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Research article

Some new generalizations of reversed Minkowski's inequality for several functions via time scales

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Abstract: In this paper, we introduce novel extensions of the reversed Minkowski inequality for various functions defined on time scales. Our approach involves the application of Jensen's and Hölder's inequalities on time scales. Our results encompass the continuous inequalities established by Benaissa as special cases when the time scale $\mathbb{T}$ corresponds to the real numbers (when $\mathbb{T} = \mathbb{R}$). Additionally, we derive distinct inequalities within the realm of time scale calculus, such as cases $\mathbb{T} = \mathbb{N}$ and $q^{\mathbb{N}}$ for $q > 1$. These findings represent new and significant contributions for the reader.

Keywords: Hölder's inequality; inequalities on time scales; Jensen's inequality
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1. Introduction

In 1889, Hölder [1] proved that

$$\sum_{k=1}^{\xi} x_k y_k \leq \left(\sum_{k=1}^{\xi} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\xi} y_k^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where $\{x_k\}_{k=1}^{\xi}$ and $\{y_k\}_{k=1}^{\xi}$ are positive sequences and $p > 1$ and $1/p + 1/q = 1$. The inequality (1.1) is reversed if $p < 0$ or $q < 0$. The integral form of (1.1) is

$$\int_{\epsilon}^{\iota} \lambda(\xi)\omega(\xi)d\xi \leq \left[\int_{\epsilon}^{\iota} \lambda^{\gamma}(\xi)d\xi\right]^{\frac{1}{\gamma}} \left[\int_{\epsilon}^{\iota} \omega^{\nu}(\xi)d\xi\right]^{\frac{1}{\nu}}, \tag{1.2}$$

where $\epsilon, \iota \in \mathbb{R}$, $\gamma > 1$, $1/\gamma + 1/\nu = 1$, and $\lambda, \omega \in C([a,b], \mathbb{R})$. If $0 < \gamma < 1$, then (1.2) is reversed. For more informations about the applications of Hölder's inequality, see [2–8].

In particular, Minkowski's inequality is considered as an application of Hölder's inequality, which states that, for $\delta \geq 1$, if $\mathcal{G}, \mathcal{W}$ are nonnegative continuous functions on $[\check{c}, \breve{a}]$ such that

$$0 < \int_{\check{c}}^{\breve{a}} \mathcal{G}^{\delta}(\tau)d\tau < \infty \text{ and } 0 < \int_{\check{c}}^{\breve{a}} \mathcal{W}^{\delta}(\tau)d\tau < \infty,$$

then

$$\left(\int_{\check{c}}^{\breve{a}} (\mathcal{G}(\tau) + \mathcal{W}(\tau))^{\delta} d\tau\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \mathcal{G}^{\delta}(\tau)d\tau\right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{W}^{\delta}(\tau)d\tau\right)^{\frac{1}{\delta}}. \tag{1.3}$$

Sulaiman [9] introduced the following outcome pertaining to the reversed Minkowski inequality: If $\mathcal{G}, \mathcal{W} > 0$, $\delta \geq 1$, and

$$1 < B \leq \frac{\mathcal{G}(\zeta)}{\mathcal{W}(\zeta)} \leq \ell,$$

for all $\zeta \in [\check{c}, \breve{a}]$, then

$$\begin{aligned} & \frac{\ell+1}{\ell-1}\left(\int_{\check{c}}^{\breve{a}} (\mathcal{G}(\zeta) - \mathcal{W}(\zeta))^{\delta} d\zeta\right)^{\frac{1}{\delta}} \\ \leq\ & \left(\int_{\check{c}}^{\breve{a}} \mathcal{G}^{\delta}(\zeta)d\zeta\right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{W}^{\delta}(\zeta)d\zeta\right)^{\frac{1}{\delta}} \\ \leq\ & \frac{B+1}{B-1}\left(\int_{\check{c}}^{\breve{a}} (\mathcal{G}(\zeta) - \mathcal{W}(\zeta))^{\delta} d\zeta\right)^{\frac{1}{\delta}}. \end{aligned} \tag{1.4}$$

Sroysang [10] showed that, if $\delta \geq 1$ and $\mathcal{G}, \mathcal{W} > 0$ with

$$0 < E < B \leq \frac{\mathcal{G}(\zeta)}{\mathcal{W}(\zeta)} \leq \ell,$$

for all $\zeta \in [\check{c}, \breve{a}]$, then

$$\begin{aligned} & \frac{\ell+1}{\ell-E}\left(\int_{\check{c}}^{\breve{a}} (\mathcal{G}(\zeta) - E\mathcal{W}(\zeta))^{\delta} d\zeta\right)^{\frac{1}{\delta}} \\ \leq\ & \left(\int_{\check{c}}^{\breve{a}} \mathcal{G}^{\delta}(\zeta)d\zeta\right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{W}^{\delta}(\zeta)d\zeta\right)^{\frac{1}{\delta}} \\ \leq\ & \frac{B+1}{B-E}\left(\int_{\check{c}}^{\breve{a}} (\mathcal{G}(\zeta) - E\mathcal{W}(\zeta))^{\delta} d\zeta\right)^{\frac{1}{\delta}}. \end{aligned}$$

Benaissa [11] introduced a novel finding concerning the inverse Minkowski inequality, proposing the following: If $\mathcal{G}, \mathcal{W} > 0$, $\alpha > 0$, and $\delta \geq 1$ such that

$$0 < E < B \leq \frac{\alpha \mathcal{G}(\zeta)}{\mathcal{W}(\zeta)} \leq \ell,$$

for all $\zeta \in [\check{c}, \breve{a}]$, then

$$\begin{aligned} &\frac{\ell + \alpha}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{\delta} d\zeta \right)^{\frac{1}{\delta}} \\ \leq & \left(\int_{\check{c}}^{\breve{a}} \mathcal{G}^{\delta}(\zeta) d\zeta \right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{W}^{\delta}(\zeta) d\zeta \right)^{\frac{1}{\delta}} \\ \leq & \frac{B + \alpha}{\alpha(B - E)} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{\delta} d\zeta \right)^{\frac{1}{\delta}}. \end{aligned} \tag{1.5}$$

