\varepsilon+\varpi)} |\lambda| \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph. \end{aligned}$$

$$\begin{aligned} G_4 &= \Big| \sum_{\mathfrak{k}=1}^{m} \tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-)) - \sum_{\mathfrak{k}=1}^{m} \tilde{I}_\mathfrak{k}(\bar{y}(\mathfrak{r}_\mathfrak{k}^-)) \Big| \\ &\leq \sum_{\mathfrak{k}=1}^{m} |\tilde{I}_\mathfrak{k}(y(\mathfrak{r}_\mathfrak{k}^-)) - \tilde{I}_\mathfrak{k}(\bar{y}(\mathfrak{r}_\mathfrak{k}^-))| \\ &\leq \mathfrak{n}\wp \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph. \end{aligned}$$

Thus, $\forall\ \mathfrak{r} \in \mathfrak{I}$, by using G_1, G_2, G_3, G_4 we get

$$\begin{aligned} & |\Upsilon^*(u)(\mathfrak{r}) - \Upsilon^*(u')(\mathfrak{r})| \\ \leq \ & \frac{\mathsf{T}^{\varpi+\varepsilon} m}{\Gamma(\varpi+\varepsilon+1)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{|\lambda| m}{\Gamma(\varpi+\varepsilon+1)} \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph \\ & + \frac{\mathsf{T}^{\varpi+\varepsilon}}{\Gamma(\varpi+\varepsilon+1)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{|\lambda|}{\Gamma(\varpi+\varepsilon+1)} \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph \\ & + \frac{\mathsf{T}^{\varepsilon+\varpi}}{\varepsilon\Gamma(\varepsilon+\varpi)} \left[k_1 + k_2 H^* \frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right] \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph + \frac{\mathsf{T}^{\varepsilon+\varpi}}{\varepsilon\Gamma(\varepsilon+\varpi)} |\lambda| \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph \\ & + \mathfrak{n}\wp \|\tilde{u}_\mathfrak{r} - \tilde{u'}_\mathfrak{r}\|_\aleph \end{aligned}$$

$$\leq \left[\left(\frac{(1+\mathfrak{n})}{\Gamma(\varpi+\varepsilon+1)}+\frac{1}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\mathsf{T}^{\varpi+\varepsilon}\left(k_1+k_2H^*\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}+|\lambda|\right)+\mathfrak{n}\wp\right]\|\tilde{u}_{\mathfrak{r}}-\tilde{u'}_{\mathfrak{r}}\|_{\aleph}$$
$$\leq \nabla\|\tilde{u}_{\mathfrak{r}}-\tilde{u'}_{\mathfrak{r}}\|_{\aleph}$$
$$\leq \left[\left(\frac{(1+\mathfrak{n})}{\Gamma(\varpi+\varepsilon+1)}+\frac{1}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\mathsf{T}^{\varpi+\varepsilon}\left(k_1+k_2H^*\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}+|\lambda|\right)+\mathfrak{n}\wp\right]\|\tilde{u}_{\mathfrak{r}}-\tilde{u'}_{\mathfrak{r}}\|_{\aleph}$$
$$\leq \nabla\|\tilde{u}-\tilde{u'}\|_{\mathsf{T}}.$$

Thus

$$|\Upsilon^*(u)(\mathfrak{r})-\Upsilon^*(u')(\mathfrak{r})|\leq\nabla\|\tilde{u}-\tilde{u'}\|_{\mathsf{T}}.$$

From (3.1), Π *is a contraction. The unique solution for the problem* (1.1) *is the fixed point of the operator* Π, *according to the Banach contraction theorem. The proof is now complete.*

