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Research article

Classes of completely monotone and Bernstein functions defined by convexity properties of their spectral measures

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Abstract: We were interested in Bernstein and Lévy measures having certain convexity-type properties. The convexity-type properties were an extension of the harmonic convexity property considered in [9]. We characterized the corresponding completely monotone and Bernstein functions. We hope this paper can aid with understanding the analogous properties and open questions presented in [8, 9].

Keywords: completely monotone function; Bernstein function; β-convex; β-concave tail; harmonic convexity
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1. Introduction

A function $f : (0, \infty) \to [0, \infty)$ is called *completely monotone*, if it is C^∞ and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, \qquad \text{for all } x > 0 \text{ and } n \in \mathbb{N}.$$

Completely monotone (CM) functions find many applications in analysis and probability and an excellent introduction into their properties can be found in the monographs [7, 11]. A function $g : (0, \infty) \to [0, \infty)$ is called *Bernstein*, if it is C^∞ and satisfies

$$(-1)^{n-1} g^{(n)}(x) \geq 0, \qquad \text{for all } x > 0 \text{ and } n \in \mathbb{N}.$$

We see from the definition that if g is a Bernstein function (BF), then g' is CM. These two classes of functions have classic integral representations, which are useful for our developments.

Theorem 1 (Bernstein)**.** *A function f is CM, if and only if it can be expressed as a Laplace transform*

$$f(x) = \int_{[0,\infty)} e^{-xt}\,\mu(dt), \tag{1.1}$$

where μ is a Radon measure on $[0,\infty)$, such that the integral converges for all $x > 0$.

The measure μ in the Bernstein representation will be called the *Bernstein measure* of f.

Theorem 2 (de Finetti-Lévy-Khintchine)**.** *A function g is BF if and only if it can be represented as*

$$g(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})\,\nu(dt) \tag{1.2}$$

for a Radon measure ν on $(0,\infty)$ and some constants $a, b \geq 0$. The measure ν satisfies

$$\int_{(0,\infty)} (1 \wedge t)\,\nu(dt) < \infty. \tag{1.3}$$

The triplet (a, b, ν) uniquely determines the Bernstein function g, and vice-versa.

The measure ν in this representation is usually called the *Lévy measure* of the Bernstein function g, and (a, b, ν) is called the *Lévy triplet* of g. The constants a and $b \geq 0$ are called the *killing rate* and the *drift term* respectively.

Recently, research has focused on different subclasses of CM and BFs. In [8], the authors investigated CM and BFs with measures that satisfy certain convexity properties. A measure μ on $[0,\infty)$ is called *harmonically convex* if $x \mapsto x\mu[0, x]$ is a convex function on $(0,\infty)$. A measure ν on $(0,\infty)$ is said to have *harmonically concave tail* if $x \mapsto x\nu(x,\infty)$ is a concave function on $(0,\infty)$. Among the main results in [8] are the following:

Theorem 3. *For any CM function f and a number $\alpha \in (0, 2/3]$, there exists a unique harmonically convex measure μ_α on $[0,\infty)$, such that*

$$f(x^\alpha) = \int_{[0,\infty)} e^{-xt}\,\mu_\alpha(dt).$$

Theorem 4. *For any Bernstein function g and a number $\alpha \in (0, 2/3]$, there exists a unique triplet (a, b, ν_α), such that*

$$g(x^\alpha) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})\,\nu_\alpha(dt),$$

where $a, b \geq 0$ are constants, and ν_α is a measure on $(0,\infty)$ with harmonically concave tail. The measure ν_α satisfies the integrability condition

$$\int_{(0,\infty)} (1 \wedge t)\,\nu_\alpha(dt) < \infty.$$

One of the open problems formulated in [8] was to find the largest possible value of r for which Theorems 3 and 4 hold for all values of α in the interval $(0, r]$. This question was successfully answered in [2]. It was shown there, see [2, Theorem 6.3], that Theorems 3 and 4 hold for all $\alpha \in (0, \alpha_*]$, where

$$\alpha_* := \inf_{x>0}\Big(\frac{\log(1 + e^x - e^{-x}) - \log(2 - e^{-x})}{x}\Big) \approx 0.717461058844...$$

and α_* is the largest value for which Theorems 3 and 4 hold. Theorem 3 suggests that it is natural to consider the set, denoted $\mathcal{H}_{CM}$, of all BFs h, such that the composition $f \circ h$ is a CM function having a harmonically convex measure for any CM function f. Analogously, Theorem 4 suggests to consider the set, denoted $\mathcal{H}_{BF}$, of all BFs h, such that the composition $g \circ h$ is a BF with measure that has harmonically concave tail for any BF g. In this way, the results in [2] show that $\{x^\alpha : \alpha \in [0, \alpha_*]\} \subset \mathcal{H}_{CM} \cap \mathcal{H}_{BF}$, so the latter two sets are non-empty. These two sets of functions have surprising properties, see Section 7 in [8]:

(1) We have $\mathcal{H}_{CM} = \mathcal{H}_{BF}$;
(2) A BF $g \in \mathcal{H}_{BF}$ if and only if $x \mapsto 1 - e^{-tg(x)}(1 + txg'(x))$ is a BF for all $t \geq 0$;
(3) For any Bernstein function f and $g \in \mathcal{H}_{BF}$, one has $f \circ g \in \mathcal{H}_{BF}$;
(4) The set $\mathcal{H}_{BF}$ is closed with respect to point-wise convergence.

Apart from these properties, very little is known about the set $\mathcal{H}_{BF}$. Is it a convex set? What are its generators? (A function $f \in \mathcal{H}_{BF}$ is called a *generator* for the class $\mathcal{H}_{BF}$ if it cannot be represented as a composition $g \circ h$ for some non-affine BF g and some $h \in \mathcal{H}_{BF}$.)

A characterization of the BFs, having Lévy measure with harmonically concave tail was proven in [8, Lemma 6.1]. It states that a BF g has Lévy measure ν with harmonically concave tail if and only if $g(x) - xg'(x)$ is a BF. The feat in [2] was accomplished by relaxing this property and considering the class $\mathcal{BF}_s$ of all BFs g, such that $sg(x) - xg'(x)$ is Bernstein, for some $s > 0$. Then, the set $\mathcal{H}_{BF}$ was extended to $\mathcal{BF}^*_s$, the later being the set of all BFs g, such that $1 - e^{-tg(x)} \in \mathcal{BF}_s$, for all $t > 0$. See Definition 1.4 in [2], where for technical reasons the killing rate and the drift term are removed from g. In particular, [2, Theorem 6.3], shows that

$$e^{-\lambda^\alpha}(1 + \alpha\lambda^\alpha) \text{ is completely monotone if and only if } \alpha \in [0, \alpha_*].$$

The latter is related to a problem on the unimodality of reciprocal positive stable distributions raised by Simon in [10].

In the current work, we hope to shed more light into these classes of CM and BFs, by relaxing the notion of harmonic convexity, see Definition 1. For a value of a parameter $\beta \in [0, 1]$, we say that a function $h : (0, \infty) \to \mathbb{R}$ is *β-convex (β-concave)* if $x^\beta h(x)$ is convex (concave) on $(0, \infty)$. Thus, we consider Bernstein measures that are β-convex and Lévy measures with β-concave tail, see Definition 2. The main results may be succinctly summarized as follows and they parallel those in [8, 9].

Suppose f is CM with measure μ and define

$$F(x) := \beta(\beta - 1)\frac{f(x)}{x} - 2(\beta - 1)f'(x) + xf''(x) - \beta(\beta - 1)\frac{\mu(\{0\})}{x}.$$

Then, as shown in Table 1, we have the following characterization of β-convexity (β-concavity) of the measure μ:

Table 1. Summary of Theorem 6.

Property of μ	Characterization	Reference
β-convex	F is completely monotone	Theorem 6 a)
β-concave	$-F$ is completely monotone	Theorem 6 b)

Similarly, suppose g is Bernstein with Lévy triplet (a, b, ν). Define

$$G(x) := \beta(\beta-1)\frac{g(x)}{x} - 2(\beta-1)g'(x) + xg''(x) - \beta(\beta-1)\frac{a}{x} - (\beta-1)(\beta-2)b.$$

Then, as shown in Table 2, we have the following characterization of β-convexity (β-concavity) of the tail of the measure ν:

Table 2. Summary of Theorem 7.

Property of ν	Characterization	Reference
β-convex tail	G is completely monotone	Theorem 7 a)
β-concave tail	$-G$ is completely monotone	Theorem 7 b)

The paper is organized as followed: Section 2 introduces the background concepts, notions, and some useful preliminary results. In Section 3, we characterize the CM functions having a β-convex (β-concave) measure and the BFs having a Lévy measure with β-convex (β-concave) tail. Section 4 contains several corollaries from the results in Section 3. Finally, the Appendix collects several classical properties of the Lebesgue-Stieltjes integral that are difficult to find in the formulation that we need.

