Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 11335–11351.
DOI: 10.3934/math.2024556
Received: 26 January 2024
Revised: 06 March 2024
Accepted: 18 March 2024
Published: 22 March 2024

Research article

Some fixed point and stability results in b-metric-like spaces with an application to integral equations on time scales

Zeynep Kalkan[1], **Aynur Şahin**[1], **Ahmad Aloqaily**[2,3] **and Nabil Mlaiki**[2,*]

[1] Department of Mathematics, Faculty of Sciences, Sakarya University, Sakarya, 54050, Turkey
[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
[3] School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney, 2150, Australia

* **Correspondence:** Email: nmlaiki2012@gmail.com, nmlaiki@psu.edu.sa.

Abstract: This paper presents the stability theorem for the T-Picard iteration scheme and establishes the existence and uniqueness theorem for fixed points concerning T-mean nonexpansive mappings within b-metric-like spaces. The outcome of our fixed point theorem substantiated the existence and uniqueness of solutions to the Fredholm-Hammerstein integral equations defined on time scales. Additionally, we provided two numerical examples from distinct time scales to support our findings empirically.

Keywords: stability; fixed point; b-metric-like space; T-mean nonexpansive mapping; Hammerstein integral equation; time scale
Mathematics Subject Classification: 47H10, 45B05

1. Introduction

Fixed point (FP) theory has been applied in many domains of science, including approximation theory, biology, chemistry, dynamic systems, economics, engineering, fractals, game theory, logic programming, optimization problems, and physics. Undoubtedly, the FP theory is an active area in mathematics and beautifully combines analysis, topology, and geometry. In the past few decades, it has been clear that the FP theory is a very effective and significant instrument for investigating nonlinear processes. Its application relies on the existence of solutions to mathematical problems that are based on the contraction principle. After Banach [1] presented his principle, the FP theory approaches became more successful and appealing to scientists, see [2–10].

Now, let's start with the definition of FP and the Banach contraction principle.

Let (W, D) be a metric space. A point $\zeta \in W$ is called a FP of a mapping $S : W \to W$, if

$$S(\zeta) = \zeta.$$

Theorem 1. *(Banach contraction principle) Let S be a self-mapping on a complete metric space (W, D). If there exists a constant $\theta \in [0, 1)$ such that*

$$D(S(\zeta), S(\xi)) \leq \theta D(\zeta, \xi) \tag{1.1}$$

for all $\zeta, \xi \in W$, then S has a unique FP.

The term "nonexpansive mapping" refers to a mapping S satisfying the condition (1.1) for $\theta = 1$. Nonexpansive mappings emerged as a direct extension of the concept of contraction mappings initially introduced by Banach [1]. In 1965, Browder [11] and Göhde [12] established an FP theorem for nonexpansive mappings within uniformly convex Banach spaces.

In 1975, Zhang [13] introduced the broader category of mean nonexpansive mappings, encompassing nonexpansive mappings, and demonstrated the existence and uniqueness of FPs for this class of mappings within Banach spaces exhibiting normal structural properties.

Definition 1. *[13] Let (W, D) be a metric space. A mapping $S : W \to W$ is called mean nonexpansive if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that*

$$D(S(\zeta), S(\xi)) \leq \alpha D(\zeta, \xi) + \beta D(\zeta, S(\xi)), \quad \forall \zeta, \xi \in W. \tag{1.2}$$

In a recent development, Mebawondu et al. [14] introduced the notion of T-mean nonexpansive mapping within the framework of metric spaces. Subsequently, they provided rigorous proof establishing the existence of a unique FP for this particular mapping.

Definition 2. *[14] Let T, S be self-mappings on a metric space (W, D). S is said to be a T-mean nonexpansive mapping if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that*

$$D(T(S(\zeta)), T(S(\xi))) \leq \alpha D(T(\zeta), T(\xi)) + \beta D(T(\zeta), T(S(\xi)), \qquad \forall \zeta, \xi \in W. \tag{1.3}$$

Remark 1. *If we take $T = I$, where I is the identity function, in (1.3), then we obtain that S is a mean nonexpansive mapping defined by (1.2).*

Also, Mebawondu et al. [14] presented the following example with T-mean nonexpansive mapping but not mean nonexpansive mapping in metric spaces.

Example 1. *[14, Examples 2.1 and 2.2] (i) Let $W = [1, \infty)$ and $D(\zeta, \xi) = |\zeta - \xi|$ for all $\zeta, \xi \in W$. Let $T, S : W \to W$ be defined by $T(\zeta) = 1 + \ln \zeta$ and*

$$S(\zeta) = \begin{cases} \sqrt[10]{\zeta}, & \text{if } \zeta \neq 2, \\ 1, & \text{if } \zeta = 2. \end{cases}$$

(ii) Let $W = [0, 2]$ and D be as in (i). Let $T, S : W \to W$ be defined by

$$T(\zeta) = \begin{cases} 1 - \zeta, & \text{if } \zeta \in [0, 1], \\ 2 - \zeta, & \text{if } \zeta \in (1, 2], \end{cases}$$

and

$$S(\zeta) = \begin{cases} 1, & \text{if } \zeta \in [0,1), \\ 2, & \text{if } \zeta \in [1,2]. \end{cases}$$

Then, in the above examples, S is T-mean nonexpansive, but not mean nonexpansive.

The oldest known iteration method is the Picard iteration, which is used in the Banach contraction principle. In 2009, Morales and Rojas [15] offered a novel iteration scheme called the T-Picard iteration, which is defined as follows.

Definition 3. *[15] Let T, S be self-mappings on a metric space (W, D) and $\zeta_0 \in W$. The sequence $\{T(\zeta_n)\} \subset W$ defined by*

$$T(\zeta_{n+1}) = T(S(\zeta_n)) = T(S^n(\zeta_0)), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{1.4}$$

is called the T-Picard iteration associated with S.

Remark 2. *If we take $T = I$ in (1.4), then we obtain the Picard iteration.*

Another domain where FP theory can be extended pertains to spatial considerations. Recently, significant advancements have been made in this direction, particularly within the realm of metric spaces. The concept of b-metric space was introduced by Bakhtin [16] and Czerwik [17] in different periods. In a different line of development, Matthews [18] introduced the notion of partial metric space and adapted the Banach contraction principle for applications in program verification.

In 2012, Amini-Harandi [19] introduced the concept of metric-like space, wherein a point's self-distance need not be equal to zero. Within these novel spaces, Amini-Harandi [19] presented various FP theorems that extended and enhanced existing results in both partial metric and b-metric spaces. Subsequently, in the following year, Alghamdi et al. [20] introduced the notion of b-metric-like space, thereby generalizing the concepts of b-metric and metric-like spaces, and established various associated FP results. Since then, numerous outcomes related to FPs of mappings under specific contractive conditions within these spaces have been obtained, as exemplified by works such as [21–24].