Theorem 3.2. *Consider the hypotheses (H1) and (H2) are hold. If*

$$\left\{\left(\frac{\mathsf{T}^{\varpi+\varepsilon}(\mathfrak{n}+1)}{\Gamma(\varpi+\varepsilon+1)}+\frac{\mathsf{T}^{\varpi+\varepsilon}}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\left[K_1+K_2H^*\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}+|\lambda|\right]+\mathfrak{n}\wp\right\}\left(M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}\right)<1,$$

then (1.1) *has at least one solution on* $(-\infty,\mathsf{T}]$.
Proof. Consider $\Upsilon^*:\Lambda_0\to\Lambda_0$.
Consider $\bar{\mathcal{P}}>0$ *and*

$$\bar{\mathcal{P}}\geq\max\left\{\|\rho\|_{\bar{C}_\wr(\mathcal{P}\tilde{C}_\wr,\mathcal{R})}\frac{|\ell|\mathsf{T}+|\ell_1|+\mathfrak{n}\wp}{1-\left[\left(\frac{(\mathfrak{n}+1)}{\Gamma(\varpi+\varepsilon+1)}+\frac{1}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\mathsf{T}^{\varpi+\varepsilon}\left(K_1+K_2H\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}+|\lambda|\right)\right]}\right\}.$$

Denote the ball

$$\mathcal{B}_{\bar{\mathcal{P}}}=\{y\in\bar{C}_\wr(\mathfrak{I},\mathfrak{R}),\ \|y\|_{\mathsf{T}}\leq\bar{\mathcal{P}}\}.$$

Here, the operator $\Upsilon^*:\mathcal{B}_{\bar{\mathcal{P}}}\to\mathcal{B}_{\bar{\mathcal{P}}}$ *fulfills all conditions of Lemma 2.3. The proof would be presented in few steps.*
Step 1 : Υ^* *is continuous.*
Take the sequence u_m *such that* $u_m\to u$ *in* $\mathcal{B}_{\bar{\mathcal{P}}}$. $\forall\ \mathfrak{r}\in\mathfrak{I}$, *we have*

$$\begin{aligned}\|\Upsilon(u_m)(\mathfrak{r})-\Upsilon(u)(\mathfrak{r})\| \leq\ & \frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_{\wr m}(\upsilon)-\mathfrak{h}_\wr(\upsilon)|d\upsilon\\ &+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_{\mathfrak{r}_m}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_{\wr m}(\upsilon)-\mathfrak{h}_\wr(\upsilon)|d\upsilon\\ &+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_{m\upsilon}-y_\upsilon|d\upsilon\\ &+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\int_{\mathfrak{r}_{\mathfrak{n}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|y_{m\upsilon}-y_\upsilon|d\upsilon\\ &+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|\mathfrak{h}_{\wr m}(\upsilon)-\mathfrak{h}_\wr(\upsilon)|d\upsilon\end{aligned}$$

$$+\frac{t^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}|\lambda|\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|y_{m\upsilon}-y_\upsilon|d\upsilon$$
$$+\sum_{\mathfrak{k}=1}^{\mathfrak{n}}|\tilde{I}_{\mathfrak{k}}(y_m(\mathfrak{r}_{\mathfrak{k}}^-))-\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))|,$$

where $\mathfrak{h}_{\wr m}, \mathfrak{h}_\wr \in \bar{\mathcal{C}}_\wr(\mathfrak{I},\mathcal{R})$ *like that*

$$\mathfrak{h}_{\wr m}(\mathfrak{r})=\mathfrak{P}(\mathfrak{r},\tilde{u}_{m\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u}_m(\mathfrak{r})+\tilde{z}(\mathfrak{r})))\text{ and }\mathfrak{h}_\wr(\mathfrak{r})=\mathfrak{P}(\mathfrak{r},\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I(u(\mathfrak{r})+\tilde{z}(\mathfrak{r})),$$

$$y_{m\mathfrak{r}}=\tilde{u}_{m\mathfrak{r}}+\tilde{z}_{m\mathfrak{r}}\text{ and }y_{\mathfrak{r}}=\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}}.$$

Here, $\|y_m-y\|_T\to 0$ *as* $m\to\infty$ *and* $\mathfrak{P},\mathfrak{h}_\wr$ *and* $\mathfrak{h}_{\wr m}$ *are continuous then by the Lebesgue dominated convergence theorem*

$$\|\Upsilon^*(u_m)-\Upsilon^*(u)\|_{\mathsf{T}}\to 0\text{ as }m\to\infty.$$

Hence, Υ^* *is continuous.*

Step 2 : $\Upsilon^*(\mathcal{B}_{\bar{\mathcal{P}}})\subset\mathcal{B}_{\bar{\mathcal{P}}}$.