2. Definitions, background results and technical lemmas

2.1. β-convexity and β-concavity

A function $h : I \to \mathbb{R}$ is *convex* on the convex interval I if

$$h(\alpha x + (1-\alpha)y) \le \alpha\, h(x) + (1-\alpha)\, h(y), \quad \text{for } x, y \in I, \text{ and } \alpha \in [0, 1].$$

The function h is *concave* if the opposite inequality holds. If h is twice differentiable in an open interval I, then h is convex on I if and only if its second order derivative is non-negative on I. Convex functions are continuous (in fact locally Lipschitz) on the interior of their domain. The directional derivatives exist (both left and right, in the extended sense) for every $x \in I$. The right directional derivative, denoted $h'_+(x)$, is right-continuous, while the left directional derivative, denoted $h'_-(x)$, is left-continuous. When h is convex, then both $h'_+(x)$ and $h'_-(x)$ are non-decreasing functions in x, see [6, Theorem 24.1]. Moreover, for any x, y in the interior of I we have

$$h(y) - h(x) = \int_{(x,y)} h'_+(t)\, dt = \int_{(x,y)} h'_-(t)\, dt, \tag{2.1}$$

see [6, Corollary 24.2.1] for details. In addition, if h is convex and $y > x$, then

$$h'_+(x) \leq \frac{f(y) - f(x)}{y - x}. \tag{2.2}$$

Definition 1. *Let $\beta \in [0, 1]$. A function $h : (0, \infty) \to \mathbb{R}$ is called β-convex (β-concave) if $x^\beta h(x)$ is convex (concave) on $(0, \infty)$.*

We consider $\beta \in [0, 1]$ in the following content without further notice. A function h is 0-convex if it is convex; and it is 1-convex precisely when $h(1/x)$ is convex. The latter equivalence follows from Lemma 2.2 in [5], that we state for completeness.

Lemma 1. *A function $h : (0, \infty) \to R$ is convex (concave) if and only if $xh(1/x)$ is convex (concave).*

When $h(1/x)$ is convex, we say that h is *harmonically convex*, since it satisfies the inequality

$$h\Big(\frac{2}{1/x + 1/y}\Big) \leq \frac{h(x) + h(y)}{2}$$

for every $x, y > 0$. Such functions are also called *reciprocally convex* in [5]. Thus, β-convexity connects the notions of convexity and harmonic/reciprocal convexity.

The following equivalence is an immediate consequence from Lemma 1.

Corollary 1. *A function h is β-convex (β-concave) precisely when $h(1/x)$ is $(1 - \beta)$-convex ($(1 - \beta)$-concave).*

If $h : (0, \infty) \to \mathbb{R}$ is β-convex, then the directional derivatives of $h(x)$ exist for all $x > 0$. More precisely, it can be shown that

$$h'_+(x) = -\beta x^{-1} h(x) + x^{-\beta} (x^\beta h(x))'_+ \quad \text{and} \quad h'_-(x) = -\beta x^{-1} h(x) + x^{-\beta} (x^\beta h(x))'_-.$$

The *cumulative distribution function* for measure μ on $[0, \infty)$ is denoted by

$$F_\mu(x) := \mu[0, x],$$

while the *tail of measure* ν on $(0, \infty)$ is denoted by

$$\bar{\nu}(x) := \nu(x, \infty).$$

Note that $\bar{\nu}(x)$ is non-increasing and a right-continuous function.

Definition 2. *Let μ and ν be measures on $[0, \infty)$ and $(0, \infty)$, respectively.*

a) We say that μ is β-convex (β-concave), if F_μ is β-convex (β-concave) on $[0, \infty)$.
b) We say that ν has β-convex (β-concave) tail, if $\bar{\nu}$ is β-convex (β-concave) on $(0, \infty)$.

The next examples illustrate this concept.

Example 1. *a) Consider the CM function $f(x) = x^{-\alpha}$ for $\alpha \in (0, 1)$. Its measure μ has cumulative distribution function*

$$F_\mu(x) = \frac{x^\alpha}{\alpha\Gamma(\alpha)}.$$

Since $x \mapsto x^\beta F_\mu(x) = x^{\alpha+\beta}/(\alpha\Gamma(\alpha))$, μ is a β-convex measure if $1 - \alpha \leq \beta \leq 1$, and it is β-concave if $0 \leq \beta \leq 1 - \alpha$.

b) Consider the BF $g(x) = x^\alpha$ for $\alpha \in (0, 1)$. Its Lévy measure ν has a tail given by

$$\bar{\nu}(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

Since $x \mapsto x^\beta \bar{\nu}(x) = x^{\beta-\alpha}/\Gamma(1-\alpha)$, ν is a measure with a β-convex tail if $0 \leq \beta \leq \alpha$, and it is a measure with a β-concave tail if $\alpha \leq \beta \leq 1$.

2.2. *Inverse formula for the Laplace transformation*

A CM function f uniquely determines its Bernstein measure μ. Indeed, for all $k \in \mathbb{N}$ and $t > 0$, define the operator

$$L_k(f(x);t) := (-1)^k x^{k+1} f^{(k)}(x)\Big|_{x=k/t}. \tag{2.3}$$

The following is an inversion formula for the Laplace-Stieltjes integrals, see [11, Chapter VII, Theorems 6a and 7a].

Theorem 5 (Inversion formula)**.** *Suppose f is a CM function with measure μ.*

a) For every $t > 0$,

$$\lim_{k\to\infty} \int_{(0,t]} L_k(f(x);u)\,du = \frac{\mu[0,t] + \mu[0,t)}{2} - \mu(\{0\}). \tag{2.4}$$

b) If measure μ has density $u(t)$, then for every $t > 0$ in the Lebesgue set of $u(t)$,

$$\lim_{k\to\infty} L_k(f(x);t) = u(t).$$

Note that the Lebesgue set of a function contains its points of continuity. If $t > 0$ is a point of continuity of F_μ, the right-hand side of (2.4) becomes $F_\mu(t) - F_\mu(0)$.

In order to deal with the higher order derivatives in the inversion formula we need the following identity which can be proved by induction (see also [1, Lemma 2.7.12] for a more general case). For any integer $k \geq 0$ and a C^{k+1} function r on $(0, \infty)$, the following identity holds:

$$\Big(x^{k+1}\Big(\frac{r(x)}{x}\Big)^{(k)}\Big)' = x^k r^{(k+1)}(x). \tag{2.5}$$

2.3. *Limiting properties for measures with β-convexity type properties*

Suppose f is CM function with measure μ. An integration by parts in (1.1) leads to

$$f(x) = x\int_{(0,\infty)} e^{-xt} F_\mu(t)\,dt \quad \text{and} \quad \lim_{t\to\infty} e^{-xt} F_\mu(t) = 0, \tag{2.6}$$

for any $x > 0$. A direct consequence of the definition of a Bernstein function is $\mu(\{0\}) = f(\infty)$. With the latter, one can also represent f as

$$f(x) = \mu(\{0\}) + \int_{(x,\infty)} (-f'(t))\, dt,$$

where $-f'(t)$ is CM. As a non-increasing function, that is integrable at infinity, is $o(1/t)$ as t approaches infinity, we obtain

$$\lim_{x\to\infty} x f'(x) = 0. \tag{2.7}$$

Lemma 2. *Suppose f is CM with $\mu(\{0\}) = 0$. Then*

$$\lim_{x\to\infty} x^{k+1}\Big(\frac{f(x)}{x}\Big)^{(k)} = 0, \tag{2.8}$$

for any $k \geq 1$.

Proof. The proof uses the inequality

$$e^{-x}x^k \leq (k+1)^k e^{-x/(k+1)} \text{ for all } k \geq 1 \text{ and } x \geq 0,$$

which follows from

$$\max_{x\in(0,\infty)} \frac{x^k}{(k+1)^k} e^{-xk/(k+1)} = e^{-k} \leq 1.$$

Indeed, by (2.6), we have

$$\begin{aligned}
\Big| x^{k+1}\Big(\frac{f(x)}{x}\Big)^{(k)} \Big| &= x \int_{(0,\infty)} e^{-xt}(xt)^k F_\mu(t)\, dt \\
&\leq (k+1)^k x \int_{(0,\infty)} e^{-xt/(k+1)} F_\mu(t)\, dt \\
&= (k+1)^{k+1}\Big(- e^{-xt/(k+1)} F_\mu(t)\Big|_{t=0}^{\infty} + \int_{(0,\infty)} e^{-xt/(k+1)}\, dF_\mu(t)\Big) \\
&= (k+1)^{k+1} F_\mu(0) + (k+1)^{k+1} f\Big(\frac{x}{k+1}\Big).
\end{aligned}$$

The fact that μ has no mass at zero implies $F_\mu(0) = \lim_{x\to\infty} f(x)$=0, and (2.8) follows. □

Lemma 3. *Suppose f is CM. If its measure μ is β-convex (or β-concave), then*

$$\lim_{t\to 0} t^{2-\beta}(t^\beta F_\mu(t))'_+ e^{-xt} = \lim_{t\to\infty} t^{2-\beta}(t^\beta F_\mu(t))'_+ e^{-xt} = 0, \tag{2.9}$$

for any $x > 0$.

