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Research article

On the approximation of analytic functions by infinite series of fractional Ruscheweyh derivatives bases

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Abstract: This paper presented a new Ruscheweyh fractional derivative of fractional order in the complex conformable calculus sense. We applied the constructed complex conformable Ruscheweyh derivative (CCRD) on a certain base of polynomials (BPs) in different regions of convergence in Fréchet spaces (F-spaces). Accordingly, we investigated the relation between the approximation properties of the resulting base and the original one. Moreover, we deduced the mode of increase (the order and type) and the $\mathbb{T}_\rho$-property of the polynomial bases defined by the CCRD. Some bases of special polynomials, such as Bessel, Chebyshev, Bernoulli, and Euler polynomials, have been discussed to ensure the validity of the obtained results.

Keywords: complex conformable derivative; Ruscheweyh derivative; bases of polynomials; effectiveness; order and type; Fréchet space
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1. Introduction

Fractional calculus is a branch of mathematics that deals with generalizations of the concepts of differentiation and integration to noninteger orders. In traditional calculus, differentiation and integration are defined for integer orders. Fractional calculus extends these operations to include noninteger orders, often involving fractional or real numbers. This field has found applications in various scientific and engineering disciplines. Some key aspects and applications of fractional calculus are fractional derivatives and integrals, partial differential equations and their applications in physics, modeling complex phenomena, anomalous diffusion, signal processing, electromagnetics, biology and medicine, finance, optimal control, mechanical engineering, and approximation theory.

Recently, in [1], there has been a comprehensive introduction to the theory of bases of polynomials (BPs) and the theory of fractional calculus. To know more details about the theory of

BPs, see [2–5]. Moreover, the emphasis of [6–10] is directed to the topic of fractional calculus. In [1], the authors have studied the approximation of analytic functions by a series of complex conformable derivative bases (CCDBs) and complex conformable integral bases (CCIBs) in F-spaces that have links with some special functions such as Euler, Bernoulli, Chebyshev, and Bessel polynomials. In [11], the authors studied the approximation properties of Cliffordian functions by hypercomplex Ruscheweyh derivative bases in Fréchet modules that have related to Cliffordian special polynomials. In the very recent paper [12], the authors derived a new base in F-modules in Clifford analysis, named the equivalent base, and discussed its convergence properties.

Studying the approximation properties of some special polynomials is a significant topic in the theory of polynomial bases. Consequently, the authors of [13] proved that Bernoulli and Euler's polynomials were not found to be effective anywhere. Furthermore, they determined that each of these polynomials is of order 1. In [14, 15], Bessel polynomials were shown to be everywhere effective. Additionally, the authors of [16] studied the effectiveness of the Chebyshev polynomials in the unit disk. As we will conclude in this paper, the current study retains the same approximation properties for certain bases constructed in terms of the previously mentioned polynomials.

The well-known Ruscheweyh operator was defined for analytic functions using the convolutional technique [17] with some interesting geometric properties for this class, such as coefficient bound, convexity radii of starlikeness, extreme point, and distortion bounds. Aligned with the intensive development in employing various existing definitions of the fractional order derivatives, several authors have presented several generalizations of the Ruscheweyh operator from different aspects. In [18], the authors generalized the Ruscheweyh operator to fractional order (which is called the Ruscheweyh-Goyal differential operator) using the Srivastava-Saigo fractional differential operator involving hypergeometric functions. In 2020, the convexity and starlikeness properties of the Ruscheweyh-Goyal derivative of a fractional order were studied in [19]. The authors of [20] studied a certain class of analytic functions by employing generalized Ruscheweyh derivatives which involve a fractional derivative operator.

Motivated by the above discussion, we establish a generalized Ruscheweyh derivative involving a general complex conformable derivative operator proposed in [21], which is called the complex conformable Ruscheweyh derivative (CCRD). Consequently, we apply the constructed operator on a base of polynomials to derive the CCRD base of polynomials. Knowing the region of the effectiveness of the original base, we investigate the required conditions to preserve the effectiveness of the CCRDBPs correspondingly. The upper bounds for the order and type of the CCRDBPs are determined and shown to be attainable. Furthermore, the $\mathbb{T}\rho$-property of the CCRDBPs is deduced. According to the results, several applications examining the effectiveness of some special polynomials are explained.

This paper is formulated in a nutshell as follows. We begin by introducing a short presentation of some essential definitions, notations, and results of previous work on the topic under investigation. In the following Section 3, we define a new base called the CCRDBPs. Section 4 deals with the approximation properties of CCRDBPs in F-space. Section 5 is devoted to the study of effectiveness properties of the CCRDBs in F-spaces in different regions including closed discs, open discs, and open discs enclosing closed discs. In Section 6, we deduce bounds for the order and type and the $\mathbb{T}_{\rho_Q}$-property of CCRDBPs. Some examples are given showing that the resulting bounds are attainable. Section 7 exhibits some applications of CCRDBPs for some special functions such as Euler, Bernoulli, Chebyshev, and Bessel polynomials. Concluding remarks and open problems for possible future work are displayed in the last section.

2. Basic notations and preliminaries

Now, we recall the definition and the effectiveness of F-space (see [1, 11, 22]).

Definition 2.1. *An F-space E over $\mathbb{C}$ is a complete Hausdorff topological vector space by countable family of semi-norms $\mathfrak{S} = \{||.||_s\}_{s\geq 0}$, such that $s < t \Rightarrow ||u||_s \leq ||u||_t$; $(u \in E)$. Hence, $U \subset E$ is open if and only if $\forall\ u \in U,\ \exists\ \epsilon > 0,\ M \geq 0$ such that the set of all open disks of the form $B^s_\epsilon(u) = \{v \in E : ||u - v||_s < \epsilon\} \subset U,\ \forall s \leq M$.*

Definition 2.2. *(Convergent sequences in F-spaces) Let E be the F-space. A sequence $\{u_n\}$ in E converges to v in E if*

$$\lim_{n\to\infty} ||u_n - v||_s = 0$$

for all $||.||_s \in \mathfrak{S}$.

The domains of representation adopted here are the open disk $C(R)$, the closed disk $\overline{C}(R)$, and $C_+(R)$; $R > 0$, where $C_+(R)$ is any open disk enclosing closed disks. These sets are defined as:

$$C(R) = \{z \in \mathbb{C} : |z| < R\},$$

$$\overline{C}(R) = \{z \in \mathbb{C} : |z| \leq R\},$$

$$C_+(R) = \{z \in \mathbb{C} : |z| < R^+\}.$$

Remark 2.1.

(1) *Let $\mathcal{J}[C(R)]$, $\mathcal{J}[\overline{C}(R)]$, $\mathcal{J}[C_+(R)]$, and $\mathcal{J}[0^+]$ denote the classes of functions, which are analytic in $C(R)$, $\overline{C}(R)$, $C_+(R)$, and at the origin, respectively. The countable family of semi-norms of each of the classes mentioned above are given, respectively, by:*

$$\begin{aligned}
||u||_r &= \sup_{\overline{C}(r)} |u(z)|,\ \forall r < R,\ u \in \mathcal{J}[C(R)],\\
||u||_R &= \sup_{\overline{C}(R)} |u(z)|,\ \forall u \in \mathcal{J}[\overline{C}(R)],\\
||u||_r &= \sup_{\overline{C}(r)} |u(z)|\ \forall R < r,\ u \in \mathcal{J}[C_+(R)],\\
||u||_\epsilon &= \sup_{\overline{C}(\epsilon)} |u(z)|,\ \epsilon > 0\ \ \forall u \in \mathcal{J}[0^+],
\end{aligned}$$

making these classes into an F-space.