Consider $y\in\mathcal{B}_{\bar{\mathcal{P}}}$, $\forall\ \mathfrak{r}\in\mathfrak{I}$ *and from* (H_2), *we get*

$$\begin{aligned}|\mathfrak{h}_\wr(\mathfrak{r})| &\le |\mathfrak{P}(\mathfrak{r},\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}},I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r}))|\\
&\le K_1\|\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}}\|_{\aleph}+K_2|I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r}))|\\
&\le K_1\|\tilde{u}_{\mathfrak{r}}\|_{\aleph}+K_1\|\tilde{z}_{\mathfrak{r}}\|_{\aleph}+K_2H^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\|\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r})\|\\
&\le K_1M^*\bar{\mathcal{P}}+K_1K^*\|\rho\|_{\aleph}+K_2H^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}[M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}]\\
&\le \left[K_1++K_2H^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right]\left(M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}\right)\end{aligned}$$

then

$$\|\mathfrak{h}_\wr\|\le\left[K_1+K_2H^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right]\left(M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}\right)$$

and

$$|y_{\mathfrak{r}}|=|\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}}|\le\left(M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}\right).$$

Thus,

$$\begin{aligned}|(\Upsilon^*u)(\mathfrak{r})| &\le \frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_\wr(\upsilon)|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_\upsilon|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_{\mathfrak{r}_{\mathfrak{n}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_\wr(\upsilon)|d\upsilon\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\int_{\mathfrak{r}_{\mathfrak{n}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|y_{\upsilon}|d\upsilon\\
&+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon\\
&+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}|\lambda|\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|y_{\upsilon}|d\upsilon\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{n}}|\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^{-}))|\\
&\le\left\{\left(\frac{\mathsf{T}^{\varpi+\varepsilon}(\mathfrak{n}+1)}{\Gamma(\varpi+\varepsilon+1)}+\frac{\mathsf{T}^{\varpi+\varepsilon}}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\left[K_1+K_2H^*\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}+|\lambda|\right]+\mathfrak{n}\wp\right\}\\
&\quad\left(M^*\bar{\mathcal{P}}+K^*\|\rho\|_{\aleph}\right)\\
&\le\bar{\mathcal{P}}.
\end{aligned}
$$

Hence,

$$\|\Upsilon^*(u)\|_{\aleph}\le\bar{\mathcal{P}}.$$

Consequently, $\Pi(\mathcal{B}_{\bar{\mathcal{P}}})\subset\mathcal{B}_{\bar{\mathcal{P}}}$.

Step 3 : $\Upsilon^*(\mathcal{B}_{\bar{\mathcal{P}}})$ *is equicontinuous.*

For $0\le\mathfrak{r}_{\mathfrak{k}-1}\le\mathfrak{r}_{\mathfrak{k}}\le\mathsf{T}$ *and* $y\in\mathcal{B}_{\bar{\mathcal{P}}}$, *we have*