Proof. As a product of two CM functions, $f(x)/x$ is also CM and so is $-(f(x)/x)'$. By (2.6), we have

$$-x\Big(\frac{f(x)}{x}\Big)' = x \int_{(0,\infty)} e^{-xt} t F_\mu(t)\, dt = -\int_{(0,\infty)} t F_\mu(t)\, d(e^{-xt})$$

$$
\begin{aligned}
&= -\lim_{t\to\infty} tF_\mu(t)e^{-xt} + \lim_{t\to 0} tF_\mu(t)e^{-xt} + \int_{(0,\infty)} e^{-xt}\, d(tF_\mu(t)) \\
&= \int_{(0,\infty)} e^{-xt}\, d(t^\beta F_\mu(t) t^{1-\beta}) \\
&= (1-\beta)\int_{(0,\infty)} e^{-xt} F_\mu(t)\, dt + \int_{(0,\infty)} e^{-xt} t^{1-\beta}\, d(t^\beta F_\mu(t)) \\
&= (1-\beta)\Big(\frac{f(x)}{x}\Big) + \int_{(0,\infty)} e^{-xt} t^{1-\beta}\, d(t^\beta F_\mu(t)),
\end{aligned}
$$

where in the penultimate equality we used Lemma 5. This shows that the last integral has to be finite. Since μ is β-convex (or β-concave), using (2.1), we obtain

$$
\int_{(0,\infty)} e^{-xt} t^{1-\beta}\, d(t^\beta F_\mu(t)) = \int_{(0,\infty)} t^{1-\beta}(t^\beta F_\mu(t))'_+ e^{-xt}\, dt < \infty. \tag{2.10}
$$

Next, note that since $t^\beta F_\mu(t)$ is non-decreasing, then $(t^\beta F_\mu(t))'_+$ is non-negative. Hence

$$
\int_{(0,1)} t^{1-\beta}(t^\beta F_\mu(t))'_+\, dt \le e^x \int_{(0,1)} t^{1-\beta}(t^\beta F_\mu(t))'_+ e^{-xt}\, dt < \infty,
$$

since $e^{-x} \le e^{-xt}$ for all $t \in (0,1)$.

If μ is β-convex, then $t^\beta F_\mu(t)$ is convex and non-decreasing. Thus, $(t^\beta F_\mu(t))'_+$ is non-decreasing and non-negative, so is $t^{2-\beta}(t^\beta F_\mu(t))'_+$. Suppose

$$
\lim_{t\to 0} t^{2-\beta}(t^\beta F_\mu(t))'_+ = c \ge 0.
$$

We have

$$
\int_{(0,1)} \frac{c}{t}\, dt \le \int_{(0,1)} \frac{t^{2-\beta}(t^\beta F_\mu(t))'_+}{t}\, dt = \int_{(0,1)} t^{1-\beta}(t^\beta F_\mu(t))'_+\, dt < \infty.
$$

Therefore, c has to be zero. The first limit in (2.9) follows.

To see the second limit, for a fixed $\epsilon > 0$, use (2.2) to bound

$$
\begin{aligned}
0 \le t^{2-\beta}(t^\beta F_\mu(t))'_+ e^{-xt} &\le \frac{1}{\epsilon} t^{2-\beta} e^{-xt}((t+\epsilon)^\beta F_\mu(t+\epsilon) - t^\beta F_\mu(t)) \\
&= \frac{1}{\epsilon} t^{2-\beta} e^{-xt/2}(e^{x\epsilon/2} e^{-x(t+\epsilon)/2}(t+\epsilon)^\beta F_\mu(t+\epsilon) - e^{-xt/2} t^\beta F_\mu(t)).
\end{aligned} \tag{2.11}
$$

Then, using (2.6) one can see that the last expression converges to zero as t approaches infinity.

If μ is β-concave, then $t^\beta F_\mu(t)$ is concave and non-decreasing. Thus, $(t^\beta F_\mu(t))'_+$ is non-increasing and non-negative. Notice

$$
\int_{(0,1)} (t^\beta F_\mu(t))'_+\, d(t^{2-\beta}) = (2-\beta)\int_{(0,1)} t^{1-\beta}(t^\beta F_\mu(t))'_+\, dt < \infty,
$$

we conclude $(t^\beta F_\mu(t))'_+$ is $o(1/t^{2-\beta})$ as t approaches zero by Lemma 8. The first limit in (2.9) follows from here.

For the second limit, note that $t^{1-\beta}e^{-xt}$ is a decreasing function for large enough t. Thus, $t^{1-\beta}(t^\beta F_\mu(t))'_+ e^{-xt}$ is decreasing and we can apply Lemma 7 to the second integral in (2.10). □

Condition (1.3) on the measure ν implies

$$\lim_{t\to\infty} \bar{\nu}(t) = 0.$$

Fubini's theorem applied to the Lévy-Khintchine representation (1.2) gives

$$g(x) = a + bx + x\int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\,dt, \text{ implying that } \int_{(0,1)} \bar{\nu}(t)\,dt < \infty. \tag{2.12}$$

A non-increasing function that is integrable at zero is $o(1/t)$ as t approaches zero, thus

$$\lim_{t\to 0} t\bar{\nu}(t) = 0 \text{ and } \lim_{t\to\infty} e^{-xt}\bar{\nu}(t) = 0 \text{ for any } x > 0. \tag{2.13}$$

(For more details about (2.12) and (2.13), refer to (3.3) and (3.6) in [7].) Integration by parts, using the facts that $\int_0^\infty te^{-xt}\,dt = x^{-2}$ and $\int_0^\infty e^{-xt}\,dt = x^{-1}$, shows that

$$g(x) = x^2\int_{(0,\infty)} e^{-xt}\kappa(t)\,dt, \tag{2.14}$$

where $\kappa(t) := at + b + \int_0^t \bar{\nu}(s)\,ds$ is positive, non-decreasing and concave, see [7, p.23, (3.4)].

Lemma 4. *If the Lévy measure ν, of a BF g, has β-convex (or β-concave) tail, then*

$$\lim_{t\to 0} t^{\beta-1}(t^{1-\beta}\bar{\nu}(1/t))'_+e^{-x/t} = \lim_{t\to\infty} t^{\beta-1}(t^{1-\beta}\bar{\nu}(1/t))'_+e^{-x/t} = 0, \quad \textit{for any } x > 0. \tag{2.15}$$

Proof. Without loss of generality, we can assume $a = b = 0$. As g is a Bernstein function, $g(x)/x$ is CM, and so is $-(g(x)/x)'$. By (2.12) and (2.13), we obtain

$$\begin{aligned}
-x\Big(\frac{g(x)}{x}\Big)' &= x\int_{(0,\infty)} e^{-xt}t\bar{\nu}(t)\,dt = -\int_{(0,\infty)} t\bar{\nu}(t)\,d(e^{-xt}) \\
&= -\lim_{t\to\infty} t\bar{\nu}(t)e^{-xt} + \lim_{t\to 0} t\bar{\nu}(t)e^{-xt} + \int_{(0,\infty)} e^{-xt}\,d(t\bar{\nu}(t)) \\
&= -\int_{(0,\infty)} e^{-x/t}\,d(t^{1-\beta}\bar{\nu}(1/t)t^{\beta-2}) \\
&= (2-\beta)\int_{(0,\infty)} e^{-x/t}t^{-2}\bar{\nu}(1/t)\,dt - \int_{(0,\infty)} e^{-x/t}t^{\beta-2}\,d(t^{1-\beta}\bar{\nu}(1/t)),
\end{aligned}$$

where we used Lemma 6 in the penultimate equality and Lemma 5 in the last equality. Now, applying Lemma 6 again, we conclude that

$$-x\Big(\frac{g(x)}{x}\Big)' = (2-\beta)\Big(\frac{g(x)}{x}\Big) - \int_{(0,\infty)} e^{-x/t}t^{\beta-2}\,d(t^{1-\beta}\bar{\nu}(1/t)).$$

This shows that the last integral has to be finite for all $x > 0$. Since ν has β-convex (or β-concave) tail, using Corollary 1 and (2.1), we obtain

$$\int_{(0,\infty)} e^{-x/t}t^{\beta-2}\,d(t^{1-\beta}\bar{\nu}(1/t)) = \int_{(0,\infty)} t^{\beta-2}(t^{1-\beta}\bar{\nu}(1/t))'_+e^{-x/t}\,dt < \infty.$$

Denote $h(t) := (t^{1-\beta}\bar{\nu}(1/t))'_+$. Applying Lemma 6 again, we have

$$\int_{(0,\infty)} t^{-\beta} h(1/t) e^{-xt}\, dt = \int_{(0,\infty)} t^{\beta-2} h(t) e^{-x/t}\, dt < \infty. \tag{2.16}$$

Next, observe that for all $x > 0$, we have

$$\int_{(0,1)} t^{-\beta} h(1/t)\, dt \le e^x \int_{(0,1)} t^{-\beta} h(1/t) e^{-xt}\, dt \le e^x \int_{(0,\infty)} t^{-\beta} h(1/t) e^{-xt}\, dt < \infty. \tag{2.17}$$

If ν has β-convex tail, then $t^{1-\beta}\bar{\nu}(1/t)$ is convex and non-decreasing, hence h is non-decreasing and non-negative. By (2.2), for an $\epsilon > 0$, we obtain

$$0 \le t^{\beta-1}(t^{1-\beta}\bar{\nu}(1/t))'_+ e^{-x/t} \le \frac{1}{\epsilon} t^{\beta-1} e^{-x/t}((t+\epsilon)^{1-\beta}\bar{\nu}(1/(t+\epsilon)) - t^{1-\beta}\bar{\nu}(1/t)).$$

Analogously to (2.11), one can see that the right-hand side converges to 0 as t approaches zero, showing the first limit in (2.15).