Motivated by these seminal results, we investigated the existence and uniqueness of FPs for T-mean nonexpansive mappings and provided a stability theorem for T-Picard iteration within the framework of b-metric-like spaces. Furthermore, we explored the existence and uniqueness of solutions to the Fredholm-Hammerstein integral equations on time scales as an application of our primary theorem. Finally, we presented two illustrative numerical examples from different time scales. It is noteworthy that our results were significant as they generalized the corresponding findings within b-metric and metric-like spaces, both of which are special instances of b-metric-like spaces.

2. Preliminaries

2.1. Some basic definitions in b-metric-like spaces

In this subsection, several well-known definitions and two lemmas in b-metric-like spaces are listed.

Definition 4. *[20, Definition 2.4] Let W be a nonempty set and $k \geq 1$ be a given real number. A function $D_{bl} : W \times W \to \mathbb{R}^+$ is b-metric-like (in brief, bl) if the following conditions are satisfied:*

$(D_{bl}1)$ $D_{bl}(\zeta, \xi) = 0 \Longrightarrow \zeta = \xi$ *(indistancy implies equality);*

($D_{bl}2$) $D_{bl}(\zeta,\xi) = D_{bl}(\xi,\zeta)$ (symmetry);
($D_{bl}3$) $D_{bl}(\zeta,\xi) \leq k[D_{bl}(\zeta,\eta) + D_{bl}(\eta,\xi)]$ (weakened triangularity)
for all $\zeta,\xi,\eta \in W$. The pair (W, D_{bl}) is called a bl space.

In a *bl* space (W, D_{bl}), if $\zeta,\xi \in W$ and $D_{bl}(\zeta,\xi) = 0$, then $\zeta = \xi$; but the converse may not be true since $D_{bl}(\zeta,\zeta)$ may be positive for some $\zeta \in W$.

Remark 3. *As it is excepted, each bl space forms a metric-like space by letting $k = 1$. On the other hand, every b-metric space is a bl space with the same parameter k. Hence, the class of bl spaces is bigger than the class of metric-like spaces (see [19]) or b-metric spaces (see [17, 25–27]). However, the inverse implications are not true in general.*

Example 2. *[20, Examples 2.5 and 2.6] Let $W = [0,\infty)$. Define a function $D_{bl} : W^2 \to [0,\infty)$ by $D_{bl}(\zeta,\xi) = (\zeta+\xi)^2$ or $D_{bl}(\zeta,\xi) = (\max\{\zeta,\xi\})^2$. Then (W, D_{bl}) is a bl space with the parameter $k = 2$. Clearly, (W, D_{bl}) is not a b-metric space or metric-like space.*

Definition 5. *[20, Definition 2.9] Let (W, D_{bl}) be a bl space, $\{\zeta_n\}$ be a sequence in W, and $\zeta \in W$. We say that*

(i) $\{\zeta_n\}$ is said to be a convergent sequence if $\lim_{n\to\infty} D_{bl}(\zeta_n,\zeta) = D_{bl}(\zeta,\zeta)$ and a Cauchy sequence if $\lim_{n,m\to\infty} D_{bl}(\zeta_n,\zeta_m)$ exists and is finite;

(ii) (W, D_{bl}) is called complete if, for every Cauchy sequence $\{\zeta_n\}$ in W, there exists $\zeta \in W$ such that $\lim_{n,m\to\infty} D_{bl}(\zeta_n,\zeta_m) = D_{bl}(\zeta,\zeta) = \lim_{n\to\infty} D_{bl}(\zeta_n,\zeta)$.

Remark 4. *In a bl space, the limit for a convergent sequence is not unique in general. However, if $\{\zeta_n\}$ is a Cauchy sequence with $\lim_{n,m\to\infty} D_{bl}(\zeta_n,\zeta_m) = 0$ in the complete bl space (W, D_{bl}), then the limit of such a sequence is unique. Indeed, in such a case, if $\zeta_n \to \zeta$ ($D_{bl}(\zeta_n,\zeta) \to D_{bl}(\zeta,\zeta)$) as $n \to \infty$ we get that $D_{bl}(\zeta,\zeta) = 0$.*

In 2009, Beiranvand et al. [28] defined the concept of sequentially convergent mapping in metric spaces. Following this, many researchers have worked on this concept (see [14, 29–32]). Inspired by these results, we provide the following definition in *bl* spaces.

Definition 6. *Let (W, D_{bl}) be a bl space. A mapping $T : W \to W$ is said to be sequentially convergent if we have, for every sequence $\{\xi_n\}$, if $\{T(\xi_n)\}$ is convergent then $\{\xi_n\}$ is also convergent.*

Lemma 1. *[33, Lemma 1.10] Let T be a self-mapping on a bl space (W, D_{bl}). If T is continuous at $u \in W$ then, for every sequence $\{\zeta_n\}$ in W such that $\zeta_n \to u$, we have $T(\zeta_n) \to T(u)$, that is,*

$$\lim_{n\to\infty} D_{bl}(T(\zeta_n), T(u)) = D_{bl}(T(u), T(u)).$$

In 2023, Calderón et al. [34] studied the concept of (T, S)-stability in metric spaces. Now, we define this concept in the setting of *bl* spaces as follows.

Definition 7. *Let T, S be self-mappings on a bl space (W, D_{bl}) and ζ^* be the FP of S. Let $\{T(\zeta_n)\}$ be a sequence generated by an iteration scheme, that is,*

$$T(\zeta_{n+1}) = h(T, S, \zeta_n), \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $\zeta_0 \in W$ *is the initial point and* h *is a function. Assume that* $\{T(\zeta_n)\}$ *converges to* $T(\zeta^*)$*. Let* $\{T(\xi_n)\} \subset W$ *be an arbitrary sequence, and set* $\rho_n = D_{bl}(T(\xi_{n+1}), h(T, S, \xi_n))$*,* $n \in \mathbb{N}_0$*. Then, the iteration scheme (2.1) is said to be* (T, S)*-stable if and only if*

$$\lim_{n\to\infty} \rho_n = 0 \Longleftrightarrow \lim_{n\to\infty} T(\xi_n) = T(\zeta^*).$$

If we take $T = I$ in Definition 7, it is reduced to the concept of the stability of an iteration scheme, which was defined by Harder and Hicks [35].

Lemma 2. *[36, p. 14] Let* $\{\xi_n\}$*,* $\{\rho_n\}$ *be sequences of non-negative numbers and* $0 \leq \theta < 1$*, so that*

$$\xi_{n+1} \leq \theta \xi_n + \rho_n,$$

for all $n \in \mathbb{N}$*. If* $\lim_{n\to\infty} \rho_n = 0$*, then* $\lim_{n\to\infty} \xi_n = 0$*.*

2.2. Basic information on time scales

In 1988, Hilger [37] introduced the theory of time scales, which has recently garnered a lot of attention, in his Ph. D. thesis to unify continuous and discrete analysis. In this subsection, we will go over several fundamental concepts related to time scales.

