Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 8610–8629.
DOI: 10.3934/math.2024417
Received: 06 January 2024
Revised: 05 February 2024
Accepted: 07 February 2024
Published: 29 February 2024

Research article

Existence results for Schrödinger type double phase variable exponent problems with convection term in $\mathbb{R}^N$

Shuai Li[1,*], **Tianqing An**[1] **and Weichun Bu**[2]

[1] School of Mathematics, Hohai University, Nanjing 210098, China
[2] School of Mathematics and Statistics, Fuyang Normal University, Fuyang 236037, China

* **Correspondence:** Email: lishuaihhu@yeah.net.

Abstract: This paper was concerned with a new class of Schrödinger equations involving double phase operators with variable exponent in $\mathbb{R}^N$. We gave the corresponding Musielak-Orlicz Sobolev spaces and proved certain properties of the double phase operator. Moreover, our main tools were the topological degree theory and Galerkin method, since the equation contained a convection term. By using these methods, we derived the existence of weak solution for the above problems. Our result extended some recent work in the literature.

Keywords: variable exponent; double phase problem; convection term; Galerkin method; topological degree
Mathematics Subject Classification: 35D30, 35J10, 35J91, 46E35

1. Introduction

The study of differential equations and variational problems with double phase operators is a new and interesting topic. Originally, in order to investigate the Lavrentiev phenomenon from strongly anisotropic materials, Zhikov [1] introduced the following functional

$$\int_\Omega (|\nabla \upsilon|^p + \mu(x)|\nabla \upsilon|^q)\, dx,$$

where the function $\mu(\cdot)$ was used as an aid to regulate the mixture between two different materials, with power hardening of rates p and q, respectively. Since then, many scholars studied double phase problems and obtained abundant theoretical achievements.

In [2], Colasuonno and Squassina studied an eigenvalue problem of double phase variational integrals and proved some properties of the Musielak-Orlicz space for the first time. Liu and Dai [3] investigated the following problem

$$\begin{cases} -div\left(|\nabla\upsilon|^{p-2}\nabla\upsilon + a(x)|\nabla\upsilon|^{q-2}\nabla\upsilon\right) = h(x,\upsilon), & x \in \Omega, \\ \upsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

By variational methods, they verified various existence and multiplicity results. Furthermore, they also obtained some essential properties of double phase operators, which has been applied to many double phase problems. For Eq (1.1), the existence of solutions has also been studied by applying Morse theory [4]. In [5–7], the authors consider a double phase problem in $\mathbb{R}^N$ with reaction terms, which does not satisfy the Ambrosetti-Rabinowitz condition. They derived some existence results based on various variational methods. For more related results on the double phase problem, one can refer to [8–12] and references therein.

If nonlinearity h also depends on the gradient $\nabla\upsilon$, such functions are usually called convection terms. Its presence increases the difficulty of the double phase problem because the gradient dependent term is non-variational. In [13], Gasinski and Winkert considered the following convection problem

$$\begin{cases} -div\left(|\nabla\upsilon|^{p-2}\nabla\upsilon + \mu(x)|\nabla\upsilon|^{q-2}\nabla\upsilon\right) = h(x,\upsilon,\nabla\upsilon), & x \in \Omega, \\ \upsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

They discussed the existence of weak solutions by using the theory of the pseudomonotone operator. The same methodology can be found in reference [14, 15]. In addition, the methods for dealing with the existence of solutions to elliptic equations with convection terms also included Galerkin method [16, 17], Brezis theorem [18], and Leray-Schauder alternative principle [19, 20].

So far, there are only few results involving the variable exponent double phase operator. In [21], the authors considered double phase problems with variable exponent for the first time and established a suitable function spaces. Moreover, we refer to the recent results [22, 23] for the existence of constant sign solutions and the existence results in complete manifolds, and to [24, 25] for the study of the double phase problem with concave-convex nonlinearities or Baouendi-Grushin type operators. To our knowledge, no work has established the results for Schrödinger equations in $\mathbb{R}^N$, which involves double phase operators and convection terms. Enlightened by the above literatures, we discuss this kind of equation as follows

$$-div(|\nabla\upsilon|^{p(x)-2}\nabla\upsilon + \lambda(x)|\nabla\upsilon|^{q(x)-2}\nabla\upsilon) + V(x)(|\upsilon|^{p(x)-2}\upsilon + \lambda(x)|\upsilon|^{q(x)-2}\upsilon) = h(x,\upsilon,\nabla\upsilon),\ x \in \mathbb{R}^N, \tag{H_V}$$

where $V : \mathbb{R}^N \to \mathbb{R}^+$ is a potential function, $h : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, and $0 \le \lambda(\cdot) \in L^1(\mathbb{R}^N)$. $p(x), q(x) \in C_+(\mathbb{R}^N)$ such that $1 < p(x) < N$, $p(x) < q(x) < p^*(x)$ and $\frac{q(x)}{p(x)} < 1 + \frac{1}{N}$. The Sobolev critical exponent is defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N, \end{cases}$$

and also defines

$$C_+(\mathbb{R}^N) := \left\{y(x) :\ y(x) \in C(\mathbb{R}^N,\mathbb{R}),\ 1 < y^- \le y(x) \le y^+ < \infty\right\}.$$

For each $y(x) \in C_+(\mathbb{R}^N)$, we denote

$$y^- := \min_{x\in\mathbb{R}^N} y(x), \qquad y^+ := \max_{x\in\mathbb{R}^N} y(x).$$

Throughout the paper, we consider equations (H_V) under some assumptions for the potential function V and Carathéodory function h.

