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Research article

Bifurcation, chaotic behavior and soliton solutions to the KP-BBM equation through new Kudryashov and generalized Arnous methods

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Abstract: This research paper investigates the Kadomtsev-Petviashvii-Benjamin-Bona-Mahony equation. The new Kudryashov and generalized Arnous methods are employed to obtain the generalized solitary wave solution. The phase plane theory examines the bifurcation analysis and illustrates phase portraits. Finally, the external perturbation terms are considered to reveal its chaotic behavior. These findings contribute to a deeper understanding of the dynamics of the Kadomtsev-Petviashvii-Benjamin-Bona-Mahony wave equation and its applications in real-world phenomena.

Keywords: bifurcation analysis; new Kudryashov method; exact solutions; generalized Arnous method; Kadomtsev-Petviashvii-Benjamin-Bona-Mahony (KP-BBM) equation; chaotic behavior
Mathematics Subject Classification: 35A09, 35A24, 35C08

1. Introduction

Nonlinear partial differential equations (NLPDEs) assume a pivotal role in exploring diverse real-world dynamic phenomena, notably within the realms of chemical reactions, plasma physics, thermodynamics, optical fibers, chaos theory, fluid dynamics, solitary waves theory, electromagnetism, quantum mechanics, and various other applications [1–4]. Over the past few decades, researchers have dedicated considerable efforts to examining nonlinear equations. For instance, Ji et al. [5] explore rational solutions to the Camassa-Holm-Kadomtsev-Petviashvili equation, which simulates dispersion in liquid drop patterns. Almatrafi [6] study the space-time fractional symmetric regularized long wave equation and presents its solitary wave solutions using improved P-expansion and (F'/F)-expansion methods. Alharbi and Almatrafi [7,8] use various approaches to investigate the geophysical Korteweg-de Vries equation and the modified regularized long-wave equation. The adaptive moving mesh technique is also used to approximate the numerical solution, reducing error. Using the exp-function method, Abdelrahman et al. [9] established the exact solutions for the nonlinear coupled KdV equations. Some other related work is listed in [10–16].

The identification of exact and soliton solutions to these equations holds paramount importance in comprehending a myriad of real-world dynamic processes. Soliton is a unique type of solitary wave that maintains its shape, speed, and size as it travels. It possesses several intriguing characteristics that explain various forms of nonlinear phenomena. The extraction of soliton solutions from NLPDEs has become highly significant in the study of nonlinear phenomena, attracting many scientists and engineers in the applied sciences field [1–4]. In recent times, solitons have revolutionized communication systems through the utilization of wave guides. They serve as the fundamental basis for transmitting and communicating data across vast distances. Consequently, finding soliton solutions for nonlinear systems has emerged as a popular topic among mathematicians and scholars. One advantage of soliton solutions is their applicability to both integrable and non-integrable evolution equations, featuring diverse types of nonlinearities. Consequently, this pursuit constitutes a vibrant and actively researched domain, yielding promising outcomes. There are various techniques, such as the extended rational sine-cosine approach [17], the extended F-expansion method [18], the extended sinh and sine-Gordon equation expansion method [19], the tanh-coth approach [20], the balance method [21], the exp-function approach [22], and so on, which have been extensively studied in the literature for solving various types of NLPDEs. These techniques have been successful in providing soliton solutions for a wide range of NLPDEs, including the famous Korteweg-de Vries equation, the nonlinear Schrödinger equation, and the Burgers equation. However, it is worth noting that each method has its limitations and may not apply to all equations. Therefore, researchers continue to explore new methods and improve existing ones to tackle the challenges posed by different types of dynamical processes.

The standard non-dimensional Korteweg-de Vries (KdV) equation [23] is of the form

$$v_t + 6vv_x + v_{xxx} = 0, \tag{1.1}$$

where x and t stand for non-dimensional space and time, and $v(x,t)$ denotes the water velocity. It was first presented by Korteweg-de Vries in 1895 as a unidirectional nonlinear wave equation.

The KdV equation's x-direction was the only restriction on the waves that were strictly one-dimensional, but Kadomtsev and Petviashvilly [24] relaxed this, and the KdV equation was expanded to the following $(2+1)$-dimensional Kadomtsev-Petviashvilly (KP) equation

$$(v_t + 6vv_x + v_{xxx})_x + v_{yy} = 0. \tag{1.2}$$

The KP equation is commonly employed to investigate weak nonlinear dispersive waves in plasma and weakly modulated long water waves [25].

However, Benjamin et al. [26] suggested an alternative to the KdV equation that incorporates nonlinear dispersion and describes the propagation of long, weakly nonlinear one-dimensional waves as follows:

$$v_t + v_x + a(u^2)_x + bv_{xxt} = 0. \tag{1.3}$$

This equation is called the Benjamin-Bona-Mahony (BBM) equation or the regularized long-wave. The KdV and BBM equations cover physical models in the various fields: acoustic waves in anharmonic crystals, acoustic-gravity waves in compressible fluids, solitons in dispersive media, and hydromagnetic waves in cold plasma. The KdV and BBM equations also encompass the modeling of long-wavelength surface waves in liquids, providing a comprehensive framework for understanding various phenomena in different media. These equations have proven to be valuable tools in studying and analyzing the behavior of these wave types, aiding in advancing our knowledge of fluid dynamics and plasma physics.

The motive of this study is to analyze a modified form of the BBM equation formulated in the KP sense by examining the $(2+1)$-dimensional Kadomtsev-Petviashvii-Benjamin-Bona-Mahony (KP-BBM) equation [27–34] of the form

$$v_{xt} + v_{xx} + 2a_1 v_x^2 + 2a_1 v v_{xx} + b_1 v_{xxxt} + k v_{yy} = 0, \tag{1.4}$$

where $v = v(x, y, t)$ represents the unknown function, x, y, and t are real independent variables, and a_1, b_1, and k are real constants. Combining these equations into the KP-BBM equation allows us to understand nonlinear wave phenomena in diverse fields such as water waves and plasma physics. Equation (1.4) was first reported by Wazwaz [27], who used the sine-cosine and extended tanh methods for finding soliton solutions to this equation. Tanwar et al. [28,29] derived some exact soliton solutions of Eq (1.4) by Lie symmetries. Using the exp function approach, Yu and Ma [30] derived the solitary and periodic solutions of the KP-BBM equation (1.4). Using the Hirota bilinear approach, Li et al. [31] investigated the lump and its interaction with two stripe soliton solutions. Abdou [32] also contributed to the study of Eq (1.4) by obtaining exact periodic wave solutions using the extended mapping method.

After thoroughly reviewing the literature, we have observed that there is no study on the $(2 + 1)$-dimensional KP-BBM equation (1.4) through the new Kudryashov (NK) and the generalized Arnous (GA) methods. Moreover, we have determined that there is no study on the stability of the equilibrium points and chaotic behavior corresponding to the dynamical system of the $(2 + 1)$-dimensional KP-BBM equation (1.4). To fill this gap, first, we provide its generalized solutions through the use of the new Kudryashov [35–37] and the generalized Arnous [38,39] methods. Then, corresponding to the dynamical system of the $(2 + 1)$-dimensional KP-BBM equation, we investigate its stability of equilibrium points and chaotic behavior. These findings demonstrate the diverse approaches used to study Eq (1.4) and highlight the importance of exploring different solution methods to understand its behavior comprehensively. Furthermore, the obtained soliton and periodic solutions contribute to the existing literature on this equation, providing valuable insights for further research in this field.