(2) *Let $\mathcal{J}[\infty]$ be the class of function on the whole plane $\mathbb{C}$. The countable family of semi-norms*

$$||u||_n = \sup_{\overline{C}(n)} |u(z)|,\ \forall u \in \mathcal{J}[\infty],\ n < \infty$$

makes $\mathcal{J}[\infty]$ into an F-space.

Now, suppose that $\{Q_n(z)\}$ is a base of an F-space E such that

$$Q_n(z) = \sum_{k=0}^{\infty} Q_{n,k}\, z^k, \tag{2.1}$$

$$z^n = \sum_{k=0}^{\infty} \pi_{n,k}\, Q_k(z), \tag{2.2}$$

$$\|Q_n\|_R = \sup_{\overline{C}(R)} |Q_n(z)|, \tag{2.3}$$

$$\Psi(Q_n, R) = \sum_{k} |\pi_{n,k}|\, \|Q_k\|_R, \tag{2.4}$$

where the latest sum is called Cannon sum. Moreover,

$$\Psi(Q, R) = \limsup_{n\to\infty} \{\Psi(Q_n, R)\}^{\frac{1}{n}} \tag{2.5}$$

represents the Cannon function of the base $\{Q_n(z)\}$ in the closed disk $\overline{C}(R)$.

Suppose that D_n denotes the highest degree polynomial in (2.2). The following restrictions are imposed:

$$\lim_{n\to\infty} \{D_n\}^{\frac{1}{n}} = 1, \tag{2.6}$$

$$D_n = O[n^a], \quad a \geq 1, \tag{2.7}$$

$$D_n = o(n \log n). \tag{2.8}$$

If $\{Q_n(z)\}$ is a polynomial of degree d_n, then $d_n \leq D_n$ (see [23]).

Let $Q = (Q_{n,k})$ and $\Pi = (\pi_{n,k})$ be the matrices of coefficients and operators, respectively of the set $\{Q_n(z)\}$. Thus according to [23], the set $\{Q_n(z)\}$ will be base if and only if

$$Q\Pi = \Pi Q = I, \tag{2.9}$$

where I is the unit matrix.

Let $u(z) = \sum_{n=0}^{\infty} a_n\, z^n$ be any function that is analytic at the origin. Inserting for z^n from (2.2), we get the basic series:

$$u(z) \sim \sum_{n=0}^{\infty} \Pi_n\, Q_n(z) \tag{2.10}$$

where

$$\Pi_n = \sum_{k=0}^{\infty} a_k\, \pi_{k,n}. \tag{2.11}$$

The authors in [1, 11, 22] introduced the idea of effectiveness for the class $\mathcal{J}[\overline{C}(R)]$. The basic series (2.10) represents $u(z)$ in $\overline{C}(R)$ if it converges uniformly to $u(z)$ in $\overline{C}(R)$. A base $\{Q_n(z)\}$ is effective for the class $\mathcal{J}[\overline{C}(R)]$ if every function $u(z) \in \mathcal{J}[\overline{C}(R)]$ analytic in $\overline{C}(R)$ is represented there by the basic series. Similar definitions are used for effectiveness in the classes $\mathcal{J}[C(R)]$, $\mathcal{J}[C_+(R)]$, $\mathcal{J}[\infty]$, and $\mathcal{J}[0^+]$. The following result was proved in [22].

Theorem 2.1. *A base $\{Q_n(z)\}$ is effective for the classes $\mathcal{J}[\overline{C}(R)]$, $\mathcal{J}[C(R)]$, $\mathcal{J}[C_+(R)]$, $\mathcal{J}[\infty]$, or $\mathcal{J}[0^+]$ if and only if $\Psi(Q,R) = R$, $\Psi(Q,r) < R \ \forall\ r < R$, $\Psi(Q,R^+) = R$, $\Psi(Q,R) < \infty\ \forall\ R < \infty$, or $\Psi(Q,0^+) = 0$, respectively.*

Now, we give two examples that explain the approximation of analytic functions by basic series of polynomials:

Example 2.1. *Consider the BPs $\{Q_n(z)\}$ defined by*

$$Q_n(z) = \begin{cases} 1, & n = 0, \\ 1 + z^n, & n \geq 1. \end{cases}$$

We can write z^n as follows:

$$z^n = Q_n(z) - Q_0(z),$$

that is the representation is available and

$$\pi_{n,0} = 1, \quad \pi_{n,k} = 0 \quad \text{for} \ \ k \neq o, n. \tag{2.12}$$

It follows that $\Psi(Q_n, R) = 2 + R^n$. Take $R = 1$, $\Psi(Q_n, 1) = 3$, and

$$\Psi(Q, 1) = \limsup_{n\to\infty} \{\Psi(Q_n, 1)\}^{\frac{1}{n}} = 1.$$

Therefore, $\Psi(Q, 1) = 1$ and the base $\{Q_n(z)\}$ is effective for $H_{[\overline{C}(1)]}$. Suppose that $u(z) = e^z$ is function analytic in $\overline{C}(1)$, then the basic series $\sum_{n=0}^{\infty} \Pi_n(u)\ Q_n(z)$ represents this function in $\overline{C}(1)$. To find the actual form of the basic series, we substitute from (2.12) in (2.10) to obtain the following:

$$e^z \sim 2 - e + \sum_{n=1}^{\infty} \frac{Q_n(z)}{n!}.$$

Example 2.2. *Consider the BPs $\{Q_n(z)\}$ defined by*

$$Q_n(z) = \begin{cases} 1, & n = 0, \\ z^n - n\, z^{n-1}, & n \geq 1. \end{cases}$$

It is clear that

$$z^n = n! \sum_{j=0}^{n} \frac{1}{j!} Q_j(z).$$

Hence,

$$\Psi(Q_n, R) = n! \sum_{j=0}^{n} \frac{1}{j!} \|Q_j\|_R > n!\ \|Q_0\|_R = n!$$

Therefore,

$$\Psi(Q,R) = \limsup_{n\to\infty} \{\Psi(Q_n,R)\}^{\frac{1}{n}} \geq \limsup_{n\to\infty} \{n!\}^{\frac{1}{n}} = \infty,$$

which means that the base $\{Q_n(z)\}$ is not effective in the closed disk $\overline{C}(R)$ for any value of R. We can find a function analytic in $\overline{C}(R)$ not represented by the basic series (2.10) in $\overline{C}(R)$. The formula (2.11) for the coefficients of basic series (2.10) takes the form

$$\Pi_n(u) = -\sum_{k=0}^{\infty} u^{(k)}(0).$$

Applying this to e^z, an analytic function in $\overline{C}(R)$, we obtain $\Pi_n(u) = -(1 + 1 + \dots)$. Since the formula (2.11) fails to define coefficients for the basic series (2.10), it is customary to say that the basic series does not exist and certainly does not represent e^z.

