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Research article

Weighted spectral geometric means and matrix equations of positive definite matrices involving semi-tensor products

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Abstract: We characterized weighted spectral geometric means (SGM) of positive definite matrices in terms of certain matrix equations involving metric geometric means (MGM) $\sharp$ and semi-tensor products $\ltimes$. Indeed, for each real number t and two positive definite matrices A and B of arbitrary sizes, the t-weighted SGM $A \diamond_t B$ of A and B is a unique positive solution X of the equation

$$A^{-1} \sharp X \;=\; (A^{-1} \sharp B)^t.$$

We then established fundamental properties of the weighted SGMs based on MGMs. In addition, $(A \diamond_{1/2} B)^2$ is positively similar to $A \ltimes B$ and, thus, they have the same spectrum. Furthermore, we showed that certain equations concerning weighted SGMs and MGMs of positive definite matrices have a unique solution in terms of weighted SGMs. Our results included the classical weighted SGMs of matrices as a special case.

Keywords: spectral geometric mean; metric geometric mean; positive definite matrix; semi-tensor product
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1. Introduction

In mathematics, we are familiar with the notion of geometric mean for positive real numbers. This notion was generalized to that for positive definite matrices of the same dimension in many ways. The metric geometric mean (MGM) of two positive definite matrices A and B is defined as

$$A \sharp B \;=\; A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}. \tag{1.1}$$

This mean was introduced by Pusz and Woronowicz [1] and studied in more detail by Ando [2]. Algebraically, $A \sharp B$ is a unique solution to the algebraic Riccati equation $XA^{-1}X = B$; e.g., [3]. Geometrically, $A \sharp B$ is a unique midpoint of the Riemannian geodesic interpolated from A to B, called the weighted MGM of A and B:

$$A \sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}, \quad 0 \leqslant t \leqslant 1. \tag{1.2}$$

Remarkable properties of the mean $\sharp_t$, where $t \in [0, 1]$, are monotonicity, concavity, and upper semi-continuity (according to the famous Löwner-Heinz inequality); see, e.g., [2,4] and a survey [5, Sect. 3]. Moreover, MGMs play an important role in the Riemannian geometry of the positive definite matrices; see, e.g., [6, Ch. 4].

Another kind of geometric means of positive definite matrices is the spectral geometric mean (SGM), first introduced by Fiedler and Pták [7]:

$$A \diamond B = (A^{-1} \sharp B)^{1/2} A (A^{-1} \sharp B)^{1/2}. \tag{1.3}$$

Note that the scalar consistency holds, i.e., if $AB = BA$, then

$$A \diamond B = A \sharp B = A^{1/2}B^{1/2}.$$

Since the SGM is based on the MGM, the SGM satisfies many nice properties as those for MGMs, for example, idempotency, homogeneity, permutation invariance, unitary invariance, self duality, and a determinantal identity. However, the SGM does not possess the monotonicity, the concavity, and the upper semi-continuity. A significant property of SGMs is that $(A \diamond B)^2$ is similar to AB and, they have the same spectrum; hence, the name "spectral geometric mean". The work [7] also established a similarity relation between the MGM $A \sharp B$ and the SGM $A \diamond B$ when A and B are positive definite matrices of the same size. After that, Lee and Kim [8] investigated the t-weighted SGM, where t is an arbitrary real number:

$$A \diamond_t B = (A^{-1} \sharp B)^t A (A^{-1} \sharp B)^t. \tag{1.4}$$

Gan and Tam [9] extended certain results of [7] to the case of the t-weighted SGMs when $t \in [0, 1]$. Many research topics on the SGMs have been widely studied, e.g., [10, 11]. Lim [12] introduced another (weighted) geometric mean of positive definite matrices varying over Hermitian unitary matrices, including the MGM as a special case. The Lim's mean has an explicit formula in terms of MGMs and SGMs.

There are several ways to extend the classical studies of MGMs and SGMs. The notion of MGMs can be defined on symmetric cones [8, 13] and reflection quasigroups [14] via algebraic-geometrical perspectives. In the framework of lineated symmetric spaces [14] and reflection quasigroups equipped with a compatible Hausdorff topology, we can define MGMs of arbitrary reals weights. The SGMs were also investigated on symmetric cones in [8]. These geometric means can be extended to those for positive (invertible) operators on a Hilbert space; see, e.g., [15, 16]. The cancellability of such means has significant applications in mean equations; see, e.g., [17, 18].

Another way to generalize the means (1.2) and (1.4) is to replace the traditional matrix multiplications (TMM) by the semi-tensor products (STP) $\ltimes$. Recall that the STP is a generalization of the TMM, introduced by Cheng [19]; see more information in [20]. To be more precise, consider a matrix pair $(A, B) \in \mathbb{M}_{m,n} \times \mathbb{M}_{p,q}$ and let $\alpha = \operatorname{lcm}(n, p)$. The STP of A and B allows the two matrices to participate the TMM through the Kronecker multiplication (denoted by $\otimes$) with certain identity matrices:

$$A \ltimes B \;=\; (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}) \;\in\; \mathbb{M}_{\frac{\alpha m}{n}, \frac{\alpha q}{p}}.$$

For the factor-dimension condition $n = kp$, we have

$$A \ltimes B \;=\; A(B \otimes I_k).$$

For the matching-dimension condition $n = p$, the product reduces to $A \ltimes B = AB$. The STP occupies rich algebraic properties as those for TMM, such as bilinearity and associativity. Moreover, STPs possess special properties that TMM does not have, for example, pseudo commutativity dealing with swap matrices, and algebraic formulations of logical functions. In the last decade, STPs were beneficial to developing algebraic state space theory, so the theory can integrate ideas and methods for finite state machines to those for control theory; see a survey in [21].