In [12], Benaissa showed that, if $\alpha > 0$, $0 < p \leq \eta$, $\mathcal{G}, \mathcal{W} > 0$, $\mathcal{U}$ is a weight function, and

$$0 < E < B \leq \frac{\alpha \mathcal{G}(\zeta)}{\mathcal{W}(\zeta)} \leq \ell \quad \text{for all } \zeta \in [\check{c}, \breve{a}], \tag{1.6}$$

then

$$\begin{aligned} &\frac{\ell + \alpha}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} \mathcal{U}(\zeta) d\zeta \right)^{\frac{p-\eta}{p\eta}} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{p} \mathcal{U}(\zeta) d\zeta \right)^{\frac{1}{p}} \\ \leq & \left(\int_{\check{c}}^{\breve{a}} \mathcal{U}(\zeta) \mathcal{G}^{\eta}(\zeta) d\zeta \right)^{\frac{1}{\eta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{U}(\zeta) \mathcal{W}^{\eta}(\zeta) d\zeta \right)^{\frac{1}{\eta}} \\ \leq & \frac{B + \alpha}{\alpha(B - E)} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{\eta} \mathcal{U}(\zeta) d\zeta \right)^{\frac{1}{\eta}}. \end{aligned} \tag{1.7}$$

Also, the author of [12] proved that, if $1 < p \leq \eta$, $\alpha > 0$, $\mathcal{G}, \mathcal{W} > 0$, and (1.6) holds, then

$$\begin{aligned} &\frac{\ell + \alpha}{\alpha(\ell - E)} (\breve{a} - \check{c})^{\frac{1-p}{\eta}} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{\frac{\eta}{p}} \mathcal{U}(\zeta) d\zeta \right)^{\frac{p}{\eta}} \\ \leq & \left(\int_{\check{c}}^{\breve{a}} \mathcal{U}(\zeta) \mathcal{G}^{\eta}(\zeta) d\zeta \right)^{\frac{1}{\eta}} + \left(\int_{\check{c}}^{\breve{a}} \mathcal{U}(\zeta) \mathcal{W}^{\eta}(\zeta) d\zeta \right)^{\frac{1}{\eta}} \\ \leq & \frac{B + \alpha}{\alpha(B - E)} \left(\int_{\check{c}}^{\breve{a}} (\alpha \mathcal{G}(\zeta) - E \mathcal{W}(\zeta))^{\eta} \mathcal{U}(\zeta) d\zeta \right)^{\frac{1}{\eta}}. \end{aligned} \tag{1.8}$$

Time scale calculus is a unification of continuous calculus and discrete calculus. Many authors have proved inequalities on time scales. For example, in 2001, Bohner and Peterson [13] presented Hölder's inequality on time scales, stating that if $\mathbb{T}$ is a time scale, $\check{c}, \breve{a} \in \mathbb{T}$, $\lambda, \omega \in C_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, \mathbb{R}^{+})$ and $\gamma > 1$ with $1/\gamma + 1/\nu = 1$, then

$$\int_{\check{c}}^{\breve{a}} \lambda(\tau) \omega(\tau) \Delta\tau \leq \left[\int_{\check{c}}^{\breve{a}} \lambda^{\gamma}(\tau) \Delta\tau \right]^{\frac{1}{\gamma}} \left[\int_{\check{c}}^{\breve{a}} \omega^{\nu}(\tau) \Delta\tau \right]^{\frac{1}{\nu}}. \tag{1.9}$$

The inequality (1.9) is reversed for $0 < \gamma < 1$ or $\gamma < 0$. In addition the authors of [13] presented Minkowski's inequality (1.3) on time scales, stating that if $a, b \in \mathbb{T}$, $\psi, \varpi \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ and $\alpha > 1$, then

$$\left(\int_a^b [\psi(\xi) + \varpi(\xi)]^\alpha \Delta\xi\right)^{\frac{1}{\alpha}} \leq \left(\int_a^b (\psi(\xi))^\alpha \Delta\xi\right)^{\frac{1}{\alpha}} + \left(\int_a^b (\varpi(\xi))^\alpha \Delta\xi\right)^{\frac{1}{\alpha}}. \tag{1.10}$$

The inequality (1.10) is reversed with the reversed sign when $\alpha < 0$ or $0 < \alpha < 1$. In [14], Lütfi proved that if $g, h : \mathbb{I} \to \mathbb{R}$ are Δ-integrable functions on $\mathbb{I} = [a,b] \in \mathbb{T}$ with $1 < l \leq g^p, h^p \leq L < \infty$ and $p > 1$, then

$$\left(\int_a^b |g(\xi)|^p \Delta\xi\right)^{\frac{1}{p}} + \left(\int_a^b |h(\xi)|^p \Delta\xi\right)^{\frac{1}{p}} \leq 2\,(L/l)^{1/p} \left(\int_a^b |g(\xi) + h(\xi)|^p \Delta\xi\right)^{\frac{1}{p}}.$$

Here, we aim to extend the inequality (1.7) and rectify (1.8) in time scale calculus which is defined in Section 2. Also, we can get some new inequalities in (continuous, discrete, and quantum) calculus.

2. Preliminaries and basic lemmas

In 2001, Bohner and Peterson introduced the backward jump and forward jump operators, denoted by $\sigma(\zeta) := \inf\{\varrho \in \mathbb{T} : \varrho > \zeta\}$ and $\rho(\zeta) := \sup\{\varrho \in \mathbb{T} : \varrho < \zeta\}$, respectively [13]. For any function $\mathcal{G} : \mathbb{T} \to \mathbb{R}$, the notations $\mathcal{G}^\sigma(\zeta)$ and $\mathcal{G}^\rho(\zeta)$ denote $\mathcal{G}(\sigma(\zeta))$ and $\mathcal{G}(\rho(\zeta))$, respectively. The time scale interval $[\check{c}, \breve{a}]_{\mathbb{T}}$ is denoted by $[\check{c}, \breve{a}] \cap \mathbb{T}$. For more information about dynamic inequalities, see for instance [15–35].