Definition 8. *[38] A time scale* $\mathbb{T}_{\mathbb{S}}$ *is an arbitrary, nonempty, and closed subset of the real numbers* $\mathbb{R}$*.*

The forward and backward jump operators are defined by $\omega(t) = \inf\{s \in \mathbb{T}_{\mathbb{S}} : s > t\}$, and $\tau(t) = \sup\{s \in \mathbb{T}_{\mathbb{S}} : s < t\}$, respectively, where $\inf \emptyset = \sup \mathbb{T}_{\mathbb{S}}$ and $\inf \mathbb{T}_{\mathbb{S}} = \sup \emptyset$. The graininess function $\mu : \mathbb{T}_{\mathbb{S}} \to [0, +\infty)$ is defined by $\mu(t) = \omega(t) - t$.

The classification of points on the time scale $\mathbb{T}_{\mathbb{S}}$ is possible with the jump operators.

Definition 9. *[38, Definition 1.1] A point* $t \in \mathbb{T}_{\mathbb{S}}$ *is said to be right-dense if* $\omega(t) = t$*, right-scattered if* $\omega(t) > t$*, left-dense if* $\tau(t) = t$*, left-scattered if* $\tau(t) < t$*, isolated if* $\tau(t) < t < \omega(t)$*, and dense if* $\tau(t) = t = \omega(t)$*.*

In the following definition, the set $\mathbb{T}_{\mathbb{S}}^{\kappa}$ that will be needed in the delta derivative is given.

Definition 10. *[38] The set* $\mathbb{T}_{\mathbb{S}}^{\kappa}$ *is defined as follows:*

$$\mathbb{T}_{\mathbb{S}}^{\kappa} = \begin{cases} \mathbb{T}_{\mathbb{S}} \backslash (\tau(\sup \mathbb{T}_{\mathbb{S}}), \sup \mathbb{T}_{\mathbb{S}}], & \text{if } \sup \mathbb{T}_{\mathbb{S}} < \infty, \\ \mathbb{T}_{\mathbb{S}}, & \text{if } \sup \mathbb{T}_{\mathbb{S}} = \infty. \end{cases}$$

Definition 11. *[38, Definition 1.10] Assume* $f : \mathbb{T}_{\mathbb{S}} \to \mathbb{R}$ *is a function and fix* $t \in \mathbb{T}_{\mathbb{S}}^{\kappa}$*. The delta derivative (also Hilger derivative)* $f^{\Delta}(t)$ *exists if, for every* $\epsilon > 0$*, there exists a neighbourhood* $U = (t - \delta, t + \delta) \cap \mathbb{T}_{\mathbb{S}}$ *for some* $\delta > 0$ *such that*

$$\left|[f(\omega(t)) - f(s)] - f^{\Delta}(t)[\omega(t) - s]\right| \leq \epsilon\, |\omega(t) - s| \qquad \text{for all } s \in U.$$

Proposition 1. *[38] Let* $f : \mathbb{T}_{\mathbb{S}} \to \mathbb{R}$ *be a function and* $t \in \mathbb{T}_{\mathbb{S}}^{\kappa}$*. If* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^{\Delta}(t) = \frac{f(\omega(t)) - f(t)}{\mu(t)}.$$

Remark 5. *[38] (i) If* $\mathbb{T}_{\mathbb{S}} = \mathbb{R}$*, then* $f^{\Delta}(t) = f'(t)$ *is the derivative used in the standard calculus.*

(ii) If $\mathbb{T}_{\mathbb{S}} = \mathbb{Z}$*, then* $f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t)$ *is the forward difference operator used in difference equations.*

Definition 12. *[38, Definition 1.1] A function* $f : \mathbb{T}_{\mathbb{S}} \to \mathbb{R}$ *is called rd-continuous if it is continuous at all right-dense points in* $\mathbb{T}_{\mathbb{S}}$ *and its left-sided limits exist (finite) at all left-dense points in* $\mathbb{T}_{\mathbb{S}}$*.*

If f is rd-continuous, then there exists a function F such that $F^{\Delta}(t) = f(t)$ (see [38, Theorem 1.74]). In this case, the (Cauchy) delta integral of f is defined by

$$\int_d^e f(t)\,\Delta t = F(e) - F(d) \quad \text{for all } d, e \in \mathbb{T}_{\mathbb{S}}.$$

Remark 6. *[38] (i) If* $\mathbb{T}_{\mathbb{S}} = \mathbb{R}$*, then*

$$\int_d^e f(t)\,\Delta t = \int_d^e f(t)\,dt$$

is the Riemann integral used in the standard calculus.

(ii) If $\mathbb{T}_{\mathbb{S}} = \mathbb{Z}$*, then*

$$\int_d^e f(t)\,\Delta t = \sum_{t=d}^{e-1} f(t) \quad \textit{for all } d, e \in \mathbb{Z} \textit{ with } d < e.$$

Lemma 3. *[39] (Cauchy-Schwarz inequality) Let* $f, g : \mathbb{T}_{\mathbb{S}} \to \mathbb{R}$ *be two rd-continuous mappings. For all* $d, e \in \mathbb{T}_{\mathbb{S}}$ *with* $d \leq e$*, we have*

$$\int_d^e |f(t)g(t)|\,\Delta t \leq \sqrt{\left\{\int_d^e |f(t)|^2\,\Delta t\right\}.\left\{\int_d^e |g(t)|^2\,\Delta t\right\}}.$$

Proposition 2. *[40] Let d and e be arbitrary points in* $\mathbb{T}_{\mathbb{S}}$*.*

(i) Every constant function $f(t) = c$ $(t \in \mathbb{T}_{\mathbb{S}})$ *is* Δ*-integrable from d to e and* $\int_d^e c\,\Delta t = c(e-d)$*.*

(ii) If f and g are Δ*-integrable on* $[d, e)$ *and* $f(t) \leq g(t)$ *for all* $t \in [d, e)$*, then*

$$\int_d^e f(t)\,\Delta t \leq \int_d^e g(t)\,\Delta t.$$

(iii) If f is Δ*-integrable on* $[d, e)$*, then* $|f|$ *is* Δ*-integrable on* $[d, e)$ *and*

$$\left|\int_d^e f(t)\,\Delta t\right| \leq \int_d^e |f(t)|\,\Delta t.$$

(iv) If f and g are Δ*-integrable on* $[d, e)$*, then their product* $f.g$ *is* Δ*-integrable on* $[d, e)$*.*

We recommend the reader to [38,41–43] for more information on time scales and their applications.

3. Some FP and stability results for T-mean nonexpansive mappings

Let's start this section with an example of T-mean nonexpansive mapping, which doesn't mean nonexpansive in a *bl* space with the same constants.