$$
\begin{aligned}
&|(\Upsilon^*u)(\mathfrak{r}_{\mathfrak{k}-1})-(\Upsilon^*u)(\mathfrak{r}_{\mathfrak{k}})|\\
\le\;&\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_0^{\mathfrak{r}_{\mathfrak{k}-1}}\left((\mathfrak{r}_{\mathfrak{k}-1}-\upsilon)^{\varpi+\varepsilon-1}-(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}\right)|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_0^{\mathfrak{r}_{\mathfrak{k}-1}}\left((\mathfrak{r}_{\mathfrak{k}-1}-\upsilon)^{\varpi+\varepsilon-1}-(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}\right)|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\sum_{0<\mathfrak{r}_{\mathfrak{k}}<\mathfrak{r}}\left(\int_0^{\mathfrak{r}_{\mathfrak{k}-1}}(\mathfrak{r}_{\mathfrak{k}-1}-\upsilon)^{\varpi+\varepsilon-1}-(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_{\upsilon}|d\upsilon+\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_{\upsilon}|d\upsilon\right)\\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}|\lambda|\left(\int_0^{\mathfrak{r}_{\mathfrak{k}-1}}(\mathfrak{r}_{\mathfrak{k}-1}-\upsilon)^{\varpi+\varepsilon-1}-(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_{\upsilon}|d\upsilon+\int_{\mathfrak{r}_{\mathfrak{k}-1}}^{\mathfrak{r}_{\mathfrak{k}}}(\mathfrak{r}_{\mathfrak{k}}-\upsilon)^{\varpi+\varepsilon-1}|y_{\upsilon}|d\upsilon\right)\\
&+\frac{\mathfrak{r}_2^{\varepsilon+\varpi-1}-\mathfrak{r}_1^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\left(\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|\mathfrak{h}_{\wr}(\upsilon)|d\upsilon+|\lambda|\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|y_{\upsilon}|d\upsilon\right)\\
&+\sum_{\mathfrak{k}=1}^{\mathfrak{n}}|\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^{-}))-\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}-1}^{-}))|\\
\le\;&\left[(\mathfrak{r}_{\mathfrak{k}}^{\varpi+\varepsilon}-\mathfrak{r}_{\mathfrak{k}-1}^{\varpi+\varepsilon})\frac{(\mathfrak{n}+1)}{\Gamma(\varpi+\varepsilon+1)}+\frac{\left(\mathfrak{r}_{\mathfrak{k}}^{\varepsilon+\varpi-1}-\mathfrak{r}_{\mathfrak{k}-1}^{\varepsilon+\varpi-1}\right)}{\varepsilon\Gamma(\varepsilon+\varpi)}\right]\left(K_1+K_2H^*\frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)}\right)+\mathfrak{n}\wp\|(y(\mathfrak{r}_{\mathfrak{k}}))-(y(\mathfrak{r}_{\mathfrak{k}-1}))\|.
\end{aligned}
$$

$|(\Upsilon^* u)(\mathfrak{r}_{\mathfrak{k}-1}) - (\Upsilon^* u)(\mathfrak{r}_{\mathfrak{k}})| \rightarrow 0$ *as* $\mathfrak{r}_{\mathfrak{k}-1} \rightarrow \mathfrak{r}_{\mathfrak{k}}$, *this gives* $\Pi(\mathcal{B}_{\bar{\mathcal{P}}})$ *is equicontinuous.* Π *is completely continuous by steps 1 to 3 along with Arzela-Ascoli theorem.*

Step 4 : *The priori bounds.*

We have to prove that the set

$$\epsilon = \{y \in \Lambda_0 : y = \varsigma\Upsilon^*(y);\ \textit{for some}\ \varsigma \in (0,1)\}$$

is bounded. Let $u \in \Lambda_0$. *Let* $y \in \Lambda_0$, *such that* $u = \varsigma\Upsilon^*(u)$; *for some* $\varsigma \in (0,1)$. *Then, for all* $\mathfrak{r} \in \mathfrak{I}$, *we get*

$$\begin{aligned}
u(\mathfrak{r}) = \varsigma(\Upsilon^* u)(\mathfrak{r}) &= \varsigma\Big\{\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon \\
&-\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon \\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon \\
&-\frac{1}{\Gamma(\varpi+\varepsilon)}\lambda\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}y_\upsilon d\upsilon \\
&-\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}\mathfrak{h}_\wr(\upsilon)d\upsilon \\
&+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}y_\upsilon d\upsilon + \sum_{\mathfrak{k}=1}^{\mathfrak{n}}\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))\Big\}. \\
|\mathfrak{h}_\wr(\mathfrak{r})| &\leq |\mathfrak{P}(\mathfrak{r},\tilde{u}_{\mathfrak{r}}+\tilde{z}_{\mathfrak{r}}, I^\theta(\tilde{u}(\mathfrak{r})+\tilde{z}(\mathfrak{r}))| \\
&\leq \left[K + +LH^*\frac{\mathsf{T}^\theta}{\Gamma(\theta+1)}\right]\Big(M^*\bar{\mathcal{P}} + K^*\|\rho\|_{\aleph}\Big) := \psi.
\end{aligned}$$

