AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 8768–8775.
DOI: 10.3934/math.2024425
Received: 23 November 2023
Revised: 30 January 2024
Accepted: 26 February 2024
Published: 29 February 2024

Research article

A new proof of a double inequality of Masjed-Jamei type

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Abstract: In this paper, we provide a new simple proof of a double inequality of Masjed-Jamei type proved by L. Zhu [1].

Keywords: inverse hyperbolic tangent function; inverse sine function; Masjed-Jamei type
Mathematics Subject Classification: 26D05, 26D15

1. Introduction

In 2010, Masjed-Jamei [2] obtained some interesting inequalities for several special functions, one of which is about the relation of the inverse tangent function $\arctan x$ and inverse hyperbolic sine function $\sinh^{-1}(x)$ as follows:

$$(\arctan x)^2 \leqslant \frac{x\sinh^{-1}(x)}{\sqrt{1+x^2}}, \quad x \in (-1,1). \tag{1.1}$$

The study related to (1.1) attracted much attention in last decade. At first, Zhu and Malešević [3] proved that (1.1) holds for any $x \in (-\infty, +\infty)$. They also obtained some refinements of (1.1) as follows:

Proposition 1.1. *[3, Theorem 1.3] For any $x \in (-\infty, +\infty)$, we have*

$$-\frac{1}{45}x^6 \leqslant (\arctan x)^2 - \frac{x\sinh^{-1}x}{\sqrt{1+x^2}} \leqslant -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{1.2}$$

$$\begin{aligned} -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} &\leqslant (\arctan x)^2 - \frac{x\sinh^{-1}x}{\sqrt{1+x^2}} \\ &\leqslant -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10395}x^{12}. \end{aligned} \tag{1.3}$$

Define

$$v_n = \frac{1}{n}\left[\frac{n!2^{n-1}}{(2n-1)!!} - \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right)\right], \ n \geqslant 3. \tag{1.4}$$

By using flexible analysis tools, Zhu and Malešević [4] extended (1.2) and (1.3) to a general form as follows:

Proposition 1.2. *[4, Theorem 1.1] For any* $x \in (-\infty, +\infty)$, *we have*

$$\sum_{n=3}^{2m+1}(-1)^n v_n x^{2n} \leqslant (\arctan x)^2 - \frac{x \sinh^{-1} x}{\sqrt{1+x^2}} \leqslant \sum_{n=3}^{2m+2}(-1)^n v_n x^{2n}. \tag{1.5}$$

Proposition 1.3. *[5, Theorem 2.1] The double inequality*

$$\frac{x \sinh^{-1}(x)}{\sqrt{1+x^2+\frac{1}{45}x^2}} < (\arctan x)^2 < \frac{x \sinh^{-1}(x)}{\sqrt{1+x^2}} \tag{1.6}$$

holds for any $x \in (0, +\infty)$ *with best constants* 0 *and* 1/45.

Please see [6, 7] for more generalizations.

Motivated by (1.1)–(1.6), Zhu and Malešević [3] also studied the relation of the inverse hyperbolic tangent function $\tanh^{-1}(x)$ and inverse sine function $\arcsin x$ as follows:

Proposition 1.4. *[3, Theorem 1.4] The inequality*

$$\left[\tanh^{-1}(x)\right]^2 < \frac{x \arcsin x}{\sqrt{1-x^2}} \tag{1.7}$$

holds for any $x \in (0, 1)$ *with the the best power number* 2.

Proposition 1.5. *[3, Theorem 1.6] The inequality*

$$\frac{x \arcsin x}{\sqrt{1-x^2}} - \left[\tanh^{-1}(x)\right]^2 < \sum_{n=3}^{N} v_n x^{2n} \tag{1.8}$$

holds for any $x \in (0, 1)$.