(V): $V(x) \in C(\mathbb{R}^N)$ and there exists $V_0 > 0$ such that

$$\inf_{x\in\mathbb{R}^N} V(x) \geq V_0, \quad \lim_{|z|\to\infty} \int_{S_1(z)} \frac{1}{V(x)} dx = 0,$$

where $S_1(z) = \{x \in \mathbb{R}^N : |x - z| \leq 1\}$, $S_a(z)$ denotes a ball of radius a with center z.

(H): There exist a nonnegative function $\gamma \in L^1(\mathbb{R}^N) \cap L^{p'(x)}(\mathbb{R}^N)$ and constants $d_1, d_2 \geq 0$ with $\max\left\{d_2 - \frac{d_2}{p^+}, \frac{d_1 p^- + d_2}{V_0 p^-}\right\} < 1$ such that

$$|h(x, u, v)| \leq \gamma(x) + d_1|u|^{p(x)-1} + d_2|v|^{p(x)-1}, \text{ for any } (x, u, v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N.$$

The condition (V) was introduced by [26] to guarantee compactness of the embedding of the Sobolev space into Lebesgue space. Another condition on function V is used in the literature [27], which satisfies

$$V \in C\left(\mathbb{R}^N, (0, +\infty)\right), \;\; meas\left(\{x \in \mathbb{R}^N : V(x) \leq L\}\right) < \infty \text{ for all } L > 0. \tag{1.3}$$

It is worth noting that the condition (1.3) is stronger than (V) (see [28]). In this paper, we will prove a new embedding theorem for the variable exponents Sobolev space in $\mathbb{R}^N$ under weaker assumption (V). In addition, we cannot implement the usual critical point theory due to the equation (H_V) not having a variational structure. Our main innovation is the first study of double phase variable exponent problems with convection terms by using Galerkin methods together with the topological degree theorem.

The outline of this article is as follows. In Section 2, we collect some necessary definitions and basic lemmas of Musielak-Orlicz space and corresponding Sobolev space. In Section 3, we present some classes of mappings and topological degree theory. We obtain the existence of strong generalized solutions and weak solutions in Sections 4 and 5, respectively. Finally, a conclusion is given in Section 6.

2. Preliminaries

In this section, we first review some known results of Lebesgue and Sobolev spaces with the variable exponent, which will be used later.

Let the variable exponent Lebesgue spaces be defined as

$$L^{p(x)}(\mathbb{R}^N) := \left\{\upsilon : \upsilon \text{ is measurable and } \int_{\mathbb{R}^N} |\upsilon(x)|^{p(x)} dx < \infty\right\},$$

endowed with the Luxemburg norm

$$|\upsilon|_{p(x)} = \inf\left\{\chi > 0 : \varrho_{p(x)}\left(\frac{\upsilon}{\chi}\right) \leq 1\right\},$$

where $\varrho_{p(x)}(\upsilon) := \int_{\mathbb{R}^N} |\upsilon|^{p(x)}dx$ is called modular and $p'(x)$ denotes the conjugate function of $p(x)$. Also, $W^{1,p(x)}(\mathbb{R}^N)$ stands for the corresponding Sobolev spaces. Define a linear subspace of $W^{1,p(x)}(\mathbb{R}^N)$ as

$$W := \left\{ \upsilon \in W^{1,p(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|\upsilon(x)|^{p(x)}dx < \infty \right\},$$

equipped with the norm

$$\|\upsilon\|_W = \inf\left\{ \chi > 0 : \int_{\mathbb{R}^N} \left(\left|\frac{\nabla \upsilon}{\chi}\right|^{p(x)} + V(x)\left|\frac{\upsilon}{\chi}\right|^{p(x)} \right) dx \le 1 \right\}.$$

The spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$, and W are separable reflexive Banach spaces (see [27, 29]).

Next, we introduce a new function space used in our study and give some of its properties. Let

$$H(x,t) = t^{p(x)} + \lambda(x)t^{q(x)}, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Obviously, $H \in N(\mathbb{R}^N)$ is locally integrable (see [24]).

The Musielak-Orlicz space $L^H(\mathbb{R}^N)$ is given by

$$L^H(\mathbb{R}^N) := \left\{ \upsilon : \upsilon \text{ is measurable and } \int_{\mathbb{R}^N} H(x,|\upsilon|)dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|\upsilon\|_H = \inf\left\{ \chi > 0 : \int_{\mathbb{R}^N} H\left(x, \left|\frac{\upsilon}{\chi}\right|\right) dx \le 1 \right\}.$$

Lemma 2.1. *[21] Suppose that $\varrho_H(\upsilon) = \int_{\mathbb{R}^N}(|\upsilon|^{p(x)} + \lambda(x)|\upsilon|^{q(x)})dx$. For $\upsilon \in L^H(\mathbb{R}^N)$, we have*
(i) $\chi = \|\upsilon\|_H$ if, and only if, $\varrho_H(\frac{\upsilon}{\chi}) = 1$;
(ii) $\|\upsilon\|_H < 1 \Rightarrow \|\upsilon\|_H^{q^+} \le \varrho_H(\upsilon) \le \|\upsilon\|_H^{p^-}$;
(iii) $\|\upsilon\|_H > 1 \Rightarrow \|\upsilon\|_H^{p^-} \le \varrho_H(\upsilon) \le \|\upsilon\|_H^{q^+}$;
(iv) $\|\upsilon\|_H < 1\ (= 1; > 1) \Leftrightarrow \varrho_H(\upsilon) < 1\ (= 1; > 1)$.

