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Research article

Numerical study of fractional phi-4 equation

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Abstract: In this paper, we established an analytical solution for the fractional phi-4 model within the Caputo derivative using the homotopy analysis method. This equation known for its nonlinear characteristics often describes various physical phenomena like solitons, wave propagation, and field theories. The fractional version introduces fractional derivatives, making it even more challenging. The homotopy analysis method can effectively handle these nonlinearities. Our objective was to illustrate the reliability and accuracy of our proposed algorithm, which we achieved through a comparative analysis against results obtained using the Yang transform decomposition method. Using the residual error to determine the optimal value of the convergence control parameter $\hbar$, the results presented underscored the remarkable efficiency and accuracy of this approach.

Keywords: phi-4 equation; Caputo fractional derivative; homotopy analysis method
Mathematics Subject Classification: 26A33, 65-XX, 55P10

1. Introduction

Fractional differential equations (FDEs) are a type of differential equation that involve fractional derivatives, providing a more accurate description of various physical, biological, and engineering phenomena. These equations have gained significant attention in recent years due to their ability to model complex processes with memory and hereditary properties. Unlike classical integer-order differential equations, fractional derivatives are nonlocal operators, incorporating information from the entire history of a system.

To solve FDEs, a variety of numerical methods have been developed, adapted, and refined. These methods bridge the gap between the theoretical framework of FDEs and practical applications, making it possible to obtain numerical solutions for a wide range of problems. Some prominent numerical methods are used for solving FDEs [1]. The phi-4 equation is a wave equation given as

$$u_{tt}(x,t) = u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t). \tag{1.1}$$

Many mathematician researchers have devoted their efforts to tackling the challenges posed by FDEs, employing various numerical methods to obtain precise and well-suited approximations. A selection of notable approaches stands out among the various methodologies that have been applied. Alquran, in their work [2], harnessed the Jacobi elliptic sine-cosine expansion method to address these equations, while Zahra presented the B-spline collocation method [3]. Bhrawy et al. [4] also proved valuable in this context. Further innovation has come from Alomari et al., who introduced the homotopy Sumudu approach [5], as well as Alquran's application of the modified residual power series method [6]. Additionally, Ehsani et al. [7] explored the homotopy perturbation method. Tariq and Akram investigated the tanh method [8]. Recently, the equation was solved by the Yang transform decomposition method, and the Yang homotopy perturbation transform method [9]. These methods, each with its unique characteristics, have made significant contributions to the expanding realm of knowledge surrounding the numerical solutions of FDEs. As researchers continue to refine and adapt these approaches, they push the boundaries of our comprehension of this vital mathematical domain.

Among the various techniques available, the homotopy analysis method (HAM) stands as one of the most prominent and versatile. It was first introduced by Liao [10–12]. HAM has found applications in solving a wide spectrum of differential equations and encompassing linear and nonlinear ones, including FDEs. For FDEs, scientists have ingeniously combined the coupled Laplace transform with HAM, resulting in a simplified algorithm tailored to this class of equations. This algorithm can be easily implemented using mathematical software such as Mathematica and Maple. Various problems have been solved via HAM such as time-fractional Korteweg-de Vries and Korteweg-de Vries-Burger's equations [13], fluid mechanic problems [14, 15], blasius flow equation [16], coupled Lane-Emden-Fowler type equation [17] and the method investigated for finding multiple solutions to boundary value problems [18]. This fusion of mathematical techniques not only expands the realm of solvable problems in fractional calculus, but also provides powerful tools for researchers and practitioners across various scientific disciplines. Overall, HAM has various features among other analytic techniques, such as containing convergent control parameters, freedom to choose some starting solution, linear operator, and ease in deriving an explicit recursive formula for the series terms. Usually, HAM can give accurate results using a few terms of the solution. On the other hand, the analytic method needs to solve a linear problem in each term, so it needs powerful software and hardware to find the higher terms of the series.