Moreover, by investigating the power series of the following function:

$$\frac{\left[\tanh^{-1}(x)\right]^2}{\frac{\arcsin x}{\sqrt{1-x^2}}} = x - \frac{1}{45}x^5 - \frac{22}{945}x^7 - \frac{61}{2835}x^9 + O(x^{10}), \tag{1.9}$$

Zhu [1] obtained the following interesting double inequality of Masjed-Jamei type.

Theorem 1.1. *[1, Theorem 1] The double inequality*

$$\frac{\left(x - x^5\right)\arcsin x}{\sqrt{1-x^2}} < \left[\tanh^{-1}(x)\right]^2 < \frac{\left(x - \frac{1}{45}x^5\right)\arcsin x}{\sqrt{1-x^2}} \tag{1.10}$$

holds for any $x \in (0, 1)$ *with best constants* -1 *and* $-\frac{1}{45}$.

The goal of this paper is to give a new and elementary proof of Theorem 1.1, which is much simpler than the proof of Zhu [1]. Zhu's proof [1] used the power series of the functions $1/\cos^n x$ and $\sin x/\cos^n x$ and properties of the Bernoulli numbers and Euler numbers. Our proof only relies on the power series of hyperbolic sine and cosine functions and some elementary computations.

2. A new proof of Theorem 1.1

We first establish two lemmas about the monotonicity of two functions.

Lemma 2.1. *Let*

$$f(x)=\frac{\left[\tanh^{-1}(x)\right]^2\sqrt{1-x^2}}{x-x^5}-\arcsin x,$$

then $f(x)$ is strictly increasing on $(0,1)$.

Proof. Let $t=\tanh^{-1}(x)\in(0,+\infty)$, then $x=\tanh(t)$. Define

$$\begin{aligned}F(t):=f(\tanh(t))&=\frac{\frac{t^2}{\cosh(t)}}{\tanh(t)-\tanh^5(t)}-\arcsin(\tanh(t))\\&=\frac{t^2\cosh^4 t}{\left(\cosh^2 t+\sinh^2 t\right)\sinh t}-\arcsin(\tanh(t)).\end{aligned}$$

In order to prove that $f(x)$ is strictly increasing on $(0,1)$, we only need to prove $F(t)$ is strictly increasing on $(0,+\infty)$. In fact,

$$\begin{aligned}\varphi(t):&=\left[\left(\cosh^2 t+\sinh^2 t\right)^2\cosh t\sinh^2 t\right]F'(t)\\&=t^2\left(2\cosh^4 t-7\cosh^2 t+4\right)\cosh^4 t+2t\left(\cosh^2 t+\sinh^2 t\right)\sinh t\cosh^5 t\\&\quad-\left(\cosh^2 t+\sinh^2 t\right)^2\sinh^2 t.\end{aligned}\tag{2.1}$$

Since

$$\begin{aligned}A_1:=&\left(2\cosh^4 t-7\cosh^2 t+4\right)\cosh^4 t\\=&\frac{1}{2^6}\left[\cosh(8t)-6\cosh(6t)-24\cosh(4t)-26\cosh(2t)-9\right]\\=&\frac{1}{2^6}\left[\sum_{n=0}^{\infty}\frac{(8t)^{2n}}{(2n)!}-6\cdot\sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n)!}-24\cdot\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!}-26\cdot\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!}-9\right],\end{aligned}$$

$$\begin{aligned}B_1:=&2\left(\cosh^2 t+\sinh^2 t\right)\sinh t\cosh^5 t\\=&\frac{1}{2^5}\left[\sinh(8t)+4\sinh(6t)+6\sinh(4t)+4\sinh(2t)\right]t\\=&\frac{1}{2^5}\left[\sum_{n=0}^{\infty}\frac{(8t)^{2n+1}}{(2n+1)!}+4\cdot\sum_{n=0}^{\infty}\frac{(6t)^{2n+1}}{(2n+1)!}+6\cdot\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!}+4\cdot\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}\right]\end{aligned}$$

