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Research article

On the number of integers which form perfect powers in the way of $x(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z^k$

Tingting Wen*

School of Mathematics, Shandong University, Jinan 250100, China

* **Correspondence:** Email: ttwen@mail.sdu.edu.cn.

Abstract: Let $k \geqslant 2$ be an integer. We studied the number of integers which form perfect k-th powers in the way of

$$x(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z^k.$$

For $k \geqslant 4$, we established a unified asymptotic formula with a power-saving error term for the number of such integers of bounded size under Lindelöf hypothesis, and we also gave an unconditional result for $k = 2$.

Keywords: counting solutions of Diophantine equations; perfect powers; double Dirichlet series
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1. Introduction

Counting the number of integral solutions of a certain Diophantine equation is an interesting project in number theory. Let $N_k(H)$ be the number of pairs of positive integers x_1, $x_2 \leqslant H$ whose product x_1x_2 is a perfect k-th power. Tolev [1] first established an asymptotic formula for $N_k(H)$. The proof combines Perron's formula with elementary ideas from the work of Heath-Brown and Moroz [2]. It is proved that for any integer $k \geqslant 2$,

$$N_k(H) = c_k H^{2/k}(\log H)^{k-1} + O\left(H^{2/k}(\log H)^{k-2}\right),$$

where $c_k > 0$ is an explicit constant depending on k. De la Bretèche et al. [3] improved this result based on the multiple Dirichlet series theory and complex analysis. They showed that there exists a constant $\theta_k > 0$ such that

$$N_k(H) = H^{2/k} Q(\log H) + O\left(H^{2/k-\theta_k+\varepsilon}\right),$$

where Q is a polynomial of degree $k - 1$ with leading coefficient c_k. In fact, they considered a more general case and counted the number of tuples of $n \geqslant 2$ integers whose product is a perfect k-th power.

Precisely, they proved that there exists a constant $\theta_{n,k} > 0$, such that

$$\begin{aligned} N_{n,k} &= \left|\{(x_1, \cdots, x_n) \in [1, H]^n \cap \mathbb{N}^n : x_1, \cdots, x_n = z^k, z \in \mathbb{N}\}\right| \\ &= H^{n/k} Q_{n,k}(\log H) + O\left(H^{n/k-\theta_{n,k}}\right), \end{aligned}$$

where $Q_{n,k}$ is a polynomial of degree $\binom{n+k-1}{k} - n$.

It is natural to consider the analogue of the above results for polynomials. Liu and Niu [4] counted pairs of polynomials whose product is a cube over finite field and obtained an asymptotic formula (see [4, Theorem 1.2]) by contour integration. It is indicated that there are some differences and challenges between polynomials and integers.

Returning to the case of integers, this problem is closely related to the integral points on algebraic surface

$$S : x_1 x_2 x_3 = x_0^3.$$

This is a split toric surface and can also be regarded as the product of three integers forming a perfect cube. It has been studied by many authors including de la Bretèche [5], Fouvry [6], Heath-Brown and Moroz [2], and Salberger [7]. Denote by $N_S(H)$ the number of primitive integral points (i.e., $\gcd(x_0, x_1, x_2, x_3) = 1$) on S satisfying $x_0 \neq 0$ and $\max_{0 \leqslant i \leqslant 3} |x_i| \leqslant H$. The sharpest unconditional result is proved by de la Bretèche [5], which states

$$N_S(H) = HP(\log H) + O\left(H^{7/8} \exp(-c(\log H)^{3/5})(\log\log H)^{-1/5})\right),$$

where P is a polynomial of degree 6 and c is a positive constant. Now, we look at the corresponding non-split toric surface

$$S' : x(y_1^2 + y_2^2) = z^3.$$

It can be observed that S' and S are isomorphic over $\mathbb{Q}(\mathrm{i})$. Denote by $N_{S'}(H)$ the number of primitive integral points on S' satisfying $z \neq 0$ and

$$\max\left\{|x|, \sqrt{y_1^2 + y_2^2}, |z|\right\} \leqslant H.$$

De la Bretèche et al. [8] studied this surface and proved that

$$N_{S'}(H) = HP'(\log H) + O\left(H^{8/9+\varepsilon}\right),$$

where P' is a cubic polynomial. Liu et al. [9] further considered the following case

$$S_4 : x(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z^3. \tag{1.1}$$

For ease of presentation, we let

$$\boldsymbol{y} = (y_1, y_2, y_3, y_4), \quad \|\boldsymbol{y}\| = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2},$$

and denote by $N_4(H)$ the number of integral tuples $(x, y_1, \cdots, y_4)$, satisfying $\max\{|x|, \|\boldsymbol{y}\|\} \leqslant H$, which form nonzero perfect cubes in the way of (1.1) since the variable z is completely determined by x and $\boldsymbol{y}$. They showed that

$$N_4(H) = c_4 H^3 (\log H)^2 + O\left(H^3 \log H\right).$$

In fact, they dealt with a more general case that the number of squares is a multiple of 4 (see [9, Theorem 7.1]). Zhai [10] improved this result by obtaining a power-saving error term that

$$N_4(H) = H^3 P_4(\log H) + O\left(H^{3-1/4+\varepsilon}\right),$$

where P_4 is a quadratic polynomial. All above results are obtained by studying the corresponding multiple Dirichlet series. Liu et al. stated in [9] that the same idea can be applied to investigate the number of integral solutions of some higher-degree Diophantine equation like

$$x^d = (y_1^2 + y_2^2 + y_3^2 + y_4^2)z^{d-2}, \quad d \geqslant 4.$$

They obtained an asymptotic formula in [11] for $d = 4$, and Wen [12] established the asymptotic formulae for any integer $d \geqslant 4$ with power-saving error terms.

We remark here that the Diophantine equations S, S', and S_4 mentioned above are homogeneous, so there is an equivalent relation between rational points in projective space and integral points in affine space up to a scalar multiplication. They are actually related to another project in number theory, which is Manin's conjecture (see [2, 5–9] for more details).

Motivated by the above work, in this paper, we mainly focus on the Diophantine equation

$$S_4^k : x(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z^k \tag{1.2}$$

for $k \geqslant 2$. Similar to (1.1), we denote by $N_4^k(H)$ the number of integral tuples $(x, y_1, \cdots, y_4)$ satisfying $\max\{|x|, \|\boldsymbol{y}\|\} \leqslant H$, which form nonzero perfect k-th powers in the way of (1.2). Recall that the Lindelöf hypothesis (LH in brief, [13, §II.3.4]) states that

$$\zeta(1/2 + \mathrm{i}t) \ll (|t| + 1)^{\varepsilon}$$

for any $\varepsilon > 0$. Our main result is as follows.