3. Complex conformable Ruscheweyh derivative bases

Recently, the authors of [21] introduced a new complex conformable derivative (CCD) with noninteger order α, which coincides with the classical complex derivative and integral for $\alpha = 1$.

Definition 3.1. *Let $u : \mathbb{C} \to \mathbb{C}$ be a complex function. For $\alpha \in (0,1]$, the conformable derivative of order α for the function u is defined as follows:*

$$T_\alpha(u)(z) = \lim_{\epsilon\to 0} \frac{u(z+\epsilon z^{1-\alpha}) - u(z)}{\epsilon}, \ \forall z \in \mathbb{C}. \tag{3.1}$$

Suppose that u and w are α-complex differentiable functions at a point $z_0 \in \mathbb{C}$. In [21], the following properties were verified.

(1) $T_\alpha(au + bw) = aT_\alpha(u) + bT_\alpha(w)$ for $a, b \in \mathbb{R}$.
(2) $T_\alpha(z^q) = qz^{q-\alpha}$ for $q \in \mathbb{R}$.
(3) If $u(z) = \beta$ where β is a constant, then $T_\alpha(u) = 0$.
(4) $T_\alpha(uw) = uT_\alpha(w) + wT_\alpha(u)$.
(5) $T_\alpha(\frac{u}{w}) = \frac{wT_\alpha(u) - uT_\alpha(w)}{w^2}, w(z) \neq 0$.
(6) If u is differentiable, then $T_\alpha(u)(z) = z^{1-\alpha}\, u'(z)$.

Remark 3.1. *Definition 3.1 is a generalization of conformable fractional derivative (CFD) and conformable fractional integral (CFI) defined by Khalil et al. [24]. Khalil's definition has a number of applications in plasma, physics, and engineering astronomy (see [25–30]). Moreover, there are several applications of Definition 3.1 in the theory of complex conformable analysis (see [31–34]).*

In 1993, Miller and Ross [35] introduced the sequential composition of fractional derivative $D^{\langle\alpha\rangle}$ in the following way:

$$D^{\langle\alpha\rangle} f(z) = D^{\alpha_1}(D^{\alpha_2}(\dots(D^{\alpha_i}(f(z))))),$$
$$\langle\alpha\rangle = \alpha_1 + \alpha_2 + \cdots + \alpha_i,$$

$$\alpha_s \in (0,1], s \in \{1,2,\ldots,i\}.$$

Combining the definitions of the sequential composition and the CCD, we aim to establish a generalized Ruscheweyh operator called the CCRD as follows:

Let $\mathcal{A}$ denote the class of functions of the form:

$$u(z) = \sum_{k=0}^{\infty} a_k\, z^k, \tag{3.2}$$

where $u(z)$ is analytic at the origin. For any two analytic functions $u(z)$ and $w(z)$ with

$$u(z) = \sum_{k=0}^{\infty} a_k\, z^k \text{ and } w(z) = \sum_{k=0}^{\infty} b_k\, z^k,$$

the convolution (Hadamard product) is given by the power series as follows:

$$(u * w)(z) = \sum_{k=0}^{\infty} a_k\, b_k\, z^k. \tag{3.3}$$

Let $u(z) \in \mathcal{A}$. Denote by $\Re^{\langle\alpha\rangle} : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$\Re^{\langle\alpha\rangle} u(z) = \frac{z}{(1-z)^{\langle\alpha\rangle+1}} * u(z) = \sum_{k=0}^{\infty} \eta_{n,\langle\alpha\rangle}\, a_k\, z^k$$

where

$$\eta_{n,\langle\alpha\rangle} = \frac{n-1+\langle\alpha\rangle}{\Gamma(\langle\alpha\rangle+1)} \prod_{j=1}^{i-1} (n-1+\langle\alpha\rangle - \sum_{s=1}^{j} \alpha_s).$$

It is obvious that

$$\Re^{\alpha_1} u(z) = \frac{z}{\Gamma(\alpha_1+1)} T_{\alpha_1}(z^{\alpha_1-1} u(z)),$$

$$\Re^{\alpha_1+\alpha_2} u(z) = \frac{z}{\Gamma(\alpha_1+\alpha_2+1)} T_{\alpha_1+\alpha_1}(z^{\alpha_1+\alpha_2-1} u(z))$$

and

$$\Re^{\langle\alpha\rangle} u(z) = \frac{z}{\Gamma(\langle\alpha\rangle+1)} T_{\langle\alpha\rangle}(z^{\langle\alpha\rangle-1} u(z)), \tag{3.4}$$

where $T_{\langle\alpha\rangle} := T_{\alpha_1}(T_{\alpha_2}(\ldots(T_{\alpha_i})))$. The operator $\Re^{\langle\alpha\rangle} u(z)$ defines the CCRD of $u(z)$.

Remark 3.2. *Observe that from (3.4), we have the following:*

$$\Re^{\langle\alpha\rangle} z^n = \eta_{n,\langle\alpha\rangle}\, z^n, \tag{3.5}$$

where

$$\lim_{n\to\infty} \{\eta_{n,\langle\alpha\rangle}\}^{\frac{1}{n}} = 1.$$

Remark 3.3. *If $\alpha_1 = \alpha_2 = \cdots = \alpha_i = \beta$, it follows that the operator in* (3.4) *can be written as*

$$T_{\beta i} = T_\beta(T_\beta(\ldots(T_\beta)))$$

that is, applying the derivative T_β, i times, then (3.5) reduces to

$$\Re^{\beta i}(z^n) = \eta_{n,\beta i}\, z^n, \ \ \beta \in (0,1]. \tag{3.6}$$

Remark 3.4. *If $\alpha_1 = \alpha_2 = \cdots = \alpha_i = 1$, then (3.5) reduces to ordinary Ruscheweyh derivative of order i (see [17])*

$$\Re^i z^n = \eta_{n,i}\, z^n, \tag{3.7}$$

where $\eta_{n,i} = \begin{pmatrix} i+n-1 \\ i \end{pmatrix}$.

By applying the operator $\Re^{\langle\alpha\rangle}$ into (2.1), we conclude the following definition.

Definition 3.2. *Let $\{Q_n(z)\}$ be a base. The CCRDB is defined as:*

$$\Re^{\langle\alpha\rangle}(Q_n)(z) = \sum_k Q_{n,k}\, \eta_{k,\langle\alpha\rangle}\, z^k. \tag{3.8}$$

For the sake of shortening notations, we write $\{\Re^{\langle\alpha\rangle}(Q_n)(z)\} = \{\Re_n^{\langle\alpha\rangle}(z)\}$.

In this paper, knowing the effectiveness of the base $\{Q_n(z)\}$ in different regions in F-spaces, we study the effectiveness properties of the derived CCRDB $\{\Re_n^{\langle\alpha\rangle}(z)\}$ in these regions. Furthermore, the upper bounds of the order and type of the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ are determined in terms of the order and type of the original base $\{Q_n(z)\}$. The conditions for which the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ has the same $\mathbb{T}_\rho$-property as $\{Q_n(z)\}$ are investigated.