Now, $h(1/t)$ is non-increasing and non-negative, and so is $t^{-\beta}h(1/t)$. As a non-increasing function which is integrable near zero, see (2.17), is $o(1/u)$ as u approaches zero, we have

$$\lim_{u\to 0} u^{1-\beta} h(1/u) = 0.$$

The second limit in (2.15) follows.

If ν has β-concave tail, then $t^{1-\beta}\bar{\nu}(1/t)$ is concave and non-decreasing, hence h is non-increasing and non-negative. The function $t^{\beta-2}e^{-x/t}$ is also non-increasing for t close to zero. Hence, from (2.16) and Lemma 7, we conclude that $t^{\beta-2}h(t)e^{-x/t}$ is $o(1/t)$ as t approaches zero. This shows the first limit in (2.15).

Next, $h(1/u)$ is non-decreasing and non-negative, and so is $u^{1-\beta}h(1/u)$. Supposing

$$\lim_{u\to 0} u^{1-\beta} h(1/u) = c \ge 0,$$

we would have

$$\int_{(0,1)} \frac{c}{u}\, du \le \int_{(0,1)} \frac{u^{1-\beta}h(1/u)}{u}\, du = \int_{(0,1)} u^{-\beta} h(1/u)\, du < \infty.$$

Therefore, c has to be zero and the second limit in (2.15) follows. □

By replacing t with $1/t$, the previous lemma can be reformulated as follows.

Corollary 2. *If the Lévy measure ν of a BF g has β-convex (or β-concave) tail, then*

$$\lim_{t\to 0} t^{1-\beta}(t^{\beta-1}\bar{\nu}(t))'_+ e^{-xt} = \lim_{t\to\infty} t^{1-\beta}(t^{\beta-1}\bar{\nu}(t))'_+ e^{-xt} = 0, \quad \textit{for any } x > 0.$$

3. Implications of the β-convexity properties of the measures μ and ν

In this section, we characterize CM and BFs with β-convexity properties on their measures. In Theorem 6, we consider CM functions with β-convex and β-concave measures. In Theorem 7, BFs whose measures have β-concave tail and β-convex tail are considered. These results extend the characterizations in [8,9]. The boundary cases of both Theorems 6 and 7, when $\beta \in \{0, 1\}$, are explored in Corollaries 7 to 14.

Theorem 6. *Suppose f is a CM function with measure μ. Consider the function*

$$F(x) = \beta(\beta-1)\frac{f(x)}{x} - 2(\beta-1)f'(x) + xf''(x) - \beta(\beta-1)\frac{\mu(\{0\})}{x}. \tag{3.1}$$

a) The measure μ is β-convex, if and only if F is CM.

b) The measure μ is β-concave, if and only if $-F$ is CM.

Proof. Notice F can be rewritten as

$$F(x) = \beta(\beta-1)\frac{f(x)-\mu(\{0\})}{x} - 2(\beta-1)f'(x) + xf''(x).$$

Thus, without loss of generality, we can assume $\mu(\{0\}) = 0$.

a) For sufficiency, suppose F is CM. Anticipating the use of the inversion formula in Theorem 5, define

$$G_k(t) := \int_{(0,t]} L_k(f(x); u)\, du, \tag{3.2}$$

where L_k is the operator defined in (2.3). We claim that for every $k \geq 2$, the function $t \mapsto t^\beta G_k(t)$ is convex on the positive reals. Indeed,

$$G_k'(t) = L_k(f(x); t) = (-1)^k x^{k+1} f^{(k)}(x)\Big|_{x=k/t},$$

and using (2.5), we have

$$\begin{aligned} G_k''(t) &= (-1)^k \frac{d}{dx}\Big(x^{k+1} f^{(k)}(x)\Big)\Big|_{x=k/t}\Big(-\frac{k}{t^2}\Big) = (-1)^{k+1} x^k (xf(x))^{(k+1)}\Big|_{x=k/t}\Big(\frac{k^2}{t^2}\Big)\frac{1}{k} \\ &= (-1)^{k+1}\frac{1}{k}\, x^{k+2}(xf(x))^{(k+1)}\Big|_{x=k/t}. \end{aligned}$$

Also notice that

$$\frac{d}{dt}\Big(x^k\Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big|_{x=k/t}\Big) = \frac{d}{dx}\Big(x^k\Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big)\Big|_{x=k/t}\Big(-\frac{k}{t^2}\Big) = -\frac{1}{k}x^{k+1}f^{(k)}(x)\Big|_{x=k/t},$$

therefore, by Lemma 2, we obtain

$$\int_{(0,t]} -\frac{1}{k}x^{k+1}f^{(k)}(x)\Big|_{x=k/u} du = x^k\Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big|_{x=k/t} - \lim_{u\to 0} x^k\Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big|_{x=k/u} = x^k\Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big|_{x=k/t}.$$

So, we have

$$G_k(t) = \int_{(0,t]} (-1)^k x^{k+1} f^{(k)}(x)\Big|_{x=k/u} du = (-1)^{k-1}k\int_{(0,t]} -\frac{1}{k}x^{k+1}f^{(k)}(x)\Big|_{x=k/u} du$$

$$= (-1)^{k-1} k x^k \Big(\frac{f(x)}{x}\Big)^{(k-1)}\Big|_{x=k/t}.$$

Putting everything together and after a trivial calculation, using that $(xf(x))^{(k+1)} = 2f^{(k)}(x) + (xf''(x))^{(k-1)}$, we obtain

$$(t^\beta G_k(t))'' = t^{\beta-2}(\beta(\beta-1)G_k(t) + 2\beta t G_k'(t) + t^2 G_k''(t)) = t^{\beta-2} x^k k(-1)^{k-1} F^{(k-1)}(x)\Big|_{x=k/t}. \tag{3.3}$$

As F is CM, we know that $(-1)^{k-1}F^{(k-1)}(x) \geq 0$, for all $x > 0$ and $k \geq 2$, which implies $(t^\beta G_k(t))'' \geq 0$. This concludes the claim.

Let F_μ be continuous at $x, y > 0$ and at the convex combination $(1-\alpha)x + \alpha y$, where $\alpha \in [0, 1]$. Then, (2.4) shows that

$$\lim_{k\to\infty} G_k(t) = F_\mu(t) - F_\mu(0) = F_\mu(t)$$

for $t \in \{x, y, (1-\alpha)x + \alpha y\}$. As $t^\beta G_k(t)$ is convex for all $k \geq 2$, we obtain

$$((1-\alpha)x + \alpha y)^\beta F_\mu((1-\alpha)x + \alpha y) \leq (1-\alpha)x^\beta F_\mu(x) + \alpha y^\beta F_\mu(y).$$

We will show that F_μ is continuous on $(0, \infty)$, thus completing the proof.

Recall that F_μ is a right-continuous, non-decreasing function. Let $u > 0$ be a jump point for F_μ, that is, $F_\mu(u-) < F_\mu(u)$. Let $\{y_n\}$ be a sequence where F_μ is continuous, that decreases and converges to u. (Recall that the points of discontinuity of F_μ is countable.) Choose a sequence $\{x_n\}$ of points of continuity of F_μ, that converges to u from the left. Synchronized with $\{x_n\}$, choose a sequence $\{\alpha_n\} \subset [1/4, 3/4]$, such that the convex combination $(1-\alpha_n)x_n + \alpha_n y_n$ is to the right of u and is a point of continuity of F_μ. By compactness, we may assume that $\{\alpha_n\}$ converges to an $\alpha \in [1/4, 3/4]$. So we have $\lim_{k\to\infty} G_k(t) = F_\mu(t)$ for every $t \in \{x_n, y_n, (1-\alpha_n)x_n + \alpha_n y_n, n \in \mathbb{N}\}$. The convexity of the functions $t^\beta G_k(t)$, in the limit gives

$$((1-\alpha_n)x_n + \alpha_n y_n)^\beta F_\mu((1-\alpha_n)x_n + \alpha_n y_n) \leq (1-\alpha_n)x_n^\beta F_\mu(x_n) + \alpha_n y_n^\beta F_\mu(y_n),$$

for each $n \in \mathbb{N}$. Letting n approach infinity, the right-continuity of F_μ, shows that

$$u^\beta F_\mu(u) \leq (1-\alpha)u^\beta F_\mu(u-) + \alpha u^\beta F_\mu(u).$$

Using that $\alpha \neq 1$, and $u > 0$, gives $F_\mu(u-) \geq F_\mu(u)$, which is a contradiction. Therefore, F_μ is continuous on $(0, \infty)$.