Theorem 1.1. *Let $k \geqslant 4$ be any integer. Assuming LH, then for any $\varepsilon > 0$, there exists a constant ϑ_k, such that*

$$N_4^k(H) = H^{2+3/k}\mathcal{P}(\log H) + O\left(H^{2+3/k-\vartheta_k+\varepsilon}\right),$$

where

$$\vartheta_k = \frac{3}{2k}\left(1 - \frac{2k/3 - 1}{[2k/3]}\right) > 0,$$

and $[\alpha]$ is the integral part of α, the implied constant only depends on k and ε, $\mathcal{P}$ is a polynomial of degree $k-1$ given by (3.14) *with leading coefficient $16\mathscr{C}_k$, and $\mathscr{C}_k$ is a positive constant given by* (3.13).

Remark 1.2. *The assumption of LH is just for simplifying the calculation. We claim that the above asymptotic formulae still holds for $k = 4$ unconditionally, since we can apply the fourth moment estimate of the Riemann zeta function instead of LH as in* (4.8).

The case of $k = 3$ has been solved in [9, 10] unconditionally as mentioned above. In fact, following the proof of Theorem 1.1, one can easily obtain an asymptotic formula for $k = 3$ with a power-saving error term $O(H^{3-1/5+\varepsilon})$. We shall leave that as an example to justify our result. We next give an unconditional result for $k = 2$.

Theorem 1.3. *Unconditionally, we have*

$$N_4^2(H) = 16H^{2+3/2}P(\log H) + O\left(H^{2+3/2-1/3+\varepsilon}\right),$$

where P is a linear polynomial given by (4.7) *and the implied constant only depends on ε.*

2. Application of Perron's formula

We first introduce the bivariate Perron's formula [10, Lemma 2.2], which plays an important role in our proof.

Lemma 2.1. *Suppose that $f(n_1, n_2)$ is a bivariate arithmetic function and its Dirichlet series*

$$F(s_1, s_2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{f(n_1, n_2)}{n_1^{s_1} n_2^{s_2}}$$

is absolutely convergent for $\Re(s_j) > \sigma_j$ $(j = 1, 2)$ with some σ_1, $\sigma_2 > 0$. Let x_1, x_2, T_1, $T_2 \geqslant 10$ be parameters such that $x_j \notin \mathbb{N}$, and define

$$b_j = \sigma_j + 1/\log x_j, \quad j = 1, 2.$$

We have

$$\sum_{n_1 \leqslant x_1} \sum_{n_2 \leqslant x_2} f(n_1, n_2) = \frac{1}{(2\pi \mathrm{i})^2} \int_{b_1 - \mathrm{i}T_1}^{b_1 + \mathrm{i}T_1} \int_{b_2 - \mathrm{i}T_2}^{b_2 + \mathrm{i}T_2} \frac{F(s_1, s_2) x_1^{s_1} x_2^{s_2}}{s_1 s_2} \, \mathrm{d}s_2 \, \mathrm{d}s_1 + O(x_1^{\sigma_1} x_2^{\sigma_2} E),$$

where

$$E := \sum_{j=1}^{2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{|f(n_1, n_2)|}{n_1^{b_1} n_2^{b_2}} \min\left\{1, \frac{1}{T_j \left|\log \frac{x_j}{n_j}\right|}\right\}.$$

Recalling the definition of $N_4^k(H)$ before, we see that

$$N_4^k(H) = \left|\{(x, \boldsymbol{y}, z) \in \mathbb{Z}^6 \cap S_4^k : \max\{|x|, ||\boldsymbol{y}||\} \leqslant H, z \neq 0\}\right|.$$

Let $r_4(n)$ be the number of representations of a positive integer n as the sum of four squares

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2$$

with

$$(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4.$$

It is well known that

$$r_4(n) = 8 r_4^*(n) \quad \text{with} \quad r_4^*(n) = \sum_{\substack{d|n \\ d \not\equiv 0 \,(\mathrm{mod}\, 4)}} d, \tag{2.1}$$

and $r_4^*(n)$ is a multiplicative arithmetic function. Let $\mathbb{1}_k$ denote an indicator function of perfect k-th power defined by

$$\mathbb{1}_k(n) = \begin{cases} 1, & \text{if } n \text{ is a perfect } k\text{-th power}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

In view of the above problem, we can write

$$\begin{aligned} N_4^k(H) &= 2 \sum_{1 \leqslant m \leqslant H} \sum_{1 \leqslant n \leqslant H^2} r_4(n) \mathbb{1}_k(n) \\ &= 16 \sum_{1 \leqslant m \leqslant H} \sum_{1 \leqslant n \leqslant H^2} r_4^*(n) \mathbb{1}_k(n). \end{aligned} \tag{2.3}$$

In order to deal with this double sum, we define the corresponding Dirichlet series

$$\mathcal{F}_k(s,w) = \sum_{m,n=1}^{\infty} \frac{r_4^*(n)\mathbb{1}_k(mn)}{m^s n^w}. \tag{2.4}$$

The following proposition gives the expression and convergence of $\mathcal{F}_k(s,w)$, which allows us to extend the double Dirichlet series to a suitable large region.

Proposition 2.2. *Let $k \geqslant 2$. If $\Re(s) > 1/k$ and $\Re(w) > 1 + 1/k$, then*

$$\mathcal{F}_k(s,w) = \prod_{j=0}^{k} \zeta\left((k-j)s + j(w-1)\right)\mathcal{H}_k(s,w), \tag{2.5}$$

where $\mathcal{H}_k(s,w)$ is an Euler product given by (2.16)*, which is absolutely convergent if s and w satisfy the conditions*

$$\min_{1\leqslant i\leqslant k-1} \Re((k-i)s + i(w-1)) \geqslant 1/2 + \varepsilon \ \text{ and } \ \Re(w) \geqslant 1 + \varepsilon. \tag{2.6}$$

Furthermore, we have

$$\mathcal{H}_k(s,w) \ll 1 \tag{2.7}$$

in the above region, and the implied constant is absolute.

Proof. Note that $r_4^*(n)$ is multiplicative, and (2.1) yields $r_4^*(1) = 1$ and

$$r_4^*(p^\nu) = \begin{cases} \dfrac{1-p^{\nu+1}}{1-p}, & \text{if } p \geqslant 3, \\ 3, & \text{if } p = 2, \end{cases} \tag{2.8}$$

for any integer $\nu \geqslant 1$, then we can rewrite $\mathcal{F}_k(s,w)$ in (2.4) as the Euler product

$$\begin{aligned} \mathcal{F}_k(s,w) &= \prod_p \sum_{\mu\geqslant 0}\sum_{\nu\geqslant 0} \frac{r_4^*(p^\nu)\mathbb{1}_k(p^{\mu+\nu})}{p^{\mu s+\nu w}} \\ &= \prod_p \sum_{d\geqslant 0} p^{-kds} \sum_{0\leqslant\nu\leqslant kd} \frac{r_4^*(p^\nu)}{p^{\nu(w-s)}} \\ &=: \prod_p \mathcal{F}_{k,p}(s,w). \end{aligned} \tag{2.9}$$