Example 3. *Let $W = [0,1]$ and $D_{bl}(\zeta,\xi) = (\max\{\zeta,\xi\})^2$ for all $\zeta,\xi \in W$. Then (W, D_{bl}) is a bl space with the parameter $k = 2$. Let $T, S : W \to W$ be defined by*

$$T(\zeta) = \begin{cases} \frac{3\zeta}{8}, & \text{if } \zeta \in [0,1), \\ 0, & \text{if } \zeta = 1, \end{cases}$$

and

$$S(\zeta) = \begin{cases} 1, & \text{if } \zeta \in [0,1), \\ 0, & \text{if } \zeta = 1. \end{cases}$$

Let $\alpha = 0$ and $\beta = 1$. Then, we get

$$D_{bl}(T(S(\zeta)), T(S(\xi))) = 0 \le \alpha D_{bl}(T(\zeta), T(\xi)) + \beta D_{bl}(T(\zeta), T(S(\xi)))$$

for all $\zeta,\xi \in W$. As a result, S is T-mean nonexpansive. To demonstrate that S does not mean nonexpansive, we assume that $\zeta = 0$ and $\xi = 1$. In this case, we have

$$D_{bl}(S(\zeta), S(\xi)) = 1 > 0 = 0.1 + 1.0 = \alpha D_{bl}(\zeta,\xi) + \beta D_{bl}(\zeta, S(\xi)).$$

Hence, S does not mean nonexpansive.

Now, let's give the FP result of the paper.

Theorem 2. *Let (W, D_{bl}) be a complete bl space and $T : W \to W$ be a continuous, one-to-one, and sequentially convergent mapping. If $S : W \to W$ is a T-mean nonexpansive mapping such that $k(\alpha + 2\beta k) < 1$, then S has a unique FP. Morever, the sequence $\{T(\zeta_n)\}$ defined by (1.4) converges strongly to $T(\zeta^*)$, where ζ^* is the FP of S.*

Proof. Using (1.3), (1.4), and $(D_{bl}3)$, we get

$$\begin{aligned} D_{bl}(T(\zeta_{n+1}), T(\zeta_n)) &= D_{bl}(T(S(\zeta_n)), T(S(\zeta_{n-1}))) \\ &\le \alpha D_{bl}(T(\zeta_n), T(\zeta_{n-1})) + \beta D_{bl}(T(\zeta_n), T(S(\zeta_{n-1}))) \\ &= \alpha D_{bl}(T(\zeta_n), T(\zeta_{n-1})) + \beta D_{bl}(T(\zeta_n), T(\zeta_n)) \\ &\le \alpha D_{bl}(T(\zeta_n), T(\zeta_{n-1})) + \beta k[D_{bl}(T(\zeta_n), T(\zeta_{n-1})) + D_{bl}(T(\zeta_{n-1}), T(\zeta_n))] \\ &= (\alpha + 2\beta k) D_{bl}(T(\zeta_n), T(\zeta_{n-1})). \end{aligned} \tag{3.1}$$

Also, we obtain

$$\begin{aligned} D_{bl}(T(\zeta_n), T(\zeta_{n-1})) &= D_{bl}(T(S(\zeta_{n-1})), T(S(\zeta_{n-2}))) \\ &\le \alpha D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})) + \beta D_{bl}(T(\zeta_{n-1}), T(S(\zeta_{n-2}))) \\ &= \alpha D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})) + \beta D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-1})) \\ &\le \alpha D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})) + \beta k[D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})) + D_{bl}(T(\zeta_{n-2}), T(\zeta_{n-1}))] \\ &= (\alpha + 2\beta k) D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})). \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$D_{bl}(T(\zeta_{n+1}), T(\zeta_n)) \leq (\alpha + 2\beta k)^2 D_{bl}(T(\zeta_{n-1}), T(\zeta_{n-2})),$$

and inductively, we get

$$D_{bl}(T(\zeta_{n+1}), T(\zeta_n)) \leq (\alpha + 2\beta k)^{n-1} D_{bl}(T(\zeta_2), T(\zeta_1)). \tag{3.3}$$

Furthermore, for $n > m$, we have

$$\begin{aligned} D_{bl}(T(\zeta_m), T(\zeta_n)) &\leq k\left[D_{bl}(T(\zeta_m), T(\zeta_{m+1})) + D_{bl}(T(\zeta_{m+1}), T(\zeta_n))\right] \\ &\leq kD_{bl}(T(\zeta_m), T(\zeta_{m+1})) + k^2\left[D_{bl}(T(\zeta_{m+1}), T(\zeta_{m+2})) + D_{bl}(T(\zeta_{m+2}), T(\zeta_n))\right] \\ &\vdots \\ &\leq kD_{bl}(T(\zeta_m), T(\zeta_{m+1})) + k^2 D_{bl}(T(\zeta_{m+1}), T(\zeta_{m+2})) \\ &\quad +... + k^{n-m-1} D_{bl}(T(\zeta_{n-2}), T(\zeta_{n-1})) + k^{n-m} D_{bl}(T(\zeta_{n-1}), T(\zeta_n)). \end{aligned} \tag{3.4}$$

Now, (3.3) and (3.4) imply that

$$\begin{aligned} D_{bl}(T(\zeta_m), T(\zeta_n)) &\leq [k(\alpha + 2\beta k)^{m-1} + k^2(\alpha + 2\beta k)^m + k^3(\alpha + 2\beta k)^{m+1} \\ &\quad +... + k^{n-m}(\alpha + 2\beta k)^{n-2}] D_{bl}(T(\zeta_2), T(\zeta_1)) \\ &= k(\alpha + 2\beta k)^{m-1}[1 + k(\alpha + 2\beta k) + (k(\alpha + 2\beta k))^2 \\ &\quad +... + (k(\alpha + 2\beta k))^{n-m-1}] D_{bl}(T(\zeta_2), T(\zeta_1)) \\ &\leq \frac{k(\alpha + 2\beta k)^{m-1}}{1 - k(\alpha + 2\beta k)} D_{bl}(T(\zeta_2), T(\zeta_1)). \end{aligned}$$

It follows that $\{T(\zeta_n)\}$ is a Cauchy sequence with $\lim_{n,m\to\infty} D_{bl}(T(\zeta_n), T(\zeta_m)) = 0$ and since (W, D_{bl}) is a complete *bl* space, there exists $\xi_0 \in W$ such that

$$\lim_{n\to\infty} D_{bl}(T(\zeta_n), \xi_0) = D_{bl}(\xi_0, \xi_0) = 0. \tag{3.5}$$

Since the mapping T is sequentially convergent and the sequence $\{T(\zeta_n)\}$ is convergent, then the sequence $\{\zeta_n\}$ is convergent. So, there exists $\zeta^* \in W$ such that

$$\lim_{n\to\infty} D_{bl}(\zeta_n, \zeta^*) = D_{bl}(\zeta^*, \zeta^*). \tag{3.6}$$

Because T is continuous, by Lemma 1 and (3.6), we have

$$\lim_{n\to\infty} D_{bl}(T(\zeta_n), T(\zeta^*)) = D_{bl}(T(\zeta^*), T(\zeta^*)). \tag{3.7}$$