Theorem 2.1. *[13] Let $\mathcal{G}, \mathcal{W} : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable at $\tau \in \mathbb{T}$. Then we have the following at τ:*

(1) The sum $\mathcal{G} + \mathcal{W} : \mathbb{T} \to \mathbb{R}$ is differentiable with

$$(\mathcal{G} + \mathcal{W})^\Delta(\tau) = \mathcal{G}^\Delta(\tau) + \mathcal{W}^\Delta(\tau).$$

(2) $\alpha\mathcal{G} : \mathbb{T} \to \mathbb{R}$ is differentiable for any constant α with

$$(\alpha\mathcal{G})^\Delta(\tau) = \alpha\mathcal{G}^\Delta(\tau).$$

(3) The product $\mathcal{G}\mathcal{W} : \mathbb{T} \to \mathbb{R}$ is differentiable and the product rule is defined by

$$\begin{aligned}(\mathcal{G}\mathcal{W})^\Delta(\tau) &= \mathcal{G}^\Delta(\tau)\mathcal{W}(\tau) + \mathcal{G}(\sigma(\tau))\mathcal{W}^\Delta(\tau) \\ &= \mathcal{G}(\tau)\mathcal{W}^\Delta(\tau) + \mathcal{G}^\Delta(\tau)\mathcal{W}(\sigma(\tau)).\end{aligned}$$

(4) If $\mathcal{G}(\tau)\mathcal{G}(\sigma(\tau)) \neq 0$, then $1/\mathcal{G} : \mathbb{T} \to \mathbb{R}$ is differentiable and

$$\left(\frac{1}{\mathcal{G}}\right)^\Delta(\tau) = -\frac{\mathcal{G}^\Delta(\tau)}{\mathcal{G}(\tau)\mathcal{G}(\sigma(\tau))}.$$

(5) If $\mathcal{W}(\tau)\mathcal{W}(\sigma(\tau)) \neq 0$, then the quotient $\mathcal{G}/\mathcal{W} : \mathbb{T} \to \mathbb{R}$ is differentiable and the quotient rule is defined as

$$\left(\frac{\mathcal{G}}{\mathcal{W}}\right)^{\Delta}(\tau) = \frac{\mathcal{G}^{\Delta}(\tau)\mathcal{W}(\tau) - \mathcal{G}(\tau)\mathcal{W}^{\Delta}(\tau)}{\mathcal{W}(\tau)\mathcal{W}(\sigma(\tau))}.$$

Definition 2.1. [13] A function $F : \mathbb{T} \to \mathbb{R}$ is said to an antiderivative of $\mathcal{G} : \mathbb{T} \to \mathbb{R}$ if

$$F^{\Delta}(\tau) = \mathcal{G}(\tau), \quad \forall \tau \in \mathbb{T}.$$

In this case, the Cauchy integral of $\mathcal{G}$ is defined as

$$\int_r^x \mathcal{G}(\tau)\Delta\tau = F(x) - F(r), \ \ \forall r, \ x \in \mathbb{T}.$$

Definition 2.2. [13] A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous provided that it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Theorem 2.2. *[36] Assume that $\check{c}, \breve{a} \in \mathbb{T}$ and $\mathcal{G} \in C_{rd}(\mathbb{T},\mathbb{R})$. Then the next features satisfy the following conditions:*
(1) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_{\check{c}}^{\breve{a}} \mathcal{G}(\tau)\Delta\tau = \int_{\check{c}}^{\breve{a}} \mathcal{G}(\tau)d\tau.$$

(2) *If $[\check{c}, \breve{a}]$ consists of only isolated points, then*

$$\int_{\check{c}}^{\breve{a}} \mathcal{G}(\tau)\Delta\tau = \sum_{\tau\in[\check{c},\breve{a})} \mu(\tau)\mathcal{G}(\tau).$$

(3) *If $\mathbb{T} = \mathbb{Z}$, then*

$$\int_{\check{c}}^{\breve{a}} \mathcal{G}(\tau)\Delta\tau = \sum_{\tau=\check{c}}^{\breve{a}-1} \mathcal{G}(\tau).$$

(4) *If $\mathbb{T} = \delta\mathbb{Z}$, $\delta > 0$, then*

$$\int_{\check{c}}^{\breve{a}} \mathcal{G}(\tau)\Delta\tau = \sum_{x=0}^{\frac{\breve{a}-\check{c}-\delta}{\delta}} \mathcal{G}(\check{c} + x\delta)\delta.$$

The AM-GM inequality yields that

$$\left(\prod_{i=1}^{\varrho} \psi_i(\xi)\right)^{\frac{1}{\varrho}} \leq \frac{\sum_{i=1}^{\varrho} \psi_i(\xi)}{\varrho}, \tag{2.1}$$

where $\psi_i(\xi)$, $i = 1, 2, ..., \varrho$ are nonnegative functions.

Lemma 2.1. *Assume that $\check{c}, \breve{a} \in \mathbb{T}$, $\breve{a} > \check{c}$, $0 < p \leq \delta$, and $h, \lambda \in C_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, \mathbb{R}^{+})$. Then,*

$$\left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^{p}(\varsigma)\Delta\varsigma\right)^{\frac{1}{p}} \leq \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)\Delta\varsigma\right)^{\frac{\delta-p}{p\delta}} \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^{\delta}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{2.2}$$

Proof. Applying (1.9) with $\gamma = \delta/p > 1$ and $\nu = \delta/(\delta - p)$, then

$$\begin{aligned}\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^p(\varsigma)\Delta\varsigma &= \int_{\check{c}}^{\breve{a}} (\lambda(\varsigma))^{\frac{\delta-p}{\delta}} (\lambda(\varsigma))^{\frac{p}{\delta}} h^p(\varsigma)\Delta\varsigma \\ &\leq \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)\Delta\varsigma\right)^{\frac{\delta-p}{\delta}} \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^{\delta}(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}}.\end{aligned}$$

Thus,

$$\left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^p(\varsigma)\Delta\varsigma\right)^{\frac{1}{p}} \leq \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)\Delta\varsigma\right)^{\frac{\delta-p}{p\delta}} \left(\int_{\check{c}}^{\breve{a}} \lambda(\varsigma)h^{\delta}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}},$$

which is (2.2). □

Lemma 2.2. *(Jensen's inequality [36]) Let $\check{c}, \breve{a} \in \mathbb{T}$ with $\check{c} < \breve{a}$ and $c, d \in \mathbb{R}$. Assuming that $\phi \in C_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, (c, d))$, $\varkappa \in C_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, \mathbb{R}^+)$.*

If $F \in C((c, d), \mathbb{R})$ is convex, then

$$F\left(\frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\phi(\varsigma)\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}\right) \leq \frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)F(\phi(\varsigma))\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}. \tag{2.3}$$

The inequality (2.3) is reversed if F is concave.