Here, we let $\mu + \nu = kd$ for $d \geqslant 0$, according to the definition of $\mathbb{1}_k$ in (2.2). On the other hand, a simple formal calculation shows

$$\begin{aligned} &\sum_{d\geqslant 0} x^d \sum_{0\leqslant\nu\leqslant kd} y^\nu \frac{1-z^{kd+1}}{1-z} = \frac{1}{1-z}\sum_{d\geqslant 0} x^d \left(\frac{1-y^{kd+1}}{1-y} - z\frac{1-(yz)^{kd+1}}{1-yz}\right) \\ &= \frac{1}{1-z}\left(\frac{1}{1-y}\left(\frac{1}{1-x} - \frac{y}{1-xy^k}\right) - \frac{z}{1-yz}\left(\frac{1}{1-x} - \frac{yz}{1-x(yz)^k}\right)\right) \\ &= \frac{G_k(x,y,z)}{(1-x)(1-xy^k)(1-x(yz)^k)} \end{aligned} \tag{2.10}$$

with

$$G_2(x,y,z) = 1 + xy(1+z) + xy^2 z$$

and

$$\begin{aligned} G_k(x,y,z) = 1 &+ xy(1+z) + xy^2(1+z+z^2) + \cdots\cdots + xy^{k-1}(1+z+\cdots+z^{k-1}) \\ &+ xy^k(z+z^2+\cdots+z^{k-1}) + x^2y^{k+1}(z^2+z^3+\cdots+z^{k-1}) \\ &+ x^2y^{k+2}(z^3+z^4+\cdots+z^{k-1}) + \cdots\cdots + x^2y^{2k-2}z^{k-1} \end{aligned}$$

for $k \geqslant 3$. Similarly, we have

$$\begin{aligned} 1 + \sum_{d\geqslant 1} x^d \left(1 + 3\sum_{1\leqslant \nu\leqslant kd} y^\nu\right) &= \frac{1}{1-x} + \frac{3}{1-y}\sum_{d\geqslant 1} x^d\left(y - y^{kd+1}\right) \\ &= \frac{1}{1-x} + \frac{3(xy - xy^{k+1})}{(1-y)(1-x)(1-xy^k)} \\ &= \frac{1 + 3xy(1+y+\cdots+y^{k-2}) + 2xy^k}{(1-x)(1-xy^k)}. \end{aligned} \tag{2.11}$$

When $p \geqslant 3$, in view of (2.8), we apply (2.10) with

$$(x,y,z) = (p^{-ks}, p^{-(w-s)}, p)$$

to deduce that

$$\mathcal{F}_{k,p}(s,w) = \prod_{0\leqslant j\leqslant k}\left(1 - \frac{1}{p^{(k-j)s+j(w-1)}}\right)^{-1}\mathcal{H}_{k,p}(s,w), \tag{2.12}$$

where

$$\mathcal{H}_{k,p}(s,w) = G_k\left(p^{-ks}, p^{-(w-s)}, p\right)\left(1 - \frac{1}{p^{kw}}\right)^{-1}\prod_{1\leqslant j\leqslant k-1}\left(1 - \frac{1}{p^{(k-j)s+j(w-1)}}\right). \tag{2.13}$$

Meanwhile, for $p = 2$, the formula (2.11) with

$$(x,y) = (2^{-ks}, 2^{-(w-s)})$$

gives us

$$\mathcal{F}_{k,2}(s,w) = \prod_{0\leqslant j\leqslant k}\left(1 - \frac{1}{2^{(k-j)s+j(w-1)}}\right)^{-1}\mathcal{H}_{k,2}(s,w), \tag{2.14}$$

where

$$\mathcal{H}_{k,2}(s,w) = \left(1 + \frac{3}{2^{(k-1)s+w}}\left(1 + \frac{1}{2^{w-s}} + \cdots + \frac{1}{2^{(k-2)(w-s)}}\right) + \frac{1}{2^{kw-1}}\right)\times\left(1 - \frac{1}{2^{kw}}\right)^{-1}\prod_{1\leqslant j\leqslant k}\left(1 - \frac{1}{2^{(k-j)s+j(w-1)}}\right). \tag{2.15}$$

It can be observed that $\mathcal{F}_{k,p}(s,w)/\mathcal{H}_{k,p}(s,w)$ will give the Euler product of Riemann zeta functions in (2.12) and (2.14). Combining (2.9) and (2.12)–(2.15), we get (2.5) with

$$\mathcal{H}_k(s,w) = \mathcal{H}_{k,2}(s,w)\prod_{p\geqslant 3}\mathcal{H}_{k,p}(s,w). \tag{2.16}$$

Next, we shall discuss the convergence. In view of the expression of $\mathcal{H}_{k,p}(s,w)$ in (2.13) and (2.15), when expanding $\mathcal{H}_{k,p}(s,w)$ into 1 plus some monomials about p, we see that the power of p in the denominator of each monomial is great than 1 if

$$\min_{1\leqslant i\leqslant k-1}\Re((k-i)s+i(w-1))\geqslant 1/2+\varepsilon \text{ and } \Re(w)\geqslant 1+\varepsilon,$$

so we have

$$\mathcal{H}_{k,p}(s,w)=1+O(p^{-1-\varepsilon}),$$

and $\mathcal{H}_k(s,w)$ is absolutely convergent in this region, which implies (2.7). This completes the proof. □

We next make use of bivariate Perron's formula. Suppose $H\notin\mathbb{N}$ and let $T\in[10,H^{1/2}]$ be a parameter to be chosen later. The analytic property of $\mathcal{F}_k(s,w)$ allows us to set

$$x_1=H,\ x_2=H^2,\ \ s_1=s=\sigma+\mathrm{i}t,\ \ s_2=w=u+\mathrm{i}v,\ \ b_1=1/k+\varepsilon,\ \ b_2=1/k+1+\varepsilon$$

in Lemma 2.1, then we get

$$N_4^k(H)=\frac{16}{(2\pi\mathrm{i})^2}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T}\int_{b_2-10\mathrm{i}T}^{b_2+10\mathrm{i}T}\frac{\mathcal{F}_k(s,w)H^{s+2w}}{ws}\,\mathrm{d}w\,\mathrm{d}s+O\Big(H^{2+\frac{3}{k}+\varepsilon}(\mathfrak{J}_1(H,T)+\mathfrak{J}_2(H,T))\Big)$$

with

$$\mathfrak{J}_1(H,T):=\sum_{m,n=1}^{\infty}\frac{r_4^*(n)\mathbb{1}_k(mn)}{m^{b_1}n^{b_2}}\min\left(1,\frac{1}{T|\log\frac{H}{m}|}\right),$$
$$\mathfrak{J}_2(H,T):=\sum_{m,n=1}^{\infty}\frac{r_4^*(n)\mathbb{1}_k(mn)}{m^{b_1}n^{b_2}}\min\left(1,\frac{1}{T|\log\frac{H^2}{n}|}\right).$$