4. Effectiveness of the CCRDBPs for the class $\mathcal{J}[\overline{C}(R)]$

This section discusses the base property of CCRD $\{\Re_n^{\langle\alpha\rangle}(z)\}$ and the conditions on the base $\{Q_n(z)\}$ to possess the effectiveness of CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ in the space $\mathcal{J}[\overline{C}(R)]$.

Theorem 4.1. *If $\{Q_n(z)\}$ is a base, then the CCRD set $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is also a base.*

Proof. By constructing the coefficient matrix $\Re^{\langle\alpha\rangle}$ and applying the CCRD $\Re^{\langle\alpha\rangle}$ into (2.1), we have

$$\Re^{\langle\alpha\rangle} Q_n(z) = \Re_n^{\langle\alpha\rangle}(z) = \sum_k Q_{n,k}\, \eta_{k,\langle\alpha\rangle} z^k.$$

Hence, the coefficients matrix $\Re^{\langle\alpha\rangle}$ is given by the following:

$$\Re^{\langle\alpha\rangle} = (\Re_{n,k}^{\langle\alpha\rangle}) = (\eta_{k,\langle\alpha\rangle}\, Q_{n,k}).$$

Also, the operators matrix $\Pi^{\langle\alpha\rangle}$ follows from the effect $\Re^{\langle\alpha\rangle}$ on both sides of the representation (2.2) where

$$z^n = \frac{1}{\eta_{n,\langle\alpha\rangle}} \sum_k \pi_{n,k}\, \Re_k^{\langle\alpha\rangle}(z)$$

and

$$\Pi^{\langle\alpha\rangle} = (\pi_{n,k}^{\langle\alpha\rangle}) = \left(\frac{1}{\eta_{n,\langle\alpha\rangle}}\ \pi_{n,k}\right).$$

Consequently,

$$\Re^{\langle\alpha\rangle}\Pi^{\langle\alpha\rangle} = \left(\sum_k R_{n,k}^{\langle\alpha\rangle}\pi_{k,h}^{\langle\alpha\rangle}\right) = \left(\sum_k Q_{n,k}\pi_{k,h}\right) = (\eta_{n,h}) = I.$$

Moreover,

$$\Pi^{\langle\alpha\rangle}\Re^{\langle\alpha\rangle} = \left(\sum_k \pi_{n,k}^{\langle\alpha\rangle}\Re_{k,h}^{\langle\alpha\rangle}\right) = \left(\sum_k \frac{1}{\eta_{n,\langle\alpha\rangle}}\pi_{n,k}\ \eta_{h,\langle\alpha\rangle}\ Q_{k,h}\right) = \left(\frac{\eta_{h,\langle\alpha\rangle}}{\eta_{n,\langle\alpha\rangle}}\eta_{n,h}\right) = I.$$

We easily obtain from (2.9) that the set $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is a base. □

Theorem 4.2. *The BPs $\{Q_n(z)\}$ for which the condition (2.6) is satisfied and its CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ have the same region of effectiveness for the class $\mathcal{J}[\overline{C}(R)]$.*

Proof. To obtain a fundamental inequality concerning the Cannon sum $\Psi\left(\Re_n^{\langle\alpha\rangle}, R\right)$ of the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$, we write

$$\begin{aligned}
\|\Re_n^{\langle\alpha\rangle}\|_R &\le \sum_j |Q_{n,j}|\eta_{j,\langle\alpha\rangle}R^j \\
&\le \|Q_n\|_R \sum_j \eta_{j,\langle\alpha\rangle} \\
&\le \|Q_n\|_R\ \eta_{d_n,\langle\alpha\rangle}\left(\eta_{d_n,\langle\alpha\rangle} + 1\right).
\end{aligned} \tag{4.1}$$

Applying (2.4) and (4.1), it follows that

$$\begin{aligned}
\Psi(\Re_n^{\langle\alpha\rangle}, R) &= \sum_k |\pi_{n,k}^{\langle\alpha\rangle}|\ \|R_k^{\langle\alpha\rangle}\|_R \\
&\le \frac{1}{\eta_{n,\langle\alpha\rangle}} \sum_k |\pi_{n,k}|\ \|Q_k\|_R\ \eta_{d_k,\langle\alpha\rangle}\left(\eta_{d_k,\langle\alpha\rangle} + 1\right) \\
&\le \frac{1}{\eta_{n,\langle\alpha\rangle}}\eta_{D_n,\langle\alpha\rangle}\left(\eta_{D_n,\langle\alpha\rangle} + 1\right)\Psi(Q_n, R).
\end{aligned} \tag{4.2}$$

A combination of (2.5), (2.6), and (4.2) gives $\Psi(\Re^{\langle\alpha\rangle}, R) \le \Psi(Q, R) \le R$, but $\Psi(\Re^{\langle\alpha\rangle}, R) \ge R$. We finally deduce that $\Psi(\Re^{\langle\alpha\rangle}, R) = R$ and the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is effective for $\mathcal{J}[\overline{C}(R)]$. □

We now show that Theorem 4.2 may be false if the condition (2.6) is not satisfied.

Example 4.1. *Define*

$$Q_n(z) = \begin{cases} z^n, & n \text{ is even,} \\ z^n + z^b,\ b = 2^n, & n \text{ is odd.} \end{cases}$$

In the case when n is even, we have $z^n = Q_n(z)$ and, hence, $\Psi(Q_n, R) = R^n$. Thus, if $R = 1$, then $\Psi(Q_n, 1) = 1$ and $\lim_{n\to\infty}\{\Psi(Q_{2n}, 1)\}^{\frac{1}{2n}} = 1$.

Now, if n is odd, then $z^n = Q_n(z) - Q_b(z)$ and

$$\Psi(Q_n, R) = R^n + 2\,R^b.$$

Thus, putting $R = 1$ implies that $\Psi(Q_n, 1) = 3$ and we obtain

$$\lim_{n\to\infty} \{\Psi(Q_{2n+1}, 1)\}^{\frac{1}{2n+1}} = 1.$$

Consequently, $\Psi(Q, 1) = \limsup_{n\to\infty} \{\Psi(Q_n, 1)\}^{\frac{1}{n}} = 1$ *and the base* $\{Q_n(z)\}$ *is effective for* $\mathcal{J}[\overline{C}(1)]$.

By forming the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$*, it follows that*

$$\Re_n^{\langle\alpha\rangle}(z) = \begin{cases} \eta_{n,\langle\alpha\rangle}\, z^n, & n \geq 2 \text{ is even and} \\ \eta_{n,\langle\alpha\rangle}\, z^n + \eta_{b,\langle\alpha\rangle}\, z^b, & n \text{ is odd.} \end{cases}$$

Since $z^n = (1 \setminus \eta_{n,\langle\alpha\rangle})\, \Re_n^{\langle\alpha\rangle}(z)$*, then when n is even, we obtain* $\Psi(\Re_n^{\langle\alpha\rangle}, R) = R^n$*. Putting* $R = 1$ *implies that* $\Psi(\Re_n^{\langle\alpha\rangle}, 1) = 1$*. Hence,*

$$\lim_{n\to\infty} \{\Psi(\Re_{2n}^{\langle\alpha\rangle}, 1)\}^{\frac{1}{2n}} = 1.$$

Moreover, when n is odd, we have $z^n = (1 \setminus \eta_{n,\langle\alpha\rangle})[\Re_n^{\langle\alpha\rangle}(z) - \eta_{b,\langle\alpha\rangle}\Re_b^{\langle\alpha\rangle}(z)]$*. Therefore,* $\Psi(\Re_n^{\langle\alpha\rangle}, R) = (1 \setminus \eta_{n,\langle\alpha\rangle})[\eta_{n,\langle\alpha\rangle}R^n + 2\, \eta_{b,\langle\alpha\rangle}R^b]$*. Consider* $R = 1$*, then*

$$\Psi(\Re^{\langle\alpha\rangle}, 1) = \limsup_{n\to\infty} \{\Psi(\Re_{2n+1}^{\langle\alpha\rangle}, 1)\}^{\frac{1}{2n+1}} = 2 > 1$$

and the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ *is not effective for* $\mathcal{J}[\overline{C}(1)]$.