Recently, the work [22] extended the MGM notion (1.1) to any pair of positive definite matrices, where the matrix sizes satisfied the factor-dimension condition:

$$A \sharp B \;=\; A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^{1/2} \ltimes A^{1/2}. \tag{1.5}$$

In fact, $A \sharp B$ is a unique positive-definite solution of the semi-tensor Riccati equation $X \ltimes A^{-1} \ltimes X = B$. After that, the MGMs of arbitrary weight $t \in \mathbb{R}$ were studied in [23]. In particular, when $t \in [0, 1]$, the weighted MGMs have remarkable properties, namely, the monotonicity and the upper semi-continuity. See Section 2 for more details.

The present paper is a continuation of the works [22, 23]. Here we investigate SGMS involving STPs. We start with the matrix mean equation:

$$A^{-1} \sharp X \;=\; (A^{-1} \sharp B)^t,$$

where A and B are given positive definite matrices of different sizes, $t \in \mathbb{R}$, and X is an unknown square matrix. Here, $\sharp$ is defined by the formula (1.5). We show that this equation has a unique positive definite solution, which is defined to be the t-weighted SGM of A and B. Another characterization of weighted SGMs are obtained in terms of certain matrix equations. It turns out that this mean satisfies various properties as in the classical case. We establish a similarity relation between the MGM and the SGM of two positive definite matrices of arbitrary dimensions. Our results generalize the work [7] and relate to the work [8]. Moreover, we investigate certain matrix equations involving weighted MGMs and SGMs.

The paper is organized as follows. In Section 2, we set up basic notation and give basic results on STPs, Kronecker products, and weighted MGMs of positive definite matrices. In Section 3, we characterize the weighted SGM for positive definite matrices in terms of matrix equations, then we provide fundamental properties of weighted SGMs in Section 4. In Section 5, we investigate matrix equations involving weighted SGMs and MGMs. We conclude the whole work in Section 6.

2. Preliminaries

Throughout, let $\mathbb{M}_{m,n}$ be the set of all $m \times n$ complex matrices and abbreviate $\mathbb{M}_{n,n}$ to $\mathbb{M}_n$. Define $\mathbb{C}^n = \mathbb{M}_{n,1}$ as the set of n-dimensional complex vectors. Denote by A^T and A^* the transpose and conjugate transpose of a matrix A, respectively. The $n \times n$ identity matrix is denoted by I_n. The general linear group of $n \times n$ complex matrices is denoted by $\mathbb{GL}_n$. Let us denote the set of $n \times n$ positive definite matrices by $\mathbb{P}_n$. A matrix pair $(A, B) \in \mathbb{M}_{m,n} \times \mathbb{M}_{p,q}$ is said to satisfy a factor-dimension condition if $n \mid p$ or $p \mid n$. In this case, we write $A \succ_k B$ when $n = kp$, and $A \prec_k B$ when $p = kn$.

2.1. Kronecker and STPs of matrices

Recall that for any matrices $A = [a_{ij}] \in \mathbb{M}_{m,n}$ and $B \in \mathbb{M}_{p,q}$, their Kronecker product is defined by

$$A \otimes B = [a_{ij}B] \in \mathbb{M}_{mp,nq}.$$

The Kronecker operation $(A, B) \mapsto A \otimes B$ is bilinear and associative.

Lemma 2.1 (e.g. [5]). *Let $(A, B) \in \mathbb{M}_{m,n} \times \mathbb{M}_{p,q}$, $(C, D) \in \mathbb{M}_{n,r} \times \mathbb{M}_{q,s}$, and $(P, Q) \in \mathbb{M}_m \times \mathbb{M}_n$, then*

(i) $(A \otimes B)^* = A^* \otimes B^*$.
(ii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
(iii) If $(P, Q) \in \mathbb{GL}_m \times \mathbb{GL}_n$, then $(P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}$.
(iv) If $(P, Q) \in \mathbb{P}_m \times \mathbb{P}_n$, then $P \otimes Q \in \mathbb{P}_{mn}$ and $(P \otimes Q)^{1/2} = P^{1/2} \otimes Q^{1/2}$.

Lemma 2.2 (e.g. [20]). *Let $(A, B) \in \mathbb{M}_{m,n} \times \mathbb{M}_{p,q}$ and $(P, Q) \in \mathbb{M}_m \times \mathbb{M}_n$, then*

(i) $(A \ltimes B)^* = B^* \ltimes A^*$.
(ii) If $(P, Q) \in \mathbb{GL}_m \times \mathbb{GL}_n$, then $(P \ltimes Q)^{-1} = Q^{-1} \ltimes P^{-1}$.
(iii) $\det(P \ltimes Q) = (\det P)^{\alpha/m}(\det Q)^{\alpha/n}$ *where* $\alpha = \operatorname{lcm}(m, n)$.

Lemma 2.3 ([23]). *For any $S \in \mathbb{P}_m$ and $X \in \mathbb{M}_n$, we have $X^* \ltimes S \ltimes X \in \mathbb{P}_\alpha$, where $\alpha = \operatorname{lcm}(m, n)$.*

2.2. Weighted MGMs of positive definite matrices

Definition 2.4. Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $\alpha = \operatorname{lcm}(m, n)$. For any $t \in \mathbb{R}$, the t-weighted MGM of A and B is defined by

$$A \sharp_t B = A^{1/2} \ltimes \left(A^{-1/2} \ltimes B \ltimes A^{-1/2}\right)^t \ltimes A^{1/2} \in \mathbb{P}_\alpha. \tag{2.1}$$

Note that $A \sharp_0 B = A \otimes I_{\alpha/m}$ and $A \sharp_1 B = B \otimes I_{\alpha/n}$. We simply write $A \sharp B = A \sharp_{1/2} B$. We clearly have $A \sharp_t B > 0$ and $A \sharp_t A = A$.

Lemma 2.5 ([22]). *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ be such that $A \prec_k B$, then the Riccati equation*

$$X \ltimes A^{-1} \ltimes X = B$$

has a unique solution $X = A \sharp B \in \mathbb{P}_n$.