The outline of this article is as follows: Section 2 provides a concise overview of both the NK and GA methods. It presents a brief explanation of these methods and their applications in solving mathematical problems. Moving on to Section 3, the paper shows the mathematical analysis and solutions of the KP-BBM equation using the NK and GA methods. It elaborates on how these methods are utilized to obtain solutions for the equation and presents the results. Section 4 discusses the physical interpretation of graphs. Additionally, in Section 5, the bifurcation analysis is carried out. In Section 6, on adding the external disturbance term, the chaotic behaviour of the system is seen. Finally, in Section 7, the conclusion of the paper is given.

2. Methodology

The new Kudryashov method [35] is an analytical method that is a valuable tool for securing exact solutions to a wide range of nonlinear PDEs. The generalized Arnous method is a generalization of the Arnous technique that has been modified by Malik and Kumar [38]. Both methods are valuable tools for securing the exact solutions to a wide range of NLPDEs and are used by many researchers [35–41]. This section analyzes the NK and GA methods to determine the soliton solutions of NLPDEs.

Consider the NLPDE

$$\Psi\left(v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial x \partial t}, \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 v}{\partial y \partial t}, ...\right) = 0, \tag{2.1}$$

where Ψ represents a polynomial of $v(x, y, t)$ and its partial derivatives. We use the wave transformation $v(x, y, t)$ to $V(\Omega)$ as

$$v(x, y, t) = V(\Omega) = V(x - \theta y - \rho t), \quad \Omega = x - \theta y - \rho t, \tag{2.2}$$

where θ and ρ are arbitrary constants. Equation (2.1) is reduced to an ordinary differential equation (ODE) in the form

$$P(V, V', V'', V''', ...) = 0, \tag{2.3}$$

where $(')$ denotes the derivative w.r.t. Ω.

2.1. The NK method

An overview of the basic steps in the new Kudryashov method is given in this subsection.

Step 1: The NK method gives the solution of Eq (2.3) as

$$V(\Omega) = \sum_{j=0}^{N} A_j G^j(\Omega), \tag{2.4}$$

where the coefficients A_j $(j = 0, 1, 2, ..., N)$ are constants to be determined such that $A_N \neq 0$, and $G(\Omega) = \frac{1}{aA^{\alpha\Omega} + bA^{-\alpha\Omega}}$ is the solution of the following non-linear ODE:

$$\begin{aligned} G'(\Omega)^2 &= (\alpha \ln(A) G(\Omega))^2 (1 - 4abG^2(\Omega)), \\ G''(\Omega) &= \alpha^2 \ln(A)^2 G(\Omega)(1 - 8abG^2(\Omega)), \end{aligned} \tag{2.5}$$

where the constants a, b, α and A are non-zero with $A > 0$ and $A \neq 1$.

Step 2: By balancing the highest-order derivative and nonlinear terms that appear in ODE (2.3), the positive integer N can be obtained by applying the homogeneous balance principle. Particularly, we define the degree of $V(\Omega)$ as $D[V(\Omega)] = N$, and it yields the degree of the other expressions in the following way:

$$D\left[\frac{d^p V(\Omega)}{dV(\Omega)^p}\right] = N + p, \qquad D\left[V^q \left(\frac{d^p V(\Omega)}{dV(\Omega)^p}\right)^s\right] = qN + s(N + p). \tag{2.6}$$

As a result, we are able to determine N.

Step 3: After putting Eq (2.4) into Eq (2.3), and since $G(\Omega) \neq 0$, we equate all the coefficients of $G^j(\Omega)$ to zero. Then, after solving the non-linear algebraic system we derive the particular values for a, b and the A_j's. By substituting these values in Eq (2.4), and then with the help of transformation (2.2), we can obtain a solution for Eq (2.1).

2.2. The generalized Arnous method

The central proceedings of the generalized Arnous method are as follows:

Step 1: The generalized Arnous method gives the solution of Eq (2.3) as

$$V(\Omega) = \alpha_0 + \sum_{i=1}^{N} \frac{\alpha_i + \beta_i G'(\Omega)^i}{G(\Omega)^i}, \tag{2.7}$$

where $\alpha_0, \alpha_i, \beta_i$ $(i = 1, 2, ..., N)$ are constants and the function $G(\Omega)$ satisfy the relation

$$[G'(\Omega)]^2 = [G(\Omega)^2 - \sigma] \ln(A)^2, \tag{2.8}$$

with

$$G^{(n)}(\Omega) = \begin{cases} G(\Omega) \ln(A)^n, & \text{n is even,} \\ G'(\Omega) \ln(A)^{n-1}, & \text{n is odd,} \end{cases} \tag{2.9}$$

where $n \geq 2$, $0 < A \neq 1$.

Equation (2.8) has solutions of the form

$$G(\Omega) = \kappa \ln(A) A^{\Omega} + \frac{\sigma}{4\kappa \ln(A) A^{\Omega}}, \tag{2.10}$$

where κ and σ are arbitrary parameters.

Step 2: By balancing the non-linear term and highest order derivative term in Eq (2.3), the positive integer N is determined of for (2.7).

Step 3: After putting Eqs (2.7)–(2.9) in Eq (2.3) and since $G(\Omega) \neq 0$, as a result of this substitution we receive a polynomial of $\frac{1}{G(\Omega)} \left(\frac{G'(\Omega)}{G(\Omega)} \right)$. Now, gather all terms of the same power and equate them to zero. Then, after solving the non-linear algebraic system for θ, ρ, κ, σ, α_0, α_i and β_i $(i = 1, 2, ..., N)$, and with the help of (2.8) and (2.2), the solutions of (2.1) can be derived.

3. Mathematical analysis

In this section, the NK method and the GA method are applied for securing the soliton solutions of the KP-BBM equation (1.4).

Assume that Eq (1.4) has a traveling wave solution of the form

$$v(x, y, t) = V(\Omega), \qquad \Omega = x - \theta y - \rho t. \tag{3.1}$$

Inserting Eq (3.1) in Eq (1.4), we get the non-linear ODE

$$(1 - \rho + k\theta^2)V'' + 2a_1(VV')' - b_1\rho V'''' = 0. \tag{3.2}$$

Integrating Eq (3.2) twice w.r.t. Ω gives

$$(1 - \rho + k\theta^2)V + a_1 V^2 - b_1\rho V'' = 0, \tag{3.3}$$

with the integral constant being treated as zero. Now, balancing the largest degree of the nonlinear term V^2 with highest order derivative term V'', we get $N = 2$.