From (3.5) and (3.7), we obtain $\xi_0 = T(\zeta^*)$. So

$$\begin{aligned} D_{bl}(T(S(\zeta^*)), T(\zeta^*)) &\leq k\left[D_{bl}(T(S(\zeta^*)), T(S(\zeta_n))) + D_{bl}(T(S(\zeta_n)), T(\zeta^*))\right] \\ &\leq k[\alpha D_{bl}(T(\zeta^*), T(\zeta_n)) + \beta D_{bl}(T(\zeta^*), T(S(\zeta_n)))] + kD_{bl}(T(S(\zeta_n), T(\zeta^*)) \\ &= k\alpha D_{bl}(T(\zeta^*), T(\zeta_n)) + k(\beta + 1) D_{bl}(T(\zeta^*), T(\zeta_{n+1})) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $D_{bl}(T(S\zeta^*), T(\zeta^*)) = 0$, which implies that $T(S(\zeta^*)) = T(\zeta^*)$. Since T is one-to-one, we have that

$$S(\zeta^*) = \zeta^*.$$

Hence, ζ^* is an FP of S. To show that ζ^* is a unique FP of S, we suppose on the contrary that there exists another FP, say ξ^*, such that $\zeta^* \neq \xi^*$. That is, $S(\zeta^*) = \zeta^*$ and $S(\xi^*) = \xi^*$. Thus, we get

$$\begin{aligned} D_{bl}(T(\zeta^*), T(\xi^*)) &= D_{bl}(T(S(\zeta^*)), T(S(\xi^*))) \\ &\leq \alpha D_{bl}(T(\zeta^*), T(\xi^*)) + \beta D_{bl}(T(\zeta^*), T(S(\xi^*))) \\ &= (\alpha + \beta) D_{bl}(T(\zeta^*), T(\xi^*)), \end{aligned}$$

which implies that

$$(1 - (\alpha + \beta)) D_{bl}(T(\zeta^*), T(\xi^*)) \leq 0.$$

Since $k(\alpha + 2\beta k) < 1$ and $k \geq 1$, we have $\alpha + \beta < 1$. That is, we obtain

$$D_{bl}(T(\zeta^*), T(\xi^*)) = 0,$$

which implies that $T(\zeta^*) = T(\xi^*)$. Since T is one-to-one, we get $\zeta^* = \xi^*$. □

Remark 7. *Theorem 2 can be regarded as an extension of Theorem 2.3 in [14] from a metric space to a bl space.*

If we choose $T = I$ in Theorem 2, we obtain the below result, which is new in the literature.

Corollary 8. *Let (W, D_{bl}) be a complete bl space and $S : W \to W$ be a mean nonexpansive mapping with $k(\alpha + 2\beta k) < 1$. Then S has a unique FP.*

We obtain the following stability result using Definition 7.

Theorem 3. *Under the hypotheses of Theorem 2, the sequence $\{T\zeta_n\}$ defined by (1.4) is (T, S)-stable.*

Proof. Suppose that $\{T(\xi_n)\}$ is an arbitrary sequence in W and $\rho_n = D_{bl}(T(\xi_{n+1}), T(S(\xi_n)))$. Let $\lim_{n\to\infty} \rho_n = 0$. Then, by (1.3) and $(D_{bl}3)$, we obtain

$$\begin{aligned} D_{bl}(T(\xi_{n+1}), T(\zeta^*)) &\leq k[D_{bl}(T(\xi_{n+1}), T(S(\xi_n))) + D_{bl}(T(S(\xi_n)), T(\zeta^*))] \\ &= k[D_{bl}(T(\xi_{n+1}), T(S(\xi_n))) + D_{bl}(T(S(\xi_n)), T(S(\zeta^*)))] \\ &\leq k\rho_n + k(\alpha + \beta) D_{bl}(T(\xi_n), T(\zeta^*)). \end{aligned}$$

By the assumption $\lim_{n\to\infty} \rho_n = 0$, it follows from Lemma 2 that $\lim_{n\to\infty} T(\xi_n) = T(\zeta^*)$.
Conversely, $\lim_{n\to\infty} D_{bl}(T(\xi_{n+1}), T(\zeta^*)) = 0$. Then, using Lemma 2, we have

$$\begin{aligned} \rho_n &= D_{bl}(T(\xi_{n+1}), T(S(\xi_n))) \\ &\leq k[D_{bl}(T(\xi_{n+1}), T(\zeta^*)) + D_{bl}(T(\zeta^*), T(S(\xi_n)))] \\ &= k[D_{bl}(T(\xi_{n+1}), T(\zeta^*)) + D_{bl}(T(S(\xi_n)), T(S(\zeta^*)))] \\ &\leq kD_{bl}(T(\xi_{n+1}), T(\zeta^*)) + k(\alpha + \beta) D_{bl}(T(\xi_n), T(\zeta^*)) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $\{T(\zeta_n)\}$ defined by (1.4) is (T, S)-stable. □

4. An application to the Fredholm-Hammerstein integral equations on time scales

The study of the existence and uniqueness of differential and integral equation solutions is essential in exploring various types of nonlinear analysis and engineering mathematics. One of the most important tools developed in this field is the FP method. Using the FP result in the previous part, we investigated the existence and uniqueness of solutions for the Fredholm-Hammerstein integral equations on time scales in this section.

First, let's give the definition of the Fredholm-Hammerstein integral equation.

Definition 13. *[44] A Fredholm-Hammerstein integral equation of the second kind is defined as*

$$\zeta(t) = f(t) + \lambda \int_d^e K(t,s)u\left(s, \zeta(s)\right) ds, \qquad d \le t, s \le e, \quad d, e \in \mathbb{R}, \tag{4.1}$$

where the kernel function $K(t,s)$ and the function $f(t)$ are given, the unknown function $\zeta(t)$ must be determined, $\lambda \in \mathbb{R}$ is a non-zero constant, and the known function u is continuous and nonlinear respect to the variable ζ. It is called homogeneous if $f(t) = 0$ for all $t \in [d,e]$ in the Eq (4.1).

Now, we provide the homogeneous Fredholm-Hammerstein integral equation of the second kind on the time scale as follows.

Definition 14. *Let $\mathbb{T}_{\mathbb{S}}$ be a time scale with the delta derivative operator Δ, and let $d, e \in \mathbb{T}_{\mathbb{S}}$. A homogeneous Fredholm-Hammerstein integral equation of the second kind on the time scale $\mathbb{T}_{\mathbb{S}}$ is defined as*

$$\zeta(t) = \lambda \int_d^e K(t,s)u\left(s, \zeta(s)\right) \Delta s, \qquad t, s \in [d,e]_{\mathbb{T}_{\mathbb{S}}} = [d,e] \cap \mathbb{T}_{\mathbb{S}}, \tag{4.2}$$

where $u : [d,e]_{\mathbb{T}_{\mathbb{S}}} \times \mathbb{R} \to \mathbb{R}$ and $K : [d,e]_{\mathbb{T}_{\mathbb{S}}} \times [d,e]_{\mathbb{T}_{\mathbb{S}}} \to \mathbb{R}$ are given functions, $\zeta : [d,e]_{\mathbb{T}_{\mathbb{S}}} \to \mathbb{R}$ is an unknown function, and $\lambda \in \mathbb{R} - \{0\}$ is a parameter.