Thus, $\forall\ \mathfrak{r} \in \mathfrak{I}$, *we attain*

$$\begin{aligned}
u(\mathfrak{r}) &\leq \frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_\wr(\upsilon)|d\upsilon \\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\sum_{i=1}^{\mathfrak{k}}\lambda\int_{\mathfrak{r}_{i-1}}^{\mathfrak{r}_i}(\mathfrak{r}_i-\upsilon)^{\varpi+\varepsilon-1}|y_\upsilon|d\upsilon \\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|\mathfrak{h}_\wr(\upsilon)|d\upsilon \\
&+\frac{1}{\Gamma(\varpi+\varepsilon)}\lambda\int_{\mathfrak{r}_{\mathfrak{k}}}^{\mathfrak{r}}(\mathfrak{r}-\upsilon)^{\varpi+\varepsilon-1}|y_\upsilon|d\upsilon \\
&+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|\mathfrak{h}_\wr(\upsilon)|d\upsilon \\
&+\frac{\mathfrak{r}^{\varepsilon+\varpi-1}}{\mathsf{T}^{\varepsilon-1}\Gamma(\varepsilon+\varpi)}\lambda\int_0^{\mathsf{T}}(\mathsf{T}-\upsilon)^{\varepsilon-1}|y_\upsilon|d\upsilon + \sum_{\mathfrak{k}=1}^{\mathfrak{n}}|\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^-))|
\end{aligned}$$

$$\leq \left[\frac{\mathfrak{n}+1}{\Gamma(\varpi+\varepsilon+1)}+\frac{1}{\varepsilon\Gamma(\varepsilon+\varpi)}\right]\psi\mathsf{T}^{\varpi+\varepsilon}+\mathfrak{n}\wp$$
$$\leq \psi'. \tag{3.6}$$

Hence,

$$\|u\|_{\mathsf{T}} \leq \psi'.$$

The set ϵ is bounded. As a result of Lemma 2.6, the operator Υ^* has at least one fixed point $y \in \mathcal{B}_{\bar{\mathcal{P}}}$, which is a solution of the problem (1.1) *on* $(-\infty, \mathsf{T}]$.

Remark 3.1. *The study of the fractional relaxation differential equation with initial and boundary conditions has been developed by the authors (see reeferences [24, 25]). They have proved the existence by using Krasnoselskii's fixed-point theorem and Schauder fixed-point theorem. In our paper, we developed the new system of fractional integro-relaxation differential equations, which include the impulse and delay term with integral boundary conditions, and also proved the existence and uniqueness of the same by using the Schaefer's and Banach fixed-point theorems.*

4. Example

Consider the fractional relaxation impulsive integro-differential equation

$$\begin{cases} D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}y(\mathfrak{r})+\frac{1}{4}y(\mathfrak{r})=\mathfrak{P}\left(\mathfrak{r},y(\mathfrak{r}),I^{\frac{1}{30}}y(\mathfrak{r})\right),\ \mathfrak{r}\neq\mathfrak{r}_{\mathfrak{k}}\ \mathfrak{r}\in[0,1],\\ \Delta y(\mathfrak{r}_{\mathfrak{k}})=\tilde{I}_{\mathfrak{k}}(y(\mathfrak{r}_{\mathfrak{k}}^{-})),\ \mathfrak{k}=1,2,...,\mathfrak{n},\\ y(\mathfrak{r})=\mathfrak{r}:\ \mathfrak{r}\in(-\infty,0],\\ {}^{LC}D^{\frac{1}{2}}y(0)={}^{LC}\ D^{\frac{1}{2}}y(1)=0,\ y(0)=\frac{1}{10}\int_0^1 y(s)ds+2. \end{cases} \tag{4.1}$$