Remark 4.1. *In Example 4.1, it was shown the the CCRDBPs* $\{\Re_n(z)\}$ *is not effective for* $\mathcal{J}[\bar{C}(1)]$*. With the same idea as Example 2.2, we can find an analytic function not represented by the basic series* (2.10) *in* $\bar{C}(1)$.

For a simple base of polynomials (SBPs) ($D_n = n$) (see [23]), we get the following result.

Corollary 4.1. *When the SBPs* $\{Q_n(z)\}$ *is effective for* $\mathcal{J}[\overline{C}(R)]$*, then so will be the CCRDBPs* $\{\Re_n^{\langle\alpha\rangle}(z)\}$.

5. Effectiveness of the CCRDBs for the classes $\mathcal{J}[C(R)]$, $\mathcal{J}[0^+]$, $\mathcal{J}[\infty]$, and $\mathcal{J}[C_+(R)]$

This section discusses the conditions on the base $\{Q_n(z)\}$ to possess the effectiveness of CCRDB $\{\Re_n^{\langle\alpha\rangle}(z)\}$ in the spaces $\mathcal{J}[C(R)]$, $\mathcal{J}[0^+]$, $\mathcal{J}[\infty]$, and $\mathcal{J}[C_+(R)]$.

Theorem 5.1. *The base* $\{Q_n(z)\}$ *and its CCRDB* $\{\Re_n^{\langle\alpha\rangle}(z)\}$ *have the same region of effectiveness for the classes* $\mathcal{J}[C(R)]$, $\mathcal{J}[0^+]$, $\mathcal{J}[\infty]$, *and* $\mathcal{J}[C_+(R)]$.

Proof. The base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ satisfies that

$$\Psi\left(\Re_n^{\langle\alpha\rangle}, r\right) = \sum_k |\pi_{n,k}^{\langle\alpha\rangle}|\, \|\Re_k^{\langle\alpha\rangle}\|_r.$$

Since

$$\|\Re_n^{\langle\alpha\rangle}\|_r = \sup_{\overline{C}(r)} |\Re_n^{\langle\alpha\rangle}(z)|$$

$$
\begin{aligned}
&= \sup_{\overline{C}(r)} \left| \sum_j Q_{n,j} \eta_{j,\langle\alpha\rangle} z^j \right| \\
&\leq \sum_j \frac{\|Q_n\|_R}{R^j} \eta_{j,\langle\alpha\rangle} r^j \\
&= \|Q_n\|_R \sum_j \eta_{j,\langle\alpha\rangle} \left(\frac{r}{R}\right)^j \\
&= k_1 \|Q_n\|_R
\end{aligned}
$$

for all $r < R$ where $k_1 = \sum_j \eta_{j,\langle\alpha\rangle} \left(\frac{r}{R}\right)^j$, it follows that

$$\Psi\left(\Re_n^{\langle\alpha\rangle}, r\right) \leq \frac{k_1}{\eta_{n,\langle\alpha\rangle}} \sum_k |\pi_{n,k}| \, \|Q_k\|_R = \frac{k_1}{\eta_{n,\langle\alpha\rangle}} \Psi\left(Q_n, R\right). \tag{5.1}$$

Therefore,

$$\Psi\left(\Re^{\langle\alpha\rangle}, r\right) = \lim_{R\to\infty} \sup\{\Psi\left(\Re_n^{\langle\alpha\rangle}, r\right)\}^{\frac{1}{n}} \leq \Psi(Q, R) \tag{5.2}$$

for all $r < R$.

Now, suppose that the base $\{Q_n(z)\}$ is effective for $\mathcal{J}[C(R)]$. We can apply Theorem 2.1 and we have

$$\Psi(Q, r) < R, \text{ for all } r < R. \tag{5.3}$$

Hence, there is a number r_1 such that $r < r_1 < R$. Therefore, using (5.2) and (5.3), we deduce that

$$\Psi(\Re^{\alpha}, r) \leq \Psi(Q, r_1) < R, \text{ for all } r < R,$$

that is, to say the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is effective for $\mathcal{J}[C(R)]$.

By the effectiveness of the base $\{Q_n(z)\}$ for $\mathcal{J}[0^+]$ and applying Theorem 2.1, it follows that $\Psi(Q, 0^+) = 0$. By making R and r tend to zero from the right in (5.2), we have $\Psi(\Re^{\alpha}, 0^+) \leq \Psi(Q, 0^+) = 0$, but we know that $\Psi(\Re^{\alpha}, 0^+) \geq 0$, thus, $\Psi(\Re^{\alpha}, 0^+) = 0$. Therefore, the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is effective for $\mathcal{J}[0^+]$.

Now, suppose that the base $\{Q_n(z)\}$ is effective for $\mathcal{J}[\infty]$. Applying Theorem 2.1, we conclude that

$$\Psi(Q, r) < \infty \text{ for all } r < \infty. \tag{5.4}$$

Thus, if we choose the number r_2 such that $r < r_2 < R$, making $R \to \infty$ in (5.2), then by using (5.4), we obtain that

$$\Psi(\Re^{\langle\alpha\rangle}, r) \leq \Psi(Q, r_2) < \infty \text{ for all } r < \infty \tag{5.5}$$

and the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is effective for $\mathcal{J}[\infty]$.

If the base $\{Q_n(z)\}$ is effective for $\mathcal{J}[C_+(R)]$ and r_3 is any positive number such that $r_3 < r$, we can apply Theorem 2.1 and we obtain

$$\Psi(Q, r_3^+) = r_3, \quad r_3 < r < R. \tag{5.6}$$

Making $R \to r_3^+$ in (5.2), we easily obtain from (5.6) that $\Psi(\Re^{\langle\alpha\rangle}, r_3^+) \leq \Psi(Q, r_3^+) = r_3$. However, $\Psi(\Re^{\langle\alpha\rangle}, r_3^+) \geq r_3$, which implies that $\Psi(\Re^{\langle\alpha\rangle}, r_3^+) = r_3$ and, consequently, the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is effective for $\mathcal{J}[C_+(R)]$. □

6. Order, type, and the $\mathbb{T}_\rho$-property of the CCRDBPs

The authors [4, 5] introduced the idea of the order and type of the BPs $\{Q_n(z\}$ as follows:

Definition 6.1. *The order and type of the BPs $\{Q_n(z\}$ are given, respectively, by*

$$\rho_Q = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log\ \Psi(Q_n, R)}{n\ \log\ n} \tag{6.1}$$

and

$$\tau_Q = \lim_{R\to\infty} \frac{e}{\rho} \limsup_{n\to\infty} \frac{\{\Psi(Q_n, R)\}^{\frac{1}{n\rho}}}{n}. \tag{6.2}$$

Importantly, if the base $\{Q_n(z)\}$ has finite order ρ_Q and finite type τ_Q, then it can represent every entire function of order less than $\frac{1}{\rho_Q}$ and type less than $\frac{1}{\tau_Q}$ in any finite disk.