If $F(\varsigma) = \varsigma^{\lambda}$, then

$$\left(\frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\phi(\varsigma)\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}\right)^{\lambda} \leq \frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\phi^{\lambda}(\varsigma)\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}, \quad \lambda > 1, \tag{2.4}$$

and

$$\left(\frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\phi(\varsigma)\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}\right)^{\lambda} \geq \frac{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\phi^{\lambda}(\varsigma)\Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \varkappa(\varsigma)\Delta\varsigma}, \quad 0 < \lambda < 1. \tag{2.5}$$

3. Main results

During this study, we assume the existence of the integrals under consideration.

Theorem 3.1. *Assume that $\check{c}, \breve{a} \in \mathbb{T}$, $\breve{a} > \check{c}$, $0 < p \leq \delta < \infty$, $\alpha > 0$, and $\mathcal{G}_x, \mathcal{W}_x, \mathcal{U}_x \in \mathrm{C}_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, \mathbb{R}^+)$, $x = 1, 2, ..., \iota$, with*

$$0 < E < B \leq \frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)} \leq \ell, \ \textit{for} \ \ x = 1, 2, ..., \iota. \tag{3.1}$$

Then,

$$\begin{aligned}&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-\delta}{p\delta}} \\ &\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} [(\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))]^{\frac{p}{\iota}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{p}}\end{aligned}$$

$$\leq \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}$$

$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.2}$$

Proof. From (3.1), we see that

$$0 < \frac{1}{E} - \frac{1}{B} \leq \frac{1}{E} - \frac{\mathcal{W}_x(\varsigma)}{\alpha\mathcal{G}_x(\varsigma)} \leq \frac{1}{E} - \frac{1}{\ell}.$$

Then,

$$\frac{\ell}{\ell - E} \leq \frac{\alpha\mathcal{G}_x(\varsigma)}{\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)} \leq \frac{B}{B-E}. \tag{3.3}$$

Since $0 < p \leq \delta$, we have from (3.3) that

$$\prod_{x=1}^{\iota}\left[\frac{\ell}{\alpha(\ell-E)}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))\right]^{\frac{p}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}$$

$$\leq \prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{p}{\iota}}(\varsigma), \tag{3.4}$$

and

$$\sum_{x=1}^{\iota}\left[\frac{B}{\alpha(B-E)}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))\right]^{\delta}\mathcal{U}_x(\varsigma) \geq \sum_{x=1}^{\iota}\mathcal{G}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma). \tag{3.5}$$

Integrating (3.4) and (3.5) over ς from $\check{c}$ to $\breve{a}$, we see that

$$\frac{\ell}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}[(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))]^{\frac{p}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{p}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{p}}, \tag{3.6}$$

and

$$\left(\int_{\check{c}}^{\breve{a}}\sum_{x=1}^{\iota}\mathcal{G}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}} \leq \frac{B}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}}\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.7}$$

By applying (2.2) with $h(\varsigma) = \prod_{x=1}^{\iota}\mathcal{G}_x^{\frac{1}{\iota}}(\varsigma)$ and $\lambda(\varsigma) = \prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}$, we see that

$$\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{p}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{\delta-p}{p\delta}}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}.$$

Then, from (3.6), we have

$$\frac{\ell}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p-\delta}{p\delta}}$$
$$\times\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left[\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)\right]^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$
$$\leq\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.8}$$

Using (3.1), we deduce that

$$0<B-E\leq\frac{\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)}{\mathcal{W}_x(\varsigma)}\leq\ell-E.$$

Therefore

$$\frac{1}{\ell-E}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)\leq\mathcal{W}_x(\varsigma)\leq\frac{1}{B-E}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right). \tag{3.9}$$

Since $0<p\leq\delta$, we have from (3.9) that

$$\frac{1}{\ell-E}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}} \tag{3.10}$$
$$\leq\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\mathcal{W}_x^{\frac{p}{\iota}}(\varsigma)\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}},$$

and

$$\left(\int_{\check{c}}^{\check{a}}\sum_{x=1}^{\iota}\mathcal{W}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}\leq\frac{1}{B-E}\left(\int_{\check{c}}^{\check{a}}\sum_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\delta}\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.11}$$

By applying (2.2) with $h(\varsigma)=\prod_{x=1}^{\iota}\mathcal{W}_x^{\frac{1}{\iota}}(\varsigma)$ and $\lambda(\varsigma)=\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}$, we have

$$\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\mathcal{W}_x^{\frac{p}{\iota}}(\varsigma)\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$
$$\leq\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{\delta-p}{p\delta}}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}.$$

Thus, (3.10) gives

$$\frac{1}{\ell-E}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p-\delta}{p\delta}} \tag{3.12}$$

$$\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$
$$\leq\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}.$$

From (3.8) and (3.12), we deduce that

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p-\delta}{p\delta}}$$
$$\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left[\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)\right]^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$
$$\leq\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}} \tag{3.13}$$
$$+\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}.$$

Applying (2.1), we see that

$$\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\leq\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota},$$

and

$$\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\leq\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}.$$

Thus, (3.13) becomes

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p-\delta}{p\delta}}$$
$$\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}\left[\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)\right]^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}$$
$$\leq\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.14}$$