and

$$\begin{aligned}C_1 :=& -\left(\cosh^2 t+\sinh^2 t\right)^2 \sinh^2 t \\ =&\frac{1}{2^3}\left[\cosh(6t)-2\cosh(4t)+3\cosh(2t)-2\right] \\ =&\frac{1}{2^3}\left[\sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n)!}-2\cdot\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!}+3\cdot\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!}-2\right],\end{aligned}$$

we have

$$\varphi(t)=A_1t^2+B_1t+C_1=\frac{1}{2^6}\sum_{n=1}^{\infty}a_{2n+2}t^{2n+2}, \tag{2.2}$$

where

$$\begin{aligned}a_{2n+2} =&\frac{8^{2n}}{(2n)!}-6\cdot\frac{6^{2n}}{(2n)!}-24\cdot\frac{4^{2n}}{(2n)!}-26\cdot\frac{2^{2n}}{(2n)!} \\ &+2\cdot\frac{8^{2n+1}}{(2n+1)!}+8\cdot\frac{6^{2n+1}}{(2n+1)!}+12\cdot\frac{4^{2n+1}}{(2n+1)!}+8\cdot\frac{2^{2n+1}}{(2n+1)!} \\ &-8\cdot\frac{6^{2n+2}}{(2n+2)!}+16\cdot\frac{4^{2n+2}}{(2n+2)!}-24\cdot\frac{2^{2n+2}}{(2n+2)!}.\end{aligned}$$

It is easy to check that

$$a_4=0,\ a_6=\frac{2816}{9},\ a_8=\frac{4224}{5}. \tag{2.3}$$

When $n\geqslant 4$,

$$\begin{aligned}a_{2n+2} >&\frac{8^{2n}}{(2n)!}\left[1-6\cdot\left(\frac{3}{4}\right)^{2n}-24\cdot\left(\frac{1}{2}\right)^{2n}-26\cdot\left(\frac{1}{4}\right)^{2n}\right] \\ &+\frac{6^{2n+1}}{(2n+1)!}\left(1-\frac{3}{n+1}\right)+8\cdot\frac{2^{2n+1}}{(2n+1)!}\left(1-\frac{3}{n+1}\right) \\ >&\frac{8^{2n}}{(2n)!}\left[1-6\cdot\left(\frac{3}{4}\right)^{8}-24\cdot\left(\frac{1}{2}\right)^{8}-26\cdot\left(\frac{1}{4}\right)^{8}\right] \\ =&\frac{8^{2n}}{(2n)!}\cdot\frac{625}{2^{11}} \\ >&0.\end{aligned} \tag{2.4}$$

Combining (2.2)–(2.4), we obtain that $\varphi(t)>0$ for any $t\in(0,+\infty)$, which implies

$$F'(t)>0 \text{ for any } t\in(0,+\infty).$$

So, $F(t)$ is strictly increasing on $(0,+\infty)$. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Let*

$$g(x) = \arcsin x - \frac{\left[\tanh^{-1}(x)\right]^2 \sqrt{1-x^2}}{x - \frac{1}{45}x^5},$$

then $g(x)$ *is strictly increasing on* $(0,1)$.

Proof. Let $t = \tanh^{-1}(x) \in (0, +\infty)$, then $x = \tanh(t)$. Define

$$\begin{aligned} G(t) := g(\tanh(t)) &= \arcsin(\tanh(t)) - \frac{\frac{t^2}{\cosh(t)}}{\tanh(t) - \frac{1}{45}\tanh^5(t)} \\ &= \arcsin(\tanh(t)) - \frac{45t^2\cosh^4 t}{\left(45\cosh^4 t - \sinh^4 t\right)\sinh t}. \end{aligned}$$