Noticing that

$$r_4^*(n)\ll n\tau(n),$$

and following the arguments in [10], one can easily get

$$\begin{aligned}\mathfrak{J}_1(H,T)&\ll\sum_{m,n=1}^{\infty}\frac{\tau(n)\mathbb{1}_k(mn)}{(mn)^{b_1}}\min\left(1,\frac{1}{T|\log\frac{H}{m}|}\right)\\&\ll\frac{H^{\varepsilon}}{T},\end{aligned}$$

and the same result holds for $\mathfrak{J}_2(H,T)$. It follows that

$$N_4^k(H)=\frac{16}{(2\pi\mathrm{i})^2}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T}\int_{b_2-10\mathrm{i}T}^{b_2+10\mathrm{i}T}\frac{\mathcal{F}_k(s,w)H^{s+2w}}{ws}\,\mathrm{d}w\,\mathrm{d}s+O\Big(\frac{H^{2+3/k+\varepsilon}}{T}\Big).\tag{2.17}$$

It suffices to evaluate the double integral on the righthand side of (2.17).

3. Proof of the Theorem 1.1

In this section, we shall prove the main theorem. The proof is divided into three steps. We shall first turn the double integral into some usual single integrals by Cauchy's residue theorem and then deal with the single integrals. The final step is choosing the suitable parameters.

3.1. Application of Cauchy's residue theorem

In this subsection, we shall apply Cauchy's residue theorem to evaluate the inner integral over w in (2.17) and derive the following result.

Lemma 3.1. *Let $k \geqslant 2$ and $N_4^k(H)$ be defined as in* (2.3). *Assuming LH, we have*

$$N_4^k(H) = 16 \sum_{j=1}^{k} I_j(H,T) + O\left(H^{2+3/k+\varepsilon}/T + H^{2+2/k+\varepsilon}\right),$$

where

$$I_j(H,T) := \frac{1}{2\pi \mathrm{i}} \int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \operatorname*{Res}_{w=w_j} \frac{\mathcal{F}_k(s,w)}{ws} H^{s+2w_j} \, \mathrm{d}s \tag{3.1}$$

with

$$w_j = (1-(k-j)s)/j + 1$$

for $1 \leqslant j \leqslant k$.

Proof. In terms of the inner integral in (2.17), we consider the domain formed by four points

$$w = b_2 \pm 10\mathrm{i}T, \quad w = 1/2k + 1 \pm 10\mathrm{i}T.$$

In this domain, from Proposition 2.2, we easily see that the integrand function $\mathcal{F}_k(s,w)/(ws)$ has k simple poles:

$$w_j = \tfrac{1}{j}(1-(k-j)s) + 1$$

for $1 \leqslant j \leqslant k$.

Using the residue theorem for the variable w, we get

$$N_4^k(H) = 16 \sum_{j=1}^{k} I_j(H,T) + R_1(H,T) + R_2(H,T) - R_3(H,T) + O\left(H^{2+3/k+\varepsilon}/T\right), \tag{3.2}$$

where $I_j(H,T)$ is given by (3.1) and

$$R_1(H,T) := \frac{16}{(2\pi\mathrm{i})^2} \int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \int_{\frac{1}{2k}+1+10\mathrm{i}T}^{b_2+10\mathrm{i}T} \frac{\mathcal{F}_k(s,w)H^{s+2w}}{ws} \, \mathrm{d}w \, \mathrm{d}s,$$

$$R_2(H,T) := \frac{16}{(2\pi\mathrm{i})^2} \int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \int_{\frac{1}{2k}+1-10\mathrm{i}T}^{\frac{1}{2k}+1+10\mathrm{i}T} \frac{\mathcal{F}_k(s,w)H^{s+2w}}{ws} \, \mathrm{d}w \, \mathrm{d}s,$$

$$R_3(H,T) := \frac{16}{(2\pi\mathrm{i})^2} \int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \int_{\frac{1}{2k}+1-10\mathrm{i}T}^{b_2-10\mathrm{i}T} \frac{\mathcal{F}_k(s,w)H^{s+2w}}{ws} \, \mathrm{d}w \, \mathrm{d}s.$$

Set $s = b_1 + \mathrm{i}t$ with $-T \leqslant t \leqslant T$ and $w = u + 10\mathrm{i}T$ with $1/2k + 1 \leqslant u \leqslant b_2$. By Proposition 2.2, one has $\mathcal{H}_k(s,w) \ll 1$, since the conditions in (2.6) are satisfied clearly. On the other hand, recall that LH states

$$\zeta(1/2+it) \ll (|t|+1)^{\varepsilon},$$

then we deduce from the Phragmén-Lindelöf principle that the subconvexity bound of the Riemann zeta function under LH satisfies

$$\zeta(\sigma+\mathrm{i}t) \ll \begin{cases} 1, & \text{if } \sigma > 1, \\ (|t|+1)^{\max\{\frac{1}{2}-\sigma,0\}+\varepsilon}, & \text{if } 0 < \sigma \leqslant 1. \end{cases} \tag{3.3}$$

This will be used several times below, and it follows that

$$\mathcal{F}_k(b_1+\mathrm{i}t, u+10\mathrm{i}T) \ll T^{\varepsilon}$$

holds uniformly for $1/2k+1 \leqslant u \leqslant b_2$. This implies

$$\begin{aligned} R_1(H,T) &\ll \int_{-T}^{T} \frac{\mathrm{d}t}{|t|+1} \int_{\frac{1}{2k}+1}^{b_2} T^{-1+\varepsilon} H^{b_1+2u}\,\mathrm{d}u \\ &\ll H^{2+3/k+\varepsilon}/T, \end{aligned}$$

and the same bound holds for $R_3(H,T)$. Similarly, for $u = 1/2k+1$, we have

$$\mathcal{F}_k(b_1+\mathrm{i}t, 1/2k+1+\mathrm{i}v) \ll (|t|+|v|+1)^{\varepsilon},$$

and $R_2(H,T)$ can be estimated as

$$\begin{aligned} R_2(H,T) &\ll H^{b_1+2(1/2k+1)} \int_{-T}^{T} (|t|+1)^{-1+\varepsilon}\,\mathrm{d}t \int_{-10T}^{10T} (|v|+1)^{-1+\varepsilon}\,\mathrm{d}v \\ &\ll H^{2+2/k+\varepsilon}. \end{aligned}$$

Inserting the upper bounds of $R_i(H,T)$ $(i=1,2,3)$ into (3.2), we obtain the required formula. □

3.2. Evaluation of $I_j(H,T)$

In this subsection, we shall evaluate $I_j(H,T)$ for $1 \leqslant j \leqslant k$, and our main idea is Cauchy's residue theorem.