From (3.7) and (3.11), we have

$$\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{G}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{W}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}$$
$$\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p-\delta}{p\delta}}\\
&\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}[(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))]^{\frac{p}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{p}}\\
&\leq\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}\\
&\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}},
\end{aligned}$$

which is (3.2). □

Corollary 3.1. *Taking* $\mathbb{T}=\mathbb{R}$ *and* $\iota=1$, *then we get (1.7).*

Corollary 3.2. *Taking* $\mathbb{T}=\mathbb{N}$, $\check{c},\breve{a}\in\mathbb{N}$, $0<p\leq\delta<\infty$, $\alpha>0$, *and* $\{\mathcal{G}_x\}_{x=1}^{\iota}$, $\{\mathcal{W}_x\}_{x=1}^{\iota}$, $\{\mathcal{U}_x\}_{x=1}^{\iota}$ *are positive sequences such that*

$$0<E<B\leq\frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)}\leq\ell,\quad \textit{for}\ \ \varsigma\in\mathbb{N}.$$

Then, $\sigma(\varsigma)=\varsigma+1$, $\mu(\varsigma)=1$, $\rho(\breve{a})=\breve{a}-1$, *and*

$$\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p-\delta}{p\delta}}\\
&\times\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}[(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))]^{\frac{p}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{1}{p}}\\
&\leq\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}}+\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}}\\
&\leq\frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{1}{\delta}}.
\end{aligned}$$

Remark 3.1. If $\mathbb{T}=q^{\mathbb{N}}$ for $q>1$, $\check{c},\breve{a}\in\mathbb{T}$, $0<p\leq\delta<\infty$, $\alpha>0$, and $\{\mathcal{G}_x\}_{x=1}^{\iota}$, $\{\mathcal{W}_x\}_{x=1}^{\iota}$, $\{\mathcal{U}_x\}_{x=1}^{\iota}$ are positive sequences such that, for $\varsigma\in\mathbb{T}$,

$$0<E<B\leq\frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)}\leq\ell.$$

Then, $\sigma(\varsigma)=q\varsigma$, $\mu(\varsigma)=\sigma(\varsigma)-\varsigma=(q-1)\varsigma$, $\rho(\breve{a})=\breve{a}/q$, and

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p-\delta}{p\delta}}$$

$$\times\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}\left[\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)\right]^{\frac{p}{\iota}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\right)^{\frac{1}{p}}$$

$$\leq\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}\frac{q-1}{\iota}\left(\varsigma\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)\right)\right)^{\frac{1}{\delta}}+\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}\frac{q-1}{\iota}\left(\varsigma\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)\right)\right)^{\frac{1}{\delta}}$$

$$\leq\frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}\frac{q-1}{\iota}\left(\varsigma\sum_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\delta}\mathcal{U}_x(\varsigma)\right)\right)^{\frac{1}{\delta}}.$$

Example 3.1. In Theorem 3.1, assume that $\mathbb{T}=\mathbb{R}$, $\check{c}=0, \breve{a}\in\mathbb{R}$, ι, $p=1$, $\delta=2$, $B=2$, $E=1$, and $\alpha=1$. In addition, if $\mathcal{G}, \mathcal{W}, \mathcal{U}\in C([\check{c},\breve{a}]_{\mathbb{T}},\mathbb{R}^{+})$ such that $\mathcal{U}(\varsigma)=\varsigma$, $\mathcal{W}(\varsigma)=\varsigma$, $\mathcal{G}(\varsigma)=3\varsigma$, and $\ell=5$, then

$$\begin{aligned}&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{p-\delta}{p\delta}}\\&\times\left(\int_{\check{c}}^{\breve{a}}\left[\left(\alpha\mathcal{G}(\varsigma)-E\mathcal{W}(\varsigma)\right)\right]^{p}\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\\&\leq\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\mathcal{G}^{\delta}(\varsigma)\,d\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\mathcal{W}^{\delta}(\varsigma)d\varsigma\right)^{\frac{1}{\delta}}\\&\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}}\left(\alpha\mathcal{G}(\varsigma)-E\mathcal{W}(\varsigma)\right)^{\delta}\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{1}{\delta}}.\end{aligned}\tag{3.16}$$

Proof. Using the hypothesis, the left-hand side of (3.16) can be written as follows:

$$\begin{aligned}&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{p-\delta}{p\delta}}\\&\times\left(\int_{\check{c}}^{\breve{a}}\left[\left(\alpha\mathcal{G}(\varsigma)-E\mathcal{W}(\varsigma)\right)\right]^{p}\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\\&=\frac{\ell+1}{\ell-1}\left(\int_{0}^{\breve{a}}\varsigma d\varsigma\right)^{\frac{-1}{2}}\int_{0}^{\breve{a}}(3\varsigma-\varsigma)\,\varsigma d\varsigma\\&=\frac{\ell+1}{\ell-1}\left(\int_{0}^{\breve{a}}\varsigma d\varsigma\right)^{\frac{-1}{2}}\int_{0}^{\breve{a}}2\varsigma^{2}d\varsigma\\&=\frac{2}{3}\frac{\ell+1}{\ell-1}\left(\frac{\breve{a}^{2}}{2}\right)^{\frac{-1}{2}}\breve{a}^{3}=\frac{2}{3}\frac{\ell+1}{\ell-1}\left(\frac{1}{2}\right)^{\frac{-1}{2}}\breve{a}^{2}\\&=\left(\frac{2}{3}\right)\left(\frac{6}{4}\right)2^{\frac{1}{2}}\breve{a}^{2}=2^{\frac{1}{2}}\breve{a}^{2}.\end{aligned}\tag{3.17}$$

Also, we see that

$$\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\mathcal{G}^{\delta}(\varsigma)\,d\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)\mathcal{W}^{\delta}(\varsigma)d\varsigma\right)^{\frac{1}{\delta}}$$

$$= \left(9\int_0^{\breve{a}} \varsigma^3 d\varsigma\right)^{\frac{1}{2}} + \left(\int_0^{\breve{a}} \varsigma^3 d\varsigma\right)^{\frac{1}{2}}$$
$$= \left(9\frac{\breve{a}^4}{4}\right)^{\frac{1}{2}} + \left(\frac{\breve{a}^4}{4}\right)^{\frac{1}{2}} = 2\breve{a}^2. \tag{3.18}$$