Now we show necessity. Suppose μ is a β-convex measure, we prove that F is CM. First, by (2.6) and [4, Theorem A.5.2], we have

$$f'(x) = \int_{(0,\infty)} e^{-xt} F_\mu(t)\,dt - x \int_{(0,\infty)} e^{-xt} t F_\mu(t)\,dt$$

and

$$f''(x) = -2 \int_{(0,\infty)} e^{-xt} t F_\mu(t)\,dt + x \int_{(0,\infty)} e^{-xt} t^2 F_\mu(t)\,dt.$$

To simplify the notation, denote

$$A_n(x) := \int_{(0,\infty)} e^{-xt} t^n F_\mu(t)\, dt \quad \text{and} \quad B_m(x) := \int_{(0,\infty)} e^{-xt} t^m\, d(t^\beta F_\mu(t)).$$

It is not difficult to see that both are well defined. With that notation, we can rewrite

$$\frac{f(x)}{x} = A_0(x), \quad f'(x) = A_0(x) - xA_1(x), \quad xf''(x) = -2xA_1(x) + x^2 A_2(x).$$

By (2.6) and Lemma 3, using Fubini's theorem, we have

$$\begin{aligned} xA_1(x) &= x\int_{(0,\infty)} e^{-xt} tF_\mu(t)\, dt = -\int_{(0,\infty)} tF_\mu(t)\, d(e^{-xt}) = -tF_\mu(t)e^{-xt}\Big|_{t=0}^{\infty} + \int_{(0,\infty)} e^{-xt}\, d(t^\beta F_\mu(t) t^{1-\beta}) \\ &= \int_{(0,\infty)} e^{-xt} t^{1-\beta}\, d(t^\beta F_\mu(t)) + (1-\beta)\int_{(0,\infty)} e^{-xt} F_\mu(t)\, dt \\ &= B_{1-\beta}(x) + (1-\beta)A_0(x) \end{aligned}$$

and

$$\begin{aligned} x^2 A_2(x) &= x^2 \int_{(0,\infty)} e^{-xt} t^2 F_\mu(t)\, dt = -xt^2 F_\mu(t) e^{-xt}\Big|_{t=0}^{\infty} + x\int_{(0,\infty)} e^{-xt}\, d(t^\beta F_\mu(t) t^{2-\beta}) \\ &= (2-\beta)x\int_{(0,\infty)} e^{-xt} tF_\mu(t)\, dt + x\int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t)) \\ &= (2-\beta)xA_1(x) + x\int_{(0,\infty)} e^{-xt} t^{2-\beta} (t^\beta F_\mu(t))'_+\, dt \\ &= (2-\beta)xA_1(x) - t^{2-\beta}(t^\beta F_\mu(t))'_+ e^{-xt}\Big|_{t=0}^{\infty} + \int_{(0,\infty)} e^{-xt}\, d(t^{2-\beta}(t^\beta F_\mu(t))'_+) \\ &= (2-\beta)xA_1(x) + \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t))'_+ + (2-\beta)\int_{(0,\infty)} e^{-xt} t^{1-\beta} (t^\beta F_\mu(t))'_+\, dt \\ &= (2-\beta)xA_1(x) + \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t))'_+ + (2-\beta)B_{1-\beta}(x). \end{aligned}$$

(Note that the above equations also hold if μ is β-concave.) To summarize, we have

$$\begin{aligned} xA_1(x) &= B_{1-\beta}(x) + (1-\beta)A_0(x), \\ x^2 A_2(x) &= 2(2-\beta)B_{1-\beta}(x) + (2-\beta)(1-\beta)A_0(x) + \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t))'_+. \end{aligned}$$

Therefore, it can be shown that

$$\begin{aligned} F(x) &= \beta(\beta-1)\frac{f(x)}{x} - 2(\beta-1)f'(x) + xf''(x) \\ &= \beta(\beta-1)A_0(x) - 2(\beta-1)(A_0(x) - xA_1(x)) - 2xA_1(x) + x^2 A_2(x) \\ &= \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t))'_+. \end{aligned}$$

The convexity of $t^\beta F_\mu(t)$ implies that $(t^\beta F_\mu(t))'_+$ is right-continuous and non-decreasing. Thus, the last integral is the Laplace transform of $t^{2-\beta}\, d(t^\beta F_\mu(t))'_+$ on $(0,\infty)$. By the Bernstein representation theorem, $F(x)$ is CM.

b) The proof is very much analogous to the proof for part a), so we will only address the differences.

For sufficiency, suppose $-F$ is CM. Define $G_k(t)$ as (3.2). Without any further assumptions, analogously to (3.3), we have

$$(t^\beta G_k(t))'' = t^{\beta-2} k x^k (-1)^{k-1} F^{(k-1)}(x)\Big|_{x=k/t}.$$

As $-F$ is CM, we know that $t^\beta G_k(t)$ is concave. Analogous proof by contradiction applies to verify the continuity of F_μ. Therefore, the measure μ is β-concave, as $G_k(t)$ converges to $F_\mu(t)$ for all $t > 0$ and G_k is concave for all $k \geq 2$.

For necessity, suppose μ is β-concave, we prove $-F$ is CM. Using the notation A_n and B_m from part a), we have

$$\frac{f(x)}{x} = A_0(x), \quad f'(x) = A_0(x) - xA_1(x), \quad xf''(x) = -2xA_1(x) + x^2 A_2(x),$$

$$x\, A_1(x) = B_{1-\beta}(x) + (1-\beta)A_0(x),$$

$$x^2\, A_2(x) = 2(2-\beta)B_{1-\beta}(x) + (2-\beta)(1-\beta)A_0(x) + \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d(t^\beta F_\mu(t))'_+.$$

Thus, we obtain

$$-F(x) = \int_{(0,\infty)} e^{-xt} t^{2-\beta}\, d\big(-(t^\beta F_\mu(t))'_+\big).$$

The concavity of $t^\beta F_\mu(t)$ implies that $-(t^\beta F_\mu(t))'_+$ is right-continuous and non-decreasing. The last integral is the Laplace transform of $t^{2-\beta}\, d\big(-(t^\beta F_\mu(t))'_+\big)$ on $(0,\infty)$. Hence, $-F$ is a CM function. □

Theorem 7. *Suppose g is BF with Lévy triplet (a, b, ν). Consider the function*

$$G(x) := \beta(\beta-1)\frac{g(x)}{x} - 2(\beta-1)g'(x) + xg''(x) - \beta(\beta-1)\frac{a}{x} - (\beta-1)(\beta-2)b. \tag{3.4}$$

a) The measure ν has β-convex tail, if and only if G is CM.

b) The measure ν has β-concave tail, if and only if $-G$ is CM.

Proof. Without loss of generality, we can assume $a = b = 0$. By (2.12) we have

$$g(x) = x \int_{(0,\infty)} e^{-xt} \bar{\nu}(t)\, dt. \tag{3.5}$$

a) We show sufficiency first. Suppose G is CM. Anticipating the use of the inversion formula in Theorem 5, define

$$G_k(t) := L_k\Big(\frac{g(x)}{x}; t\Big) = (-1)^k x^{k+1} \Big(\frac{g(x)}{x}\Big)^{(k)}\Big|_{x=k/t}, \tag{3.6}$$

where the operator L_k is defined in (2.3). We claim that $t^\beta G_k(t)$ is convex on $(0,\infty)$ for every $k \geq 1$. Notice that by (2.5),

$$G_k'(t) = (-1)^k \frac{d}{dx}\Big(x^{k+1}\Big(\frac{g(x)}{x}\Big)^{(k)}\Big)\Big|_{x=k/t}\Big(-\frac{k}{t^2}\Big) = (-1)^{k+1}\frac{1}{k}\, x^{k+2} g(x)^{(k+1)}\Big|_{x=k/t},$$

and

$$G_k''(t) = (-1)^{k+1}\frac{1}{k}\frac{d}{dx}\Big(x^{k+2}g(x)^{(k+1)}\Big)\Big|_{x=k/t}\Big(-\frac{k}{t^2}\Big) = (-1)^{k+2}\frac{1}{k^2}\, x^{k+3}\big(xg(x)\big)^{(k+2)}\Big|_{x=k/t}.$$

So we have

$$(t^\beta G_k(t))'' = t^{\beta-2}(\beta(\beta-1)G_k(t) + 2\beta t G_k(t)' + t^2 G_k(t)'') = t^{\beta-2}x^{k+1}(-1)^k G^{(k)}(x)\Big|_{x=k/t}.$$

As G is CM, we know $(-1)^k G^{(k)}(x) \geq 0$ for all $x > 0$ and $k \geq 1$, which implies $t^\beta G_k(t)$ is convex. This closes the claim.

Let x and y belong to the interval $(0,\infty)$ and let the function $\bar{\nu}$ be continuous at these points as well as at the convex combination $(1-\alpha)x+\alpha y$, where α is in $[0,1]$. By Theorem 5, as k approaches infinity, the limit of G_k equals $\bar{\nu}(t)$ for any t taken from the set $\{x, y, (1-\alpha)x+\alpha y\}$. Furthermore, since $t^\beta G_k(t)$ is convex for all $k \geq 1$, we have the inequality:

$$((1-\alpha)x+\alpha y)^\beta\ \bar{\nu}((1-\alpha)x+\alpha y) \leq (1-\alpha)x^\beta\ \bar{\nu}(x) + \alpha y^\beta\ \bar{\nu}(y).$$

To conclude the proof, we need to establish the continuity of $\bar{\nu}$ on $(0,\infty)$.