3.1. Analysis of solutions via the NK method

In this section, the KP-BBM equation is solved using the NK method. The NK method suggests the solution of Eq (3.3) is of the form

$$V(\Omega) = \sum_{j=0}^{2} A_j G^j(\Omega) = A_0 + A_1 G(\Omega) + A_2 G^2(\Omega), \tag{3.4}$$

where A_0, A_1 and A_2 are arbitrary constants with $A_2 \neq 0$. Substituting Eq (3.4) into Eq (3.3) and equating the coefficients of $G^j(\Omega)$ to zero, give

$$\begin{aligned}
0 &= (1 - \rho + k\theta^2)A_0 + A_0^2 a_1,\\
0 &= (1 - \rho + k\theta^2)A_1 + 2A_0A_1a_1 - b_1 \ln(A)^2\alpha^2\rho A_1,\\
0 &= (1 - \rho + k\theta^2)A_2 + (2A_0A_2 + A_1^2)a_1 - 4b_1 \ln(A)^2\alpha^2\rho A_2,\\
0 &= 8b_1 \ln(A)^2\alpha^2\rho ab A_1 + 2A_1A_2a_1,\\
0 &= 24b_1 \ln(A)^2\alpha^2\rho ab A_2 + A_2^2 a_1.
\end{aligned} \tag{3.5}$$

Solving system (3.5), yields the following sets of solutions:

Set 1:

$$A_0 = -\frac{4b_1 \ln(A)^2\alpha^2(1 + k\theta^2)}{a_1(4b_1 \ln(A)^2\alpha^2 - 1)},\ A_1 = 0,\ A_2 = \frac{24abb_1 \ln(A)^2\alpha^2(1 + k\theta^2)}{a_1(4b_1 \ln(A)^2\alpha^2 - 1)},\ \rho = -\frac{1 + k\theta^2}{4b_1 \ln(A)^2\alpha^2 - 1}. \tag{3.6}$$

By substituting Eq (3.6) into Eq (3.4), we get the solution of Eq (3.3) as

$$V_1(\Omega) = -\frac{4b_1 \ln(A)^2\alpha^2(1 + k\theta^2)}{a_1(4b_1 \ln(A)^2\alpha^2 - 1)}\left[1 - 6ab\left(\frac{1}{aA^{\alpha\Omega} + bA^{-\alpha\Omega}}\right)^2\right]. \tag{3.7}$$

From Eq (3.1) and Eq (3.7), the combo bright-singular soliton solution of Eq (1.4) can be written as

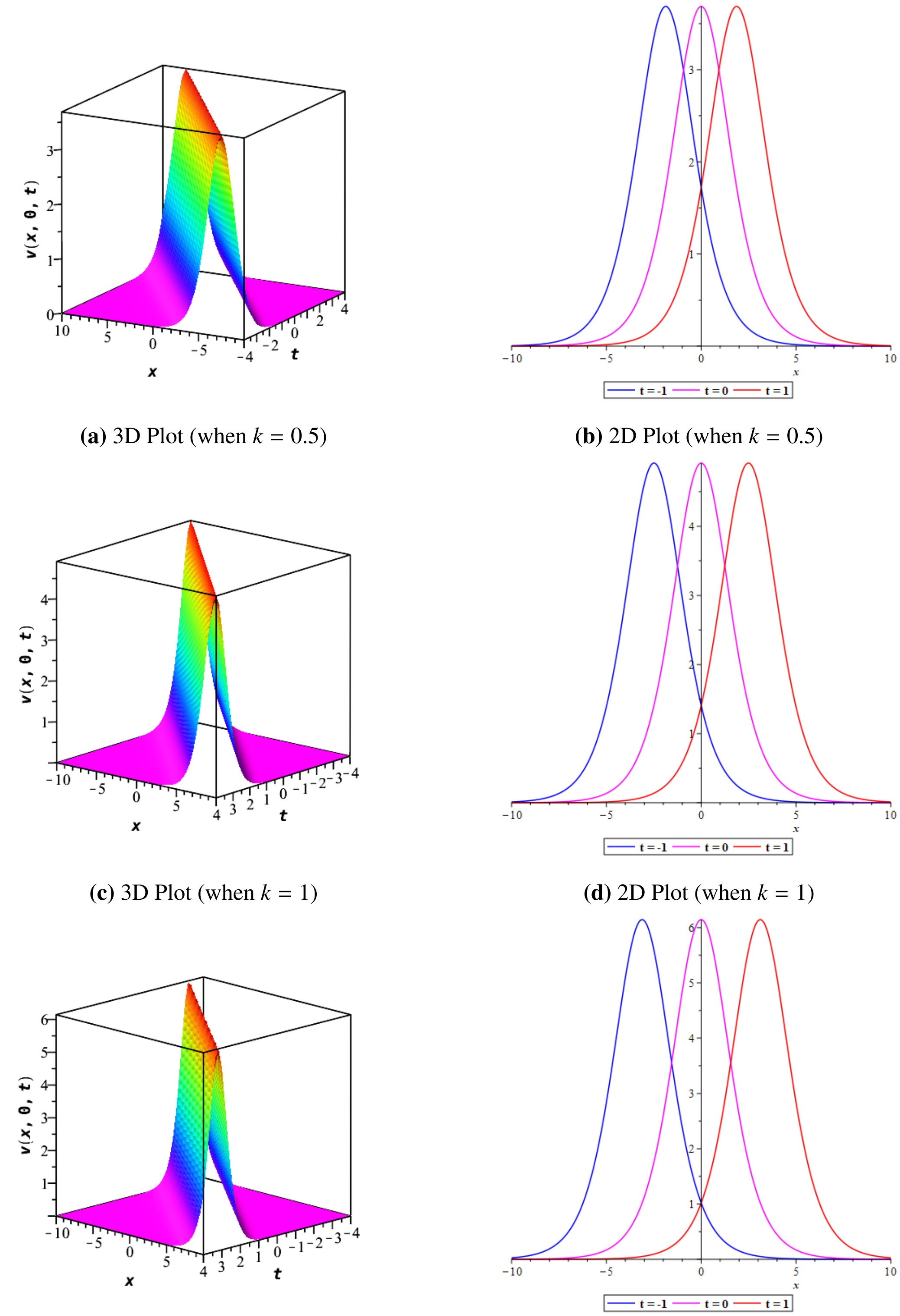
$$v_1(x, y, t) = -\frac{4b_1 \ln(A)^2\alpha^2(1 + k\theta^2)}{a_1(4b_1 \ln(A)^2\alpha^2 - 1)}\left[1 - 6ab\left(\frac{1}{(a + b)\cosh(\ln(A)\,\alpha\Omega) + (a - b)\sinh(\ln(A)\,\alpha\Omega)}\right)^2\right], \tag{3.8}$$

where Ω is given by $\Omega = x - \theta y + \frac{1+k\theta^2}{4b_1(\ln(A))^2\alpha^2-1}t$. In particular, upon choosing $a = \pm b$ and $A = e$, one arrives at bright and singular soliton solutions respectively, as

$$v_1^{(1)}(x, y, t) = -\frac{4b_1\alpha^2(1 + k\theta^2)}{a_1(4b_1\alpha^2 - 1)}\left[1 - \frac{3}{2}\left(\operatorname{sech}\left(\alpha\left(x - \theta y + \frac{1 + k\theta^2}{4b_1\alpha^2 - 1}t\right)\right)\right)^2\right], \tag{3.9}$$

and

$$v_1^{(2)}(x, y, t) = -\frac{4b_1\alpha^2(1 + k\theta^2)}{a_1(4b_1\alpha^2 - 1)}\left[1 + \frac{3}{2}\left(\operatorname{csch}\left(\alpha\left(x - \theta y + \frac{1 + k\theta^2}{4b_1\alpha^2 - 1}t\right)\right)\right)^2\right]. \tag{3.10}$$