Now, let $\{Q_n(z)\}$ be a BPs of order ρ_Q and type τ_Q, and the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is of order $\rho_{\Re^{\langle\alpha\rangle}}$ and type $\tau_{\Re^{\langle\alpha\rangle}}$.

The following theorem provides the relation between the orders and types of the original base $\{Q_n(z)\}$ and the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$.

Theorem 6.1. *Let ρ_Q and τ_Q be the order and type of the BPs $\{Q_n(z)\}$ satisfying the condition (2.7), then the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ has order $\rho_{\Re^{\langle\alpha\rangle}} \le \rho_Q$ and type $\tau_{\Re^{\langle\alpha\rangle}} \le \tau_Q$ whenever $\rho_{\Re^{\langle\alpha\rangle}} = \rho_Q$. The values of ρ_Q and type τ_Q are attainable.*

Proof. The proof of this theorem is denoted on the inequality (4.2) since

$$\Psi(\Re_n^{\langle\alpha\rangle}, R) \le \frac{1}{\eta_{n,\langle\alpha\rangle}} \eta_{D_n,\langle\alpha\rangle} \left(\eta_{D_n,\langle\alpha\rangle} + 1\right) \Psi(Q_n, R)$$

then

$$\lim_{R\to\infty} \lim_{n\to\infty} \frac{\log \Psi(\Re_n^{\langle\alpha\rangle}, R)}{n \log n} \le \lim_{R\to\infty} \lim_{n\to\infty} \frac{\log \eta_{D_n,\langle\alpha\rangle}(\eta_{D_n,\langle\alpha\rangle} + 1) + \log \Psi(Q_n, R)}{n \log n}. \tag{6.3}$$

In view of (6.1), it follows that the order of the CCRDBPs is at most ρ_Q. If $\rho_{\Re^{\langle\alpha\rangle}} = \rho_Q$, we have

$$\lim_{R\to\infty} \frac{e}{\rho_{\Re^{\langle\alpha\rangle}}} \limsup_{n\to\infty} \frac{[\Psi(\Re_n^{\langle\alpha\rangle}, R)]^{\frac{1}{n\rho_{\Re^{\langle\alpha\rangle}}}}}{n} \le \lim_{R\to\infty} \frac{e}{\rho_P} \limsup_{n\to\infty} \frac{[\Psi(Q_n, R)]^{\frac{1}{n\rho_P}}}{n}. \tag{6.4}$$

Therefore, the type of the CCRDBPs is at most τ_Q. □

Note that the upper bound in Theorem 6.1 is attainable and the following example explains this fact.

Example 6.1. *Suppose that $\{Q_n(z)\}$ is the BPs defined by $Q_n(z) = n^n + z^n, \quad Q_0(z) = 1,$ for which*

$$\Psi(Q_n, R) = n^n \left[2 + \left(\frac{R}{n}\right)^n\right].$$

Simple steps for the base $\{Q_n(z)\}$ is of order $\rho_P = 1$ and type $\tau_P = e$. Now, Construct the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ such that

$$\Re_n^{\langle\alpha\rangle}(z) = n^n + \eta_{n,\langle\alpha\rangle}\, z^n, \ \ \Re_0^{\langle\alpha\rangle}(z) = 1.$$

Hence,

$$\Psi(\Re_n^{\langle\alpha\rangle}, R) = \frac{n^n}{\eta_{n,\langle\alpha\rangle}} \left[2 + \frac{\eta_{n,\langle\alpha\rangle}}{R^{\langle\alpha\rangle}} \left(\frac{R}{n}\right)^n\right].$$

Therefore, $\rho_{\Re^{\langle\alpha\rangle}} = 1$ *and type* $\tau_{\Re^{\langle\alpha\rangle}} = e$ *for the base* $\{\Re_n^{\langle\alpha\rangle}(z)\}$.

The following example shows the importance of condition (2.7) in Theorem 6.1.

Example 6.2. *Let the BPs* $\{Q_n(z)\}$ *be defined by*

$$Q_n(z) = \begin{cases} z^n, & n \text{ is even,} \\ \mu\ (\frac{z}{b})^{2\mu} + z^n, & \mu = n^n,\ b > 1 \text{ and } n \text{ is odd.} \end{cases}$$

Hence,

$$z^n = Q_n(z) - \frac{\mu}{b^{2\mu}} Q_{2\mu}(z)$$

and

$$\Psi(Q_n, R) = R^n + 2\mu \left(\frac{R}{b}\right)^{2\mu}.$$

Applying the definition of the order, we get $\rho_Q = 1$.

Now, we construct the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ *as follows:*

$$\Re_n^{\langle\alpha\rangle}(z) = \begin{cases} \eta_{n,\langle\alpha\rangle}\ z^n, & n \text{ is even,} \\ \eta_{n,\langle\alpha\rangle}\ z^n + \frac{\mu}{b^{2\mu}}\ \eta_{2\mu,\langle\alpha\rangle}\ z^{2\mu}, & n \text{ is odd.} \end{cases}$$

Thus,

$$z^n = \frac{1}{\eta_{n,\langle\alpha\rangle}} \Re_n^{\langle\alpha\rangle}(z) - \frac{\mu}{b^{2\mu}}\ \frac{\eta_{2\mu<\alpha>}}{\eta_{n,\langle\alpha\rangle}}\ \Re_{2\mu}^{\langle\alpha\rangle}(z).$$

Consequently,

$$\Psi(\Re_n^{\langle\alpha\rangle}, R) = R^n + \frac{2\mu}{b^{\langle\alpha\rangle}}\ \frac{\eta_{2\mu,\langle\alpha\rangle}}{\eta_{n,\langle\alpha\rangle}}\ \left(\frac{R}{b}\right)^{2\mu}.$$

Therefore, $\rho_{\Re^{\langle\alpha\rangle}} = 2$ *and* $\rho_{\Re^{\langle\alpha\rangle}} > \rho_P$ *as required.*

If the base of polynomials $\{Q_n(z)\}$ is a simple $(D_n = n)$ (see [23]), then the following corollary is a special case of Theorem 6.1.

Corollary 6.1. *When the SBPs* $\{Q_n(z)\}$ *is of type* τ_Q *and order* ρ_Q*, then the CCRDBPs* $\{\Re_n^{\langle\alpha\rangle}(z)\}$ *will be of type* $\tau_{\Re^{\langle\alpha\rangle}} \leq \tau_P$ *and order* $\rho_{\Re^{\langle\alpha\rangle}} \leq \rho_P$ *whenever* $\rho_{\Re^{\langle\alpha\rangle}} = \rho_P$.