Lemma 3.2. *Let $k \geqslant 4$ and $I_j(H,T)$ for $1 \leqslant j \leqslant k$ be defined as in* (3.1)*, then we have the following estimates under LH for each $I_j(H,T)$:*

$$\begin{aligned} I_j(H,T) &\ll H^{2+\frac{2}{k}+\frac{j-2}{2k(k-j-1)}+\varepsilon} + H^{2+\frac{3}{k}+\varepsilon}/T, \quad \text{for } 1 \leqslant j < 2k/3, \\ I_{2k/3}(H,T) &= C_{2k/3} H^{2+3/k} + O\left(H^{2+\frac{3}{k}+\varepsilon}/T\right), \\ I_j(H,T) &= H^{2+3/k} P_j(\log H) + O\left(H^{2+\frac{3}{2k}+\frac{2k-3}{2k(j-1)}+\varepsilon} + H^{2+\frac{3}{k}+\varepsilon}/T\right), \quad \text{for } 2k/3 < j < k, \\ I_k(H,T) &= H^{2+3/k} P_k(\log H) + O\Big(H^{2+\frac{2}{k}+\frac{k-2}{2k(k-1)}+\varepsilon} T^{\frac{1}{2(k-1)}} + H^{2+\frac{3}{k}+\varepsilon}/T\Big), \end{aligned}$$

where $C_{3k/2}$, $P_j(\log H)$ for $2k/3 < j < k$ and $P_k(\log H)$ are defined in (3.6), (3.9), *and* (3.12), *respectively.*

Remark 3.3. *In fact, the above result still holds for $k = 3$ apart from the third formula, since the third case vanishes if $k = 3$.*

Proof. Recall the definition of $I_j(H,T)$ in (3.1). We deduce from Proposition 2.2 that

$$\operatorname*{Res}_{w=w_j}\frac{\mathcal{F}_k(s,w)}{ws}=\frac{\mathcal{H}_k(s,w_j)}{j\cdot w_j s}\prod_{\substack{0\leqslant\ell\leqslant k\\ \ell\neq j}}\zeta\left(\frac{\ell}{j}+\left(1-\frac{\ell}{j}\right)ks\right)=:G_j(s).$$

It follows that

$$I_j(H,T)=\frac{1}{2\pi\mathrm{i}}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T}G_j(s)H^{2+\frac{2}{j}+(3-\frac{2k}{j})s}\mathrm{d}s.$$

The proof of this lemma is divided into four parts based on whether $3-2k/j$ is positive, negative, or zero. Keep in mind that the letter j is always an integer.

If $1\leqslant j<2k/3$, the condition (2.6) becomes

$$\min_{1\leqslant i\leqslant k-1}\Re((k-i)s+i(w_j-1))\geqslant 1/2+\varepsilon,$$

which implies

$$\Re(s)<(2k-2-j)/2k(k-1-j).$$

So, we shift the line of integration from b_1 to

$$\Re(s)=\sigma_j=(2k-2-j)/2k(k-1-j)-\varepsilon.$$

By Proposition 2.2, the Euler product $\mathcal{H}_k(s,w_j)$ is absolutely convergent in the region $b_1\leqslant\Re(s)\leqslant\sigma_j$, and we have $\mathcal{H}_k(s,w_j)\ll 1$. It follows from this and (3.3) that

$$\begin{aligned}G_j(\sigma+\mathrm{i}t)&\ll\frac{|\mathcal{H}_k(s,w_j)|}{(|t|+1)^2}\prod_{\substack{0\leqslant\ell\leqslant k\\ \ell\neq j}}\left|\zeta\left(\frac{\ell}{j}+\left(1-\frac{\ell}{j}\right)k(\sigma+\mathrm{i}t)\right)\right|\\&\ll\left|\zeta\left(\tfrac{k}{j}+(1-\tfrac{k}{j})k(\sigma+\mathrm{i}t)\right)\right|/(|t|+1)^{2-\varepsilon}\\&\ll(|t|+1)^{1/2(k-1-j)-2+\varepsilon}\\&\ll(|t|+1)^{-3/2+\varepsilon}\end{aligned}\tag{3.4}$$

for $b_1\leqslant\sigma\leqslant\sigma_j$, since the real part of each zeta function is

$$\Re(\ell/j+(1-\ell/j)ks)>1/2$$

for $0\leqslant\ell\leqslant k-1$ and $\ell\neq j$. The above bound allows us to extend the integral of $I_j(H,T)$ on $[-T,T]$ to $(-\infty,\infty)$, so we have

$$I_j(H,T)=\frac{1}{2\pi\mathrm{i}}\int_{(b_1)}G_j(s)H^{2+\frac{2}{j}-(\frac{2k}{j}-3)s}\,\mathrm{d}s+O\left(H^{2+\frac{3}{k}+\varepsilon}/T\right),$$

where $\int_{(b_1)}$ means $\int_{b_1-i\infty}^{b_1+i\infty}$. Note that there is no pole in the region $b_1\leqslant\Re(s)\leqslant\sigma_j$. The residue theorem tells us that

$$I_j(H,T)=\frac{1}{2\pi\mathrm{i}}\int_{(\sigma_j)}G_j(s)H^{2+\frac{2}{j}-(\frac{2k}{j}-3)s}\,\mathrm{d}s+O\left(H^{2+\frac{3}{k}+\varepsilon}/T\right).\tag{3.5}$$

By (3.4), the integral on the righthand side can be estimated as

$$H^{2+\frac{2}{j}-(\frac{2k}{j}-3)\sigma_j}\int_{-\infty}^{\infty}|G_j(\sigma_j+\mathrm{i}t)|\,\mathrm{d}t \ll H^{2+\frac{2}{k}+\frac{j-2}{2k(k-j-1)}+\varepsilon}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(|t|+1)^{\frac{3}{2}-\varepsilon}}$$
$$\ll H^{2+\frac{2}{k}+\frac{j-2}{2k(k-j-1)}+\varepsilon}.$$

Combining this with (3.5), we obtain the first assertion in Lemma 3.2.

If $j = 2k/3$ is an integer, we find that

$$I_j(H,T) = \frac{1}{2\pi\mathrm{i}}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} G_j(s)H^{2+\frac{2}{j}}\mathrm{d}s.$$

Otherwise, this case does not exist. Note that

$$G_{2k/3}(s) \ll (|t|+1)^{-(2-\varepsilon)}$$

for $s = b_1 + \mathrm{i}t$, which shows that $G_{2k/3}(b_1+\mathrm{i}t,\chi)$ has a good convergence as a function in t. It follows that

$$\begin{aligned} I_{2k/3}(H,T) &= \frac{1}{2\pi\mathrm{i}}\int_{(b_1)} G_{2k/3}(s)H^{2+\frac{3}{k}}\,\mathrm{d}s + O\Big(H^{2+3/k+\varepsilon}/T\Big) \\ &=: C_{2k/3}H^{2+3/k} + O\Big(H^{2+3/k+\varepsilon}/T\Big), \end{aligned}$$

where

$$C_{2k/3} = \frac{1}{2\pi\mathrm{i}}\int_{(\frac{1}{k}+\varepsilon)} G_{2k/3}(s)\,\mathrm{d}s \tag{3.6}$$

is an absolute constant. This gives the second assertion.