(a) 3D Plot (when $k = 0.5$)

(b) 2D Plot (when $k = 0.5$)

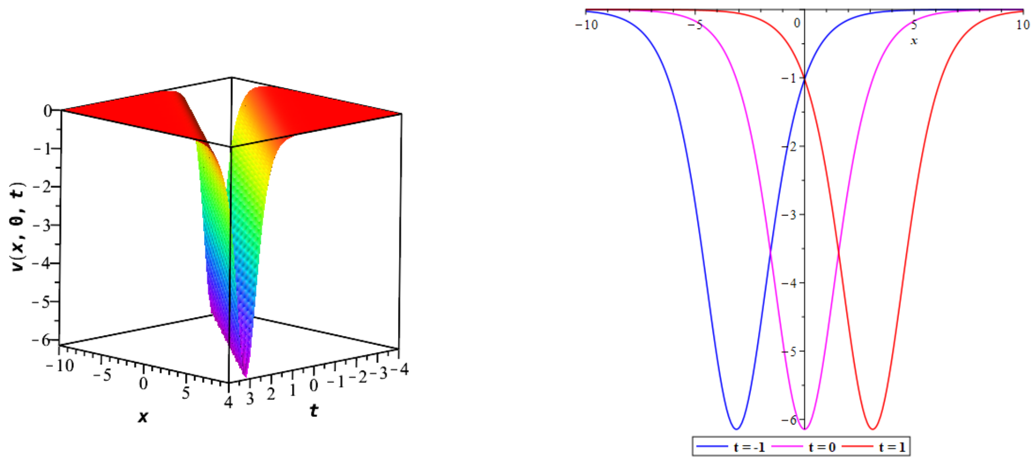
(c) 3D Plot (when $k = 1$)

(d) 2D Plot (when $k = 1$)

(e) 3D Plot (when $k = 1.5$)

(f) 2D Plot (when $k = 1.5$)

Figure 1. 3D and 2D graphical structure of bright soliton solution $v_1(x, y, t)$ (3.13) of the KP-BBM equation (1.4) with $-10 \leq x \leq 10$, $-5 \leq t \leq 5$ for $a = 2$, $b = 2$, $a_1 = 0.15$, $b_1 = -0.2$, $\theta = 1$, $\alpha = 0.5$, $y = 0$ and $A = 2.7$.



(a) 3D Plot **(b)** 2D Plot

Figure 2. 3D and 2D graphical structure of dark soliton solution $v_1(x,y,t)$ (3.13) of the KP-BBM equation (1.4) with $-10 \le x \le 10$, $-5 \le t \le 5$ for $a = 2$, $b = 2$, $a_1 = -0.15$, $b_1 = -0.2$, $k = 1.5$, $\theta = 1$, $\alpha = 0.5$, $y = 0$ and $A = 2.7$.

Set 2:

$$A_0 = A_1 = 0, \quad A_2 = -\frac{24abb_1 \ln(A)^2\alpha^2(1+k\theta^2)}{a_1(4b_1\ln(A)^2\alpha^2+1)}, \quad \rho = \frac{1+k\theta^2}{4b_1\ln(A)^2\alpha^2+1}. \tag{3.11}$$

By substituting Eq (3.11) into Eq (3.4), we get the solution of Eq (3.3) as

$$V_2(\Omega) = -\frac{24abb_1 \ln(A)^2\alpha^2(1+k\theta^2)}{a_1(4b_1\ln(A)^2\alpha^2+1)}\left(\frac{1}{aA^{\alpha\Omega}+bA^{-\alpha\Omega}}\right)^2. \tag{3.12}$$

From Eq (3.1) and Eq (3.12), the bright-singular soliton solution of Eq (1.4) can be written as

$$v_2(x,y,t) = -\frac{24abb_1 \ln(A)^2\alpha^2(1+k\theta^2)}{a_1(4b_1\ln(A)^2\alpha^2+1)}\left(\frac{1}{(a+b)\cosh(\ln(A)\,\alpha\Omega)+(a-b)\sinh(\ln(A)\,\alpha\Omega)}\right)^2, \tag{3.13}$$

where Ω is given by $\Omega = x - \theta y - \frac{1+k\theta^2}{4b_1(\ln(A))^2\alpha^2+1}t$. Graphically the representation of solution (3.20) has been given in Figures 1 and 2. In particular, upon choosing $a = \pm b$ and $A = \mathrm{e}$, one arrives at bright and singular soliton solutions, respectively, as

$$v_2^{(1)}(x,y,t) = -\frac{6b_1\alpha^2(1+k\theta^2)}{a_1(4b_1\alpha^2+1)}\left(\operatorname{sech}\left(\alpha\left(x-\theta y-\frac{1+k\theta^2}{4b_1\alpha^2+1}t\right)\right)\right)^2, \tag{3.14}$$

and

$$v_2^{(2)}(x,y,t) = \frac{6b_1\alpha^2(1+k\theta^2)}{a_1(4b_1\alpha^2+1)}\left(\operatorname{csch}\left(\alpha\left(x-\theta y-\frac{1+k\theta^2}{4b_1\alpha^2+1}t\right)\right)\right)^2. \tag{3.15}$$

3.2. Analysis of solutions via the GA method

The GA method is used in this section to establish the solitary wave solutions to the KP-BBM equation (1.4). In the GA method, the assumed solution can be written as

$$V(\Omega) = \alpha_0 + \sum_{i=1}^{N}\frac{\alpha_i + \beta_i G'(\Omega)^i}{G(\Omega)^i}.$$

We have $N = 2$, and, therefore, GA method suggest solution of Eq (3.3) in the form

$$V(\Omega) = \alpha_0 + \frac{\alpha_1 + \beta_1 G'(\Omega)}{G(\Omega)} + \frac{\alpha_2 + \beta_2 G'(\Omega)^2}{G(\Omega)^2}. \tag{3.16}$$

By substituting Eq (3.16) into Eq (3.3) along with Eqs (2.8) and (2.9), we receive a polynomial in term of $\frac{1}{G(\Omega)}\left(\frac{G'(\Omega)}{G(\Omega)}\right)$. Now, we get a system of algebraic equations, collecting all terms of the same power and equating them to zero, as shown below:

$$\begin{aligned}
0 &= \beta_1\left(2\beta_2 \ln(A)^2 a_1 + k\theta^2 + 2a_1\alpha_0 - \rho + 1\right),\\
0 &= 2\beta_1\alpha_1 a_1,\\
0 &= -2\beta_1(\sigma(a_1\beta_2 - b_1\rho)\ln(A)^2 + \alpha_2 a_1),\\
0 &= (\ln(A)^4 a_1\beta_2^2 + ((\beta_1^2 + 2\beta_2\alpha_0)a_1 + (k\theta^2 - \rho + 1)\ln(A)^2 + \alpha_0(k\theta^2 + a_1\alpha_0 - \rho + 1),\\
0 &= \left((2a_1\beta_2 - b_1\rho)\ln(A)^2 + k\theta^2 + 2a_1\alpha_0 - \rho + 1\right)\alpha_1,\\
0 &= -2\alpha_1(\sigma(a_1\beta_2 - b_1\rho)\ln(A)^2 - \alpha_2 a_1),\\
0 &= (\beta_2\ln(A)^2\sigma - \alpha_2)(\sigma(a_1\beta_2 - 6b_1\rho)\ln(A)^2 - \alpha_2 a_1),\\
0 &= (-2\beta_2\sigma(a_1\beta_2 - 2b_1\rho)\ln(A)^4 + (((-2\sigma\alpha_0 + 2\alpha_2)\beta_2 - \sigma\beta_1^2)a_1 - \sigma(k\theta^2 - \rho + 1)\beta_2\\
&\quad - 4\alpha_2\rho b_1)\ln(A)^2 + (2\alpha_0\alpha_2 + \alpha_1^2)a_1 + \alpha_2(k\theta^2 - \rho + 1)).
\end{aligned} \tag{3.17}$$