In this paper, we applied the HAM for [2, 3, 5],

$${}^{C}D_t^{\alpha} u = u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t), \tag{1.2}$$

subject to the initial conditions $u(x,0) = f(x), u_t(x,0) = g(x)$, where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative (Cfd) of order α $(1 < \alpha \leq 2)$. Finding a convergent series solution for the fractional phi-4 equation with easy computational terms and analyzing these results in terms of accuracy and convergence will be a great effort in this field. By HAM implementation, a recursive formula for finding the series terms is derived. Also, we proved the convergence of series solutions and made a comparison with the previously published algorithm. The obtained numerical results via the HAM algorithm are more accurate than q-HAM, Yang transforms decomposition method, and Yang homotopy perturbation transform method (YHPM).

2. Some ideas of fractional calculus

In this section, we provide fundamental definitions of the fractional calculus theory utilized in this paper.

Definition 2.1. *We note the function*

$$k(t) \in C_{\mu,\mu\in\mathbb{R}} \; if \, \exists p > \mu : k(t) = t^p k_1(t),$$

where $k_1(t) \in C(0, \infty)$.

Definition 2.2. *The Riemann-Liouville fractional operator of* $k \in C_\mu, \mu \geq -1$ *of order* $\alpha \geq 0$ *is [1],*

$$I^\alpha k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} k(\tau) d\tau, \alpha > 0,$$
$$I^0 k(t) = k(t).$$

We also require the following properties:

For $k \in C_\mu, \mu \geq -1$, $\alpha, \beta \geq 0$, *and* $\gamma \geq -1$*:*

$$I^\alpha I^\beta k(t) = I^{\alpha+\beta} k(t),$$
$$I^\alpha I^\beta k(t) = I^\beta I^\alpha k(t),$$
$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

Definition 2.3. *The Cfd of k,* $h \in C_{-1}^m$ *is*

$${}^C D^\alpha k(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\varsigma)^{(m-\alpha-1)} k^{(m)}(\varsigma) d\varsigma,$$

where $m - 1 < \alpha < m, m \in \mathbb{N}$.

For $m - 1 < \alpha \leq m$, $n \in \mathbb{N}$ *and* $k \in C_\mu^m$, $\mu \geq -1$, *then*

$$I^\alpha D^\alpha k(t) = k(t) - \sum_{n=0}^{m-1} k^{(n)}(0^+) \frac{t^n}{n!}.$$

3. The HAM

Let's begin by introducing the fundamental principles of HAM. To illustrate the application of HAM in solving FDEs, we consider the following fractional differential equation:

$${}^C D^\alpha v(x,t) + \Re v(x,t) + \mathfrak{N} v(x,t) = h(x,t), \quad 1 < \alpha \leq 2. \tag{3.1}$$

${}^{C}D_t^{\alpha}v(x,t)$ is the Cfd of v. In our study, $\Re$ and $\mathfrak{N}$ represent the linear and nonlinear operators, respectively, with h serving as the source term. We apply HAM, as elaborated in [10–12], to define the nonlinear operator

$$N[\psi(x,t,q)] = {}^{C}D^{\alpha}v(x,t) + \Re v(x,t) + \mathfrak{N}v(x,t) - h(x,t), \tag{3.2}$$

where ψ is a real function $q \in [0,1]$.

The zeroth order deformation [11, 12] is

$$(1-q)L[\psi(x,t,q) - v_0(x,t)] = \hbar q H(x,t) N[\psi(x,t,q)]. \tag{3.3}$$

In this context, $\hbar$ is a nonzero auxiliary parameter, $H(x,t)$ is a nonzero auxiliary function that can be chosen as 1, v_0 serves as the initial guess for v, and ψ represents an unknown function.

It is evident that $\psi|_{q\to 0} = v_0(x,t)$ and $\psi|_{q\to 1} = v(x,t)$. To proceed, we expand ψ in a Taylor series

$$\psi(x,t,q) = \sum_{i=0}^{n} v_i(x,t) q^i,$$

where

$$v_i(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t,q)}{\partial q^m}|_{q=0}.$$

The m-th order deformation equation is

$$L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar R_m(v_{m-1}(x,t)). \tag{3.4}$$

Thus,

$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar L^{-1}[R_m(v_{m-1}(x,t))], \tag{3.5}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Now, we define the nonlinear operator for (1.2) as:

$$N\left[\psi(x,t,q)\right] = {}^{C}D^{\alpha}\psi - \left(\frac{\partial^2 \psi}{\partial x^2} - m^2\psi - \lambda\psi^3\right). \tag{3.6}$$