The corresponding Sobolev spaces are given by

$$W^{1,H}(\mathbb{R}^N) := \left\{ \upsilon \in L^H(\mathbb{R}^N) : |\nabla \upsilon| \in L^H(\mathbb{R}^N) \right\},$$

endowed with the norm

$$\|\upsilon\|_{1,H} = \|\nabla \upsilon\|_H + \|\upsilon\|_H.$$

Moreover, in order to study problems (H_V), we consider the following space

$$E = \left\{ |\nabla \upsilon| \in L^H(\mathbb{R}^N), \int_{\mathbb{R}^N} V(x)H(x,|\upsilon|)dx < \infty \right\},$$

with the equivalent norm

$$\|\upsilon\| = \inf\left\{ \chi > 0 : \varrho\left(\frac{\upsilon}{\chi}\right) \le 1 \right\}.$$

The modular $\varrho : E \to \mathbb{R}$ is given by

$$\varrho(\upsilon) = \int_{\mathbb{R}^N} \left(|\nabla\upsilon|^{p(x)} + \lambda(x)|\nabla\upsilon|^{q(x)}\right) + V(x)\left(|\upsilon|^{p(x)} + \lambda(x)|\upsilon|^{q(x)}\right) dx.$$

Analogy to the proof of Proposition 2.13 in [21], we have the following connection between modular and norm $\|\cdot\|$.

Lemma 2.2. Suppose that $\upsilon_n, \upsilon \in E$, then
(i) $\chi = \|\upsilon\|$ if, and only if, $\varrho(\frac{\upsilon}{\chi}) = 1$;
(ii) $\|\upsilon\| < 1 \Rightarrow \|\upsilon\|^{q^+} \leq \varrho(\upsilon) \leq \|\upsilon\|^{p^-}$;
(iii) $\|\upsilon\| > 1 \Rightarrow \|\upsilon\|^{p^-} \leq \varrho(\upsilon) \leq \|\upsilon\|^{q^+}$;
(iv) $\|\upsilon\| < 1\ (= 1; > 1) \Leftrightarrow \varrho(\upsilon) < 1\ (= 1; > 1)$);
(v) $\lim_{n\to\infty} |\upsilon_n - \upsilon| = 0 \Leftrightarrow \lim_{n\to\infty} \varrho(\upsilon_n - \upsilon) = 0$.

Theorem 2.1. $L^H(\mathbb{R}^N)$, $W^{1,H}(\mathbb{R}^N)$, and E are separable reflexive Banach spaces.

Proof. Since $H \in N(\mathbb{R}^N)$ is locally integrable and the Lebesgue measure on $\mathbb{R}^N$ is σ-finite and separable, then $L^H(\mathbb{R}^N)$ is a separable Banach space that follows from ([30], Theorems 7.7 and 7.10). By Proposition 2.12 of [21], we know that H is uniformly convex. Note that

$$H(x, 2t) \leq 2^{q^+} H(x, t),$$

which satisfies the condition (Δ_2). Thus, $L^H(\mathbb{R}^N)$ is uniformly convex, follows from ([30], Theorem 11.6), and is a reflexive space based on the Milman-Pettis theorem. Similar to the proof of Theorem 2.7 (ii) in [5], we obtain $W^{1,H}(\mathbb{R}^N)$ as a separable reflexive Banach space and E as a closed subspace of $W^{1,H}(\mathbb{R}^N)$.

We present the following embedding results. For convenience, the notation $\rightharpoonup(\to)$ is means weak (strong) convergence and the symbol $\hookrightarrow(\hookrightarrow\hookrightarrow)$ denotes the continuous (compact) embedding, respectively.

Theorem 2.2. Assume that (V) holds and $\mu(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \leq \mu(x) < p^*(x)$, then the spaces W are continuously compact embedded in $L^{\mu(x)}(\mathbb{R}^N)$.

Proof. (i) First, we discuss the case $\mu(x) = p(x)$ and suppose that $\upsilon_n \rightharpoonup 0$ in W. If (V) holds, the embedding $W \hookrightarrow\hookrightarrow L^{p(x)}(S_R(0))$ holds ([27], Proposition 2.4). So, we only show that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\geq R} |\upsilon_n|^{p(x)} dx \leq \varepsilon, \text{ for any } n \in \mathbb{N}.$$

Note that $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a bounded sequence in W. Set $\rho = \|\upsilon_n\|_W^{p^+} + \|\upsilon_n\|_W^{p^-}$ and choose an arbitrary number $s \in (1, \frac{N}{N-p^-})$, then $p(x) < sp(x) < p^*(x)$. Using Proposition 2.4 of [27] again, there is a constant $Q > 0$ such that

$$\left(\int_{\mathbb{R}^N} |\upsilon_n|^{sp(x)} dx\right)^{\frac{1}{s}} \leq Q. \tag{2.1}$$

Let $\{z_i\}_i \subset \mathbb{R}^N$ such that $\bigcup_{i=1}^{\infty} S_1(z_i) = \mathbb{R}^N$ and every $x \in \mathbb{R}^N$ is covered by at most 2^N such balls. Denote

$$X(z_i) = \left\{x \in \mathbb{R}^N : \frac{1}{V(x)} < b\right\} \cap S_1(z_i), \quad Y(z_i) = \left\{x \in \mathbb{R}^N : \frac{1}{V(x)} > b\right\} \cap S_1(z_i).$$