In order to prove that $g(x)$ is strictly increasing on $(0,1)$, we only need to prove $G(t)$ is strictly increasing on $(0, +\infty)$. Define

$$\begin{aligned} \psi(t) :=& \left[\left(45\cosh^4 t - \sinh^4 t\right)^2 \cosh t \sinh^2 t\right] G'(t) \\ =& t^2\left(1980\cosh^6 t - 90\cosh^4 t + 315\cosh^2 t - 180\right)\cosh^4 t \\ &- 90t\left(45\cosh^4 t - \sinh^4 t\right)\sinh t\cosh^5 t + \left(45\cosh^4 t - \sinh^4 t\right)^2 \sinh^2 t. \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} A_2 :=& \left(1980\cosh^6 t - 90\cosh^4 t + 315\cosh^2 t - 180\right)\cosh^4 t \\ =& \frac{1}{2^7}\left[495\cosh(10t) + 4860\cosh(8t) + 22815\cosh(6t) + 61560\cosh(4t)\right. \\ &\left. + 106290\cosh(2t) + 63180\right] \\ =& \frac{1}{2^7}\left[495 \cdot \sum_{n=0}^{\infty}\frac{(10t)^{2n}}{(2n)!} + 4860 \cdot \sum_{n=0}^{\infty}\frac{(8t)^{2n}}{(2n)!} + 22815 \cdot \sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n)!} + 61560 \cdot \sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!}\right. \\ &\left. + 106290 \cdot \sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!} + 63180\right], \end{aligned}$$

$$\begin{aligned} B_2 :=& -90\left(45\cosh^4 t - \sinh^4 t\right)\sinh t\cosh^5 t \\ =& \frac{1}{2^7}\left[-990\sinh(10t) - 8100\sinh(8t) - 27450\sinh(6t) - 48600\sinh(4t) - 42300\sinh(2t)\right] \\ =& \frac{1}{2^7}\left[-990 \cdot \sum_{n=0}^{\infty}\frac{(10t)^{2n+1}}{(2n+1)!} - 8100 \cdot \sum_{n=0}^{\infty}\frac{(8t)^{2n+1}}{(2n+1)!} - 27450 \cdot \sum_{n=0}^{\infty}\frac{(6t)^{2n+1}}{(2n+1)!}\right. \\ &\left. -48600 \cdot \sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!} - 42300 \cdot \sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}\right], \end{aligned}$$

and

$$C_2 := \left(45\cosh^4 t - \sinh^4 t\right)^2 \sinh^2 t$$
$$=\frac{1}{2^7}\left[484\cosh(10t) + 3080\cosh(8t) + 6660\cosh(6t) + 3840\cosh(4t) - 7080\cosh(2t) - 6984\right]$$
$$=\frac{1}{2^7}\Bigg[484\cdot\sum_{n=0}^{\infty}\frac{(10t)^{2n}}{(2n)!} + 3080\cdot\sum_{n=0}^{\infty}\frac{(8t)^{2n}}{(2n)!} + 6660\cdot\sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n)!} + 3840\cdot\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!}$$
$$-7080\cdot\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!} - 6984\Bigg],$$

we have

$$\psi(t) = A_2t^2 + B_2t + C_2 = \frac{1}{2^7}\sum_{n=1}^{\infty} b_{2n+2}t^{2n+2}, \tag{2.6}$$

where

$$b_{2n+2} =495\cdot\frac{10^{2n}}{(2n)!} + 4860\cdot\frac{8^{2n}}{(2n)!} + 22815\cdot\frac{6^{2n}}{(2n)!} + 61560\cdot\frac{4^{2n}}{(2n)!} + 106290\cdot\frac{2^{2n}}{(2n)!}$$
$$- 990\cdot\frac{10^{2n+1}}{(2n+1)!} - 8100\cdot\frac{8^{2n+1}}{(2n+1)!} - 27450\cdot\frac{6^{2n+1}}{(2n+1)!} - 48600\cdot\frac{4^{2n+1}}{(2n+1)!}$$
$$- 42300\cdot\frac{2^{2n+1}}{(2n+1)!} + 484\cdot\frac{10^{2n+2}}{(2n+2)!} + 3080\cdot\frac{8^{2n+2}}{(2n+2)!} + 6660\cdot\frac{6^{2n+2}}{(2n+2)!}$$
$$+ 3840\cdot\frac{4^{2n+2}}{(2n+2)!} - 7080\cdot\frac{2^{2n+2}}{(2n+2)!}.$$