If $2k/3 < j < k$, we shift the line of integration from b_1 to

$$\Re(s) = \sigma_j' = (j-2)/2k(j-1)+\varepsilon.$$

It is easy to check that (2.6) is exactly satisfied, so we have $\mathcal{H}_k(s,w_j) \ll 1$. It follows from (3.3) again that

$$\begin{aligned} G_j(\sigma+\mathrm{i}t) &\ll \frac{|\mathcal{H}_k(s,w_j)|}{|w_js|}\prod_{\substack{0\leqslant\ell\leqslant k\\ \ell\neq j}}\left|\zeta\left(\frac{\ell}{j}+\left(1-\frac{\ell}{j}\right)k(\sigma+\mathrm{i}t)\right)\right| \\ &\ll \frac{|\zeta(k(\sigma+\mathrm{i}t))|}{(|t|+1)^2} \\ &\ll (|t|+1)^{\max\{1/2-k\sigma,0\}-2+\varepsilon} \\ &\ll (|t|+1)^{-3/2+\varepsilon} \end{aligned} \tag{3.7}$$

for $\sigma_j' < \sigma \leqslant b_1$, which yields

$$I_j(H,T) = \frac{1}{2\pi\mathrm{i}}\int_{(b_1)} G_j(s)H^{2+\frac{2}{j}+(3-\frac{2k}{j})s}\,\mathrm{d}s + O\Big(H^{2+\frac{3}{k}+\varepsilon}/T\Big).$$

Note that

$$0 < \Re(\ell/j + (1 - \ell/j)ks) < 1$$

for $0 \leqslant \ell < j$, so there is only a pole $s = 1/k$ of order j given by $G_j(s)$ in the strip $\sigma'_j \leqslant \sigma \leqslant b_1$. The residue theorem gives us

$$I_j(H,T) = \operatorname*{Res}_{s=1/k} G_j(s) H^{2+3/k} + \frac{1}{2\pi \mathrm{i}} \int_{(\sigma'_j)} G_j(s) H^{2+\frac{2}{j}+(3-\frac{2k}{j})s} \,\mathrm{d}s + O\Big(H^{2+\frac{3}{k}+\varepsilon}/T\Big). \tag{3.8}$$

We can compute the residue in the main term as

$$\begin{aligned} \operatorname*{Res}_{s=1/k} G_j(s) &= \frac{1}{(j-1)!} \lim_{s\to 1/k} \frac{\mathrm{d}^{j-1}}{\mathrm{d}s^{j-1}} \Big((s - \tfrac{1}{k})^j G_j(s) H^{s-\frac{1}{k}}\Big) \\ &=: P_j(\log H). \end{aligned} \tag{3.9}$$

This is a polynomial of degree $j-1$ with leading coefficient

$$\frac{\mathcal{H}_k(\frac{1}{k}, \frac{1}{k}+1)(j/k)^{j-2}}{(k+1)((j-1)!)^2}.$$

The integral in (3.8) can be bounded by

$$H^{2+\frac{2}{j}+(3-\frac{2k}{3})\sigma'_j} \int_{-\infty}^{\infty} |G_j(\sigma'_j + \mathrm{i}t)| \,\mathrm{d}t \ll H^{2+\frac{3}{2k}+\frac{2k-3}{2k(j-1)}+\varepsilon}.$$

Inserting (3.9) and the above estimate into (3.8), we get the third formula in Lemma 3.2.

Finally, it remains to deal with the case $j = k$. Now, we shift the line of integration from b_1 to

$$\Re(s) = \sigma_k = (k-2)/2k(k-1) + \varepsilon.$$

Noticing that $w_k = 1/k + 1$ is no longer dependent on s, it follows from (3.3) that

$$\begin{aligned} G_k(\sigma + \mathrm{i}t) &= \frac{\mathcal{H}_k(s, w_k)}{k \cdot w_k s} \prod_{0 \leqslant \ell < k} \zeta\left(\frac{\ell}{k} + \left(1 - \frac{\ell}{k}\right) k(\sigma + \mathrm{i}t)\right) \\ &\ll \frac{|\zeta(k(\sigma + \mathrm{i}t))|}{(|t|+1)^{1-\varepsilon}} \\ &\ll \begin{cases} (|t|+1)^{(1/2-k\sigma)-1+\varepsilon}, & \text{if } \sigma_k \leqslant \sigma \leqslant 1/2k, \\ (|t|+1)^{-1+\varepsilon}, & \text{if } 1/2k < \sigma \leqslant b_1, \end{cases} \end{aligned} \tag{3.10}$$

since $\mathcal{H}_k(s, w_k)$ is absolutely convergent for $\sigma_k \leqslant \Re(s) \leqslant b_1$ according to (2.6). Note that there is only one pole $s = 1/k$ of order k given by $G_k(s)$ in the rectangle formed by $\sigma_k \pm \mathrm{i}T$ and $b_1 \pm \mathrm{i}T$. It follows from the residue theorem that

$$\begin{aligned} I_k(H,T) &= \operatorname*{Res}_{s=1/k} G_k(s) H^{2+3/k} + \left(\int_{b_1-\mathrm{i}T}^{\sigma_k-\mathrm{i}T} + \int_{\sigma_k-\mathrm{i}T}^{\sigma_k+\mathrm{i}T} + \int_{\sigma_k+\mathrm{i}T}^{b_1+\mathrm{i}T}\right) G_k(s) H^{2+\frac{2}{k}+s} \,\mathrm{d}s \\ &= H^{2+3/k} P_k(\log H) + K_1(H,T) + K_2(H,T) + K_3(H,T), \end{aligned} \tag{3.11}$$

where $K_i(H,T)$ for $i = 1,2,3$ corresponds to the above three integrals, respectively, and P_k is a polynomial of degree $k-1$ given by

$$P_k(\log H) = \frac{1}{(k-1)!}\lim_{s\to 1/k}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}\left((s-\tfrac{1}{k})^k G_k(s)H^{s-\frac{1}{k}}\right) \tag{3.12}$$

with leading coefficient

$$\mathscr{C}_k = \frac{\mathcal{H}_k(\frac{1}{k}, \frac{1}{k}+1)}{(k+1)((k-1)!)^2}. \tag{3.13}$$