Recall that $\bar{\nu}$ is non-increasing and right-continuous. Suppose there exists a point $u > 0$, where the function exhibits a jump, that is $\bar{\nu}(u-) > \bar{\nu}(u)$. Let $\{y_n\}$ be a sequence where $\bar{\nu}$ is continuous, that decreases and converges to u. Choose a sequence $\{x_n\}$ of points of continuity of $\bar{\nu}$ that converges to u from the left. Together with it, choose a sequence $\{\alpha_n\}$ in the interval $[1/4, 3/4]$, such that the convex combination $(1-\alpha_n)x_n + \alpha_n y_n$ is to the left of u and is a point of continuity of $\bar{\nu}$. By compactness, we may assume that $\{\alpha_n\}$ converges to an α in $[1/4, 3/4]$. Thus, we can conclude that the limit of $\lim_{k\to\infty} G_k(t)$ equals $\bar{\nu}$ for every t in the set $\{x_n, y_n, (1-\alpha_n)x_n+\alpha_n y_n, n \in \mathbb{N}\}$. Given the convexity of the functions $t^\beta G_k(t)$ for every $k \geq 1$ and each $n \in \mathbb{N}$, we obtain:

$$((1-\alpha_n)x_n + \alpha_n y_n)^\beta \bar{\nu}((1-\alpha_n)\ x_n + \alpha_n\ y_n) \leq (1-\alpha_n)\ x_n^\beta\ \bar{\nu}(x_n) + \alpha_n\ y_n^\beta\ \bar{\nu}(y_n).$$

Letting n approach infinity, the right-continuity of $\bar{\nu}$ implies that

$$u^\beta \bar{\nu}(u-) \leq (1-\alpha)u^\beta \bar{\nu}(u-) + \alpha u^\beta \bar{\nu}(u).$$

Using that $\alpha \neq 0$ and $u > 0$, we arrive at $\bar{\nu}(u-) \leq \bar{\nu}(u)$, which is a contradiction. Therefore, $\bar{\nu}$ is continuous on $(0,\infty)$.

Now we show necessity. Suppose measure ν has β-convex tail. As a result, function $t^\beta \bar{\nu}(t)$ is convex and $s^{1-\beta}\bar{\nu}(1/s)$ is also convex by Corollary 1. We prove that G is CM. By (3.5) and change of variable $s = 1/t$,

$$\frac{g(x)}{x} = \int_{(0,\infty)} e^{-xt}\bar{\nu}(t)\,dt = \int_{(0,\infty)} e^{-x/s}s^{-2}\bar{\nu}(1/s)\,ds.$$

Therefore, by [4, Theorem A.5.2], we have

$$g'(x) = \int_{(0,\infty)} e^{-x/s} s^{-2} \bar{\nu}(1/s)\, ds - x \int_{(0,\infty)} e^{-x/s} s^{-3} \bar{\nu}(1/s)\, ds,$$
$$g'(x) = -2 \int_{(0,\infty)} e^{-x/s} s^{-3} \bar{\nu}(1/s)\, ds + x \int_{(0,\infty)} e^{-x/s} s^{-4} \bar{\nu}(1/s)\, ds,$$

To simplify the notation, denote

$$C_n(x) = \int_{(0,\infty)} e^{-x/s} s^{-2-n} \bar{\nu}(1/s)\, ds, \qquad D_m(x) = \int_{(0,\infty)} e^{-x/s} s^{-2+m}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big).$$

With these notations, we can rewrite

$$\frac{g(x)}{x} = C_0(x), \qquad g'(x) = C_0(x) - xC_1(x), \qquad xg''(x) = -2xC_1(x) + x^2 C_2(x).$$

By (2.13) and Lemma 4, we have

$$\begin{aligned}
xC_1(x) &= x \int_{(0,\infty)} e^{-x/s} s^{-3} \bar{\nu}(1/s)\, ds = \int_{(0,\infty)} s^{-1} \bar{\nu}(1/s)\, d(e^{-x/s}) \\
&= s^{-1} \bar{\nu}(1/s) e^{-x/s} \Big|_{s=0}^{\infty} - \int_{(0,\infty)} e^{-x/s}\, d\big(s^{1-\beta} \bar{\nu}(1/s) s^{\beta-2}\big) \\
&= - \int_{(0,\infty)} e^{-x/s} s^{\beta-2}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big) + (2-\beta) \int_{(0,\infty)} e^{-x/s} s^{-2} \bar{\nu}(1/s)\, ds \\
&= -D_\beta(x) + (2-\beta) C_0(x),
\end{aligned}$$

and

$$\begin{aligned}
x^2 C_2(x) &= x^2 \int_{(0,\infty)} e^{-x/s} s^{-4} \bar{\nu}(1/s)\, ds \\
&= x s^{-2} \bar{\nu}(1/s) e^{-x/s} \Big|_{s=0}^{\infty} - x \int_{(0,\infty)} e^{-x/s}\, d\big(s^{1-\beta} \bar{\nu}(1/s) s^{\beta-3}\big) \\
&= (3-\beta) x \int_{(0,\infty)} e^{-x/s} s^{-3} \bar{\nu}(1/s)\, ds - x \int_{(0,\infty)} e^{-x/s} s^{\beta-3}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big) \\
&= (3-\beta) xC_1(x) - x \int_{(0,\infty)} e^{-x/s} s^{\beta-3} \big(s^{1-\beta} \bar{\nu}(1/s)\big)'_+\, ds \\
&= (3-\beta) xC_1(x) - s^{\beta-1} \big(s^{1-\beta} \bar{\nu}(1/s)\big)' e^{-x/s} \Big|_{s=0}^{\infty} + \int_{(0,\infty)} e^{-x/s}\, d\big(s^{\beta-1} \big(s^{1-\beta} \bar{\nu}(1/s)\big)'_+\big) \\
&= (3-\beta) xC_1(x) + \int_{(0,\infty)} e^{-x/s} s^{\beta-1}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big)'_+ + (\beta-1) \int_{(0,\infty)} e^{-x/s} s^{\beta-2}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big) \\
&= (3-\beta) xC_1(x) + \int_{(0,\infty)} e^{-x/s} s^{\beta-1}\, d\big(s^{1-\beta} \bar{\nu}(1/s)\big)'_+ + (\beta-1) D_\beta.
\end{aligned}$$

(Note that the above equations also hold if ν has β-concave tail.) To summarize,

$$xC_1(x) = -D_\beta(x) + (2-\beta) C_0(x),$$

$$x^2C_2(x) = 2(\beta-2)D_\beta + (3-\beta)(2-\beta)C_0(x) + \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d\big(s^{1-\beta}\bar{\nu}(1/s)\big)'_+.$$

Therefore, it can be shown that

$$\begin{aligned} G(x) &= \beta(\beta-1)\frac{g(x)}{x} - 2(\beta-1)g'(x) + xg''(x) \\ &= \beta(\beta-1)C_0(x) - 2(\beta-1)(C_0(x) - xC_1(x)) - 2xC_1(x) + x^2C_2(x) \\ &= \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d\big(s^{1-\beta}\bar{\nu}(1/s)\big)'_+. \end{aligned}$$

As $s^{1-\beta}\bar{\nu}(1/s)$ is convex, $\big(s^{1-\beta}\bar{\nu}(1/s)\big)'_+$ is non-decreasing. It defines a Radon measure. One can see G is CM by definition.

b) The proof is very much analogous to the proof for part a), so we will only address the difference.

For sufficiency, suppose $-G$ is CM. Define G_k as (3.6). Without any further assumption,

$$(t^\beta G_k(t))'' = t^{\beta-2}x^{k+1}(-1)^kG^{(k)}(x)\Big|_{x=k/t}.$$

As $-G$ is CM, we know that $t^\beta G_k(t)$ is concave. Analogous proof by contradiction applies to verify the continuity of $\bar{\nu}(t)$. As $G_k(t)$ converges to $\bar{\nu}(t)$ for all $t > 0$, and as G_k has β-concave tail for all $k \geq 1$, the tail of ν is β-concave.

To show the necessity, suppose that the tail of ν is β-concave, we prove $-G$ is CM. Following the notation C_n and D_m in part a), we also have

$$\frac{g(x)}{x} = C_0(x), \quad g'(x) = C_0(x) - xC_1(x), \quad xg''(x) = -2xC_1(x) + x^2C_2(x),$$

$$xC_1(x) = -D_\beta(x) + (2-\beta)C_0(x),$$

$$x^2C_2(x) = (3-\beta)xC_1(x) + \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d\big(s^{1-\beta}\bar{\nu}(1/s)\big)'_+ + (\beta-1)D_\beta.$$

Thus, we obtain

$$-G(x) = \int_{(0,\infty)} e^{-x/s}s^{\beta-1}\, d\big(-(s^{1-\beta}\bar{\nu}(1/s))'\big).$$

As $s^\beta\bar{\nu}(s)$ is concave, $s^{1-\beta}\bar{\nu}(1/s)$ is concave by Corollary 1, implying $-\big(s^{1-\beta}\bar{\nu}(1/s)\big)'_+$ is non-decreasing. It defines a Radon measure and $-G$ is CM by definition. □

4. Corollaries

This section contains several corollaries of the main results.

Corollary 3. *The CM function F in Theorem 6 has the representation*

$$F(x) = \int_0^\infty e^{-xs}r'_\beta(s)ds, \tag{4.1}$$

where $r'_\beta(s) = s^{2-\beta}(s^\beta F_\mu(s))''$ for almost all $s > 0$.