Let $W = C\left([d,e]_{\mathbb{T}_{\mathbb{S}}}\right)$ be the set of all real continuous functions defined on $[d,e]_{\mathbb{T}_{\mathbb{S}}}$. We endowed W with the D_{bl}

$$D_{bl}(\zeta, \xi) = \sup_{t \in [d,e]_{\mathbb{T}_{\mathbb{S}}}} \left(|\zeta(t)| + |\xi(t)|\right)^2, \qquad \forall \zeta, \xi \in W.$$

It was shown in [20, p. 21] that (W, D_{bl}) is a complete *bl* space with the parameter $k = 2$.

In the following theorem, we proved the existence and uniqueness of the integral Eq (4.2) on the complete *bl* space $C\left([d,e]_{\mathbb{T}_{\mathbb{S}}}\right)$.

Theorem 4. *Consider the Fredholm-Hammerstein integral Eq (4.2) such that the functions K and u are Δ-integrable on $[d,e]_{\mathbb{T}_{\mathbb{S}}}$. Assume that*

$$\left(|u(t, \zeta(t))| + |u(t, \xi(t))|\right)^2 \le \left(|\zeta(t)| + |\xi(t)|\right)^2 \tag{4.3}$$

for all $\zeta, \xi \in C\left([d,e]_{\mathbb{T}_{\mathbb{S}}}\right)$ and

$$\sup_{t \in [d,e]_{\mathbb{T}_{\mathbb{S}}}} \left(\int_d^e K^2(t,s)\, \Delta s \right) \le L, \tag{4.4}$$

where $L \in \left(0, \frac{1}{2\lambda^2(e-d)}\right)$ is a constant. Then the Eq (4.2) has a unique solution in $C\left([d,e]_{\mathbb{T}_{\mathbb{S}}}\right)$.

Proof. We define the mappings T and S as follows:

$$T(\zeta(t)) = \frac{2}{3}\zeta(t)$$

and

$$S(\zeta(t)) = \lambda \int_d^e K(t,s)u(s,\zeta(s))\,\Delta s, \quad t \in [d,e]_{\mathbb{T}_{\mathbb{S}}}.$$

Since the functions K and u are Δ-integrable, then, clearly S is a self-mapping on $C([d,e]_{\mathbb{T}_{\mathbb{S}}})$, that is, $S : C([d,e]_{\mathbb{T}_{\mathbb{S}}}) \to C([d,e]_{\mathbb{T}_{\mathbb{S}}})$. By the Cauchy-Schwarz inequality in Lemma 3, Proposition 2, and (4.3), for all $\zeta, \xi \in C([d,e]_{\mathbb{T}_{\mathbb{S}}})$, we obtain

$$\begin{aligned}
(|T(S(\zeta(t)))| + |T(S(\xi(t)))|)^2 &= \left(\left|\frac{2\lambda}{3}\int_d^e K(t,s)u(s,\zeta(s))\,\Delta s\right| + \left|\frac{2\lambda}{3}\int_d^e K(t,s)u(s,\xi(s))\,\Delta s\right|\right)^2 \\
&\le \frac{4\lambda^2}{9}\left(\int_d^e |K(t,s)|\,.\,|u(s,\zeta(s))|\,\Delta s + \int_d^e |K(t,s)|\,.\,|u(s,\xi(s))|\,\Delta s\right)^2 \\
&= \frac{4\lambda^2}{9}\left(\int_d^e |K(t,s)|\,.(|u(s,\zeta(s))| + |u(s,\xi(s))|)\,\Delta s\right)^2 \\
&\le \frac{4\lambda^2}{9}\left(\int_d^e K^2(t,s)\,\Delta s\right)\left(\int_d^e (|u(s,\zeta(s))| + |u(s,\xi(s))|)^2\,\Delta s\right) \\
&\le \frac{4\lambda^2}{9}\left(\int_d^e K^2(t,s)\,\Delta s\right)\left(\int_d^e (|\zeta(s)| + |\xi(s)|)^2\,\Delta s\right) \\
&\le \frac{4\lambda^2}{9}\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}\left(\int_d^e K^2(t,s)\,\Delta s\right)(e-d)\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}(|\zeta(t)| + |\xi(t)|)^2,
\end{aligned}$$

which implies that

$$\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}(|T(S(\zeta(t)))| + |T(S(\xi(t)))|)^2 \le \frac{4\lambda^2}{9}\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}\left(\int_d^e K^2(t,s)\Delta s\right)(e-d)\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}(|\zeta(t)| + |\xi(t)|)^2.$$

Taking the advantage of (4.4), we have

$$\sup_{t\in[d,e]_{\mathbb{T}_{\mathbb{S}}}}(|T(S(\zeta(t)))| + |T(S(\xi(t)))|)^2 \le \frac{4\lambda^2}{9}L(e-d)D_{bl}(\zeta,\xi).$$

Thus, if we say $\lambda^2 L(e-d) = \alpha$, we obtain

$$D_{bl}(T(S(\zeta)), T(S(\xi))) \le \alpha D_{bl}(T(\zeta), T(\xi)) \le \alpha D_{bl}(T(\zeta), T(\xi)) + \beta D_{bl}(T(\zeta), T(S(\xi))).$$

Since $L < \frac{1}{2\lambda^2(e-d)}$, then we have $\alpha < \frac{1}{2}$. Then S is a T-mean nonexpansive mapping on $C([d,e]_{\mathbb{T}_{\mathbb{S}}})$ with $\alpha < \frac{1}{2}$ and $\beta = 0$ implying that $2(\alpha + 4\beta) < 1$. Then it is clear that T is a continuous, one-to-one, and sequentially convergent mapping. Therefore, all conditions of Theorem 2 are satisfied, and so S has a unique FP that is a unique solution of the integral Eq (4.2) for $|\lambda| < \frac{1}{\sqrt{2L(e-d)}}$.

The following examples illustrate the result of Theorem 4.

Example 4. *Let* $\mathbb{T}_{\mathbb{S}} = 2^{\mathbb{N}_0} = \{1, 2, 4, ...\}$. *For all* $t \in 2^{\mathbb{N}_0}$, $\omega(t) = 2t$. *Consider the following integral equation*

$$\zeta(t) = \lambda \int_1^4 \frac{\zeta(s)}{s^2}\,\Delta s, \quad \forall t \in [1,4]_{\mathbb{T}_{\mathbb{S}}}.$$

Here, $K(t,s) = s^{-2}$ *and* $u(t,\zeta(t)) = \zeta(t)$ *for all* $\zeta \in C([1,4]_{\mathbb{T}_{\mathbb{S}}})$. *Thus, by Proposition 1, we obtain*

$$\begin{aligned} \int_1^4 K^2(t,s)\,\Delta s &= \int_1^4 \left(s^{-2}\right)^2 \Delta s \\ &= \int_1^4 s^{-4}\,\Delta s \\ &= \int_1^4 \left(-\frac{8}{7}s^{-3}\right)^{\Delta} \Delta s \\ &= -\frac{8}{7}(s^{-3})|_{s=1}^{s=4} \\ &= \frac{9}{8}. \end{aligned}$$

Therefore, the condition (4.4) holds with $L = \frac{9}{8}$. *Clearly, we get*

$$(|u(t,\zeta(t))| + |u(t,\xi(t))|)^2 = (|\zeta(t)| + |\xi(t)|)^2.$$

Thus, the condition (4.3) is provided. By Theorem 4, the given integral equation has a unique solution for $L < \frac{1}{6\lambda^2}$, *that is,* $|\lambda| < \frac{2}{3\sqrt{3}}$. *Consequently, we conclude that this integral equation has a unique solution, which is the trivial solution* $\zeta(t) \equiv 0$.