This is a positive constant depending on k. By (3.10), using the same method as before, we can estimate $K_1(H,T)$ by

$$\begin{aligned} K_1(H,T) &\ll \int_{\sigma_k}^{\frac{1}{2k}} \frac{H^{2+2/k+\sigma}}{T^{1/2+k\sigma-\varepsilon}}\,\mathrm{d}\sigma + \int_{\frac{1}{2k}}^{b_1} \frac{H^{2+2/k+\sigma}}{T^{1-\varepsilon}}\,\mathrm{d}\sigma \\ &\ll H^{2+\frac{2}{k}+\frac{k-2}{2k(k-1)}+\varepsilon}/T^{\frac{2k-3}{2(k-1)}} + H^{2+\frac{3}{k}+\varepsilon}/T, \end{aligned}$$

so does $K_3(H,T)$. As for $K_2(H,T)$, noting that

$$G_k(\sigma_k + \mathrm{i}t) \ll (|t|+1)^{-(2k-3)/2(k-1)+\varepsilon},$$

we get

$$\begin{aligned} K_2(H,T) &\ll H^{2+\frac{2}{k}+\sigma_k}\int_{-T}^{T}(|t|+1)^{-(2k-3)/2(k-1)+\varepsilon}\,\mathrm{d}t \\ &\ll H^{2+\frac{2}{k}+\frac{k-2}{2k(k-1)}+\varepsilon}T^{\frac{1}{2(k-1)}}. \end{aligned}$$

The last formula in Lemma 3.2 is obtained followed from the above two estimates and (3.11). This completes the proof. □

3.3. Completion the proof of Theorem 1.1

Combining Lemmas 3.1 and 3.2, we get

$$\begin{aligned} \frac{N_4^k(H)}{H^2} &= 16H^{\frac{3}{k}}\Bigg(\mathbb{1}_{j=2k/3}C_{2k/3} + \sum_{2k/3<j\leqslant k} P_j(\log H)\Bigg) + O\left(H^{\frac{2}{k}+\varepsilon} + H^{\frac{3}{k}+\varepsilon}/T\right) \\ &\quad + O\Bigg(\sum_{1\leqslant j<2k/3} H^{\frac{2}{k}+\frac{j-2}{2k(k-j-1)}+\varepsilon} + \sum_{2k/3<j<k} H^{\frac{3}{2k}+\frac{2k-3}{2k(j-1)}+\varepsilon} + H^{\frac{2}{k}+\frac{k-2}{2k(k-1)}+\varepsilon}T^{\frac{1}{2(k-1)}}\Bigg) \end{aligned}$$

for $k \geqslant 4$. It can be simplified to

$$N_4^k(H)/H^2 = H^{\frac{3}{k}}\mathcal{P}(\log H) + O\left(H^{\frac{3}{k}+\varepsilon}/T + H^{\frac{3}{2k}+\frac{2k-3}{2k[2k/3]}+\varepsilon} + H^{\frac{2}{k}+\frac{k-2}{2k(k-1)}+\varepsilon}T^{\frac{1}{2(k-1)}}\right),$$

where $[\alpha]$ is the integral part of α and

$$\mathcal{P}(\log H) = 16\Bigg(\mathbb{1}_{j=2k/3}C_{2k/3} + \sum_{2k/3<j\leqslant k} P_j(\log H)\Bigg). \tag{3.14}$$

This is a polynomial of degree $k-1$ with leading coefficient $16\mathscr{C}_k$. It suffices to choose a suitable T to balance the error terms. It can be observed that the error term is actually dominated by $H^{\frac{3}{2k}+\frac{2k-3}{2k[2k/3]}+\varepsilon}$ for $k \geqslant 4$. Just letting $H^{9/4k^2} \ll T \ll H^{1/k}$, one can get the second formula in Theorem 1.1.

4. Proof of Theorem 1.3

Following the proof of Theorem 1.1, we give a sketch of the proof for $k = 2$. According to Proposition 2.2, the corresponding Dirichlet series can be written as

$$\mathcal{F}_2(s,w) = \zeta(2s)\zeta(s+w-1)\zeta(2(w-1))\mathcal{H}_2(s,w)$$

with

$$\mathcal{H}_2(s,w) = \left(1 - \frac{1}{2^{2w-2}}\right)\prod_p \left(1 + \frac{1}{p^{s+w}} + \frac{1}{p^{s+w-1}} + \frac{1}{p^{2w-1}}\right)\left(1 - \frac{1}{p^{2w}}\right)^{-1}\left(1 - \frac{1}{p^{s+w-1}}\right),$$

which is absolutely convergent and satisfies $\mathcal{H}_2(s,w) \ll 1$ in the region

$$\Re(s+w) \geqslant 3/2 + \varepsilon, \quad \Re(w) \geqslant 1 + \varepsilon. \tag{4.1}$$

Applying Perron's formula, we get

$$N_4^2(H) = \frac{16}{(2\pi \mathrm{i})^2}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T}\int_{b_2-10\mathrm{i}T}^{b_2+10\mathrm{i}T} \frac{\mathcal{F}_2(s,w)H^{s+2w}}{ws}\,\mathrm{d}w\,\mathrm{d}s + O\Big(\frac{H^{2+3/2+\varepsilon}}{T}\Big) \tag{4.2}$$

for $b_1 = 1/2 + \varepsilon$ and $b_2 = 3/2 + \varepsilon$.

Following the arguments in Section 3, we still shift the path of integration over w to $\Re(w) = 5/4$. Clearly, (4.1) is satisfied. Applying the residue theorem to evaluate the inner integral in (4.2), we can get (3.2) for $k = 2$. Note that the assumption of LH is used to bound $\mathcal{F}_k(s,w)$ and $G_j(s)$ in some region before. Unconditionally, we have the following well-known estimate for the Riemann zeta function (see [13, Theorem II.3.8] for example):

$$\zeta(\sigma + \mathrm{i}t) \ll \begin{cases} 1, & \text{if } \sigma > 1, \\ (|t|+1)^{\frac{1-\sigma}{3}+\varepsilon}, & \text{if } 1/2 \leqslant \sigma \leqslant 1, \\ (|t|+1)^{\frac{3-4\sigma}{6}+\varepsilon}, & \text{if } 0 < \sigma < 1/2. \end{cases} \tag{4.3}$$

Just replacing (3.3) by (4.3) and using the same method as in §3.1, we can get

$$N_4^2(H) = 16(I_1(H,T) + I_2(H,T)) + O\left(H^{2+\frac{3}{2}+\varepsilon}/T + H^{3+\varepsilon}T^{\frac{1}{2}}\right) \tag{4.4}$$

with

$$I_1(H,T) = \frac{1}{2\pi \mathrm{i}}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \frac{\zeta(2s)\zeta(2-2s)}{(2-s)s}\mathcal{H}_2(s,2-s)H^{4-s}\,\mathrm{d}s,$$
$$I_2(H,T) = \frac{1}{2\pi \mathrm{i}}\int_{b_1-\mathrm{i}T}^{b_1+\mathrm{i}T} \frac{\zeta(2s)\zeta(s+1/2)}{3s}\mathcal{H}_2(s,3/2)H^{3+s}\,\mathrm{d}s.$$