Proof. We omit the proof since it is almost identical to the one for the next corollary. One just needs to replace $\bar{\nu}(s)$ with $F_\mu(s)$ and $\nu(ds)$ with $\mu(ds)$. □

Corollary 4. *The CM function G in Theorem 7 has the representation*

$$G(x) = \int_0^\infty e^{-xs} r'_\beta(s) ds, \tag{4.2}$$

where $r'_\beta(s) = s^{2-\beta}(s^\beta \bar{\nu}(s))''$ *for almost all* $s > 0$.

Proof. Using Fubini's theorem, several times, function G can be rewritten as

$$\begin{aligned} G(x) &= \beta(\beta-1)\frac{g(x)-a-bx}{x} - 2(\beta-1)(g'(x)-b) + xg''(x) \\ &= \beta(\beta-1)\int_0^\infty e^{-xt}\bar{\nu}(t)\,dt - 2(\beta-1)\int_{(0,\infty)} e^{-xt}\,t\,\nu(dt) - x\int_{(0,\infty)} e^{-xt}\,t^2\,\nu(dt) \\ &= \beta(\beta-1)x\int_0^\infty \bar{\nu}(t)\Big(\int_t^\infty e^{-xs}\,ds\Big)\,dt - 2(\beta-1)x\int_{(0,\infty)}\Big(\int_t^\infty e^{-xs}\,ds\Big)\,t\,\nu(dt) \\ &\quad -x\int_{(0,\infty)} e^{-xt}\,t^2\,\nu(dt) \\ &= \beta(\beta-1)x\int_0^\infty e^{-xs}\Big(\int_0^s \bar{\nu}(t)\,dt\Big)\,ds - 2(\beta-1)x\int_{(0,\infty)} e^{-xs}\Big(\int_0^s t\nu(dt)\Big)\,ds \\ &\quad -x\int_{(0,\infty)} e^{-xs}\,s^2\,\nu(ds) \\ &= x\int_0^\infty e^{-xs}\rho_\beta(ds), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \rho_\beta(ds) &:= \left[\beta(\beta-1)\int_0^s \bar{\nu}(t)\,dt - 2(\beta-1)\int_0^s t\,\nu(dt)\right] ds - s^2\,\nu(ds) \\ &= \left[\beta(\beta-1)\int_0^s \bar{\nu}(t)\,dt - 2(\beta-1)\Big(\int_0^s \bar{\nu}(t)\,dt - s\bar{\nu}(s)\Big)\right] ds - s^2\,\nu(ds) \\ &= (1-\beta)\left[(2-\beta)\int_0^s \bar{\nu}(t)\,dt - 2s\bar{\nu}(s)\right] ds - s^2\,\nu(ds). \end{aligned} \tag{4.4}$$

Comparing (4.3) with (2.6) we make the following observation: G (resp. $-G$) is CM precisely when ρ_β (resp. $-\rho_\beta$) has a non-negative, non-decreasing density function r_β. In that case, solving for $\nu(ds)$ in (4.4) shows that necessarily ν has a density functionand and without loss of generality we may assume that it is of the form $m(s)/s^{\beta+1}$, $s > 0$. Substituting it in (4.4), we obtain

$$\begin{aligned} r_\beta(s) &= (1-\beta)\left[(2-\beta)\int_0^s \bar{\nu}(t)\,dt - 2s\bar{\nu}(s)\right] - s^{1-\beta}\,m(s) \\ &= (1-\beta)(2-\beta)\int_0^s \bar{\nu}(t)\,dt - (2-\beta)s\bar{\nu}(s) + s^{2-\beta}\,(s^\beta\bar{\nu}(s))'_+ \end{aligned} \tag{4.5}$$

$$= (1-\beta)(2-\beta)\int_0^s \bar{\nu}(t)\,dt - (1-\beta)s\bar{\nu}(s) + s^{3-\beta}\,(s^{\beta-1}\bar{\nu}(s))'_+. \tag{4.6}$$

Using that $s^{\beta}\bar{\nu}(s)$ is convex (resp. concave), the derivative of r_β exists almost everywhere and differentiating (4.5) gives

$$r'_\beta(s) = s^{2-\beta}\,(s^{\beta}\bar{\nu}(s))''. \tag{4.7}$$

Using L'Hopital's rule, (2.13), Corollary 2, and (4.6), one sees that

$$\lim_{s\to 0+} r_\beta(s) = 0 = \lim_{s\to\infty} e^{-xs}\, r_\beta(s),\;\; x > 0.$$

An application of Fubini's theorem finally gives

$$G(x) = x\int_0^\infty e^{-xs}\rho_\beta(ds) = \int_0^\infty e^{-xs} r'_\beta(s)ds, \tag{4.8}$$

which completing the proof. □

Corollary 5. *The function G, defined in* (3.4)*, can never be a Bernstein function.*

Proof. We use the notation and representations from the proof of Corollary 4. Comparing (4.3) with (2.12) we observe that: G is a BF precisely when ρ_β has a non-negative, non-increasing density function r_β. Assuming the latter, then (4.7) shows that ν has β-concave tail. Then, Theorem 7, part b) shows that $-G$ is CM, which is a contradiction. □

A similar proof shows the next analogous corollary.

Corollary 6. *The function F, defined in* (3.1)*, can never be a BF.*

Standard facts about BFs imply that if $xG(x)$ is a BF, then G is completely monotone.

The next corollaries deal with special cases of the main results. Some of them re-derive several of the results in [9].

Corollary 7. *Suppose f is CM with measure μ. Then, μ is harmonically convex precisely when $f(x) - xf'(x)$ is CM.*

Proof. By Theorem 6 part a), with $\beta = 1$, μ is harmonically convex precisely when $xf''(x)$ is CM. We show this condition is equivalent to $f(x) - xf'(x)$ being CM. If $f(x) - xf'(x)$ is CM, then

$$xf''(x) = -(f(x) - xf'(x))'$$

is CM. Conversely, if $xf''(x)$ is CM, then, to see $f(x) - xf'(x)$ is CM, it suffices to show its non-negativity. This is trivial, because $f(x) \geq 0$ and $f'(x) \leq 0$ for all $x > 0$. □

Corollary 8. *Suppose f is CM with measure μ. Then, μ is convex precisely when $x(f(x) - \mu(\{0\}))$ is CM.*

Proof. By Theorem 6 part a), applied to the shifted function $f(x) - \mu(\{0\})$ with $\beta = 0$, the measure μ is convex precisely when $2f'(x) + xf''(x)$ is CM. We show this condition is equivalent to $x(f(x) - \mu(\{0\}))$ being CM. If $x(f(x) - \mu(\{0\}))$ is CM, then

$$2f'(x) + xf''(x) = (x(f(x) - \mu(\{0\})))''$$

is completely monotone.

Conversely, suppose $2f'(x) + xf''(x)$ is CM. To see $x(f(x) - \mu(\{0\}))$ is CM, we only have to show

$$x(f(x) - \mu(\{0\})) \geq 0 \quad \text{and} \quad xf'(x) + f(x) - \mu(\{0\}) \leq 0.$$

The first inequality holds because $f(x) - \mu(\{0\}) \geq 0$. For the second inequality, as $2f'(x) + xf''(x) \geq 0$, we know $xf'(x) + f(x) - \mu(\{0\})$ is non-decreasing. By (2.7), we obtain

$$\lim_{x\to\infty} xf'(x) + f(x) - \mu(\{0\}) = 0.$$

The second inequality follows from here. □

Corollary 9. *Suppose f is CM with measure μ. Then, μ is harmonically concave precisely when $f(x) = \mu(\{0\})$.*

Proof. Consider the shifted function $f(x) - \mu(\{0\})$. By Theorem 6 part b), with $\beta = 1$, the measure μ is harmonically concave precisely when $-xf''(x)$ is CM. We show this condition is equivalent to $f(x) = \mu(\{0\})$. If $f(x) = \mu(\{0\})$, then $-xf''(x) = 0$, which is CM. Conversely, if $-xf''(x)$ is CM, then so is $-xf''(x)(1/x) = -f''(x)$. Thus, we obtain $f''(x) \leq 0$. Notice that $f''(x) \geq 0$, because f is CM. Therefore, we have $f''(x) = 0$, and $f(x) = \mu(\{0\})$. □

Corollary 10. *Suppose f is CM with measure μ. Then, μ is concave precisely when $f(x) + xf'(x)$ is CM.*

Proof. Without loss of generality, we can assume μ has no mass at zero. By Theorem 6 part b), with $\beta = 0$, the measure μ is concave, if and only if $-2xf'(x) - xf''(x)$ is CM. We show this condition is equivalent to $f(x) + xf'(x)$ being CM. If $f(x) + xf'(x)$ is CM, then

$$-2xf'(x) - xf''(x) = -(f(x) + xf'(x))'$$

is CM. Conversely, if $-2xf'(x) - xf''(x)$ is CM, to see $f(x) + xf'(x)$ is CM, it suffices to show it is non-negative. As its derivative is non-positive, $f(x) + xf'(x)$ is non-increasing. By (2.7), we obtain

$$\lim_{x\to\infty} f(x) + xf'(x) = 0.$$

So $f(x) + xf'(x) \geq 0$. This completes the proof. □

Corollary 11. *Suppose g is a Bernstein function with Lévy triplet (a, b, ν). Then, ν has harmonically convex tail precisely when $g(x) = a + bx$.*