Thus

$$\begin{aligned}\int_{X(z_i)} |\upsilon_n|^{p(x)}dx \leq &\frac{1}{b}\int_{\mathbb{R}^N} V(x)|\upsilon_n|^{p(x)}dx \leq \frac{1}{b}\int_{\mathbb{R}^N}\left(|\nabla \upsilon_n|^{p(x)} + V(x)|\upsilon_n|^{p(x)}\right)dx\\ \leq &\frac{1}{b}\left(\|\upsilon_n\|_W^{p^+} + \|\upsilon_n\|_W^{p^-}\right) = \frac{\rho}{b}.\end{aligned}$$

By the Hölder inequality and (2.1), we get

$$\int_{Y(z_i)} |\upsilon_n|^{p(x)}dx \leq \left(\int_{Y(z_i)} |\upsilon_n|^{sp(x)}dx\right)^{\frac{1}{s}} \left(\int_{Y(z_i)} dx\right)^{\frac{s-1}{s}} = [meas(Y(z_i))]^{\frac{s-1}{s}} Q.$$

Hence,

$$\begin{aligned}\int_{|x|\geq R} |\upsilon_n|^{p(x)}dx &\leq \sum_{|z_i|\geq R-1}^{\infty} \int_{S_1(z_i)} |\upsilon_n|^{p(x)}dx = \sum_{|z_i|\geq R-1}^{\infty}\left[\int_{X(z_i)} |\upsilon_n|^{p(x)}dx + \int_{Y(z_i)} |\upsilon_n|^{p(x)}dx\right]\\ &\leq \sum_{|z_i|\geq R-1}^{\infty}\left(\frac{\rho}{b} + Q\sup_{|z_i|\geq R-1}[meas(Y(z_i))]^{\frac{s-1}{s}}\right)\\ &\leq \frac{2^N\rho}{b} + 2^N Q \sup_{|z_i|\geq R-1}[meas(Y(z_i))]^{\frac{s-1}{s}}.\end{aligned}$$

Now, we choose b large enough such that $2^{N+1}\rho \leq b\varepsilon$. As is shown in [26], the $meas\,(Y(z)) \to 0$ for $|z| \to \infty$, then we can find that $R' > 0$ satisfies

$$2^N Q \sup_{|z_i|\geq R-1}[meas(Y(z_i))]^{\frac{s-1}{s}} \leq \frac{\varepsilon}{2}.$$

For the above R',

$$\int_{|x|\geq R'} |\upsilon_n|^{p(x)}dx \leq \varepsilon.$$

Therefore, $\upsilon_n \to 0$ in $L^{p(x)}(\mathbb{R}^N)$.

(ii) For $p(x) < \mu(x) < p^*(x)$, there exists $\sigma(x) \in (0, 1)$ such that $\frac{1}{\mu(x)} = \frac{\sigma(x)}{p(x)} + \frac{1-\sigma(x)}{p^*(x)}$, then we have

$$f(x) = \frac{p(x)}{\mu(x)\sigma(x)} > 1, \quad g(x) = \frac{p^*(x)}{\mu(x)(1-\sigma(x))} > 1.$$

According to the embedding $W \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$ and $\{\upsilon_n\}$ is bounded in W, we get

$$\int_{\mathbb{R}^N} |\upsilon_n|^{p^*(x)}dx < \infty, \text{ for any } n \in \mathbb{N}. \tag{2.2}$$

From (i) and (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\upsilon_n|^{\mu(x)}dx \leq & 2\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p(x)}dx\right)^{\frac{1}{f(x)}}\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p^*(x)}dx\right)^{\frac{1}{g(x)}}\\ \leq & 2\left[\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p(x)}dx\right)^{\frac{1}{f^+}}+\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p(x)}dx\right)^{\frac{1}{f^-}}\right]\left[\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p^*(x)}dx\right)^{\frac{1}{g^+}}+\left(\int_{\mathbb{R}^N}|\upsilon_n|^{p^*(x)}dx\right)^{\frac{1}{g^-}}\right]\\ \rightarrow & 0.\end{aligned}$$

This means that $\upsilon_n \to 0$ in $L^{\mu(x)}(\mathbb{R}^N)$. The proof is complete.

Theorem 2.3. Suppose that (V) holds and $p(x) \leq \theta(x) \leq p^*(x)$ for $x \in \mathbb{R}^N$. Thus, the embedding $L^H(\mathbb{R}^N) \hookrightarrow L^{p(x)}(\mathbb{R}^N)$ and $E \hookrightarrow L^{\theta(x)}(\mathbb{R}^N)$ holds. Moreover, $E \hookrightarrow\hookrightarrow L^{\theta(x)}(\mathbb{R}^N)$ also holds whenever $p(x) \leq \theta(x) < p^*(x)$. This implies there exists $C_\theta > 0$ such that

$$|\upsilon|_{\theta(x)} \leq C_\theta\|\upsilon\|, \quad \upsilon \in E.$$

Proof. Let $H_p = t^p$, then $H_p < H$. Thus, applying Theorem 10.3 of [30], we obtain

$$L^H(\mathbb{R}^N) \hookrightarrow L^{p(x)}(\mathbb{R}^N) \text{ and } E \hookrightarrow W.$$

From Theorem 2.2, we have $W \hookrightarrow\hookrightarrow L^{\theta(x)}(\mathbb{R}^N)$, so $E \hookrightarrow\hookrightarrow L^{\theta(x)}(\mathbb{R}^N)$ for $p(x) \leq \theta(x) < p^*(x)$. Using again the Theorem 10.3 of [30], we get $E \hookrightarrow W^{1,H}(\mathbb{R}^N) \hookrightarrow W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$.

Before stating our main results, we need to present the corresponding definitions.