Solving system (3.17) with the help of the Maple software, we obtain the sets of solution as follows:

Set 1:

$$\begin{aligned}
\alpha_0 &= -\ln(A)^2\beta_2, \quad \alpha_1 = 0, \quad \beta_1 = 0,\\
\alpha_2 &= \frac{\sigma\ln(A)^2(4a_1b_1\beta_2\ln(A)^2 - 6b_1k\theta^2 + a_1\beta_2 - 6b_1)}{(4b_1\ln(A)^2 + 1)a_1},\\
\rho &= \frac{1 + k\theta^2}{4b_1\ln(A)^2 + 1}.
\end{aligned} \tag{3.18}$$

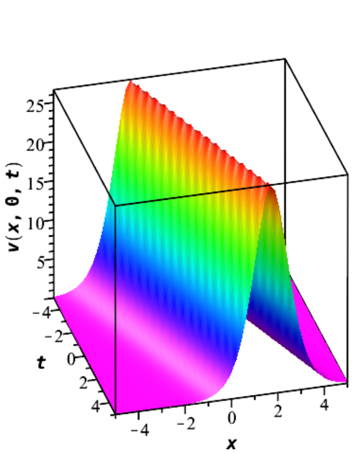
By substituting Eq (3.18) into Eq (3.16) and simplifying, we get the solution of Eq (3.3) as

$$V_1(\Omega) = -\frac{96\ln(A)^4\kappa^2 A^{2\Omega} b_1\sigma\left(k\theta^2 + 1\right)}{\left(4\kappa^2\ln(A)^2 A^{2\Omega} + \sigma\right)^2\left(4b_1\ln(A)^2 + 1\right)a_1}. \tag{3.19}$$

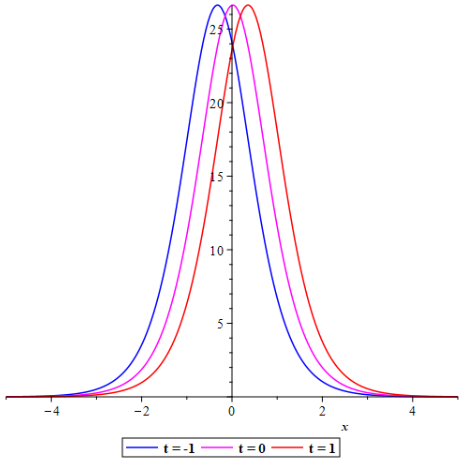
Therefore, from Eqs (3.1) and (3.19), the soliton solution of Eq (1.4) can be written as

$$v_1(x, y, t) = -\frac{96\ln(A)^4\kappa^2 A^{2\Omega} b_1\sigma\left(k\theta^2 + 1\right)}{\left(4\kappa^2\ln(A)^2 A^{2\Omega} + \sigma\right)^2\left(4b_1\ln(A)^2 + 1\right)a_1}, \tag{3.20}$$

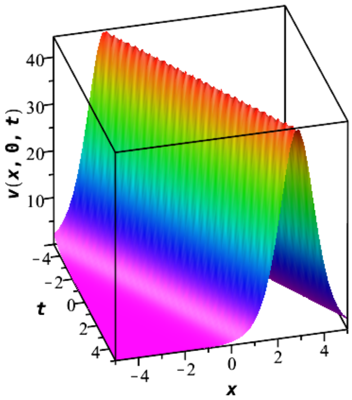
where Ω is given by $\Omega = x - \theta y - \frac{1+k\theta^2}{4b_1\ln(A)^2+1}t$. Solution (3.20) is graphically represented in Figures 3 and 4.



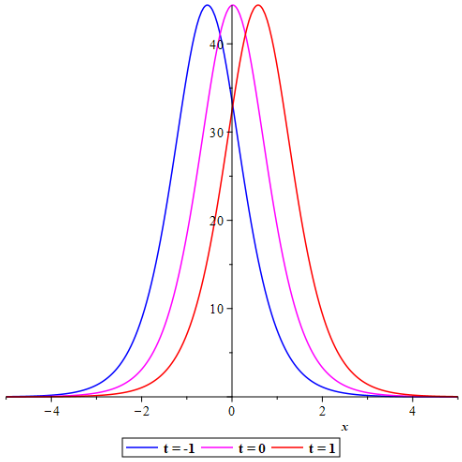
(a) 3D Plot (when $k = 0.5$)



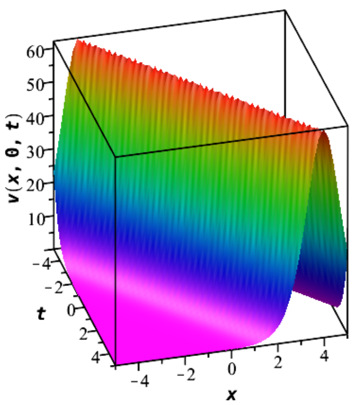
(b) 2D Plot (when $k = 0.5$)



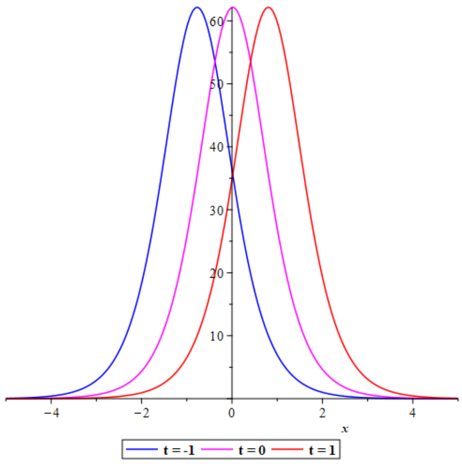
(c) 3D Plot (when $k = 1$)



(d) 2D Plot (when $k = 1$)

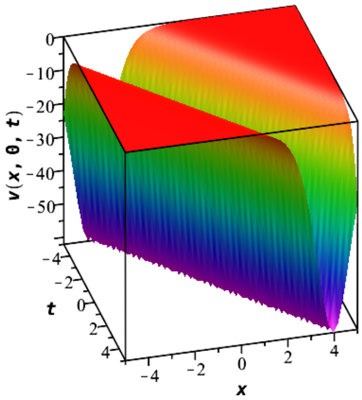


(e) 3D Plot (when $k = 1.5$)