Lemma 2.6 ([23]). *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $X, Y \in \mathbb{P}_n$. Let $t \in \mathbb{R}$ and $\alpha = \operatorname{lcm}(m, n)$, then*

(i) *Positive homogeneity: For any scalars* $a, b, c > 0$, *we have* $c(A \sharp_t B) = (cA) \sharp_t (cB)$ *and, more generally,*

$$(aA) \sharp_t (bB) \;=\; a^{1-t} b^t (A \sharp_t B). \tag{2.2}$$

(ii) *Self duality:* $(A \sharp_t B)^{-1} = A^{-1} \sharp_t B^{-1}$.
(iii) *Permutation invariance:* $A \sharp_{1/2} B = B \sharp_{1/2} A$. *More generally,* $A \sharp_t B = B \sharp_{1-t} A$.
(iv) *Consistency with scalars: If* $A \ltimes B = B \ltimes A$, *then* $A \sharp B = A^{1-t} \ltimes B^t$.
(v) *Determinantal identity:*

$$\det(A \sharp B) \;=\; \sqrt{(\det A)^{\alpha/m} (\det B)^{\alpha/n}}.$$

(vi) *Cancellability: If* $t \neq 0$, *then the equation* $A \sharp_t X = A \sharp_t Y$ *implies* $X = Y$.

3. Characterizations of weighted SGMs in terms of matrix equations

In this section, we define and characterize weighted SGMs in terms of certain matrix equations involving MGMs and STPs.

Theorem 3.1. *Let* $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. *Let* $t \in \mathbb{R}$ *and* $\alpha = \operatorname{lcm}(m, n)$, *then the mean equation*

$$A^{-1} \sharp X \;=\; (A^{-1} \sharp B)^t \tag{3.1}$$

has a unique solution $X \in \mathbb{P}_\alpha$.

Proof. Note that the matrix pair (A, X) satisfies the factor-dimension condition. Let $Y = (A^{-1} \sharp B)^t$ and consider

$$X \;=\; Y \ltimes A \ltimes Y.$$

Using Lemma 2.5, we obtain that $Y = A^{-1} \sharp X$. Thus, $A^{-1} \sharp X = (A^{-1} \sharp B)^t$. For the uniqueness, let $Z \in \mathbb{P}_\alpha$ be such that $A^{-1} \sharp Z = Y$. By Lemma 2.5, we get

$$Z \;=\; Y \ltimes A \ltimes Y \;=\; X.$$

□

We call the matrix X in Theorem 3.1 the t-weighted SGM of A and B.

Definition 3.2. Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $\alpha = \operatorname{lcm}(m, n)$. For any $t \in \mathbb{R}$, the t-weighted SGM of A and B is defined by

$$A \diamond_t B \;=\; (A^{-1} \sharp B)^t \ltimes A \ltimes (A^{-1} \sharp B)^t \;\in\; \mathbb{M}_\alpha. \tag{3.2}$$

According to Lemma 2.3, we have $A \diamond_t B \in \mathbb{P}_\alpha$. In particular, $A \diamond_0 B = A \otimes I_{\alpha/m}$ and $A \diamond_1 B = B \otimes I_{\alpha/n}$. When $t = 1/2$, we simply write $A \diamond B = A \diamond_{1/2} B$. The formula (3.2) implies that

$$A \diamond_t A \;=\; A, \quad A \diamond_t A^{-1} \;=\; A^{1-2t} \tag{3.3}$$

for any $t \in \mathbb{R}$. Note that in the case $n \mid m$, we have

$$A \lozenge_t B \;=\; (A^{-1} \sharp B)^t A (A^{-1} \sharp B)^t,$$

i.e., Eq (3.2) reduces to the same formula (1.4) as in the classical case $m = n$. By Theorem 3.1, we have

$$A^{-1} \sharp (A \lozenge_t B) \;=\; (A^{-1} \sharp B)^t \;=\; (B \lozenge_t A)^{-1} \sharp B.$$

The following theorem provides another characterization of the weighted SGMs.

Theorem 3.3. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. Let $t \in \mathbb{R}$ and $\alpha = \operatorname{lcm}(m, n)$, then the following are equivalent:*

(i) $X = A \lozenge_t B$.
(ii) There exists a positive definite matrix $Y \in \mathbb{P}_\alpha$ such that

$$X \;=\; Y^t \ltimes A \ltimes Y^t \;=\; Y^{t-1} \ltimes B \ltimes Y^{t-1}. \tag{3.4}$$

Moreover, the matrix Y satisfying (3.4) is uniquely determined by $Y = A^{-1} \sharp B$.

Proof. Let $X = A \lozenge_t B$. Set $Y = A^{-1} \sharp B \in \mathbb{P}_\alpha$. By Definition 3.2, we have $X = Y^t \ltimes A \ltimes Y^t$. By Lemma 2.5, we get $Y \ltimes A \ltimes Y = B \otimes I_{\alpha/n}$. Hence,

$$Y^{t-1} \ltimes B \ltimes Y^{t-1} \;=\; Y^t Y^{-1} \ltimes B \ltimes Y^{-1} Y^t \;=\; Y^t \ltimes A \ltimes Y^t \;=\; X.$$

To show the uniqueness, let $Z \in \mathbb{P}_\alpha$ be such that

$$X \;=\; Z^t \ltimes A \ltimes Z^t = Z^{t-1} \ltimes B \ltimes Z^{t-1}.$$

We have $Z \ltimes A \ltimes Z = B \otimes I_{\alpha/n}$. Note that the pair $(A, B \otimes I_{\alpha/n})$ satisfies the factor-dimension condition. Now, Lemma 2.5 implies that $Z = A^{-1} \sharp B = Y$.

Conversely, suppose there exists a matrix $Y \in \mathbb{P}_\alpha$ such that Eq (3.4) holds, then $Y \ltimes A \ltimes Y = B$. Applying Lemma 2.5, we have $Y = A^{-1} \sharp B$. Therefore,

$$X \;=\; (A^{-1} \sharp B)^t \ltimes A \ltimes (A^{-1} \sharp B)^t \;=\; A \lozenge_t B.$$

□

4. Fundamental properties of weighted SGMs

Fundamental properties of the weighted SGMs (3.2) are as follows.