Furthermore,

$$\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} (\alpha\mathcal{G}(\varsigma) - E\mathcal{W}(\varsigma))^{\delta}\,\mathcal{U}(\varsigma)\,d\varsigma\right)^{\frac{1}{\delta}}$$
$$= 3\left(\int_0^{\breve{a}} (2\varsigma)^2\,\varsigma d\varsigma\right)^{\frac{1}{2}} = 6\left(\frac{\breve{a}^4}{4}\right)^{\frac{1}{2}} = 3\breve{a}^2. \tag{3.19}$$

From (3.17)–(3.19), we see that the inequality (3.16) holds. The proof is complete. □

Theorem 3.2. *Assume that* $\check{c}, \breve{a} \in \mathbb{T}$, $\breve{a} > \check{c}$, $0 < \delta < \infty$, $\alpha > 0$, $\mathcal{G}_x$, $\mathcal{W}_x$, $\mathcal{U}_x \in C_{rd}([\check{c}, \breve{a}]_{\mathbb{T}}, \mathbb{R}^+)$, $x = 1, 2, ..., \iota$, *and (3.1) holds. If* $p > 1$, *then*

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\,\Delta\varsigma\right)^{\frac{1-p}{\delta}}$$
$$\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\,\Delta\varsigma\right)^{\frac{p}{\delta}}$$
$$\leq \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}$$
$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} (\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\delta}\,\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}, \tag{3.20}$$

and, if $0 < p < 1$, *then*

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\,\Delta\varsigma\right)^{\frac{p-1}{\delta}}$$
$$\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\,\Delta\varsigma\right)^{\frac{1}{\delta}}$$
$$\leq \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{p}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{p}{\delta}}$$
$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} (\alpha\mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\frac{\delta}{p}}\,\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{p}{\delta}}. \tag{3.21}$$

Proof. To prove this theorem, we have two cases:

Case 1: $p > 1$. From (3.1), we deduce that

$$\frac{\ell}{\ell - E} \leq \frac{\alpha \mathcal{G}_x(\varsigma)}{\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)} \leq \frac{B}{B-E}.$$

Then,

$$\prod_{x=1}^{\iota} \left[\frac{\ell}{\alpha(\ell - E)} (\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)) \right]^{\frac{\delta}{\iota p}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \leq \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}, \tag{3.22}$$

and

$$\sum_{x=1}^{\iota} \mathcal{G}_x^{\delta}(\varsigma) \mathcal{U}_x(\varsigma) \leq \sum_{x=1}^{\iota} \left[\frac{B}{\alpha(B-E)} (\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)) \right]^{\delta} \mathcal{U}_x(\varsigma). \tag{3.23}$$

Integrating (3.22) and (3.23) over ς from $\check{c}$ to $\breve{a}$, we see that

$$\frac{\ell}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}}, \tag{3.24}$$

and

$$\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{G}_x^{\delta}(\varsigma) \mathcal{U}_x(\varsigma) \Delta\varsigma \right)^{\frac{1}{\delta}} \leq \frac{B}{\alpha(B-E)} \left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} (\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\delta} \mathcal{U}_x(\varsigma) \Delta\varsigma \right)^{\frac{1}{\delta}}. \tag{3.25}$$

Applying (2.5) with $\lambda = 1/p < 1$, $\varkappa(\varsigma) = \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}$, and $\phi(\varsigma) = \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)$, we get

$$\left(\frac{\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma) \Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma} \right)^{\frac{1}{p}} \geq \frac{\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma}{\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma}.$$

Then, we have from (3.24) that

$$\frac{\ell}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{1-p}{\delta}} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} (\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}}$$

$$\leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma \right)^{\frac{1}{\delta}}. \tag{3.26}$$

Using (3.1), we deduce that

$$0 < B - E \leq \frac{\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)}{\mathcal{W}_x(\varsigma)} \leq \ell - E.$$

Therefore,

$$\frac{1}{\ell - E}\left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right) \leq \mathcal{W}_x(\varsigma) \leq \frac{1}{B - E}\left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right).$$

Then, we have that

$$\begin{aligned} &\frac{1}{\ell - E}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota p}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}} \\ &\leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}}, \end{aligned} \tag{3.27}$$

and

$$\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{W}_x^{\delta}(\varsigma) \mathcal{U}_x(\varsigma)\Delta\varsigma \right)^{\frac{1}{\delta}} \leq \frac{1}{B - E}\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\delta} \mathcal{U}_x(\varsigma)\Delta\varsigma \right)^{\frac{1}{\delta}}. \tag{3.28}$$

Applying (2.5) with $\lambda = 1/p < 1$, $\phi(\varsigma) = \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)$, and $\varkappa(\varsigma) = \prod_{x=1}^{\iota} (\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}$, we see that

$$\begin{aligned} &\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}} \\ &\leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p-1}{\delta}} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{1}{\delta}}. \end{aligned}$$

Thus, (3.27) becomes

$$\begin{aligned} &\frac{1}{\ell - E}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{1-p}{\delta}} \\ &\times \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota p}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{p}{\delta}} \\ &\leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{1}{\delta}}. \end{aligned} \tag{3.29}$$

Adding (3.26) and (3.29), we see that

$$\frac{\ell + \alpha}{\alpha(\ell - E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma \right)^{\frac{1-p}{\delta}}$$

$$\times\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota p}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p}{\delta}}$$

$$\leq\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}} \tag{3.30}$$

$$+\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{\delta}}.$$

Applying (2.1) to the terms $\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)\Delta\varsigma\right)^{\frac{1}{\delta}}$ and $\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1}{\delta}}$, then (3.30) becomes

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1-p}{\delta}}$$

$$\times\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota p}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p}{\delta}}$$

$$\leq\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.31}$$

From (3.25) and (3.28), we have

$$\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{G}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{W}_x^{\delta}(\varsigma)\mathcal{U}_x(\varsigma)\Delta\varsigma}{\iota}\right)^{\frac{1}{\delta}}$$

$$\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}. \tag{3.32}$$

Combining (3.31) and (3.32), we get

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{1-p}{\delta}}$$

$$\times\left(\int_{\check{c}}^{\check{a}}\prod_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota p}}\left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}\Delta\varsigma\right)^{\frac{p}{\delta}}$$

$$\leq\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}}$$

$$\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\check{a}}\frac{\sum_{x=1}^{\iota}\left(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma)\right)^{\delta}\mathcal{U}_x(\varsigma)}{\iota}\Delta\varsigma\right)^{\frac{1}{\delta}},$$

which is (3.20).