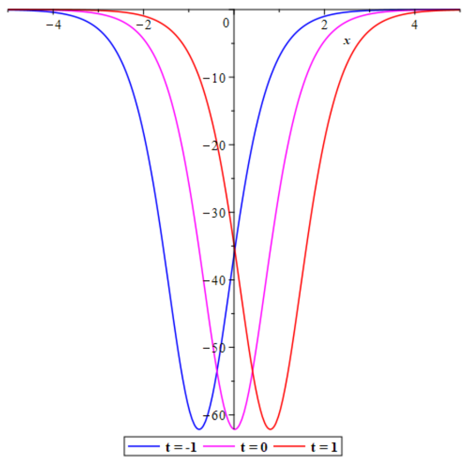


(f) 2D Plot (when $k = 1.5$)

Figure 3. 3D and 2D graphical structure of bright soliton solution $v_1(x, y, t)$ (3.20) of the KP-BBM equation (1.4) with $-5 \le x, t \le 5$ for $a_1 = 0.15$, $b_1 = 2$, $\theta = 2$, $\sigma = 0.5$, $\kappa = 0.35$, $y = 0$ and $A = 2.7$.



(**a**) 3D Plot



(**b**) 2D Plot

Figure 4. 3D and 2D graphical structure of dark soliton solution $v_1(x, y, t)$ (3.20) of the KP-BBM equation (1.4) with $-5 \leq x, t \leq 5$ for $a_1 = -0.15$, $b_1 = 2$, $\theta = 2$, $\sigma = 0.5$, $\kappa = 0.35$, $k = 1.5$, $y = 0$ and $A = 2.7$.

Set 2:

$$\begin{aligned}
\alpha_0 &= -\frac{\ln(A)^2(4a_1b_1\beta_2\ln(A)^2 + 4b_1k\theta^2 - a_1\beta_2 + 4b_1)}{(4b_1\ln(A)^2 - 1)a_1}, \\
\alpha_1 &= 0, \quad \beta_1 = 0, \\
\alpha_2 &= \frac{\sigma\ln(A)^2(4a_1b_1\beta_2\ln(A)^2 + 6b_1k\theta^2 - a_1\beta_2 + 6b_1)}{(4b_1\ln(A)^2 - 1)a_1}, \\
\rho &= -\frac{1 + k\theta^2}{4b_1\ln(A)^2 - 1}.
\end{aligned} \tag{3.21}$$

By substituting Eq (3.21) into Eq (3.16), we get the solution of Eq (3.3) as

$$V_2(\Omega) = \frac{4\left(k\theta^2 + 1\right)\left(-16A^{4\Omega}\ln(A)^4\kappa^4 + 16A^{2\Omega}\ln(A)^2\kappa^2\sigma - \sigma^2\right)\ln(A)^2 b_1}{\left(4\kappa^2\ln(A)^2A^{2\Omega} + \sigma\right)^2\left(4b_1\ln(A)^2 - 1\right)a_1}. \tag{3.22}$$

Therefore, from Eqs (3.1) and (3.22), the soliton solution of Eq (1.4) can be written as

$$v_2(x, y, t) = \frac{4\left(k\theta^2 + 1\right)\left(-16A^{4\Omega}\ln(A)^4\kappa^4 + 16A^{2\Omega}\ln(A)^2\kappa^2\sigma - \sigma^2\right)\ln(A)^2 b_1}{\left(4\kappa^2\ln(A)^2A^{2\Omega} + \sigma\right)^2\left(4b_1\ln(A)^2 - 1\right)a_1}, \tag{3.23}$$

where Ω is given by $\Omega = x - \theta y - \frac{1+k\theta^2}{4b_1\ln(A)^2 - 1}t$.

4. Physical interpretation

Graphical interpretation of some of the obtained solutions of the KP-BBM equation (1.4) will be discussed in this section. Through the use of the new Kudryashov and generalized Arnous methods, we mainly derived the generalized soliton solutions in the form of combo bright-singular soliton solutions of KP-BBM equation (1.4). By giving particular values to the parameters, bright, dark, and singular solutions can be achieved from these solutions. It is crucial to note that the outcomes from this article or extracted solutions are new and have not been reported before.

Figures 1 and 2 depict the dynamic behavior of the solution given by (3.13), where the parameters are specified as $a = 2$, $b = 2$, $a_1 = \{0.15, -0.15\}$, $k = \{0.5, 1, 1.5\}$, $b_1 = -0.2$, $\theta = 1$, $\alpha = 0.5$, $y = 0$, and $A = 2.7$. In Figure 1, it is noticeable that, as the parameter k increases, the soliton's amplitude increases, resulting in a sharper profile. This suggests that we can regulate the soliton's speed by adjusting the value of k, offering control over its characteristics. The combined insights from Figures 1 and 2 shed light on the influence of a_1 on solution (3.13). As a_1 transitions from a positive to a negative value, the solution's profile transforms from a bright soliton to a dark soliton.

Figures 3 and 4 portray the dynamic evolution of the solution governed by (3.20), with specified parameters: $a_1 = \{0.15, -0.15\}$, $k = \{0.5, 1, 1.5\}$, $b_1 = 2$, $\theta = 2$, $\sigma = 0.5$, $\kappa = 0.35$, $y = 0$, and $A = 2.7$. In Figure 1, a discernible trend emerges as the parameter k increases: the soliton's amplitude also increases, leading to a sharper profile. This observation implies that the soliton's speed can be finely controlled by adjusting the value of k, providing a means to tailor its characteristics. The combined insights gleaned from Figures 1 and 2 illuminate the impact of a_1 on the solution described by (3.20). As a_1 transitions from a positive to a negative value, the profile of the solution undergoes a transformation, shifting from a bright soliton to a dark soliton.

5. Stability of the equilibrium points

Bifurcation analysis helps to identify critical points in parameter space where the behavior of solutions undergoes qualitative changes. Understanding these points is crucial for predicting the stability and characteristics of solutions to the original PDE. Here, we perform the bifurcation analysis on Eq (1.4) by converting it into the corresponding planar dynamical system.

In Eq (3.3), by taking $V' = U$, Eq (1.4) is converted into the following planar dynamical system:

$$\begin{aligned} V' &= U, \\ U' &= V(a_2 + b_2 V), \end{aligned} \tag{5.1}$$

where $a_2 = \frac{1-\rho+k\theta^2}{b_1\rho}$ and $b_2 = \frac{a_1}{b_1\rho}$.

For the above autonomous system, the trace of the Jacobian matrix is zero. In this case, according to the bifurcation theory of planar nonlinear dynamical systems [42, 43], the equilibrium point (v, u) is a center point if $J(v, u) > 0$, and a saddle point, if $J(v, u) < 0$. $J(v, u)$ represents the Jacobian determinant of the coefficient matrix of the system. The planar dynamical system (5.1) has two equilibrium points, $V_1 = (0, 0)$ and $V_2 = (-\frac{a_2}{b_2}, 0)$.

Case 1. When $a_2 b_2 > 0$.

(i) For $a_2 > 0$, $b_2 > 0$, the Jacobian determinant, $J(V_1) < 0$ and $J(V_2) > 0$, and hence V_1 is a saddle point, whereas V_2 is a center point. See Figure 5(a).