Theorem 4.1. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$, $t \in \mathbb{R}$, and $\alpha = \operatorname{lcm}(m, n)$, then*

(i) Permutation invariance: $A \lozenge_t B = B \lozenge_{1-t} A$. In particular, $A \lozenge B = B \lozenge A$.
(ii) Positive homogeneity: $c(A \lozenge_t B) = (cA) \lozenge_t (cB)$ for all $c > 0$. More generally, for any scalars $a, b > 0$, we have

$$(aA) \lozenge_t (bB) \;=\; a^{1-t} b^t (A \lozenge_t B).$$

(iii) Self-duality: $(A \between_t B)^{-1} = A^{-1} \between_t B^{-1}$.

(iv) Unitary invariance: For any $U \in \mathbb{U}_\alpha$, *we have*

$$U^*(A \between_t B)U = (U^* \ltimes A \ltimes U) \between_t (U^* \ltimes B \ltimes U). \tag{4.1}$$

(v) Consistency with scalars: If $A \ltimes B = B \ltimes A$, *then* $A \between_t B = A^{1-t} \ltimes B^t$.

(vi) Determinantal identity:

$$\det(A \between_t B) = (\det A)^{\frac{(1-t)\alpha}{m}} (\det B)^{\frac{t\alpha}{n}}.$$

(vii) Left and right cancellability: For any $t \in \mathbb{R} - \{0\}$ *and* $Y_1, Y_2 \in \mathbb{P}_n$, *the equation*

$$A \between_t Y_1 = A \between_t Y_2$$

implies $Y_1 = Y_2$. *For any* $t \in \mathbb{R} - \{1\}$ *and* $X_1, X_2 \in \mathbb{P}_m$, *the equation* $X_1 \between_t B = X_2 \between_t B$ *implies* $X_1 = X_2$. *In other words, the maps* $X \mapsto A \between_t X$ *and* $X \mapsto X \between_t B$ *are injective for any* $t \neq 0, 1$.

(viii) $(A \between B)^2$ *is positively similar to* $A \ltimes B$ *i.e., there is a matrix* $P \in \mathbb{P}_\alpha$ *such that*

$$(A \between B)^2 = P(A \ltimes B)P^{-1}.$$

In particular, $(A \between B)^2$ *and* $A \ltimes B$ *have the same eigenvalues.*

Proof. Throughout this proof, let $X = A \between_t B$ and $Y = A^{-1} \sharp B$. From Theorem 3.3, the characteristic equation (3.4) holds.

To prove (i), set $Z = B \between_{1-t} A$ and $W = B^{-1} \sharp A$. By Theorem 3.3, we get

$$Z = W^{1-t} \ltimes B \ltimes W^{1-t} = W^{-t} \ltimes A \ltimes W^{-t}.$$

It follows from Lemma 2.6(ii) that

$$W^{-1} = B \sharp A^{-1} = A^{-1} \sharp B = Y.$$

Hence, $X = Y^t \ltimes A \ltimes Y^t = W^{-t} \ltimes A \ltimes W^{-t} = Z$, i.e., $A \between_t B = B \between_{1-t} A$.

The assertion (ii) follows directly from the formulas (3.2) and (2.2):

$$\begin{aligned} (aA) \between_t (bB) &= (a^{-1}A^{-1} \sharp bB)^t \ltimes (aA) \ltimes (a^{-1}A^{-1} \sharp bB)^t \\ &= (a^{-1} \sharp b)^t (A^{-1} \sharp B)^t \ltimes (aA) \ltimes (a^{-1} \sharp b)^t (A^{-1} \sharp B)^t \\ &= (a^{-1} \sharp b)^t a (a^{-1} \sharp b)^t (A^{-1} \sharp B)^t \ltimes A \ltimes (A^{-1} \sharp B)^t \\ &= a^{1-t} b^t (A \between_t B). \end{aligned}$$

To prove the self-duality (iii), set $W = Y^{-1} = A \sharp B^{-1}$. Observe that

$$\begin{aligned} X^{-1} &= (Y^t \ltimes A \ltimes Y^t)^{-1} = Y^{-t} \ltimes A^{-1} \ltimes Y^{-t} = W^t \ltimes A^{-1} \ltimes W^t, \\ X^{-1} &= (Y^{t-1} \ltimes B \ltimes Y^{t-1})^{-1} = Y^{1-t} \ltimes B^{-1} \ltimes Y^{1-t} = W^{t-1} \ltimes B^{-1} \ltimes W^{t-1}. \end{aligned}$$

Theorem 3.3 now implies that

$$(A \between_t B)^{-1} = X^{-1} = A^{-1} \between_t B^{-1}.$$

To prove (iv), let $U \in \mathbb{U}_\alpha$ and consider $W = U^* \ltimes Y \ltimes U$. We have

$$\begin{aligned} W^t \ltimes U^* \ltimes A \ltimes U \ltimes W^t &= U^* \ltimes Y^t \ltimes U \ltimes U^* \ltimes A \ltimes U \ltimes U^* \ltimes Y^t \ltimes U \\ &= U^* \ltimes Y^t \ltimes A \ltimes Y^t \ltimes U \\ &= U^* \ltimes X \ltimes U, \end{aligned}$$

and, similarly,

$$W^{t-1} \ltimes U^* \ltimes B \ltimes U \ltimes W^{t-1} = U^* \ltimes Y^{t-1} \ltimes B \ltimes Y^{t-1} \ltimes U = U^* \ltimes X \ltimes U.$$

By Theorem 3.3, we arrive at (4.1).