The treatments of $I_1(H,T)$ and $I_2(H,T)$ are a little different from before. There are only two Riemann zeta functions involved here, which can be treated more carefully. As for $I_1(H,T)$, we shift the line of integration from b_1 to $\sigma_1 = 1 - \varepsilon$ due to (4.1). It follows from (4.3) that

$$\zeta(2s)\zeta(2-2s)\mathcal{H}_2(s,2-s) \ll |\zeta(2-2s)| \ll (|t|+1)^{\max\{(2\sigma-1)/3,\,(8\sigma-5)/6\}+\varepsilon}$$

for $s = \sigma + \mathrm{i}t$ and $b_1 \leqslant \sigma \leqslant \sigma_1$. Noticing that there is no pole for the integrand function in this strip, similar to (3.5), we deduce from the residue theorem that

$$I_1(H,T) = \frac{1}{2\pi\mathrm{i}} \int_{(\sigma_1)} \frac{\zeta(2s)\zeta(2-2s)}{(2-s)s} \mathcal{H}_2(s, 2-s) H^{4-s}\,\mathrm{d}s + O\left(H^{2+\frac{3}{2}+\varepsilon}/T\right).$$

For $s = \sigma_1 + \mathrm{i}t$, the integral above can be bounded by

$$H^{4-\sigma_1} \int_{-\infty}^{\infty} \frac{\zeta(2\varepsilon - 2\mathrm{i}t)}{(|t|+1)^2}\,\mathrm{d}t \ll H^{3+\varepsilon} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(|t|+1)^{3/2-\varepsilon}} \ll H^{3+\varepsilon}.$$

It follows that

$$I_1(H,T) \ll H^{3+\varepsilon} + H^{2+3/2+\varepsilon}/T. \tag{4.5}$$

It suffices to deal with $I_2(H,T)$. In view of (4.1), we shift the path of integration from $b_1 \pm \mathrm{i}T$ to $\varepsilon \pm \mathrm{i}T$. Note that there is a pole $s = 1/2$ of order 2 in this region. It follows from the residue theorem that

$$I_2(H,T) = H^{2+\frac{3}{2}} P(\log H) + \left(\int_{b_1-\mathrm{i}T}^{\varepsilon-\mathrm{i}T} + \int_{\varepsilon-\mathrm{i}T}^{\varepsilon+\mathrm{i}T} + \int_{\varepsilon+\mathrm{i}T}^{b_1+\mathrm{i}T} \right) \frac{\zeta(2s)\zeta(s+1/2)}{3s} \mathcal{H}_2(s, 3/2) H^{3+s}\,\mathrm{d}s, \tag{4.6}$$

where P is a linear polynomial given by

$$\begin{aligned} P(\log H) &= \lim_{s\to 1/2} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{3s}(s-1/2)^2 \zeta(2s)\zeta(s+1/2)\mathcal{H}_2(s,3/2)H^{s-\frac{1}{2}} \right) \\ &= \frac{1}{3}\mathcal{H}_2(1/2,3/2)\log H + (\gamma - 2/3)\mathcal{H}_2(1/2,3/2) + \frac{1}{3}\mathcal{H}_2'(1/2,3/2) \end{aligned} \tag{4.7}$$

and γ is Euler's constant. We derive from (4.3) that the first and third integrals in (4.6) can be bounded by

$$\int_{\varepsilon}^{1/4} T^{\frac{2-5\sigma}{3}-1+\varepsilon} H^{3+\sigma}\mathrm{d}\sigma + \int_{1/4}^{b_1} T^{1/2-\sigma-1+\varepsilon} H^{3+\sigma}\mathrm{d}\sigma \ll H^{3+\varepsilon}/T^{\frac{1}{3}} + H^{3+\frac{1}{4}+\varepsilon}/T^{\frac{3}{4}} + H^{3+\frac{1}{2}+\varepsilon}/T.$$

Recall the fourth moment estimate of the Riemann zeta function (see [14, Theorem 5.1])

$$\int_{-T}^{T} |\zeta(\sigma + \mathrm{i}t)|^4\,\mathrm{d}t \ll T \log^4 T$$

for $1/2 \leqslant \sigma < 1$ and the functional equation, which states

$$\zeta(s) = \chi(s)\zeta(1-s) \ \text{ with } \ \chi(s) \sim (|t|+1)^{1/2-\sigma}$$

for $\sigma \leqslant 1/2$. By Hölder's inequality and partial integration, we can estimate the second integral in (4.6) as follows

$$\begin{aligned} &H^{3+\varepsilon} \int_{-T}^{T} \frac{|\zeta(2\varepsilon + 2\mathrm{i}t)\zeta(1/2+\varepsilon+\mathrm{i}t)|}{|t|+1}\,\mathrm{d}t \\ &\ll H^{3+\varepsilon} \left(\int_{-T}^{T} \frac{|\zeta(2\varepsilon+2\mathrm{i}t)|^4}{|t|+1}\,\mathrm{d}t \right)^{1/4} \left(\int_{-T}^{T} \frac{|\zeta(1/2+\varepsilon+\mathrm{i}t)|^4}{|t|+1}\,\mathrm{d}t \right)^{1/4} \left(\int_{-T}^{T} \frac{\mathrm{d}t}{|t|+1} \right)^{1/2} \\ &\ll H^{3+\varepsilon} \left(\int_{-T}^{T} \frac{\left|(|t|+1)^{1/2-2\varepsilon}\zeta(1-2\varepsilon-2\mathrm{i}t)\right|^4}{|t|+1}\,\mathrm{d}t \right)^{1/4} \\ &\ll H^{3+\varepsilon} T^{1/2}. \end{aligned} \tag{4.8}$$

Combining all the above, we get

$$I_2(H,T) = H^{2+\frac{3}{2}}P(\log H) + O\Big(H^{3+\varepsilon}T^{1/2} + H^{3+1/4+\varepsilon}/T^{3/4} + H^{3+1/2+\varepsilon}/T\Big). \tag{4.9}$$

Finally, inserting (4.5) and (4.9) into (4.2), we get

$$N_4^2(H) = 16H^{2+\frac{3}{2}}P(\log H) + O\Big(H^{2+\frac{3}{2}+\varepsilon}/T + H^{3+\varepsilon}T^{\frac{1}{2}} + H^{3+\frac{1}{4}+\varepsilon}/T^{\frac{3}{4}}\Big).$$

Choosing $T = H^{1/3}$, the error terms are balanced to $H^{2+3/2-1/3+\varepsilon}$. This gives the required formula in Theorem 1.3.

5. Conclusions

In this paper, we study the number of integers which form perfect powers in the way of

$$x(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z^k$$

and the proof relies on techniques coming from complex analysis. We point out that this problem was never done before except for the case $k = 3$. It is not easy to establish a unified asymptotic formula with power-saving error terms for large k, so we assume the Lindelöf hypothesis for $k \geqslant 4$. Theorem 1.1 gives an asymptotic formula with a power-saving error term for the number of such integers of bounded size under LH. This is the novelty of this paper. Moreover, Theorem 1.3 gives an unconditional result for $k = 2$.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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