The m-th order deformation equation can be derived by collecting the coefficients of the same power of q^m, $m = 1, 2, 3, \ldots$ in (3.3), which reads

$$L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar K_m[\overrightarrow{v}_{m-1}(x,t)], \tag{3.7}$$

$$\tag{3.8}$$

where

$$\begin{aligned} K_n[\overrightarrow{v}_{n-1}(x,t)] = {}^{C}D^{\alpha}v_{n-1} - (1-\chi_n)v_0(x,t) \\ - \left(\left[\frac{\partial^2 v_{n-1}}{\partial x^2} - m^2 v_{n-1} - \lambda\sum_{i=0}^{n-1} v_{n-1-i}\sum_{j=0}^{i} v_j u_{i-j}\right]\right). \end{aligned} \tag{3.9}$$

So, the corresponding m-th order deformation equation is

$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar L^{-1} K_m[\overrightarrow{v}_{m-1}(x,t)], \tag{3.10}$$

subject to the initial conditions should be $v_m(x,0) = 0, (v_m)_t(x,0) = 0$, where L^{-1} is the inverse operator, which can be chosen as $L^{-1} = I^\alpha$. It is worth mentioning that $v(x,t)$ can be represented as a series

$$v(x,t) = \sum_{i=0}^{\infty} v_i(x,t), \tag{3.11}$$

with,

$$v(x,0) = \tanh\left(\frac{x}{4}\right), v_t(x,0) = -\frac{3}{4}\text{sech}^2\left(\frac{x}{4}\right),$$

by choosing v_0 as

$$v_0(x,t) = \tanh\left(\frac{x}{4}\right) - \frac{3t}{4}\text{sech}^2\left(\frac{x}{4}\right).$$

We solve the above Eq (3.10) to get the series terms of the solution:

$$\begin{aligned}
v_1(x,t) = {} & \frac{1}{8\Gamma(\alpha+1)}\left(9\hbar t^\alpha \tanh\left(\frac{x}{4}\right)\text{sech}^2\left(\frac{x}{4}\right)\right) + \frac{1}{32\Gamma(\alpha+2)}\left(27\hbar t^{\alpha+1}\left(\cosh\left(\frac{x}{2}\right) - 2\right)\text{sech}^4\left(\frac{x}{4}\right)\right) \\
& - \frac{1}{8\Gamma(\alpha+3)}\left(27\hbar t^{\alpha+2}\tanh\left(\frac{x}{4}\right)\text{sech}^4\left(\frac{x}{4}\right)\right) + \frac{1}{32\Gamma(\alpha+4)}\left(81\hbar t^{\alpha+3}\text{sech}^6\left(\frac{x}{4}\right)\right). \\
v_2(x,t) = {} & \frac{9\hbar(\hbar+1)t^\alpha \tanh\left(\frac{x}{4}\right)\text{sech}^2\left(\frac{x}{4}\right)}{8\Gamma(\alpha+1)} + \frac{27\hbar(\hbar+1)t^{\alpha+1}\left(\cosh\left(\frac{x}{2}\right)-2\right)\text{sech}^4\left(\frac{x}{4}\right)}{32\Gamma(\alpha+2)} \\
& - \frac{27\hbar(\hbar+1)t^{\alpha+2}\tanh\left(\frac{x}{4}\right)\text{sech}^4\left(\frac{x}{4}\right)}{8\Gamma(\alpha+3)} + \frac{81\hbar(\hbar+1)t^{\alpha+3}\text{sech}^6\left(\frac{x}{4}\right)}{32\Gamma(\alpha+4)} \\
& + \frac{81\hbar^2 t^{2\alpha+1}\left(30\cosh\left(\frac{x}{2}\right) - 3\cosh(x) - 35\right)\text{sech}^6\left(\frac{x}{4}\right)}{512\Gamma(2\alpha+2)} \\
& - \frac{27\hbar^2 t^{2\alpha}\left(3\cosh\left(\frac{x}{2}\right) - 7\right)\tanh\left(\frac{x}{4}\right)\text{sech}^4\left(\frac{x}{4}\right)}{64\Gamma(2\alpha+1)} \\
& + \frac{81h^2 t^{2\alpha+2}\left(4\cosh\left(\frac{x}{2}\right) - 9\right)\tanh\left(\frac{x}{4}\right)\text{sech}^6\left(\frac{x}{4}\right)}{64\Gamma(2\alpha+3)} \\
& + \frac{81\hbar^2 t^{2\alpha+3}\left(25\cosh\left(\frac{x}{2}\right) - 14\right)\text{sech}^8\left(\frac{x}{4}\right)}{256\Gamma(2\alpha+4)} \\
& - \frac{729(\alpha+2)(\alpha+3)\hbar^2 t^{2\alpha+3}\left(\cosh\left(\frac{x}{2}\right) - 2\right)\text{sech}^8\left(\frac{x}{4}\right)}{512\Gamma(2\alpha+4)} \\
& - \frac{36\hbar^2 t^{2\alpha+1}\sinh^{10}\left(\frac{x}{4}\right)\text{csch}^6\left(\frac{x}{2}\right)\left(16(2\alpha+3)(\alpha+1)^2 + 3t\,\text{csch}\left(\frac{x}{2}\right)\left(36(\alpha+3)t\,\text{csch}\left(\frac{x}{2}\right) - (\alpha+2)(2\alpha+3)(2\alpha+15)\right)\right)}{\Gamma(2\alpha+4)} \\
& + \frac{18\hbar^2 t^{2\alpha+1}\sinh^{10}\left(\frac{x}{4}\right)\text{csch}^9\left(\frac{x}{2}\right)\left(\frac{81(\alpha+4)(\alpha+5)t^3}{(\alpha+2)(2\alpha+3)} + 48(\alpha+1)^2\sinh\left(\frac{x}{2}\right) + 8(\alpha+1)^2\sinh\left(\frac{3x}{2}\right) + 36(\alpha+1)^2\sinh(x)\right)}{\Gamma(2\alpha+3)}.
\end{aligned}$$