Definition 2.1. Let E be a real reflexive Banach space with dual E^*. A mapping $\mathcal{L} : E \to E^*$ is said to be
(i) of class (S_+), if for each $\{\upsilon_n\} \in E$ with $\upsilon_n \rightharpoonup \upsilon$ and $\limsup_{n\to\infty}\langle\mathcal{L}\upsilon_n, \upsilon_n - \upsilon\rangle \leq 0$, then $\upsilon_n \to \upsilon$ in E ;
(ii) quasimonotone, if for each $\{\upsilon_n\} \in E$ with $\upsilon_n \rightharpoonup \upsilon$, we have $\limsup\langle\mathcal{L}\upsilon_n, \upsilon_n - \upsilon\rangle \geq 0$.

Definition 2.2. We say that $\upsilon \in E$ is a weak solution of problems (H_V), if

$$\left\langle-\Delta_\lambda^V\upsilon, \varphi\right\rangle = \int_{\mathbb{R}^N} h(x, \upsilon, \nabla\upsilon)\varphi dx, \tag{2.3}$$

for any $\varphi \in E$, where $-\Delta_\lambda^V$ denotes the double phase type operator as

$$-\Delta_\lambda^V\upsilon = -div(|\nabla\upsilon|^{p(x)-2}\nabla\upsilon + \lambda(x)|\nabla\upsilon|^{q(x)-2}\nabla\upsilon) + V(x)(|\upsilon|^{p(x)-2}\upsilon + \lambda(x)|\upsilon|^{q(x)-2}\upsilon).$$

Define functional $J : E \to R$ as

$$J(\upsilon) = \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|\nabla\upsilon|^{p(x)} + \frac{\lambda(x)}{q(x)}|\nabla\upsilon|^{q(x)}\right)dx + \int_{\mathbb{R}^N}V(x)\left(\frac{1}{p(x)}|\upsilon|^{p(x)} + \frac{\lambda(x)}{q(x)}|\upsilon|^{q(x)}\right)dx. \tag{2.4}$$

Obviously, $J \in C^1(E, \mathbb{R})$ (see [21]). We denote $\mathcal{L} = J' : E \to E^*$, then

$$\langle\mathcal{L}\upsilon, \phi\rangle = \int_{\mathbb{R}^N}\left(|\nabla\upsilon|^{p(x)-2}\nabla\upsilon + \lambda(x)|\nabla\upsilon|^{q(x)-2}\nabla\upsilon\right)\nabla\phi dx + \int_{\mathbb{R}^N}V(x)\left(|\upsilon|^{p(x)-2}\upsilon + \lambda(x)|\upsilon|^{q(x)-2}\upsilon\right)\phi dx,\ \upsilon, \phi \in E. \tag{2.5}$$

Lemma 2.3. The operator $\mathcal{L}: E \to E^*$ has the properties, such as continuous, bounded, strictly monotone, homeomorphism, and of type (S_+).

Proof. (a) Since $\mathcal{L} = J'$ and $J \in C^1$, then $\mathcal{L}$ is continuous. For all $\eta_1, \eta_2 \in \mathbb{R}^N$ with $\eta_1 \neq \eta_2$, by the well-known inequality

$$(|\eta_1|^{\tau-2}\eta_1 - |\eta_2|^{\tau-2}\eta_2)(\eta_1 - \eta_2) > 0,\ \tau > 1, \tag{2.6}$$

we obtain

$$\langle \mathcal{L}(\eta_1) - \mathcal{L}(\eta_2), \eta_1 - \eta_2 \rangle > 0,$$