For the assertion (v), the assumption $A \ltimes B = B \ltimes A$ together with Lemma 2.6 (iv) yields

$$Y = A^{-1} \sharp B = A^{-1/2} \ltimes B^{1/2}.$$

It follows that

$$\begin{aligned} Y^t \ltimes A \ltimes Y^t &= A^{-t/2} \ltimes B^{t/2} \ltimes A \ltimes A^{-t/2} \ltimes B^{t/2} = A^{1-t} \ltimes B^t, \\ Y^{t-1} \ltimes B \ltimes Y^{t-1} &= A^{-(t-1)/2} \ltimes B^{(t-1)/2} \ltimes B \ltimes A^{-(t-1)/2} \ltimes B^{(t-1)/2} = A^{1-t} \ltimes B^t. \end{aligned}$$

Now, Theorem 3.3 implies that $A \lozenge_t B = A^{1-t} \ltimes B^t$. The determinantal identity (vi) follows directly from the formula (1.4), Lemma 2.2(iii), and Lemma 2.6(v):

$$\begin{aligned} \det(A \lozenge_t B) &= \det(A^{-1} \sharp B)^{2t} (\det A)^{\frac{\alpha}{m}} \\ &= (\det A)^{-\frac{\alpha t}{m}} (\det B)^{\frac{\alpha t}{n}} (\det A)^{\frac{\alpha}{m}} \\ &= (\det A)^{\frac{(1-t)\alpha}{m}} (\det B)^{\frac{t\alpha}{n}}. \end{aligned}$$

To prove the left cancellability, let $t \in \mathbb{R} - \{0\}$ and suppose that $A \lozenge_t Y_1 = A \lozenge_t Y_2$. We have

$$\begin{aligned} \left(A^{1/2} \ltimes (A^{-1} \sharp Y_1)^t \ltimes A^{1/2}\right)^2 &= A^{1/2} \ltimes (A \lozenge_t Y_1) \ltimes A^{1/2} \\ &= A^{1/2} \ltimes (A \lozenge_t Y_2) \ltimes A^{1/2} \\ &= \left(A^{1/2} \ltimes (A^{-1} \sharp Y_2)^t \ltimes A^{1/2}\right)^2. \end{aligned}$$

Taking the positive square root yields

$$A^{1/2} \ltimes (A^{-1} \sharp Y_1)^t \ltimes A^{1/2} = A^{1/2} \ltimes (A^{-1} \sharp Y_2)^t \ltimes A^{1/2},$$

and, thus, $(A^{-1} \sharp Y_1)^t = (A^{-1} \sharp Y_2)^t$. Since $t \neq 0$, we get $A^{-1} \sharp Y_1 = A^{-1} \sharp Y_2$. Using the left cancellability of MGM (Lemma 2.6(vi)), we obtain $Y_1 = Y_2$. The right cancellability follows from the left cancellability together with the permutation invariance (i).

For the assertion (viii), since $A \lozenge B = Y^{1/2} \ltimes A \ltimes Y^{1/2} = Y^{-1/2} \ltimes B \ltimes Y^{-1/2}$, we have

$$\begin{aligned} (A \lozenge B)^2 &= (Y^{1/2} \ltimes A \ltimes Y^{1/2})(Y^{-1/2} \ltimes B \ltimes Y^{-1/2}) \\ &= Y^{1/2}(A \ltimes B) Y^{-1/2}. \end{aligned}$$

Note that the matrix $Y^{1/2}$ is positive definite. Thus, $(A \lozenge B)^2$ is positively similar to $A \ltimes B$, so they have the same eigenvalues. □

Remark 4.2. Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. Instead of Definition 3.2, the permutation invariance (i) provides an alternative definition of $A \diamond_t B$ as follows:

$$\begin{aligned} A \diamond_t B &= (B^{-1} \sharp A)^{1-t} \ltimes B \ltimes (B^{-1} \sharp A)^{1-t} \\ &= (A \sharp B^{-1})^{1-t} \ltimes B \ltimes (A \sharp B^{-1})^{1-t}. \end{aligned}$$

In particular, if $m \mid n$, we have

$$A \diamond_t B = (A \sharp B^{-1})^{1-t} B (A \sharp B^{-1})^{1-t}.$$

The assertion (viii) is the reason why $A \diamond B$ is called the SGM.

Now, we will show that $A \sharp B$ and $A \diamond_t B$ are positively similar when A and B are positive definite matrices of arbitrary sizes. Before that, we need the following lemma.

Lemma 4.3. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. Let $t \in \mathbb{R}$ and $\alpha = \operatorname{lcm}(m, n)$, then there exists a unique $Y_t \in \mathbb{P}_\alpha$ such that*

$$A \diamond_t B = Y_t \ltimes A \ltimes Y_t \quad \text{and} \quad B \diamond_t A = Y_t^{-1} \ltimes A \ltimes Y_t^{-1}.$$

Proof. Set $Y_t = (A^{-1} \sharp B)^t$, then $Y_t \ltimes A \ltimes Y_t = A \diamond_t B$. Using Lemma 2.6, we obtain that

$$Y_t^{-1} \ltimes B \ltimes Y_t^{-1} = (B^{-1} \sharp A)^t \ltimes B \ltimes (B^{-1} \sharp A)^t = B \diamond_t A.$$

To prove the uniqueness, let $Z_t \in \mathbb{P}_\alpha$ be such that $Z_t \ltimes A \ltimes Z_t = A \diamond_t B$ and $Z_t^{-1} \ltimes A \ltimes Z_t^{-1} = B \diamond_t A$. By Lemma 2.5, we get $Z_t = A^{-1} \sharp (A \diamond_t B)$, but Theorem 3.1 says that

$$A^{-1} \sharp (A \diamond_t B) = (A^{-1} \sharp B)^t.$$

Thus, $Z_t = Y_t$. □

Theorem 4.4. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. Let $t \in \mathbb{R}$ and $\alpha = \operatorname{lcm}(m, n)$, then $A \sharp B$ is positively similar to $(A \diamond_{1-t} B)^{1/2} U (A \diamond_t B)^{1/2}$ for some unitary $U \in \mathbb{M}_\alpha$.*