The exact solution for $\alpha = 2$ is

$$v(x,t) = \tanh\left(\frac{x-3t}{4}\right).$$

4. Convergence analysis

Theorem 4.1. *If the solution series* $v(x,t) = \sum_{i=0}^{\infty} v_i(x,t)$ *converge, where* v_m *is obtained by* (3.10)*, then they must be solutions of* (1.2).

Proof 4.2. *Assume that* $\sum_{i=0}^{n} v_i(x,t)$ *converges, meaning* $\lim_{n\longrightarrow\infty} v_n(x,t) = 0$. *Referring to Eq* (3.8)*, we deduce:*

$$\begin{aligned}\hbar H\sum_{m=1}^{\infty} K_m &= \lim_{n\longrightarrow\infty}\sum_{m=0}^{n} L[v_m - \chi_m v_{m-1}]\\ &= L[\lim_{n\longrightarrow\infty}\sum_{m=0}^{n}[v_m - \chi_m v_{m-1}]]\\ &= L[\lim_{n\longrightarrow\infty} v_n].\end{aligned}$$

Here, L represents a linear operator. Given that $\sum_{k\longrightarrow\infty} u_k = 0$ *converge implies that* $\lim_{n\to\infty} u_n = 0$, *and taking into account that* $H \neq 0$ *and* $\hbar \neq 0$, *this leads to the implication that* $\sum_{m=1}^{\infty} K_m = 0$. *We can proceed by expanding* $N[\psi(x,t,q)]$ *about* $q = 0$ *and subsequently setting* $q = 1$

$$N[\psi(x,t,1)] = 0,$$

We can observe that $v(x,t) = \psi(x,t,1) = \sum_{n=0}^{\infty} v_n(x,t)$ *satisfies* (1.2).

Theorem 4.3. *[5] Let the solution terms* $v_0(x,t), v_1(x,t), v_2(x,t), ...$ *be defined as* (3.5)*. The solution* $S = \sum_{m=0}^{\infty} v_m(x,t)$*,* (3.11) *converges if there exists* $0 < \kappa < 1$ *such that* $\|v_{m+1}(x,t)\| \leq \kappa\|v_m(x,t)\|, \forall m > m_0$*, for some* $m_0 \in \mathbb{N}$.