which implies that $\mathcal{L}$ is strictly monotone. Next, we show that $\mathcal{L}$ is bounded. Let $\chi_1 = \|\upsilon\|$, $\chi_2 = \|\phi\|$ and $L = max\left\{\chi_1^{p^- -1}, \chi_1^{q^+ -1}\right\}$. From Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}
\left\langle \mathcal{L}\upsilon, \frac{\phi}{\chi_2} \right\rangle =& \int_{\mathbb{R}^N} \left(|\nabla\upsilon|^{p(x)-2}\nabla\upsilon + \lambda(x)|\nabla\upsilon|^{q(x)-2}\nabla\upsilon\right)\frac{\nabla\phi}{\chi_2}dx + \int_{\mathbb{R}^N} V(x)\left(|\upsilon|^{p(x)-2}\upsilon + \lambda(x)|\upsilon|^{q(x)-2}\upsilon\right)\frac{\phi}{\chi_2}dx \\
\leq& L\int_{\mathbb{R}^N}\left(\left|\frac{\nabla\upsilon}{\chi_1}\right|^{p(x)-1} + \lambda(x)\left|\frac{\nabla\upsilon}{\chi_1}\right|^{q(x)-1}\right)\left|\frac{\nabla\phi}{\chi_2}\right| + V(x)\left(\left|\frac{\upsilon}{\chi_1}\right|^{p(x)-1} + \lambda(x)\left|\frac{\upsilon}{\chi_1}\right|^{q(x)-1}\right)\left|\frac{\nabla\phi}{\chi_2}\right|dx \\
\leq& L\left(\int_{\mathbb{R}^N}\left|\frac{\nabla\upsilon}{\chi_1}\right|^{p(x)}dx\right)^{\frac{1}{p'(x)}}\left(\int_{\mathbb{R}^N}\left|\frac{\nabla\phi}{\chi_2}\right|^{p(x)}dx\right)^{\frac{1}{p(x)}} \\
&+ L\left(\int_{\mathbb{R}^N}\lambda(x)\left|\frac{\nabla\upsilon}{\chi_1}\right|^{q(x)}dx\right)^{\frac{1}{q'(x)}}\left(\int_{\mathbb{R}^N}\lambda(x)\left|\frac{\nabla\phi}{\chi_2}\right|^{q(x)}dx\right)^{\frac{1}{q(x)}} \\
&+ L\left(\int_{\mathbb{R}^N}V(x)\left|\frac{\upsilon}{\chi_1}\right|^{p(x)}dx\right)^{\frac{1}{p'(x)}}\left(\int_{\mathbb{R}^N}V(x)\left|\frac{\phi}{\chi_2}\right|^{p(x)}dx\right)^{\frac{1}{p(x)}} \\
&+ L\left(\int_{\mathbb{R}^N}\lambda(x)V(x)\left|\frac{\upsilon}{\chi_1}\right|^{q(x)}dx\right)^{\frac{1}{q'(x)}}\left(\int_{\mathbb{R}^N}\lambda(x)V(x)\left|\frac{\phi}{\chi_2}\right|^{q(x)}dx\right)^{\frac{1}{q(x)}} \\
\leq& \frac{L}{p'^-}\int_{\mathbb{R}^N}\left|\frac{\nabla\upsilon}{\chi_1}\right|^{p(x)}dx + \frac{L}{q'^-}\int_{\mathbb{R}^N}\lambda(x)\left|\frac{\nabla\upsilon}{\chi_1}\right|^{q(x)}dx \\
&+ \frac{L}{p'^-}\int_{\mathbb{R}^N}V(x)\left|\frac{\upsilon}{\chi_1}\right|^{p(x)}dx + \frac{L}{q'^-}\int_{\mathbb{R}^N}\lambda(x)V(x)\left|\frac{\upsilon}{\chi_1}\right|^{q(x)}dx \\
&+ \frac{L}{p^-}\int_{\mathbb{R}^N}\left|\frac{\nabla\phi}{\chi_2}\right|^{p(x)}dx + \frac{L}{q^-}\int_{\mathbb{R}^N}\lambda(x)\left|\frac{\nabla\phi}{\chi_2}\right|^{q(x)}dx \\
&+ \frac{L}{p^-}\int_{\mathbb{R}^N}V(x)\left|\frac{\phi}{\chi_2}\right|^{p(x)}dx + \frac{L}{q^-}\int_{\mathbb{R}^N}\lambda(x)V(x)\left|\frac{\phi}{\chi_2}\right|^{q(x)}dx \\
\leq& \frac{L}{q'^-}\varrho\left(\frac{\upsilon}{\chi_1}\right) + \frac{L}{p^-}\varrho\left(\frac{\phi}{\chi_2}\right) = \frac{L}{q'^-} + \frac{L}{p^-},
\end{aligned}$$

thus, we have

$$\|\mathcal{L}\upsilon\|_{E^*} = \sup_{\phi\in E,\ \|\phi\|_E\leq 1} |\langle \mathcal{L}(\upsilon), \phi\rangle| \leq \frac{L}{q'^-} + \frac{L}{p^-}.$$

Hence, $\mathcal{L}$ is bounded.

(b) Let $\{\upsilon_n\}_{n\in\mathbb{N}} \subseteq E$ such that

$$\upsilon_n \rightharpoonup \upsilon \text{ and } \limsup_{n\to\infty}\langle \mathcal{L}(\upsilon_n) - \mathcal{L}(\upsilon), \upsilon_n - \upsilon\rangle \leq 0. \tag{2.7}$$

By the monotonicity of $\mathcal{L}$, we get

$$\liminf_{n\to\infty}\langle \mathcal{L}(\upsilon_n) - \mathcal{L}(\upsilon), \upsilon_n - \upsilon\rangle \geq 0,$$

then

$$\lim_{n\to\infty}\langle \mathcal{L}(\upsilon_n) - \mathcal{L}(\upsilon), \upsilon_n - \upsilon\rangle = 0,$$

that is

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla\upsilon_n|^{p(x)-2}\nabla\upsilon_n - |\nabla\upsilon|^{p(x)-2}\nabla\upsilon\right)(\nabla\upsilon_n - \nabla\upsilon) + \lambda(x)\left(|\nabla\upsilon_n|^{q(x)-2}\nabla\upsilon_n - |\nabla\upsilon|^{q(x)-2}\nabla\upsilon\right)(\nabla\upsilon_n - \nabla\upsilon)\\
&+\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\left(|\upsilon_n|^{p(x)-2}\upsilon_n - |\upsilon|^{p(x)-2}\upsilon\right)(\upsilon_n - \upsilon) + \lambda(x)V(x)\left(|\upsilon_n|^{q(x)-2}\upsilon_n - |\upsilon|^{q(x)-2}\upsilon\right)(\upsilon_n - \upsilon)dx = 0.
\end{aligned}$$

In view of (2.6), $\nabla\upsilon_n$ and υ_n converge in measure to $\nabla\upsilon$ and υ in $\mathbb{R}^N$, respectively. Without loss of generality, let $\nabla\upsilon_n \to \nabla\upsilon$ and $\upsilon_n \to \upsilon$ a.e., on $\mathbb{R}^N$. Based on the Fatou lemma, we obtain

$$\liminf_{n\to\infty} J(\upsilon_n) \geq J(\upsilon). \tag{2.8}$$

Noting that $\lim_{n\to\infty}\langle \mathcal{L}(\upsilon), \upsilon_n - \upsilon\rangle = 0$, then $\limsup_{n\to\infty}\langle \mathcal{L}(\upsilon_n), \upsilon_n - \upsilon\rangle \leq 0$. According to Young's inequality, we also obtain