(ii) For $a_2 < 0$, $b_2 < 0$, we have $J(V_1) > 0$ and $J(V_2) < 0$, and hence V_1 is a center point, while V_2 is a saddle point. See Figure 5(b).

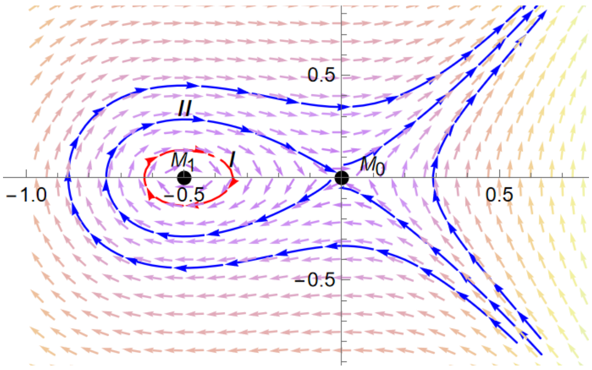
The trajactory *I* in Figure 5(a) indicates that the periodic solution exists while the trajectory *II* indicates the existence of a soliton solution [42]. Similar conclusions are shown by Figure 5(b).

Case 2. When $a_2b_2 < 0$.

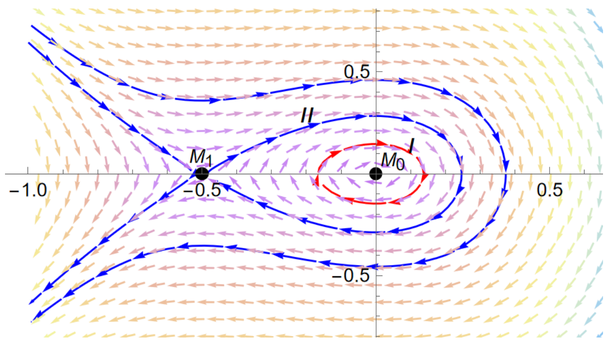
(i) For $a_2 < 0$, $b_2 > 0$, we have $J(V_1) > 0$ and $J(V_2) < 0$, and hence V_1 is a center point, while V_2 is a saddle point. See Figure 6(a).

(ii) For $a_2 > 0$, $b_2 < 0$, we have $J(V_1) < 0$ and $J(V_2) > 0$, and hence V_1 is a saddle point, while V_2 is a center point. See Figure 6(b).

The trajectories I and II in Figure 6 shows similar conclusions as in Case 1.

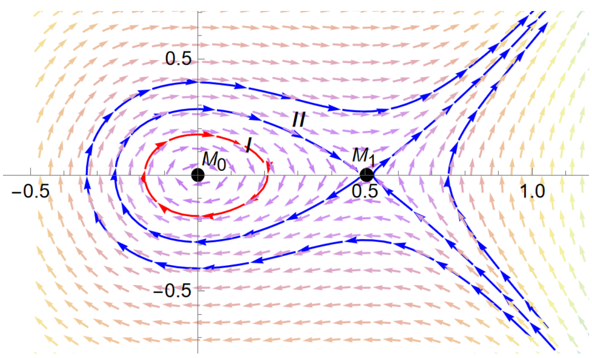


(a) When $a_2 > 0$ and $b_2 > 0$

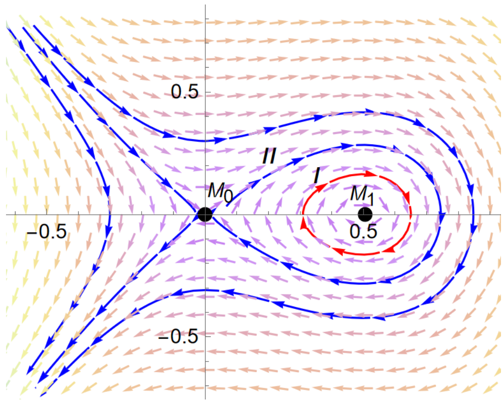


(b) When $a_2 < 0$ and $b_2 < 0$

Figure 5. Phase portrait for system (5.1) when $a_2b_2 > 0$.



(a) When $a_2 < 0$ and $b_2 > 0$



(b) When $a_2 > 0$ and $b_2 < 0$

Figure 6. Phase portrait for system (5.1) when $a_2b_2 < 0$.

6. Chaotic behavior

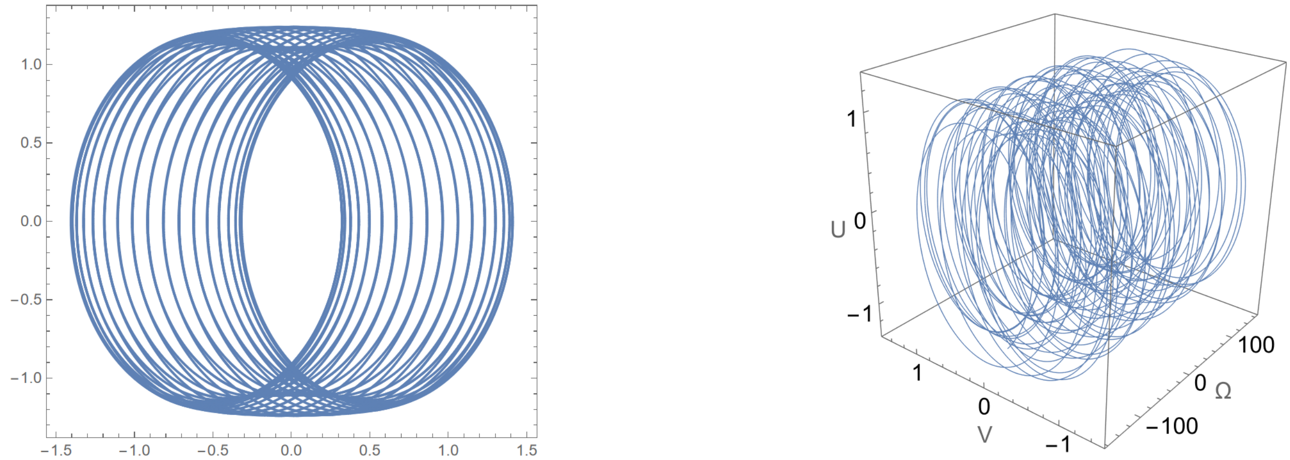
Chaotic dynamics may emerge as a result of bifurcations, leading to complex and seemingly random behavior in the solutions of the PDE. In the previous section, we see that the planar dynamical system (5.1) has no chaotic phenomenon, however we found that upon addition of an external perturbed term, the chaotic behavior does exist. The corresponding perturbation form of the planar dynamical system (5.1) is as follows:

$$\begin{aligned} V' &= U, \\ U' &= V(a_2 + b_2 V) + P(\Omega), \end{aligned} \tag{6.1}$$

where $P(\Omega)$ is perturbed term.