5. Numerical result and discussion

Figure 1 shows the 5-th order HAM solution and the absolute error. We can see from the figures that the HAM solution agreed with the exact solution presented in Figure 2. Now, we know that we can control the convergence of the series in the frame of HAM, for different values of $\hbar$. We plot the $\hbar$-curve of 10-th order HAM approximations of $v_t(0.1,0)$ for different values of $\alpha = 2, 1.9$, and $\alpha = 1.2$ to determine the influence of $\hbar$ on the convergence of the HAM solution in Figure 3. We can discover the valid region of $\hbar$ where the curve is a horizontal line and is $-1.15 \leq \hbar \leq -0.85$. The optimal value of $\hbar$ can be determined by the residual error

$$\Delta(\hbar) = \int_{\Omega} (N(v_n(x,t)))^2\, d\Omega.$$

The optimal value of $\hbar$ is given by the minimization of $\Delta(\hbar)$ using the algebraic equation

$$\frac{d\Delta(\hbar)}{d\hbar} = 0.$$

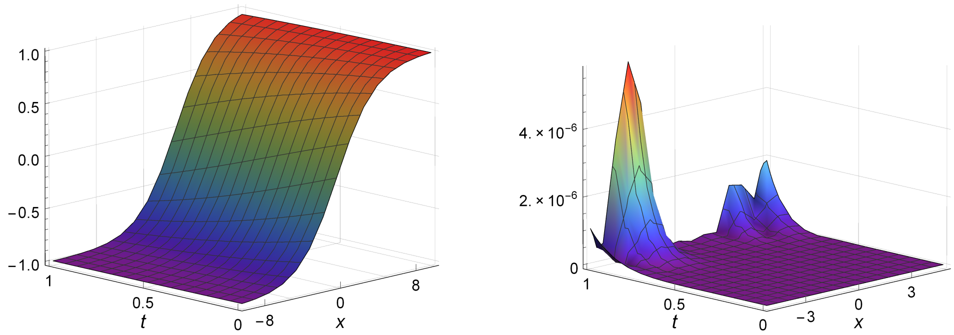

Figure 1. The 5−th HAM solution (left) and the absolute error (right) of (1.2) with $\alpha = 2$.

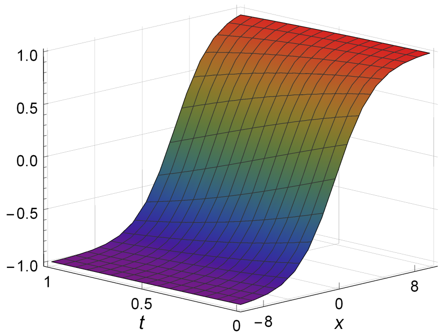

Figure 2. The exact solution of (1.2) for $\alpha = 2$.

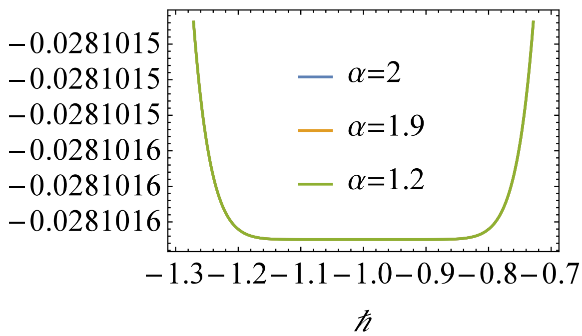

Figure 3. The $\hbar$−curve of (1.2) for $\alpha = 2, 1.9$, and 1.2.

The residual error R, and the 14-th HAM solution $\tilde{v} = \sum_{k)0}^{14} v_k(x,t)$ is represented in Figure 4,

$$R(x,t) = {}^{C}D_t^{\alpha}\tilde{v} - (\tilde{v}_{xx}(x,t) - m^2\tilde{v}(x,t) - \lambda\tilde{v}^3(x,t)),$$

for $\alpha = 1.9, 1.5$, and $\alpha = 1.2$. In the following tables, we get the absolute error of the 5-th order HAM solution, with the exact solution in Table 1 corresponding to the optimal value of $\hbar \simeq -0.968874$.