Proof. By Lemma 4.3, there exists $Y_t \in \mathbb{P}_\alpha$ such that $A \diamond_t B = Y_t \ltimes A \ltimes Y_t$ and $B \diamond_t A = Y_t^{-1} \ltimes A \ltimes Y_t^{-1}$. Using Lemmas 2.2 and 2.5, we have

$$\begin{aligned} Y_t (A \diamond_{1-t} B) Y_t &= B \otimes I_{\alpha/n} \\ &= (A \sharp B) \ltimes A^{-1} \ltimes (A \sharp B) \\ &= (A \sharp B) Y_t (A \diamond_t B)^{-1} Y_t (A \sharp B), \end{aligned}$$

then

$$\left((A \diamond_t B)^{-1/2} Y_t (A \sharp B) Y_t (A \diamond_t B)^{-1/2} \right)^2 = (A \diamond_t B)^{-1/2} Y_t^2 (A \diamond_{1-t} B) Y_t^2 (A \diamond_t B)^{-1/2}.$$

Thus,

$$A \sharp B = Y_t^{-1} (A \diamond_t B)^{1/2} \left((A \diamond_t B)^{-1/2} Y_t^2 (A \diamond_{1-t} B) Y_t^2 (A \diamond_t B)^{-1/2} \right)^{1/2} (A \diamond_t B)^{1/2} Y_t^{-1}.$$

Set $V = (A \diamond_t B)^{-1/2} Y_t^2 (A \diamond_{1-t} B)^{1/2}$ and $U = V^{-1}(VV^*)^{1/2}$. Obviously, U is a unitary matrix. We obtain

$$\begin{aligned} A \sharp B &= Y_t^{-1}(A \diamond_t B)^{1/2}(VV^*)^{1/2}(A \diamond_t B)^{1/2} Y_t^{-1} \\ &= Y_t(A\diamond_{1-t}B)^{1/2} V^{-1}(VV^*)^{1/2}(A \diamond_t B)^{1/2} Y_t^{-1} \\ &= Y_t(A\diamond_{1-t}B)^{1/2} U (A \diamond_t B)^{1/2} Y_t^{-1}. \end{aligned}$$

This implies that $(A\diamond_{1-t}B)^{1/2}U(A \diamond_t B)^{1/2}$ is positive similar to $A \sharp B$. □

In general, the MGM $A \natural_t B$ and the SGM $A \diamond_t B$ are not comparable (in the Löwner partial order). We will show that $A \natural_t B$ and $A \diamond_t B$ coincide in the case that A and B are commuting with respect to the STP. To do this, we need a lemma.

Lemma 4.5. *Let $(P, Q) \in \mathbb{P}_m \times \mathbb{P}_n$. If*

$$P \ltimes Q \ltimes P \ltimes Q^{-1} = Q \ltimes P \ltimes Q^{-1} \ltimes P, \tag{4.2}$$

then $P \ltimes Q = Q \ltimes P$.

Proof. From Eq (4.2), we have

$$\left(Q^{-1/2} \ltimes P \ltimes Q^{1/2}\right)\left(Q^{-1/2} \ltimes P \ltimes Q^{1/2}\right)^* = \left(Q^{-1/2} \ltimes P \ltimes Q^{1/2}\right)^* \left(Q^{-1/2} \ltimes P \ltimes Q^{1/2}\right).$$

This implies that $Q^{-1/2} \ltimes P \ltimes Q^{1/2}$ is a normal matrix. Since $Q^{-1/2} \ltimes P \ltimes Q^{1/2}$ and $P \otimes I_{\alpha/m}$ are similar matrices, we conclude that the eigenvalues of $Q^{-1/2} \ltimes P \ltimes Q^{1/2}$ are real and $Q^{-1/2} \ltimes P \ltimes Q^{1/2}$ is Hermitian. Hence,

$$Q^{-1/2} \ltimes P \ltimes Q^{1/2} = \left(Q^{-1/2} \ltimes P \ltimes Q^{1/2}\right)^* = Q^{1/2} \ltimes P \ltimes Q^{-1/2}.$$

Therefore, $P \ltimes Q = Q \ltimes P$. □

The next theorem generalizes [7, Theorem 5.1].

Theorem 4.6. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $t \in \mathbb{R}$. If $A \ltimes B = B \ltimes A$, then $A \natural_t B = A \diamond_t B$. In particular, $A \sharp B = A \diamond B$ if and only if $A \ltimes B = B \ltimes A$.*

Proof. Suppose $A \ltimes B = B \ltimes A$. By Lemma 2.6 and Theorem 4.1, we have

$$A \natural_t B = A^{1-t} \ltimes B^t = A \diamond_t B.$$

Next, assume that $A \sharp B = A \diamond B = X$. By Lemma 2.5, we have

$$X \ltimes A^{-1} \ltimes X = B \otimes I_{\alpha/n}.$$

Set $Y = A^{-1} \sharp B$. By Lemma 3.3, we get $X = Y^{1/2} \ltimes A \ltimes Y^{1/2} = Y^{-1/2} \ltimes B \ltimes Y^{-1/2}$. It follows that

$$\begin{aligned} Y^{1/2} \ltimes X \ltimes Y^{1/2} &= B \otimes I_{\alpha/n} = X \ltimes A^{-1} \ltimes X \\ &= X \ltimes Y^{1/2} \ltimes X^{-1} \ltimes Y^{1/2} \ltimes X. \end{aligned}$$

Thus,

$$Y^{1/2} \ltimes X \ltimes Y^{1/2} \ltimes X^{-1} = X \ltimes Y^{1/2} \ltimes X^{-1} \ltimes Y^{1/2}.$$

Lemma 4.5 implies that $X \ltimes Y^{1/2} = Y^{1/2} \ltimes X$. Hence,

$$\begin{aligned} A \ltimes B &= A \ltimes Y \ltimes A \ltimes Y = Y^{-1/2} \ltimes X^2 \ltimes Y^{1/2} \\ &= X^2 = Y^{1/2} \ltimes X^2 \ltimes Y^{-1/2} \\ &= Y \ltimes A \ltimes Y \ltimes A \\ &= B \ltimes A. \end{aligned}$$

□

Theorem 4.7. *Let* $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ *and* $\alpha = \operatorname{lcm}(m, n)$*, then the following statements are equivalent:*

(i) $A \lozenge B = I_\alpha$,
(ii) $A \otimes I_{\alpha/m} = B^{-1} \otimes I_{\alpha/n}$,
(iii) $A \sharp B = I_\alpha$.