It is easy to check that

$$b_{2n+2} > 0, \quad 1 \leqslant n \leqslant 9. \tag{2.7}$$

When $n \geqslant 10$,

$$\begin{aligned} b_{2n+2} >&\frac{10^{2n}}{(2n)!}\cdot\left(495 - \frac{990\times 10}{2n+1}\right) + \frac{8^{2n}}{(2n)!}\cdot\left(4860 - \frac{8100\times 8}{2n+1}\right)\\ &+ \frac{6^{2n}}{(2n)!}\cdot\left(22815 - \frac{27450\times 6}{2n+1}\right) + \frac{4^{2n}}{(2n)!}\cdot\left(61560 - \frac{48600\times 4}{2n+1}\right)\\ &+ \frac{2^{2n}}{(2n)!}\cdot\left(106290 - \frac{42300\times 2}{2n+1}\right) + \frac{2^{2n+2}}{(2n+2)!}\cdot\left(3840\cdot 2^{2n+2} - 7840\right)\\ &>0. \end{aligned} \tag{2.8}$$

Combining (2.6)–(2.8), we obtain that $\psi(t) > 0$ for any $t \in (0, +\infty)$, which implies

$$G'(t) > 0 \ \text{ for any } \ t \in (0, +\infty).$$

So, $G(t)$ is strictly increasing on $(0, +\infty)$. The proof of Lemma 2.2 is completed. □

Proof of Theorem 1.1. By Lemma 2.1 and

$$\lim_{x\to 0^+} f(x) = 0,$$

we get $f(x) > 0$ for any $x \in (0, 1)$, which implies

$$\left[\tanh^{-1}(x)\right]^2 > \frac{\left(x - x^5\right)\arcsin x}{\sqrt{1 - x^2}}, \quad x \in (0, 1). \tag{2.9}$$

By Lemma 2.2 and

$$\lim_{x\to 0^+} g(x) = 0,$$

we get $g(x) > 0$ for any $x \in (0, 1)$, which implies

$$\left[\tanh^{-1}(x)\right]^2 < \frac{\left(x - \frac{1}{45}x^5\right)\arcsin x}{\sqrt{1 - x^2}}, \quad x \in (0, 1). \tag{2.10}$$

Since

$$\lim_{x\to 0^+} \frac{\left[\tanh^{-1}(x)\right]^2 \cdot \sqrt{1 - x^2} - x\arcsin x}{x^5 \arcsin x} = -1, \tag{2.11}$$

$$\lim_{x\to 1^-} \frac{\left[\tanh^{-1}(x)\right]^2 \cdot \sqrt{1 - x^2} - x\arcsin x}{x^5 \arcsin x} = -\frac{1}{45}, \tag{2.12}$$

Theorem 1.1 follows from (2.9)–(2.12). □

3. Conclusions

In this paper, we give a new simple proof of a double inequality of Masjed-Jamei type proved by Zhu [1]. We believe that the technique used in this paper can be used to obtain other interesting analytic inequalities.

Based on numerical experiments and (1.9), we propose the following conjectures:

Conjecture 1.

$$\phi(x) = \frac{\sqrt{1 - x^2}\left[\tanh^{-1}(x)\right]^2 - x\arcsin x}{x^5 \arcsin x}$$

is strictly increasing on $(0, 1)$.

Conjecture 2.

$$h(x) = x - \frac{\sqrt{1 - x^2}\left[\tanh^{-1}(x)\right]^2}{\arcsin x}$$

is absolutely monotonic on $(0, 1)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author was supported by the Foundation of Hubei Provincial Department of Eduction (No. Q20233003), the Scientific Research Fund of Hubei Provincial Department of Eduction (No. B2022207) and Hubei University of Education, Bigdata Modeling and Intelligent Computing Research Institute.

Conflict of interest

The author declares no conflict of interest in this paper.

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