In fact, it can be shown that the equation has only the solution $\zeta(t) \equiv 0$ *for* $|\lambda| < \frac{2}{3\sqrt{3}}$ *by using direct computation. Let*

$$c = \int_1^4 \frac{\zeta(s)}{s^2}\,\Delta s.$$

Then we have $\zeta(t) = c\lambda$, *and so,*

$$\begin{aligned} c &= \int_1^4 \frac{c\lambda}{s^2}\,\Delta s \\ &= c\lambda \int_1^4 \frac{1}{s^2}\,\Delta s \\ &= \frac{3}{2}c\lambda. \end{aligned}$$

This equation can be written as

$$c\left(1 - \frac{3}{2}\lambda\right) = 0,$$

that is, for $\lambda \neq \frac{2}{3}$, *there is only one solution, which is* $c = 0$. *Hence, we obtain that this integral equation has a unique solution when* $\lambda \neq \frac{2}{3}$.

Example 5. *Let $\mathbb{T}_{\mathbb{S}} = \mathbb{Z}$. Consider the following integral equation*

$$\zeta(t) = \lambda \int_0^4 (s + \omega(s)) \frac{\zeta(s)}{1 + (\zeta(s))^2} \Delta s, \quad \forall t \in [0, 4]_{\mathbb{T}_{\mathbb{S}}}. \tag{4.5}$$

Here, $K(t, s) = s + \omega(s)$, $u(t, \zeta(t)) = \frac{\zeta(t)}{1+(\zeta(t))^2}$ for all $\zeta \in C([0, 4]_{\mathbb{T}_{\mathbb{S}}})$ and $\omega(t) = t + 1$ for all $t \in \mathbb{Z}$. By Remark 6 (ii), we obtain

$$\begin{aligned} \int_0^4 K^2(t, s) \Delta s &= \int_0^4 (2s + 1)^2 \Delta s \\ &= \int_0^4 \left(4s^2 + 4s + 1\right) \Delta s \\ &= \sum_{s=0}^{3} \left(4s^2 + 4s + 1\right) \\ &= 84. \end{aligned}$$

Therefore, the condition (4.4) holds with $L = 84$. Also, we get

$$(|u(t, \zeta(t))| + |u(t, \xi(t))|)^2 = \left(\left|\frac{\zeta(t)}{1 + (\zeta(t))^2}\right| + \left|\frac{\xi(t)}{1 + (\xi(t))^2}\right|\right)^2 \leq (|\zeta(t)| + |\xi(t)|)^2,$$

that is, the condition (4.3) is satisfied. By Theorem 4, the integral Eq (4.5) has a unique solution for $L < \frac{1}{8\lambda^2}$. Therefore, we have $|\lambda| < \frac{1}{4\sqrt{42}}$. We deduce that the given integral equation has a unique solution, which is the trivial solution $\zeta(t) \equiv 0$.

In fact, by using direct computation, it can be shown that the equation has a unique solution $\zeta(t) \equiv 0$. Let

$$c = \int_0^4 (s + \omega(s)) \frac{\zeta(s)}{1 + (\zeta(s))^2} \Delta s. \tag{4.6}$$

Thus, we can write

$$\zeta(t) = c\lambda. \tag{4.7}$$

By substituting the value of $\zeta(t)$ given by (4.7) into (4.6), we obtain

$$\begin{aligned} c &= \int_0^4 (s + s + 1) \frac{c\lambda}{1 + (c\lambda)^2} \Delta s \\ &= \frac{c\lambda}{1 + (c\lambda)^2} \int_0^4 (2s + 1) \Delta s \\ &= 16 \frac{c\lambda}{1 + (c\lambda)^2}, \end{aligned}$$

and from here, we get

$$c^3 \lambda^2 + c(1 - 16\lambda) = 0.$$

For $\lambda < \frac{1}{16}$, there is only one solution, which is $c = 0$. Hence, the integral Eq (4.5) has a unique solution when $\lambda < \frac{1}{16}$.

In the above examples, Theorem 4 provides a small interval for λ. However, on this interval, we can guarantee the existence and uniqueness of the solution without computing it.

5. Conclusions

We have demonstrated the existence and uniqueness of the FPs of T-mean nonexpansive mappings and provided the stability result for the T-Picard iteration in *bl* spaces. Additionally, we presented an application of the Fredholm-Hammerstein integral equations and two numerical examples on time scales. Using similar approaches in our results, the integral type of T-mean nonexpansive mappings described in [14] can be studied in *bl* spaces. Also, the types of T-mean nonexpansive mappings can be studied in different spaces, such as hyperbolic metric spaces defined by Kohlenbach [45].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors, A. Aloqaily and N. Mlaiki, would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraites et leurs applications, *Fund. Math.*, **3** (1922), 133–187. https://doi.org/10.4064/fm-3-1-133-181
2. R. P. Agarwal, Ü. Aksoy, E. Karapınar, İ. M. Erhan, F-contraction mappings on metric-like spaces in connection with integral eqautions on time scales, *RACSAM*, **114** (2020). https://doi.org/10.1007/s13398-020-00877-5
3. D. R. Kumar, Common fixed point results under w-distance with applications to nonlinear integral equations and nonlinear fractional differential equations, *Math. Slovaca*, **71** (2021), 1511–1528. https://doi.org/10.1515/ms-2021-0068
4. M. Younis, I. Altun, V. Chauhan, Graphical structure of extended b-metric spaces: An application to the transverse oscillations of a homogeneous bar, *Int. J. Nonlin. Sci. Num.*, **23** (2022), 1239–1252. https://doi.org/10.1515/ijnsns-2020-0126
5. D. R. Kumar, Common solution to a pair of nonlinear Fredholm and Volterra integral equations and nonlinear fractional differential equations, *J. Comput. Appl. Math.*, **404** (2022). https://doi.org/10.1016/j.cam.2021.113907
6. W. Shatanawi, T. A. M. Shatnawi, New fixed point results in controlled metric type spaces based on new contractive conditions, *AIMS Math.*, **8** (2023), 9314–9330. https://doi.org/10.3934/math.2023468

7. A. Z. Rezazgui, A. A. Tallafha, W. Shatanawi, Common fixed point results via $A\nu$-α-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space, *AIMS Math.*, **8** (2023), 7225–7241. https://doi.org/10.3934/math.2023363

8. M. Joshi, A. Tomar, T. Abdeljawad, On fixed points, their geometry and application to satellite web coupling problem in S-metric spaces, *AIMS Math.*, **8** (2023), 4407–4441. https://doi.org/10.3934/math.2023220

9. M. Younis, D. Singh, L. Chen, M. Metwali, A study on the solutions of notable engineering models, *Math. Model. Anal.*, **27** (2022), 492–509. https://doi.org/10.3846/mma.2022.15276

10. M. Younis, H. Ahmad, L. Chen, M. Han, Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations, *J. Geom. Phys.*, **192** (2023). https://doi.org/10.1016/j.geomphys.2023.104955

11. F. E. Browder, Nonexpansive nonlinear operators in a Banach spaces, *P. Natl. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041

12. D. Göhde, Zum Prinzip der kontraktiven Abbildung, *Math. Nachr.*, **30** (1965), 251–258. https://doi.org/10.1002/mana.19650300312

13. S. Zhang, About fixed point theory for mean nonexpansive mapping in Banach spaces, *J. Sichuan Univ.*, **2** (1975), 67–68.