For comparing purposes, the absolute error of the HAM solution via q-HAM and Yang transforms decomposition method (YTDM) is presented in Table 2, and we ignore the YHPM since it has the same values of YTDM [9]. According to this table, HAM can give more accurate results than the other considered methods. In Table 3, we give 10-th order HAM approximation for $\alpha = 2, 1.9, 1.5$, and $\alpha = 1.2$, and the optimal values of $\hbar$ corresponding are $-0.968874, -0.958194, -0.868833$, and -0.732442, respectively. Finally, to demonstrate the assumption of Theorem 4.3, we compute $\frac{v_{i+1}}{v_i}$ in the domain $x \in (-5, 5), t \in (0, 1)$ in Table 4. It obtains that $\|v_{i+1}\|_\infty \leq \kappa \|v_i\|_\infty$ and $\kappa < 1$.

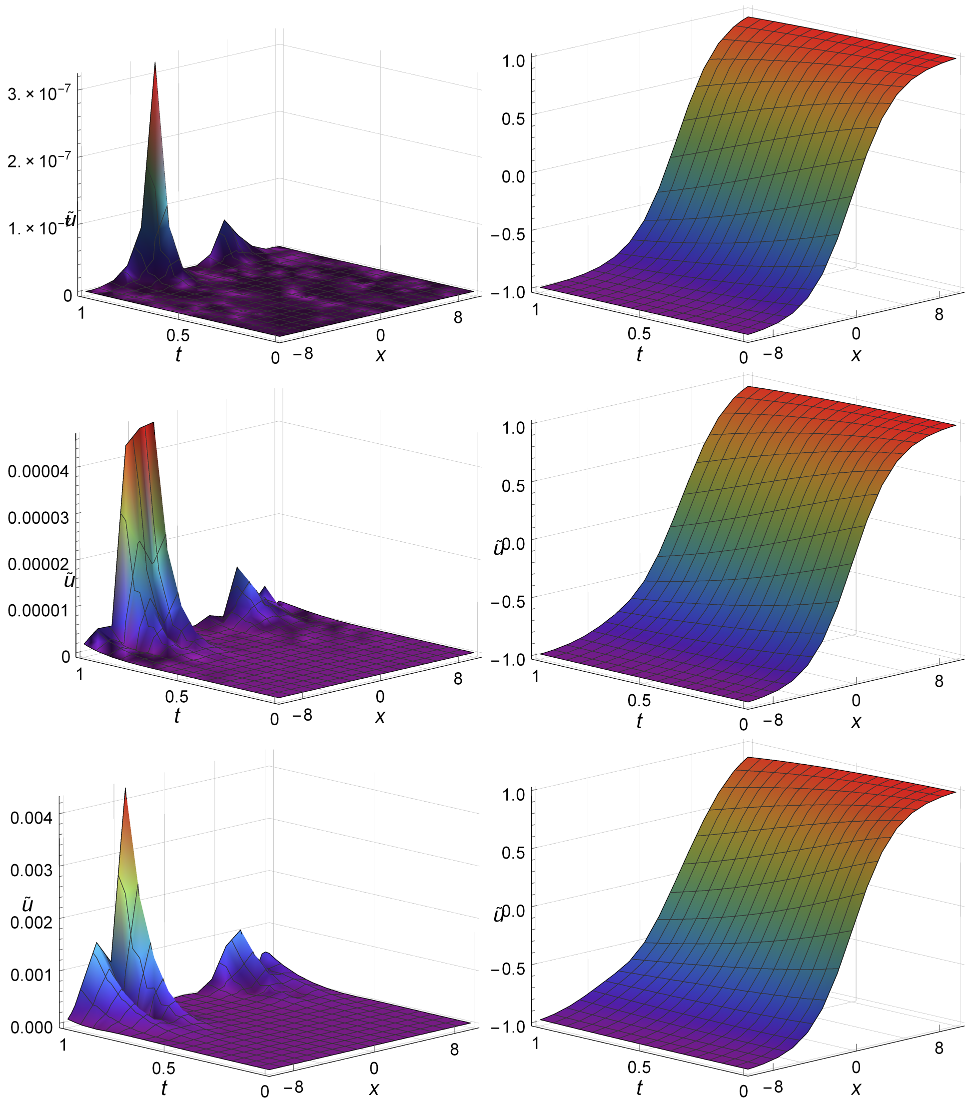

Figure 4. The residual error (left) and the 10−th HAM solution (right) of (1.2) with $\alpha = 1.9, 1.5$, and $\alpha = 1.2$.