Case 2: $0 < p < 1$. Again, by using (3.1), we see that

$$\frac{\ell}{\ell - E} \leq \frac{\alpha \mathcal{G}_x(\varsigma)}{\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)} \leq \frac{B}{B - E}.$$

Then,

$$\left(\frac{\ell}{\alpha(\ell - E)}\right)^{\delta} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \leq \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma), \tag{3.33}$$

and

$$\sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma) \mathcal{U}_x(\varsigma) \leq \sum_{x=1}^{\iota} \left[\frac{B}{\alpha(B - E)} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)\right]^{\frac{\delta}{p}} \mathcal{U}_x(\varsigma). \tag{3.34}$$

Integrating (3.33) and (3.34), we have for $p, \delta > 0$, that

$$\frac{\ell}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma) \Delta\varsigma\right)^{\frac{1}{\delta}}, \tag{3.35}$$

and

$$\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma) \mathcal{U}_x(\varsigma) \Delta\varsigma\right)^{\frac{p}{\delta}} \leq \frac{B}{\alpha(B - E)} \left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{p}} \mathcal{U}_x(\varsigma) \Delta\varsigma\right)^{\frac{p}{\delta}}. \tag{3.36}$$

Applying (2.4) with $\lambda = 1/p > 1$, $\varkappa(\varsigma) = \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}$, and $\phi(\varsigma) = \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma)$, we get

$$\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \mathcal{G}_x^{\frac{\delta}{\iota}}(\varsigma) \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1-p}{\delta}} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}}.$$

So, the inequality (3.35) can be written as follows

$$\frac{\ell}{\alpha(\ell - E)} \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-1}{\delta}} \times \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}}. \tag{3.37}$$

From (3.1), we deduce that

$$0 < B - E \leq \frac{\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)}{\mathcal{W}_x(\varsigma)} \leq \ell - E.$$

Thus,

$$\frac{1}{\ell - E}\left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right) \leq \mathcal{W}_x(\varsigma) \leq \frac{1}{B - E}\left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right).$$

Then, we have for $\delta > 0$ that

$$\frac{1}{\ell - E}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}}, \tag{3.38}$$

and

$$\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma) \mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}} \leq \frac{1}{B - E}\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{p}} \mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}}. \tag{3.39}$$

Applying (2.4) with $\lambda = 1/p > 1$, $\phi(\varsigma) = \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma)$, and $\varkappa(\varsigma) = \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}}$, we get

$$\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1-p}{\delta}} \left(\int_{\check{c}1}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}}.$$

Then, the inequality (3.38) becomes

$$\frac{1}{\ell - E}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-1}{\delta}} \times \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}}. \tag{3.40}$$

Adding (3.37) and (3.40), we get

$$\frac{\ell + \alpha}{\alpha(\ell - E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-1}{\delta}}$$

$$
\begin{aligned}
&\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \\
&\leq \left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}} \\
&+\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{\iota p}}(\varsigma) \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p}{\delta}}.
\end{aligned} \tag{3.41}
$$

Applying the inequality (2.1) on the right-hand side of (3.41), we have

$$
\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-1}{\delta}} \\
&\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \\
&\leq \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota} \Delta\varsigma\right)^{\frac{p}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota} \Delta\varsigma\right)^{\frac{p}{\delta}}.
\end{aligned} \tag{3.42}
$$

From (3.36) and (3.39), we get

$$
\begin{aligned}
&\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}} \\
\leq\;& \frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} \sum_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{p}} \mathcal{U}_x(\varsigma)\Delta\varsigma\right)^{\frac{p}{\delta}}.
\end{aligned} \tag{3.43}
$$

Combining (3.42) and (3.43), we see that

$$
\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{p-1}{\delta}} \\
&\times\left(\int_{\check{c}}^{\breve{a}} \prod_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{\iota}} \left(\mathcal{U}_x(\varsigma)\right)^{\frac{1}{\iota}} \Delta\varsigma\right)^{\frac{1}{\delta}} \\
&\leq \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota} \Delta\varsigma\right)^{\frac{p}{\delta}} + \left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\,\mathcal{U}_x(\varsigma)}{\iota} \Delta\varsigma\right)^{\frac{p}{\delta}} \\
&\leq \frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}} \frac{\sum_{x=1}^{\iota} \left(\alpha \mathcal{G}_x(\varsigma) - E\mathcal{W}_x(\varsigma)\right)^{\frac{\delta}{p}} \mathcal{U}_x(\varsigma)}{\iota} \Delta\varsigma\right)^{\frac{p}{\delta}},
\end{aligned}
$$

which is (3.21). □

Remark 3.2. Assume that $\mathbb{T}=\mathbb{R}$; $\check{c},\breve{a}\in\mathbb{R}$ with $\breve{a}>\check{c}$, $0<\delta<\infty$, $\alpha>0$; and $\mathcal{G}_x$, $\mathcal{W}_x$, $\mathcal{U}_x\in C([\check{c},\breve{a}],\mathbb{R}^+)$ with

$$0<E<B\leq\frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)}\leq\ell\quad\text{for}\quad x=1,2,...,\iota.$$