14. A. A. Mebawondu, C. Izuchukwu, H. A. Abass, O. T. Mewomo, Some results on generalized mean nonexpansive mapping in complete metric spaces, *Bol. Soc. Parana. Mat.*, **40** (2022), 1–16. https://doi.org/10.5269/bspm.44174

15. J. Morales, E. Rojas, Some results on T Zamfirescu operators, *Rev. Notas Mat.*, **5** (2009), 64–71.

16. I. A. Bakhtin, The contraction principle in quasimetric spaces, *Funct. Anal.*, **30** (1989), 26–37.

17. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1** (1993), 5–11.

18. S. G. Matthews, Partial metric topology, *Ann. NY Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

19. A. A. Harandi, Metric-like spaces, partial metric spaces and fixed points, *Fixed Point Theory A.*, **2012** (2021). https://doi.org/10.1186/1687-1812-2012-204

20. M. A. Alghamdi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, *J. Inequal. Appl.*, **2013** (2013). https://doi.org/10.1186/1029-242X-2013-402

21. N. Hussain, J. R. Roshan, V. Parvaneh, Z. Kadelburg, Fixed points of contractive mappings in b-metric-like spaces, *The Scientific World J.*, **2014** (2014). https://doi.org/10.1155/2014/981578

22. H. H. Alsulami, S. Gülyaz, E. Karapınar, İ. M. Erhan, An Ulam stability result on quasi-b-metric-like spaces, *Open Math.*, **14**, (2016), 1087–1103. https://doi.org/10.1515/math-2016-0097

23. K. Jain, J. Kaur, Some fixed point results in b-metric spaces and b-metric-like spaces with new contractive mappings, *Axioms*, **10** (2021). https://doi.org/10.3390/axioms10020055

24. S. K. Prakasam, A. J. Gnanaprakasam, Ö. Ege, G. Mani, S. Haque, N. Mlaiki, Fixed point for an $\mathbb{O}g\mathfrak{F}$-c in $\mathbb{O}$-complete b-metric-like spaces, *AIMS Math.*, **8** (2023), 1022–1039. https://doi.org/10.3934/math.2023050

25. Z. Kalkan, A. Şahin, Some new results in partial cone b-metric space, *Commun. Adv. Math. Sci.*, **3** (2020), 67–73. https://doi.org/10.33434/cams.684102

26. D. R. Kumar, M. Prabavathy, S. Radenovic, On existence and approximation of common fixed points in b-metric spaces, *Asian-Eur. J. Math.*, **15** (2022). https://doi.org/10.1142/S1793557122500309

27. I. D. Arandelovic, Z. D. Mitrovic, A. Aloqaily, N. Mlaiki, The results of common fixed points in b-metric spaces, *Symmetry*, **15** (2023), 1344. https://doi.org/10.3390/sym15071344

28. A. Beiranvand, S. Moradi, M. Omid, H. Pazandeh, Two fixed point theorems for special mappings, *arXiv Preprint*, 2009.

29. M. Öztürk, M. Başarır, On some common fixed point theorems for f-contraction mappings in cone metric spaces, *Int. J. Math. Anal.*, **5** (2011), 119–127. https://doi.org/10.1186/1687-1812-2011-93

30. C. T. Aage, P. G. Golhare, On fixed point theorems in dislocated quasi b-metric spaces, *Int. J. Adv. Math.*, **2016** (2016), 55–70. https://doi.org/10.1186/s13663-016-0565-9

31. K. Zoto, P. S. Kumari, Fixed point theorems for s-α contractions in dislocated and b-dislocated metric spaces, *Thai J. Math.*, **17** (2019), 263–276.

32. A. M. Zaki, A. O. Ismail, A note on cone metric spaces, *Curr. Sci. Int.*, **11** (2022), 319–328.

33. H. Aydi, A. Felhi, S. Sahmim, On common fixed points for $\alpha - \psi$ contractions and generalized cyclic contractions in b-metric-like spaces and consequences, *J. Nonlinear Sci. Appl.*, **9** (2016), 2492–2510. https://doi.org/10.22436/jnsa.009.05.48

34. K. Calderón, A. Padcharoen, J. M. Moreno, Some stability and strong convergence results for the algorithm with perturbations for a T-Ciric quasicontraction in CAT(0) spaces, *J. Inequal. Appl.*, **2023** (2023). https://doi.org/10.1186/s13660-022-02911-z

35. A. M. Harder, T. L. Hicks, Some stability results for fixed point iteration procedures, *Math. Japon.*, **33** (1988), 693–706.

36. V. Berinde, *Iterative approximation of fixed points*, Berlin: Springer-Verlag, 2007. https://doi.org/10.1109/SYNASC.2007.49

37. S. Hilger, Analysis on measure chains a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153

38. M. Bohner, A. Peterson, *Dynamic equations on time scales*, Boston/Berlin: Birkhauser, 2001. https://doi.org/10.1007/978-1-4612-0201-1

39. R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: A survey, *Math. Inequal. Appl.*, **4** (2001), 535–557. https://doi.org/10.7153/mia-04-48

40. G. S. Guseinov, Integration on time scales, *J. Math. Anal. Appl.*, **285** (2003), 107–127. https://doi.org/10.1016/S0022-247X(03)00361-5

41. S. Georgiev, *Integral equations on time scales*, Paris: Atlantis Press, 2016. https://doi.org/10.2991/978-94-6239-228-1

42. Z. Kalkan, A. Şahin, Some new stability results of Volterra integral equations on time scales, *Maltepe J. Math.*, **4** (2022), 44–54. https://doi.org/10.47087/mjm.1145159

43. X. Hu, Y. Li, Multiplicity result to a system of over-determined Fredholm fractional integro-differential equations on time scales, *AIMS Math.*, **7** (2022), 2646–2665. https://doi.org/10.3934/math.2022149
44. A. Pouria, The numerical solution of Fredholm-Hammerstein integral equations by combining the collocation method and radial basis functions, *Filomat*, **33** (2019), 667–682. https://doi.org/10.2298/FIL1903667A
45. U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *T. Am. Math. Soc.*, **357** (2004), 89–128. https://doi.org/10.1090/S0002-9947-04-03515-9