Table 1. Absolute error $|v(x,t) - v_5(x,t)|$ for $\alpha = 2$.

$x \vert t$	0.1	0.15	0.2	0.25	0.3
−4	5.57376×10^{-11}	1.33563×10^{-10}	2.46961×10^{-10}	3.71369×10^{-10}	4.28088×10^{-10}
−2	5.56469×10^{-11}	1.10847×10^{-10}	1.55207×10^{-10}	1.4406×10^{-10}	1.19798×10^{-11}
0	3.64594×10^{-12}	1.05828×10^{-11}	2.02059×10^{-11}	2.93718×10^{-11}	3.33951×10^{-11}
2	5.19869×10^{-11}	9.45541×10^{-11}	1.10459×10^{-10}	6.16391×10^{-11}	7.72913×10^{-11}
4	5.71094×10^{-11}	1.32993×10^{-10}	2.28284×10^{-10}	2.94917×10^{-10}	2.33266×10^{-10}

Table 2. Absolute error $|v(x,0.05) - v_5(x,0.05)|$ versus q-HAM and YTDM with $\alpha = 2$.

x	HAM	q-HAM	YTDM
−5	7.57905×10^{-10}	3.99056×10^{-02}	2.47883×10^{-03}
−3	1.19357×10^{-9}	3.84183×10^{-02}	2.77402×10^{-03}
−1	7.38443×10^{-10}	1.83324×10^{-02}	2.97842×10^{-03}
1	6.77525×10^{-10}	4.27869×10^{-02}	2.56482×10^{-03}
3	1.20073×10^{-9}	5.88089×10^{-02}	1.96730×10^{-03}
5	7.82861×10^{-10}	5.99307×10^{-02}	1.75689×10^{-03}

Table 3. HAM solutions for different values of α.

x	t	Exact	$\alpha = 2$	$\alpha = 1.9$	$\alpha = 1.5$	$\alpha = 1.2$
−2	0.1	−0.519021833	−0.519021853	−0.518235	−0.511215	−0.497473
	0.2	−0.571669966	−0.571670036	−0.569266	−0.551869	−0.525623
	0.3	−0.619996867	−0.619996994	−0.615529	−0.587005	−0.550409
0	0.1	−0.074859690	−0.074859690	−0.074800	−0.074203	−0.072856
	0.2	−0.148885033	−0.148885033	−0.148519	−0.145573	−0.140475
	0.3	−0.221278467	−0.221278467	−0.220262	−0.213085	−0.202623
1	0.1	0.173235157	0.173235168	0.172795	0.168947	0.161751
	0.2	0.099667994	0.099668022	0.098508	0.090595	0.080106
	0.3	0.024994792	0.024994816	0.023206	0.013077	0.003285
2	0.1	0.401134284	0.401134303	0.400382	0.393714	0.380852
	0.2	0.336375544	0.336375603	0.334188	0.318681	0.296387
	0.3	0.268271182	0.268271257	0.264439	0.241007	0.213929

Table 4. The values of $\frac{\|v_{i+1}\|}{\|v_i\|}$ for different values of α in the domain $x \in (-5,5), t \in (0,1)$.

i	$\alpha = 2$	$\alpha = 1.9$	$\alpha = 1.2$
1	0.063345	0.078224	0.315335
2	0.153453	0.181753	0.223146
3	0.089140	0.11385	0.535211
4	0.104045	0.119254	0.456154
5	0.103515	0.133143	0.179105

6. Conclusions

In conclusion, our application of HAM to the fractional phi-4 equation has yielded highly favorable results, as demonstrated in this work. This success underscores the effectiveness of HAM in addressing complex nonlinear equations with fractional derivatives. The method demonstrated its efficiency through its accurate numerical results compared to previous published results and applies to the convergence conditions of the series. Our findings contribute to the growing body of knowledge in this field and highlight the potential of HAM as a valuable tool for solving a wide range of mathematical and physical problems involving FDEs. Moreover, the method can also be accurate in finding convergent solutions to FDEs with multiple parameters and other definitions of fractional differential, and this could be in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Clemente Cesarano is the Guest Editor of Special Issue “Numerical Methods for Special Functions” for AIMS Mathematics. Prof. Clemente Cesarano was not involved in the editorial review and the decision to publish this article.

This work does not have any conflicts of interest.

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