For analyzing the chaotic behavior, we consider some special perturbation terms as follows:

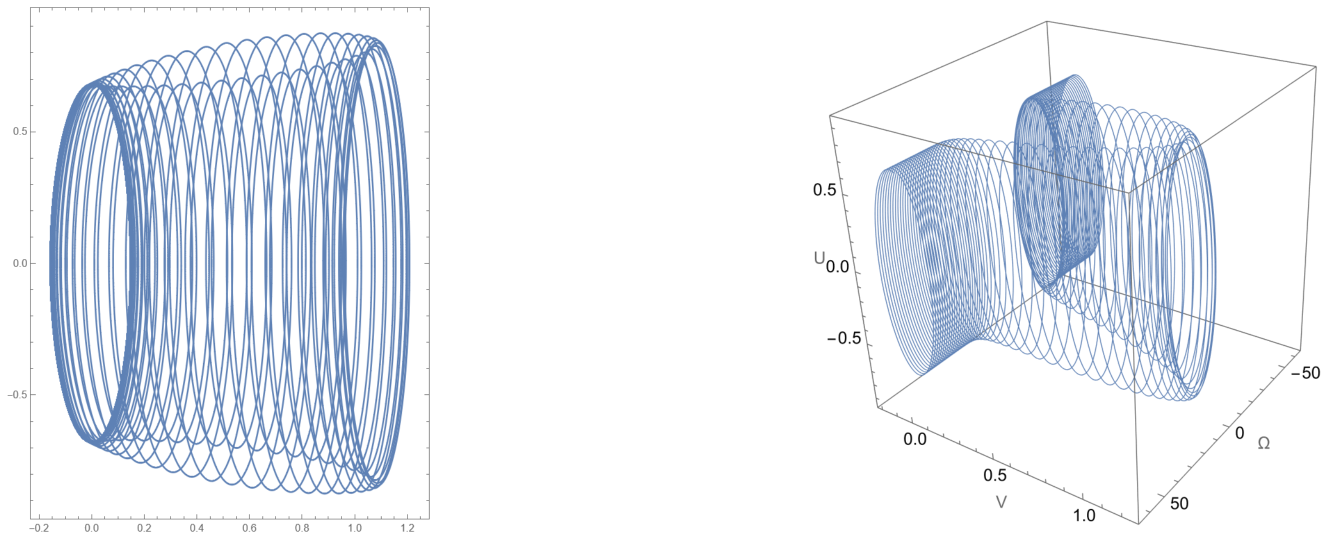
(i) Trigonometric function: Taking $P(\Omega) = \cos(0.12\Omega)$, we get the phase portraits as in Figure 7.



(a) **(b)**

Figure 7. (a) Phase portrait for system (6.1) in 2 dimension, when $a_2 = -1.85$, $b_2 = 0.01$ and $P(\Omega) = \cos(0.12\Omega)$; (b) phase portrait with perturbation term.

(ii) Gaussian function: Taking $P(\Omega) = \frac{90}{\sqrt{2\pi}} e^{-\frac{(0.12\Omega)^2}{2}}$, the phase portraits as in Figure 8 are obtained.



(a) **(b)**

Figure 8. (a) phase portrait for system (6.1) in 2 dimension, when $a_2 = -20$, $b_2 = -12$ and $P(\Omega) = \frac{90}{\sqrt{2\pi}} e^{-\frac{(0.12\Omega)^2}{2}}$; (b) phase portrait with perturbation term.

(iii) Hyperbolic function: Taking $P(\Omega) = \cosh(0.023\Omega)$, the phase portraits appears as in Figure 9.

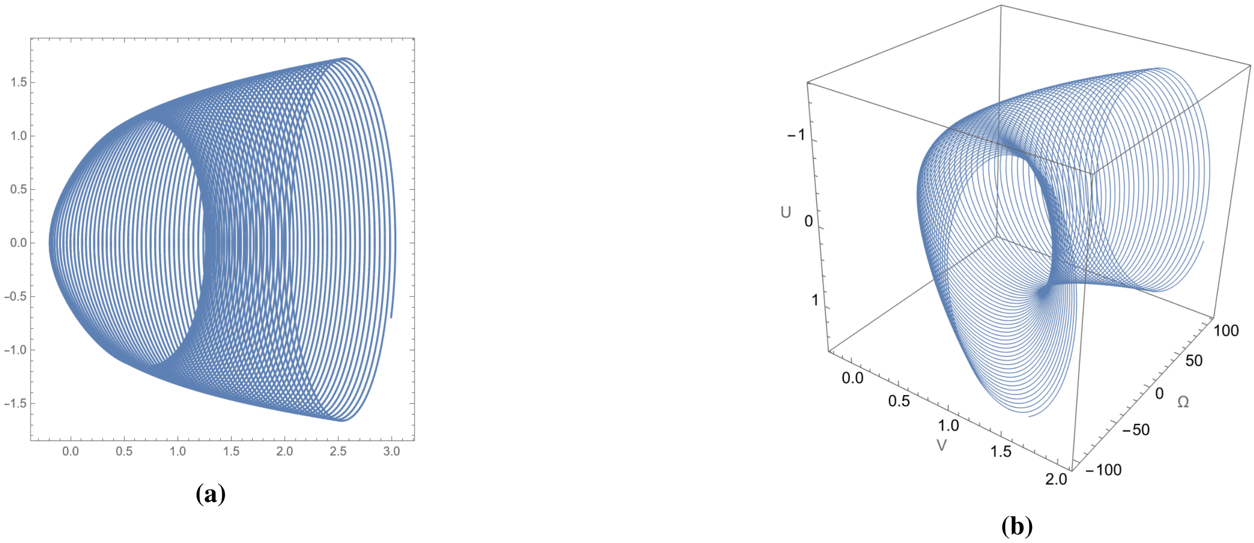


Figure 9. (a) phase portrait for system (6.1) in 2 dimension, when $a_2 = 0.3$, $b_2 = -2.5$ and $P(\Omega) = \cosh(0.023\Omega)$; (b) phase portrait with perturbation term.

From the above three cases we can conclude that the perturbed system (6.1) has different chaotic behaviors for different perturbed terms, and the phase portraits obtained are different.

7. Conclusions

In conclusion, the KP-BBM equation has been studied successfully by using the new Kudrashov and generalized Arnous method. Our study showed a range of traveling wave responses, and established the necessary conditions for the existence of these solutions, considering relevant physical factors. The approach we have proposed has generated a collection of solutions that could be useful in understanding certain physical events. A variety of solitons and other solutions such as the bright soliton solutions and dark soliton solutions were presented. The restrictions on the parameters were considered to guarantee the existence of the obtained solitons. Moreover, the obtained solutions were illustrated graphically to discuss their physical nature. Further, we have carried out bifurcation analysis, and the results of our qualitative analysis show that periodic and solitons solutions exist. The system also behaves chaotically when external disturbance terms are taken into consideration.

These findings contribute to a deeper understanding of the dynamics of the KP-BBM wave equation and its applications in real-world phenomena. The presentation of 3D and 2D graphical structures of the obtained solitary wave solutions enhances the understanding of the mathematical model. Additionally, the chaotic behavior observed in the presence of external perturbations highlights the sensitivity of the system to small changes, emphasizing the need for careful consideration when studying its behavior in practical scenarios. In the future, these methodologies can be applied to the generalized form of the KP-BBM equation, the KP-BBM equation with competing dispersion effects, and a wider range of physical phenomena. This will further enhance our ability to predict and analyze complex systems governed by nonlinear partial differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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