Proof. First, we show the equivalence between the statements (i) and (ii). Suppose that $A \lozenge B = I_\alpha$. Letting $Y = A^{-1} \sharp B$, we have by Theorem 3.3 that

$$Y^{1/2} \ltimes A \ltimes Y^{1/2} = Y^{-1/2} \ltimes B \ltimes Y^{-1/2} = I_\alpha.$$

Applying Lemma 2.1, we obtain

$$A \otimes I_{\alpha/m} = Y^{-1} = B^{-1} \otimes I_{\alpha/n}.$$

Now, suppose $A \otimes I_{\alpha/m} = B^{-1} \otimes I_{\alpha/n}$. By Lemma 2.1, we have

$$\begin{aligned} A \ltimes B &= (A \otimes I_{\alpha/m})(B \otimes I_{\alpha/n}) = (B^{-1} \otimes I_{\alpha/n})(B \otimes I_{\alpha/n}) \\ &= I_n \otimes I_{\alpha/n} = I_\alpha, \end{aligned}$$

and similarly, $B \ltimes A = I_\alpha$. Now, Theorem 4.1(v) implies that

$$\begin{aligned} A \lozenge B &= A^{1/2} \ltimes B^{1/2} \\ &= (B^{-1/2} \otimes I_{\alpha/n})(B^{1/2} \otimes I_{\alpha/n}) = I_\alpha. \end{aligned}$$

Next, we show the equivalence between (ii) and (iii). Suppose that $A \sharp B = I_\alpha$, then we have

$$(A^{-1/2} \ltimes B \ltimes A^{-1/2})^{1/2} = A^{-1/2} \ltimes I_\alpha \ltimes A^{-1/2} = A^{-1} \otimes I_{\alpha/m}.$$

This implies that

$$A^{-1/2} \ltimes B \ltimes A^{-1/2} = (A^{-1} \otimes I_\alpha)^2 = A^{-2} \otimes I_{\alpha/m}.$$

Thus, $B \otimes I_{\alpha/n} = A^{-1} \otimes I_{\alpha/m}$ or $A \otimes I_{\alpha/m} = B^{-1} \otimes I_{\alpha/n}$.

Now, suppose (iii) holds, then we get $A \ltimes B = I_\alpha = B \ltimes A$. It follows from Lemma 2.6 (iv) that $A \sharp B = A^{1/2} \ltimes B^{1/2} = I_\alpha$. □

In particular from Theorem 4.7, when $m = n$, we have that $A \lozenge B = I_n$ if and only if $A = B^{-1}$, if and only if, $A \sharp B = I_n$. This result was included in [7] and related to the work [8].

5. Matrix equations involving metric and SGMs

In this section, we investigate matrix equations involving MGMs and SGMs of positive definite matrices. In particular, recall that the work [23] investigated the matrix equation $A \sharp_t X = B$. We discuss this matrix equation when the MGM $\sharp_t$ is replaced by the SGM $\Diamond_t$ in the next theorem.

Theorem 5.1. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ where $m \mid n$. Let $t \in \mathbb{R} - \{0\}$, then the mean equation*

$$A \Diamond_t X = B, \tag{5.1}$$

in an unknown $X \in \mathbb{P}_n$, is equivalent to the Riccati equation

$$W_t \ltimes A \ltimes W_t = B \tag{5.2}$$

in an unknown $W_t \in \mathbb{P}_n$. Moreover, Eq (5.1) has a unique solution given by

$$X = A \Diamond_{1/t} B = (A \sharp B^{-1})^{1-\frac{1}{t}} B (A \sharp B^{-1})^{1-\frac{1}{t}}. \tag{5.3}$$

Proof. Let us denote $W_t = (A^{-1} \sharp X)^t$ for each $t \in \mathbb{R} - \{0\}$. By Definition 3.2, we have

$$A \Diamond_t X = (A^{-1} \sharp X)^t \ltimes A \ltimes (A^{-1} \sharp X)^t = W_t \ltimes A \ltimes W_t.$$

Note that the map $X \mapsto W_t$ is injective due to the cancellability of the MGM $\sharp_t$ (Lemma 2.6(vi)). Thus, Eq (5.1) is equivalent to the Riccati equation (5.2). Now, Lemma 2.5 implies that Eq (5.2) is equivalent to $W_t = A^{-1} \sharp B$. Thus, Eq (5.1) is equivalent to the equation

$$(A^{-1} \sharp X)^t = A^{-1} \sharp B. \tag{5.4}$$

We now solve (5.4). Indeed, we have

$$A^{-1} \sharp X = (A^{-1} \sharp B)^{1/t}.$$

According to Theorem 3.1 and Definition 3.2, this equation has a unique solution denoted by the SGM of A and B with weight $1/t$. Now, Remark 4.2 provides the explicit formula (5.3) of $A \Diamond_{1/t} B$. □

Remark 5.2. For the case $n \mid m$ in Theorem 5.1, we get a similar result. In particular to the case $m \mid n$, the mean equation

$$A \Diamond X = B \tag{5.5}$$

has a unique solution $X = (A^{-1} \sharp B) B (A^{-1} \sharp B)$.