Let $\delta > 0$ be a real constant and

$$B_\delta = \{y \in \bar{C}_\wr((-\infty,0],\mathfrak{R},) : \lim_{\eta\to\infty} e^{\delta\eta}y(\eta) \text{ exists in } \mathcal{R}\}.$$

The norm B_δ is provided by

$$\|y\|_\delta = \sup_{\eta\in(-\infty,0]} e^{\delta\eta}y(\eta).$$

Here $\varpi = \frac{1}{2}$, $\lambda = \frac{1}{4}$, $\varepsilon = \frac{3}{2}$, $\ell = \frac{1}{10}$, and $\ell_1 = 2$.

$$\mathfrak{P}\left(\mathfrak{r},y(\mathfrak{r}),I^{\frac{1}{30}}y(\mathfrak{r})\right)=\frac{sin(\mathfrak{r})}{\exp(\mathfrak{r}^2)+7}\left(\frac{1}{90(|y(\mathfrak{r})|+1)}+\frac{|I^{\frac{1}{30}}y(\mathfrak{r})|}{|1+I^{\frac{1}{30}}y(\mathfrak{r})|}\right).$$

For $y_i \in \mathfrak{R}$, $i = 1, 2$, we have

$$\begin{aligned} &|\mathfrak{P}(\mathfrak{r},y_1,y_2)-\mathfrak{P}(\mathfrak{r},y_1,y_2)|\\ =\ &\left|\frac{sin(\mathfrak{r})}{\exp(\mathfrak{r}^2)+7}\left(\left(\frac{1}{90(|y_1|+1)}-\frac{1}{90(|y_1|+1)}\right)+\left(\frac{|I^{\frac{1}{30}}y_2|}{|1+I^{\frac{1}{30}}y_2|}-\frac{|I^{\frac{1}{30}}y_2|}{|1+I^{\frac{1}{30}}y_2|}\right)\right)\right|\\ \leq\ &\frac{1}{8}\left(\frac{1}{90}\|y_1-y_1\|_{B_\delta}+\frac{1}{30}|y_2-y_2|\right). \end{aligned}$$

Hence, the hypothesis (H1) is fulfilled with $l = \frac{1}{240}$, $k = \frac{1}{720}$, $\mathsf{T} = 1$, $\theta = \frac{1}{30}$, $\mathfrak{n} = 1$ & $\wp = \frac{1}{2}$. Here $K^* = M^* = H^* = 1$, indeed

$$\begin{aligned}\nabla &= \left[\left(\frac{(\mathfrak{n}+1)}{\Gamma(\varpi+\varepsilon+1)} + \frac{1}{\varepsilon\Gamma(\varpi+\varepsilon)}\right)\mathsf{T}^{\varpi+\varepsilon}\left(K_1 + K_2 H^* \frac{\mathsf{T}^{\theta}}{\Gamma(\theta+1)} + |\lambda|\right) + \mathfrak{n}\wp\right] \\ &= \left[\left(\frac{(1+1)}{\Gamma(\frac{1}{2}+\frac{3}{2}+1)} + \frac{1}{\frac{3}{2}\Gamma(\frac{1}{2}+\frac{3}{2})}\right)\left(\frac{1}{720} + \frac{1}{240}\frac{1}{\Gamma(\frac{1}{30}+1)} + \frac{1}{4}\right) + \frac{1}{2}\right] \\ &\simeq 0.9249 < 1.\end{aligned}$$

The conditions of Theorem 3.1 are satisfied. It gives that the problem 4.1 has a solution, which is unique on $(-\infty, 1]$.

5. Conclusions

In this paper, we focused on investigating the existence and uniqueness of results for fractional relaxation differential equations with boundary conditions. Here, we defined the integral operator and proved the continuous and completely continuous functions using Arezela-Ascoli's theorem and the Lebesgue dominated convergence theorem. Under some hypothesis and Schaefer's fixed-point theorem, we proved the existence results for the system. Banach fixed-point theorem was used to prove the uniqueness of the solution of the system. The derived results have been justified by providing a suitable example. In the future, the aforesaid analysis can be extended to state-dependent delay or include the stochastic process.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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