Now, we discuss the $\mathbb{T}_{\rho_{\Re^{\langle\alpha\rangle}}}$-property of CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ in $\overline{C}(R)$, $C(R)$, and at the origin.

The $\mathbb{T}_{\rho_Q}$-property of the base $\{Q_n(z)\}$ is defined by the authors in [36] as follows:

Definition 6.2. *If the base* $\{Q_n(z)\}$ *represents all entire functions of order less than* ρ_Q *in* $\overline{C}(R)$, $C(R)$, *or at the origin, then it is said to have property* $\mathbb{T}_{\rho_Q}$ *in* $\overline{C}(R)$, $C(R)$, *or at the origin.*

Let

$$\psi(Q,R) = \limsup_{n\to\infty} \frac{\log\ \Psi(Q_n,R)}{n\ \log\ n}.$$

The following theorem concerns the property $\mathbb{T}_{\rho_Q}$ of the base $\{Q_n(z)\}$ (see [36]).

Theorem 6.2. *A base $\{Q_n(z)\}$ is to have the property $\mathbb{T}_{\rho_Q}$ for all entire functions (EFs) of order less than ρ_Q in $C(R)$, $\overline{C}(R)$, or at the origin if and only if, $\psi(Q,R) \leq \frac{1}{\rho_Q}$, $\psi(Q,r) \leq \frac{1}{\rho_Q}$ for all $r < R$ or $\psi(Q,0^+) \leq \frac{1}{\rho_Q}$.*

Next, we construct the $\mathbb{T}_{\rho_{\mathfrak{R}_n^{\langle\alpha\rangle}}}$-property of the CCHDBPs in the closed disk $\overline{C}(R)$ for $R > 0$.

Theorem 6.3. *Let $\{Q_n(z)\}$ be the BPs that have T_{ρ_Q}-property in $\overline{C}(R)$, where $R > 0$ and for which the condition (2.8) is satisfied, then the CCRDBPs $\{\mathfrak{R}_n^{\langle\alpha\rangle}(z)\}$ have the same property.*

Proof. Suppose that the function $\psi(\mathfrak{R}^{\langle\alpha\rangle},R)$ is given by:

$$\psi(\mathfrak{R}^{\langle\alpha\rangle},R) = \limsup_{n\to\infty} \frac{\log\Psi(\mathfrak{R}_n^{\langle\alpha\rangle},R)}{n\log n}, \tag{6.5}$$

where $\Psi(\mathfrak{R}_n^{\langle\alpha\rangle},R)$ is the Cannon sum of the CCRDBPs $\{\mathfrak{R}_n^{\langle\alpha\rangle}(z)\}$ then by using (2.8), (4.2), and (6.5), we obtain that

$$\begin{aligned}\psi(\mathfrak{R}^{\langle\alpha\rangle},R) &\leq \limsup_{n\to\infty} \frac{\log\eta_{D_n,\langle\alpha\rangle}(\eta_{D_n,\langle\alpha\rangle}+1)+\log\Psi(Q_n,R)}{n\log n}\\ &\leq \psi(Q,R).\end{aligned} \tag{6.6}$$

Since the base $\{Q_n(z)\}$ has the property $\mathbb{T}_{\rho_Q}$ in $\bar{C}(R)$, $R > 0$, by inequality (6.6) and Theorem 6.2, we have

$$\psi(\mathfrak{R}^{\langle\alpha\rangle},R) \leq \psi(Q,R) \leq \frac{1}{\rho_Q}.$$

and the base $\{\mathfrak{R}_n^{\langle\alpha\rangle}(z)\}$ has property $\mathbb{T}_{\rho_Q}$ in $\overline{C}(R)$, $R > 0$. □

The fact that CCRDBPs $\{\mathfrak{R}_n^{\langle\alpha\rangle}(z)\}$ does not have the property $\mathbb{T}_{\rho_Q}$ in $\overline{C}(R)$ if the condition (2.8) is not satisfied is illustrated by the following example.

Example 6.3. *Suppose that $\{Q_n(z)\}$ is defined by:*

$$Q_n(z) = \begin{cases} z^n, & n \text{ is even},\\ z^n + \frac{z^{s(n)}}{2^{n^n}}, & n \text{ is odd}\end{cases}$$

where $s(n)$ is the nearest even integer to $n\log n + n^n$. When n is odd, we obtain:

$$z^n = Q_n(z) - \frac{Q_{s(n)}(z)}{2^{n^n}}.$$

Hence,

$$\Psi(Q_n,R) = R^n + 2\frac{R^{s(n)}}{2^{n^n}}.$$

Putting $R = 2$, it follows that

$$\Psi(Q_n, 2) = 2^n + 2^{n \log n+1}$$

so that

$$\psi(Q, 2) = \limsup_{n\to\infty} \frac{log\ \Psi(Q_n, 2)}{n\ log\ n} \leq \log 2.$$

It follows that the base $Q_n(z)$ has the $\mathbb{T}_{\frac{1}{\log 2}}$-property in $\overline{C}(2)$.

The CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ is

$$\Re_n^{\langle\alpha\rangle}(z) = \begin{cases} \eta_{n,\langle\alpha\rangle} z^n, & n\ even, \\ \eta_{n,\langle\alpha\rangle} z^n + \eta_{t(n),\langle\alpha\rangle}\ \frac{z^{t(n)}}{2^{n^n}}, & n\ odd. \end{cases}$$

Hence, when n is odd, we obtain

$$\Psi(\Re_n^{\langle\alpha\rangle}, R) = R^n + 2\frac{\eta_{t(n),\langle\alpha\rangle}}{\eta_{n,\langle\alpha\rangle}}\ \frac{R^{t(n)}}{2^{n^n}},$$

so that when $R = 2$,

$$\Psi(\Re_n^{\langle\alpha\rangle}, 2) = 2^n + 2\frac{\eta_{t(n),\langle\alpha\rangle}}{\eta_{n,\langle\alpha\rangle}}\ \frac{2^{t(n)}}{2^{n^n}}.$$

Thus,

$$\psi(\Re^{\langle\alpha\rangle}, 2) = \limsup_{n\to\infty} \frac{\log\ \Psi(\Re_n^{\langle\alpha\rangle}, 2)}{n\ \log\ n} \leq 1 + log2$$

and the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ does not have the $\mathbb{T}_{\frac{1}{\log 2}}$-property in $\overline{C}(2)$, as required.

If the BPs $\{Q_n(z)\}$ is simple ($D_n = n$) (see [23]), we get the following corollary.

Corollary 6.2. *If the SBPs $\{Q_n(z)\}$ has the $\mathbb{T}\rho_Q$ -property in $\overline{C}(R)$, then the base $\{R_n^{\langle\alpha\rangle}(z)\}$, associated with it has the same property.*

In the following, we deduce that the base $\{Q_n(z)\}$ and the CCRDBPs $\{R_n^{\langle\alpha\rangle}(z)\}$ have the same $\mathbb{T}\rho_Q$ in $C(R)$, where $R > 0$ or at the origin.