$$\begin{aligned}
\langle \mathcal{L}(\upsilon_n), \upsilon_n - \upsilon\rangle &= \int_{\mathbb{R}^N}\left(|\nabla\upsilon_n|^{p(x)-2}\nabla\upsilon_n\right)(\nabla\upsilon_n - \nabla\upsilon) + \lambda(x)\left(|\nabla\upsilon_n|^{q(x)-2}\nabla\upsilon_n\right)(\nabla\upsilon_n - \nabla\upsilon)dx\\
&\quad + \int_{\mathbb{R}^N} V(x)\left(|\upsilon_n|^{p(x)-2}\upsilon_n\right)(\upsilon_n - \upsilon) + \lambda(x)V(x)\left(|\upsilon_n|^{q(x)-2}\upsilon_n\right)(\upsilon_n - \upsilon)dx\\
&= \int_{\mathbb{R}^N}\left(|\nabla\upsilon_n|^{p(x)} + \lambda(x)|\nabla\upsilon_n|^{q(x)} + V(x)|\upsilon_n|^{p(x)} + \lambda(x)V(x)|\upsilon_n|^{q(x)}\right)\\
&\quad - \int_{\mathbb{R}^N}|\nabla\upsilon_n|^{p(x)-1}|\nabla\upsilon|dx - \int_{\mathbb{R}^N}\lambda(x)|\nabla\upsilon_n|^{q(x)-1}|\nabla\upsilon|dx\\
&\quad - \int_{\mathbb{R}^N}V(x)|\upsilon_n|^{p(x)-1}|\upsilon|dx - \int_{\mathbb{R}^N}\lambda(x)V(x)|\upsilon_n|^{q(x)-1}|\upsilon|dx\\
&\geq \int_{\mathbb{R}^N}\left(|\nabla\upsilon_n|^{p(x)} + \lambda(x)|\nabla\upsilon_n|^{q(x)} + V(x)|\upsilon_n|^{p(x)} + \lambda(x)V(x)|\upsilon_n|^{q(x)}\right)\\
&\quad - \frac{1}{p'(x)}\int_{\mathbb{R}^N}|\nabla\upsilon_n|^{p(x)}dx - \frac{1}{p(x)}\int_{\mathbb{R}^N}|\nabla\upsilon|^{p(x)}dx\\
&\quad - \frac{1}{q'(x)}\int_{\mathbb{R}^N}\lambda(x)|\nabla\upsilon_n|^{q(x)}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}\lambda(x)|\nabla\upsilon|^{q(x)}dx\\
&\quad - \frac{1}{p'(x)}\int_{\mathbb{R}^N}V(x)|\upsilon_n|^{p(x)}dx - \frac{1}{p(x)}\int_{\mathbb{R}^N}V(x)|\upsilon|^{p(x)}dx\\
&\quad - \frac{1}{q'(x)}\int_{\mathbb{R}^N}\lambda(x)V(x)|\upsilon_n|^{q(x)}dx - \frac{1}{q(x)}\int_{\mathbb{R}^N}\lambda(x)V(x)|\upsilon|^{q(x)}dx
\end{aligned}$$

$$=J(\upsilon_n) - J(\upsilon).$$

From this and (2.8), we get

$$\lim_{n\to\infty} J(\upsilon_n) = J(\upsilon).$$

Let $f(\upsilon) = \frac{1}{p(x)}|\nabla\upsilon|^{p(x)} + \frac{\lambda(x)}{q(x)}|\nabla\upsilon|^{q(x)} + \frac{V(x)}{p(x)}|\upsilon|^{p(x)} + \frac{\lambda(x)V(x)}{q(x)}|\upsilon|^{q(x)}$. The Vitali theorem yields the uniform integrability of the sequence $\{f(\upsilon_n)\}_{n\in\mathbb{N}}$. On the other hand,

$$|\nabla\upsilon_n - \nabla\upsilon|^{p(x)} + \lambda(x)|\nabla\upsilon_n - \nabla\upsilon|^{q(x)} + V(x)|\upsilon_n - \upsilon|^{p(x)} + \lambda(x)V(x)|\upsilon_n - \upsilon|^{q(x)} \le 2^{q^+-1}q^+\left(f(\upsilon_n) + f(\upsilon)\right),$$

which means the sequence $\left\{|\nabla\upsilon_n - \nabla\upsilon|^{p(x)} + \lambda(x)|\nabla\upsilon_n - \nabla\upsilon|^{q(x)} + V(x)|\upsilon_n - \upsilon|^{p(x)} + \lambda(x)V(x)|\upsilon_n - \upsilon|^{q(x)}\right\}_{n\in\mathbb{N}}$ is also uniformly integrable. Applying the Vitali theorem, it follows that

$$\lim_{n\to\infty} \varrho(\upsilon_n - \upsilon) = 0.$$

Hence, $\upsilon_n \to \upsilon$ in E.

(c) Since $\mathcal{L}$ is strictly monotone, $\mathcal{L}$ is an injection, and

$$\lim_{\|\upsilon\|\to\infty} \frac{\langle \mathcal{L}\upsilon, \upsilon\rangle}{\|\upsilon\|} = \frac{\varrho(\upsilon)}{\|\upsilon\|} = +\infty,$$

$\mathcal{L}$ is coercive. In view of the Minty-Browder Theorem, $\mathcal{L}$ has an inverse mapping $\mathcal{L}^{-1} : E^* \to E$. Next, we prove that $\mathcal{L}^{-1}$ is continuous to ensure $\mathcal{L}$ is homeomorphism.