Theorem 5.3. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$. Let $t \in \mathbb{R} - \{0\}$ and $\alpha = \operatorname{lcm}(m, n)$, then the equation*

$$(A \sharp X) \sharp_t (B \sharp X) = I_\alpha \tag{5.6}$$

has a unique solution $X = A^{-1} \Diamond_t B^{-1} \in \mathbb{P}_\alpha$.

Proof. For the case $t = 0$, Lemma 2.5 tells us that the equation $A \sharp X = I_\alpha$ has a unique solution

$$X = A^{-1} \otimes I_{\alpha/m} = A^{-1} \Diamond_0 B^{-1}.$$

Now, assume that $t \neq 0$. To prove the uniqueness, let $U = A \sharp X$ and $V = B \sharp X$, then

$$U \ltimes A^{-1} \ltimes U = X = V \ltimes B^{-1} \ltimes V.$$

Since $U \sharp_t V = I_\alpha$, we obtain $(U^{-1/2} \ltimes V \ltimes U^{-1/2})^t = U^{-1}$ and, thus, $V = U^{(t-1)/t}$. It follows that

$$\begin{aligned} B \otimes I_{\alpha/n} &= V \ltimes X^{-1} \ltimes V = V \ltimes U^{-1} \ltimes A \ltimes U^{-1} \ltimes V \\ &= U^{-1/t} \ltimes A \ltimes U^{-1/t}. \end{aligned}$$

Using Lemma 2.5, we have that $U^{-1/t} = A^{-1} \sharp B$ and, thus, $U = (A^{-1} \sharp B)^{-t}$. Hence,

$$\begin{aligned} X &= (A^{-1} \sharp B)^{-t} \ltimes A^{-1} \ltimes (A^{-1} \sharp B)^{-t} \\ &= (A \sharp B^{-1})^t \ltimes A^{-1} \ltimes (A \sharp B^{-1})^t = A^{-1} \Diamond_t B^{-1}. \end{aligned}$$

□

Corollary 5.4. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $\alpha = \operatorname{lcm}(m, n)$, then the equation*

$$A \sharp X = B \sharp X^{-1} \tag{5.7}$$

has a unique solution $X = A^{-1} \Diamond B \in \mathbb{P}_\alpha$.

Proof. Equation (5.7) and Lemma 2.6 imply that

$$(A \sharp X)^{-1} = (B \sharp X^{-1})^{-1} = B^{-1} \sharp X.$$

Thus, Eq (5.7) is equivalent to the following equation:

$$(A \sharp X) \sharp_{1/2} (B^{-1} \sharp X) = I_\alpha.$$

Now, the desired solution follows from the case $t = 1/2$ in Theorem 5.3. □

In particular, when $m = n$ and $A = B$, the equation $A \sharp X = A \sharp X^{-1}$ has a unique solution $X = A \Diamond A^{-1} = A^0 = I$ by Eq (3.3).

Theorem 5.5. *Let $(A, B) \in \mathbb{P}_m \times \mathbb{P}_n$ and $\alpha = \operatorname{lcm}(m, n)$, then the equation*

$$(A \sharp X) \Diamond_t (B \sharp X) = I_\alpha \tag{5.8}$$

has a unique solution $X = A^{-1} \Diamond_t B^{-1} \in \mathbb{P}_\alpha$.

Proof. If $t = 0$, the equation $A \sharp X^{-1} = I_\alpha$ has a unique solution $X = A^{-1} \otimes I_{\alpha/m} = A^{-1} \Diamond_0 B^{-1}$. Now, consider $t \neq 0$, and let $U = A \sharp X$ and $V = B \sharp X$, then

$$U^{-1} \ltimes A \ltimes U^{-1} = X^{-1} = V^{-1} \ltimes B \ltimes V^{-1}.$$

Since $U \lozenge_t V = I_\alpha$, we have that $U = (U^{-1} \sharp V)^{-2t}$, i.e., $U^{1/(2t)} = U \sharp V^{-1}$. Applying Lemma 2.5, we get $V^{-1} = U^{1/(2t)} \ltimes U^{-1} \ltimes U^{1/(2t)} = U^{(1-t)/t}$. Hence,

$$B = V \ltimes U^{-1} \ltimes A \ltimes U^{-1} \ltimes V = U^{-1/t} \ltimes A \ltimes U^{-1/t}.$$

Using Lemma 2.5, we have $U^{-1/t} = A^{-1} \sharp B$, i.e., $U = (A^{-1} \sharp B)^{-t}$. Thus,

$$X^{-1} = (A^{-1} \sharp B)^t \ltimes A \ltimes (A^{-1} \sharp B)^t = A \lozenge_t B.$$

Hence, by the self-duality of the SGM $\lozenge_t$, we have

$$X = (A \lozenge_t B)^{-1} = A^{-1} \lozenge_t B^{-1}.$$

□

All results in this section seem to be not noticed before in the literature. In particular, from Theorems 5.3 and 5.5, when $m = n$ and $A = B$, the equation $A \sharp X = I$ has a unique solution $X = A^{-1}$.

6. Conclusions

We characterize weighted SGMs of positive definite matrices in terms of certain matrix equations involving MGMs and STPs. Indeed, for each real number t, the unique positive solution of the matrix equation $A^{-1} \sharp X = (A^{-1} \sharp B)^t$ is defined to be the t-weighted SGM of A and B. We then establish several properties of the weighted SGMs such as permutation invariance, homogeneity, self-duality, unitary invariance, cancellability, and a determinantal identity. The most significant property is the fact that $(A \lozenge B)^2$ is positively similar to $A \ltimes B$, so the two matrices have the same spectrum. The results in Sections 3 and 5 include the classical weighted SGMs of matrices as special cases. Furthermore, we show that certain equations concerning weighted SGMs and weighted MGMs of positive definite matrices have a unique solution written explicitly as weighted SGMs of associated matrices. In particular, the equation $A \lozenge_t X = B$ can be expressed in terms of the famous Riccati equation. For future works, we may investigate SGMs from differential-geometry viewpoints, such as geodesic property.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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