Theorem 6.4. *Let $\{Q_n(z)\}$ be a BPs that has the $\mathbb{T}\rho_Q$-property at the origin or in $C(R)$, $R > 0$, then the CCRDBPs $\{\Re_n^{\langle\alpha\rangle}(z)\}$ have the same property.*

Proof. Let $\{Q_n(z)\}$ have property $\mathbb{T}\rho_Q$ in $C(R)$, $R > 0$, then

$$\psi(Q, r) \leq \frac{1}{\rho_Q}\ \ \forall r < R. \tag{6.7}$$

Using (5.1), we obtain

$$\psi(\Re^{\langle\alpha\rangle}, r) = \limsup_{n\to\infty} \frac{\log\ \Psi(\Re_n^{\langle\alpha\rangle}, R)}{n \log\ n} \leq \psi(Q, r_1). \tag{6.8}$$

such that $r < r_1 < R$. Using (6.7) and (6.8), we have $\psi(\Re^{\langle\alpha\rangle}, r) \leq \frac{1}{\rho_Q}$ for all $r < R$, and the the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ has the $\mathbb{T}\rho_Q$-property in $C(R)$, $R > 0$.

If $\{Q_n(z)\}$ has the $\mathbb{T}\rho_Q$-property at the origin, then

$$\psi(Q, 0^+) \leq \frac{1}{\rho_Q}. \tag{6.9}$$

Let $r_1 \to 0^+$ in (6.8), then by (6.9), we have

$$\psi(\Re^{\langle\alpha\rangle}, 0^+) \leq \psi(Q, 0^+) \leq \frac{1}{\rho_Q}.$$

and the base $\{\Re_n^{\langle\alpha\rangle}(z)\}$ has the property $\mathbb{T}\rho_Q$ at the origin. □

7. Applications

The problem of classical special functions such as Bessel, Chebyshev, Bernoulli and Euler polynomials can be considered as an application of BPs. These polynomials are a family of orthogonal polynomials that arise in approximation theory, numerical analysis, and signal processing. Also, these polynomials have applications in number theory, combinatorics, and algebraic geometry. Recently, the authors in [14, 15] proved that the base of proper Bessel polynomials (BPBPs) $\{P_n(z)\}$ and the base of general Bessel polynomials (BGBPs) $\{G_n(z)\}$ are effective for $\mathcal{J}[\overline{C}(R)]$.

The following corollaries are immediate consequences of Theorem 4.2.

Corollary 7.1. *(1) The BPBPs $\{P_n(z)\}$ and the CCRD of BPBPs $\{\Re_n^{\langle\alpha\rangle}P_n(z)\}$ have the same region of effectiveness for the class $\mathcal{J}[\overline{C}(R)]$.*

(2) The BGBPs $\{G_n(z)\}$ and the CCRD of BGBPs $\{\Re_n^{\langle\alpha\rangle}G_n(z)\}$ have the same region of effectiveness for the class $\mathcal{J}[\overline{C}(R)]$.

Recently in [16], it is proved that the base of Chebyshev polynomial (BCPs) $\{C_n(z)\}$ is effective for $\mathcal{J}[\overline{C}(1)]$. With the application of Theorem 4.2, we get the following result:

Corollary 7.2. *The BCPs $\{C_n(z)\}$ and the CCRD of BCPs $\{\Re_n^{\langle\alpha\rangle}C_n(z)\}$ have the same region of effectiveness for the class $\mathcal{J}[\overline{C}(R)]$.*

The base of Bernoulli polynomials (BBPs) $\{B_n(z)\}$ is of type $\frac{1}{2\pi}$ and order 1, and the base of Euler polynomials (BEPs)) $\{E_n(z)\}$ is of type $\frac{1}{\pi}$ and order 1 (see [13]).

Owing to Theorem 6.1, we obtain the following results:

Corollary 7.3. *The BBPs $\{B_n(z)\}$ and the CCRD of BBPs $\{\Re_n^{\langle\alpha\rangle}B_n(z)\}$ are of the same type $\frac{1}{2\pi}$ and order* 1.

Corollary 7.4. *The BEPs $\{E_n(z)\}$ and the CCRD of BEPs $\{\Re_n^{\langle\alpha\rangle}E_n(z)\}$ are of the same type $\frac{1}{\pi}$ and order* 1.

Moreover, in [13], the BBPs $\{B_n(z)\}$ and the BEPs $\{E_n(z)\}$ have the property $\mathbb{T}_1$.

From Theorem 6.3, we conclude directly the following corollary:

Corollary 7.5. *If the BBPs $\{B_n(z)\}$ and the BEPs $\{E_n(z)\}$ have the property $\mathbb{T}_1$, then the CCRD of BBPs $\{\Re_n^{\langle\alpha\rangle}B_n(z)\}$ and BEPs $\{\Re_n^{\langle\alpha\rangle}E_n(z)\}$ have the same property, respectively.*

Suppose that $J_N(\Re_n^{\langle\alpha\rangle})$ is a polynomial of the operator $\Re_n^{\langle\alpha\rangle}$ as given in (3.4) such that

$$J_N(\Re_n^{\langle\alpha\rangle}) = \sum_{j=1}^{N} \lambda_j \, (\Re_n^{\langle\alpha\rangle})^j, \quad \lambda_i \in \mathbb{C},$$

where $(\Re_n^{\langle\alpha\rangle})^j = (\Re_n^{\langle\alpha\rangle})^{j-1}\Re_n^{\langle\alpha\rangle}$. Clearly, Theorems 4.1, 4.2, 5.1, 6.1, 6.3, and 6.4 will be valid when we replace the base $\{\Re_n^{\langle\alpha\rangle} Q_n(z)\}$ by the base $\{J_N(\Re_n^{\langle\alpha\rangle})Q_n(z)\}$.

It must be stated here that we can obtain similar results for the generalized complex conformable derivative and integral BPs $\{J_N(T_\alpha)Q_n(z)\}$ and $\{J_N(I_\alpha)Q_n(z)\}$. These results generalize the result in [1].

8. Conclusions

Owing to the increasing interest in fractional calculus and its many practical applications, this study concentrated on modeling analytic functions in terms of complex conformable fractional derivatives and integral bases in various domains in F-spaces. In the current work, we constructed a generalized Ruscheweyh derivative of a noninteger order in the complex conformable sense. We investigated the effectiveness, the growth order and type and the $\mathbb{T}_\rho$-property of CCRDBPs in F-spaces in several domains: Closed disks $\overline{C}(R)$, open disks $C(R)$, open regions surrounding closed disks $C_+(R)$, at the origin, and for all entire functions. Furthermore, some applications on the CCRDBPs of Bernoulli, Euler, Bessel, and Chebyshev polynomials were discussed. Our results are considered as a modified generalization to those given in [37–39].

It is clear that when $\langle\alpha\rangle = 1$ in Theorems 4.1, 4.2, 5.1, 6.1, 6.3, and 6.4, results obtained by [37–39] yield. In the future, it will be of great interest to study some geometric properties of a class of analytic functions associated with CCRDs. Moreover, it is possible to study some applications for partial differential equations with solutions approximated by infinite basic series using numerical methods and compare them with other methods. The convergence properties of CCRDBPs can be examined in the case of higher-dimensional spaces (greater than three) in hyperspherical regions, polycylinderical regions, hyperelliptical regions, or Faber regions, which will be a great start for further study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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