In addition, if $p>1$, then

$$\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}d\varsigma\right)^{\frac{1-p}{\delta}}\\
&\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}d\varsigma\right)^{\frac{p}{\delta}}\\
&\leq\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}d\varsigma\right)^{\frac{1}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}d\varsigma\right)^{\frac{1}{\delta}}\\
&\leq\frac{B+\alpha}{\alpha(B-E)}\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\mathcal{U}_x(\varsigma)}{\iota}d\varsigma\right)^{\frac{1}{\delta}}.
\end{aligned}\tag{3.44}$$

If $0<p<1$, then

$$\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}d\varsigma\right)^{\frac{p-1}{\delta}}\\
&\times\left(\int_{\check{c}}^{\breve{a}}\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}d\varsigma\right)^{\frac{1}{\delta}}\\
&\leq\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}d\varsigma\right)^{\frac{p}{\delta}}+\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}\mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}d\varsigma\right)^{\frac{p}{\delta}}\\
&\leq\frac{B+\alpha}{B-E}\left(\int_{\check{c}}^{\breve{a}}\frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{p}}\mathcal{U}_x(\varsigma)}{\iota}d\varsigma\right)^{\frac{p}{\delta}}.
\end{aligned}$$

Remark 3.3. If $\mathbb{T}=\mathbb{R}$, $\iota=1$, $p>1$, and $0<\delta<\infty$, then we obtain the correction and the generalization of (1.8) by replacing $\breve{a}-\check{c}$ with $\int_{\check{c}}^{\breve{a}}\mathcal{U}(\varsigma)d\varsigma$.

Remark 3.4. Suppose that $\mathbb{T}=\mathbb{N}$; $\check{c}$, $\breve{a}\in\mathbb{N}$, $0<\delta<\infty$, $\alpha>0$, and $\{\mathcal{G}_x\}_{x=1}^{\iota}$, $\{\mathcal{W}_x\}_{x=1}^{\iota}$, $\{\mathcal{U}_x\}_{x=1}^{\iota}$ are positive sequences such that

$$0<E<B\leq\frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)}\leq\ell,\quad\varsigma\in\mathbb{N}.$$

In addition, if $p>1$, then

$$\begin{aligned}
&\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{1-p}{\delta}}\\
&\times\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p}{\delta}}
\end{aligned}$$

$$\leq \left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}} + \left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}}$$

$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\,\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{1}{\delta}},$$

and, if $0 < p < 1$, then

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p-1}{\delta}}$$

$$\times\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1}\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{1}{\delta}}$$

$$\leq \left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota} \mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{p}{\delta}} + \left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota} \mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{p}{\delta}}$$

$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\breve{a}-1} \frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{p}}\,\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{p}{\delta}}.$$

Remark 3.5. Assume that $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$; $\check{c}, \breve{a} \in \mathbb{T}$, $0 < \delta < \infty$, $\alpha > 0$; and $\{\mathcal{G}_x\}_{x=1}^{\iota}$, $\{\mathcal{W}_x\}_{x=1}^{\iota}$, $\{\mathcal{U}_x\}_{x=1}^{\iota}$ are positive sequences with

$$0 < E < B \leq \frac{\alpha\mathcal{G}_x(\varsigma)}{\mathcal{W}_x(\varsigma)} \leq \ell, \quad \text{for} \quad \varsigma \in \mathbb{T}.$$

In addition, if $p > 1$, then $\rho(\breve{a}) = \breve{a}/q$, $\sigma(\varsigma) = q\varsigma$, $\mu(\varsigma) = \sigma(\varsigma) - \varsigma = (q-1)\varsigma$, and

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{1-p}{\delta}}$$

$$\times\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota p}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p}{\delta}}$$

$$\leq \left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{G}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}} + \left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\frac{\sum_{x=1}^{\iota} \mathcal{U}_x(\varsigma)\mathcal{W}_x^{\delta}(\varsigma)}{\iota}\right)^{\frac{1}{\delta}}$$

$$\leq \frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\frac{\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\delta}\,\mathcal{U}_x(\varsigma)}{\iota}\right)^{\frac{1}{\delta}},$$

and, if $0 < p < 1$, then

$$\frac{\ell+\alpha}{\alpha(\ell-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\breve{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{p-1}{\delta}}$$

$$\times\left(\sum_{\varsigma=\check{c}}^{\frac{\check{a}}{q}}(q-1)\varsigma\prod_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{\iota}}(\mathcal{U}_x(\varsigma))^{\frac{1}{\iota}}\right)^{\frac{1}{\delta}}$$

$$\leq\left(\sum_{\varsigma=\check{c}}^{\frac{\check{a}}{q}}\left(\frac{(q-1)\varsigma}{\iota}\right)\sum_{x=1}^{\iota}\mathcal{W}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)\right)^{\frac{p}{\delta}}+\left(\sum_{\varsigma=\check{c}}^{\frac{\check{a}}{q}}\left(\frac{(q-1)\varsigma}{\iota}\right)\sum_{x=1}^{\iota}\mathcal{G}_x^{\frac{\delta}{p}}(\varsigma)\mathcal{U}_x(\varsigma)\right)^{\frac{p}{\delta}}$$

$$\leq\frac{B+\alpha}{\alpha(B-E)}\left(\sum_{\varsigma=\check{c}}^{\frac{\check{a}}{q}}\left(\frac{(q-1)\varsigma}{\iota}\right)\sum_{x=1}^{\iota}(\alpha\mathcal{G}_x(\varsigma)-E\mathcal{W}_x(\varsigma))^{\frac{\delta}{p}}\mathcal{U}_x(\varsigma)\right)^{\frac{p}{\delta}}.$$

4. Conclusions

In this paper, we have established a novel extensions of the reversed Minkowski's inequality for various functions on delta calculus time scales by applying Jensen's and Hölder's inequalities on time scales. In addition, we have presented some new inequalities in different cases, like $\mathbb{T}=\mathbb{N}$ and $q^{\mathbb{N}}$ for $q>1$.

In the future, we will apply the reversed Minkowski inequality for various functions to diamond-alpha calculus and conformable calculus time scales.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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