Proof. By Theorem 7 part a), applied to the shifted function $g(x) - a - bx$ with $\beta = 1$, the measure ν has harmonically convex tail, if and only if $xg''(x)$ is completely monotone. We show this condition is equivalent to $g(x) = a + bx$. If $g(x) = a + bx$, then $xg''(x) = 0$, which is CM. Conversely, if $xg''(x)$ is CM, then so is $xf''(x)(1/x) = f''(x)$, that is $g''(x) \geq 0$. Because g is a Bernstein function, $g''(x) \leq 0$. Thus, we obtain $g''(x) = 0$, which implies $g(x) = a + bx$. □

Corollary 12. *Suppose g is a Bernstein function with Lévy triplet (a, b, ν). Then, ν has convex tail precisely when $g(x) + xg'(x)$ is a Bernstein function.*

Proof. By Theorem 7 part a), applied to the shifted BF $g(x) - a - bx$ with $\beta = 0$, the measure ν has convex tail precisely when $2g'(x) + xg''(x)$ is CM. We show this condition is equivalent to $g(x) + xg'(x)$ being a Bernstein function. If $g(x) + xg'(x)$ is a Bernstein function, then

$$2g'(x) + xg''(x) = (g(x) + xg'(x))',$$

is CM. (Note that $2g'(x) + xg''(x) \geq 0$.) Conversely, if $2g'(x) + xg''(x)$ is CM, then it suffices to show $g(x) + xg'(x) \geq 0$ to see that $g(x) + xg'(x)$ is a Bernstein function. This is trivial, because $g(x) \geq 0$ and $g'(x) \geq 0$. □

Corollary 13. *Suppose g is a Bernstein function with Lévy triplet (a, b, ν). Then, ν has harmonically concave tail precisely when $g(x) - xg'(x)$ is a Bernstein function.*

Proof. By Theorem 7 part b), applied to the shifted BF $g(x) - a - bx$ with $\beta = 1$, the measure ν has harmonically concave tail, if and only if $-xg''(x)$ is CM. We show this condition is equivalent to $g(x) - xg'(x)$ being a Bernstein function. If $g(x) - xg'(x)$ is a Bernstein function, then

$$-xg''(x) = (g(x) - xg'(x))',$$

is CM. Conversely, if $-xg''(x)$ is CM, then, to show $g(x) - xg'(x)$ is a Bernstein function, it suffices to show it is non-negative. As its derivative is non-negative, $g(x) - xg'(x)$ is non-decreasing. Noticing that $\lim_{x\to 0} xg'(x) = 0$, see [8, (2.11)], we obtain

$$\lim_{x\to 0} g(x) - xg'(x) = a \geq 0.$$

So $g(x) - xg'(x) \geq 0$, and this completes the proof. □

Corollary 14. *Suppose g is a Bernstein function with Lévy triplet (a, b, ν). Then, ν has concave tail precisely when $g(x) = a + bx$.*

Proof. Consider the shifted BF $g(x) - a - bx$. By Theorem 7 part b), with $\beta = 0$, the measure ν has concave tail precisely when $-2g'(x) - xg''(x)$ is CM. We show this condition is equivalent to $g(x) = 0$. If $g(x) = 0$, then $-2g'(x) - xg''(x) = 0$, which is CM. Conversely, if $-2g'(x) - xg''(x)$ is CM, then, $g(x) + xg'(x)$ is non-increasing, as

$$(g(x) + xg'(x))' = 2g'(x) + xg''(x) \leq 0.$$

Since

$$\lim_{x\to 0} g(x) + xg'(x) = 0,$$

we obtain that $g(x) + xg'(x) \leq 0$, and thus its anti-derivative $xg(x)$ is non-increasing. Because $g(x)$ approaches zero as x approaches zero, we know $xg(x) \leq 0$. However, g is a Bernstein function, indicating $g(x)$= 0. This concludes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

Lemma 5. *Suppose f is continuous with bounded variation on $(0,\infty)$ and g is right-continuous with bounded variation on $(0,\infty)$. Then,*

$$\int_{(0,\infty)} m(x)\, d(f(x)g(x)) = \int_{(0,\infty)} m(x)f(x)\, dg(x) + \int_{(0,\infty)} m(x)g(x)\, df(x), \tag{A.1}$$

where m is right-continuous and non-negative on $(0,\infty)$.

Proof. The sketch of proof is provided below.
Step 1: Show that (A.1) holds for increasing g on the closed interval $[a,b] \subset (0,\infty)$. For partition $a = x_0 < x_1 < \cdots < x_n = b$, we have

$$\begin{aligned}
\int_{[a,b]} m(x)\, d(f(x)g(x)) &= \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} m(x_i)[f(x_{i+1})g(x_{i+1}) - f(x_i)g(x_i)] \\
&= \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} m(x_i)f(x_i)[g(x_{i+1}) - g(x_i)] + \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} m(x_i)g(x_i)[f(x_{i+1}) - f(x_i)] \\
&\quad + \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} m(x_i)[f(x_{i+1}) - f(x_i)][g(x_{i+1}) - g(x_i)] \\
&= \int_{[a,b]} m(x)f(x)\, dg(x) + \int_{[a,b]} m(x)g(x)\, df(x) \\
&\quad + \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} m(x_i)[f(x_{i+1}) - f(x_i)][g(x_{i+1}) - g(x_i)].
\end{aligned}$$

Notice that f is continuous, thus uniformly continuous, on $[a,b]$. For any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|t-s| < \delta$, we have $|f(t) - f(s)| < \epsilon$. For any partition whose mesh is small, we obtain

$$\begin{aligned}
\lim_{\text{mesh}\to 0} \left| \sum_{i=0}^{n-1} m(x_i)[f(x_{i+1}) - f(x_i)][g(x_{i+1}) - g(x_i)] \right| &\le \epsilon \lim_{\text{mesh}\to 0} \sum_{i=0}^{n-1} \left| m(x_i)[g(x_{i+1}) - g(x_i)] \right| \\
&= \epsilon \int_{[a,b]} m(x)\, dg(x).
\end{aligned}$$

So this limit can be arbitrarily small, which indicates

$$\int_{[a,b]} m(x)\, d(f(x)g(x)) = \int_{[a,b]} m(x)f(x)\, dg(x) + \int_{[a,b]} m(x)g(x)\, df(x).$$

Step 2: (A.1) holds for g with bounded variation on $[a,b]$, as such g can be represented as the difference of two increasing functions.
Step 3: (A.1) holds on $(0,\infty)$, as the equation holds when taking the limit of $a \to 0$ and $b \to \infty$. □

The following result is Theorem 6.2.2 in [3] or Theorem A.1 in [8]. It gives integration by parts for Lebesgue-Stieltjes integrals on finite intervals. It can be extended to the interval $(0,\infty)$ by taking limits.

Theorem 8. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous and right-continuous functions, respectively, with bounded variation. Then*

$$\int_{[a,b]} f\, dg + \int_{[a,b]} g\, df = f(b)g(b) - f(a-)g(a-),$$
$$\int_{(a,b]} f\, dg + \int_{(a,b]} g\, df = f(b)g(b) - f(a)g(a),$$
$$\int_{(a,b)} f\, dg + \int_{(a,b)} g\, df = f(b-)g(b-) - f(a)g(a).$$

The following lemma is a particular case of the change of variable formula for Lebesgue-Stieltjes integrals, see [11, Theorem 11a].

Lemma 6. *Suppose f is continuous on $(0, \infty)$, and g has bounded variation on $(0, \infty)$. Then*

$$\int_{(0,\infty)} f(x)\, dg(x) = -\int_{(0,\infty)} f(1/t)\, dg(1/t).$$

Lemma 7. *Suppose $f(t) \geq 0$ is non-increasing. If $\int_{(0,\infty)} f(t)\, dt < \infty$, then $f(t)$ is $o(1/t)$ as t approaches zero or infinity.*

Lemma 8. *Suppose $f(t) \geq 0$ is non-increasing. If $\int_{(0,1)} f(t)\, d(t^p) < \infty$ for some $p > 0$. Then $f(t)$ is $o(1/t^p)$ as $t \to 0$.*

Proof. After a change of variable by $s = t^p$, we obtain $\int_{(0,1)} f(s^{1/p})\, d(s) < \infty$. Because $f(s^{1/p})$ is non-increasing for any $p > 0$, we conclude that $f(s^{1/p})$ is $o(1/s)$ as $s \to 0$. This implies $f(t)$ is $o(1/t^p)$ as $t \to 0$. □

Lemma 9. *Suppose $f(t) \geq 0$ is non-increasing and g is strictly increasing with $g(0) = 0$. If $\int_{(0,1)} f(t)\, d(g(t)) < \infty$, then $f(t)$ is $o(1/g(t))$ as $t \to 0$.*

Proof. As g is strictly increasing with $g(0) = 0$, its inverse function $g^{-1}(t)$ is also strictly increasing with $g^{-1}(0) = 0$. Change of variable by setting $t = g^{-1}(s)$, we have $f(g^{-1}(s))$ is non-increasing $\int_{(0,g(1))} f(g^{-1}(s))\, ds < \infty$. So $f(g^{-1}(s))$ is $o(1/s)$ as $s \to 0$, which implies $\lim_{t\to 0} f(t)g(t) = 0$. □