If $\varpi_n, \varpi \in E^*$, $\varpi_n \to \varpi$, let $\upsilon_n = \mathcal{L}^{-1}(\varpi_n)$, $\upsilon = \mathcal{L}^{-1}(\varpi)$, then $\mathcal{L}(\upsilon_n) = \varpi_n$, $\mathcal{L}(\upsilon) = \varpi$. Note that $\{\upsilon_n\}$ is bounded in E by the coercivity of $\mathcal{L}$. Without loss of generality, assume that $\upsilon_n \rightharpoonup \upsilon_0$. It follows from $\varpi_n \to \varpi$ that

$$\lim_{n\to\infty} \langle \mathcal{L}(\upsilon_n) - \mathcal{L}(\upsilon), \upsilon_n - \upsilon\rangle = \langle \varpi_n - \varpi, \upsilon_n - \upsilon\rangle = 0.$$

Thus, $\upsilon_n \to \upsilon_0$ in E, as $\mathcal{L}$ is of type (S_+). Moreover, form $\mathcal{L}(\upsilon_0) = \lim_{n\to\infty} \mathcal{L}(\upsilon_n) = \lim_{n\to\infty} \varpi_n = \varpi$, we have $\upsilon_n \to \upsilon$. Therefore, $\mathcal{L}^{-1}$ is continuous.

3. Some classes of mappings and topological degree

Let E be a real separable reflexive Banach space. E^* is its dual space and denote by $\langle\cdot,\cdot\rangle$ its duality pairing. For a nonempty subset M of E, $\overline{M}$ and ∂M denote the closure and the boundary of M.

Definition 3.1. Let F be another real Banach space. A mapping $\mathcal{L} : M \subset E \to F$ is called
(i) bounded, if it takes any bounded set into a bounded set,
(ii) demicontinuous, if for any $\{\upsilon_n\} \in M$, $\upsilon_n \to \upsilon$ implies $\mathcal{L}(\upsilon_n) \rightharpoonup \mathcal{L}(\upsilon)$.

Definition 3.2. Let $T : M_1 \subset E \to E^*$ be a bounded operator such that $M \subset M_1$ and any operator $\mathcal{L} : M \subset E \to E$. If for any $\{\upsilon_n\} \in M$ with $\upsilon_n \rightharpoonup \upsilon$, $\omega_n := T\upsilon_n \rightharpoonup \omega$ and $\limsup\langle \mathcal{L}\upsilon_n, \omega_n - \omega\rangle \le 0$, we have $\upsilon_n \to \upsilon$, then $\mathcal{L}$ satisfies condition $(S_+)_T$.

For each $M \subset E$, define the following types of operators

$\mathcal{L}_1(M) :=\{\mathcal{L} : M \to E^*|\mathcal{L}$ is demicontinuous, bounded, and satisfies condition$(S_+)\}$,

$\mathcal{L}_T(M) :=\{\mathcal{L} : M \to E|\mathcal{L}$ is demicontinuous and satisfies condition$(S_+)_T\}$,

$\mathcal{L}_{T,B}(M) :=\{\mathcal{L} : M \to E|\mathcal{L}$ is demicontinuous, bounded, and satisfies condition$(S_+)_T\}$,

$\mathcal{L}(E) :=\{\mathcal{L} \in \mathcal{L}_{T,B}(\overline{M})|M \in \Theta,\ T \in \mathcal{L}_1(\overline{M})\}$,

where Θ denotes the collection of all bounded open sets in E and $T \in \mathcal{L}_1(\overline{M})$ is called an essential inner map to $\mathcal{L}$.

Lemma 3.1. *[31] Let $M \subset E$ be a bounded open set. Suppose that $T \in \mathcal{L}_1(\overline{M})$ is continuous and $K : E_k \subset E^* \to E$ is demicontinuous such that $T(\overline{M}) \subset E_k$, then the following properties hold*
(i) If K is quasimonotone, then $I + K \circ T \in \mathcal{L}_T(\overline{M})$, where I stands for the identity operator.
(ii) If K is class of (S_+), then $K \circ T \in \mathcal{L}_T(\overline{M})$.

Definition 3.3. Assume that $M \subset E$ is a bounded open set, $T \in \mathcal{L}_1(\overline{M})$ is continuous, and $\mathcal{L}, K \in \mathcal{L}_T(\overline{M})$. Define affine homotopy $\mathcal{H} : [0,1] \times \overline{M} \to E$ as

$$\mathcal{H}(\eta, \upsilon) = (1-\eta)\mathcal{L}\upsilon + \eta K\upsilon \ \text{ for } \ (\eta, \upsilon) \in [0,1] \times \overline{M},$$

where it is called an admissible affine homotopy with the common continuous essential inner map T and it satisfies condition $(S_+)_T$ (see [31]).

Now, we give the topological degree for the class $\mathcal{L}(E)$.

Theorem 3.1. There exists a unique degree function

$$d : \left\{(\mathcal{L}, M, h) : M \in \Theta,\ T \in \mathcal{L}_1(\overline{M}),\ \mathcal{L} \in \mathcal{L}_{T,B}(\overline{M}),\ h \notin \mathcal{L}(\partial M)\right\} \to \mathbb{Z},$$

which satisfies the properties such as normalization, additivity, homotopy invariance, and existence (see [31, 32]).

4. Existence of weak solutions

Lemma 4.1. Under assumptions (H), the operator $K : E \to E^*$ given by

$$\langle K\upsilon, \xi\rangle = -\int_{\mathbb{R}^N} h(x, \upsilon, \nabla\upsilon)\xi dx, \quad \upsilon, \xi \in E, \tag{4.1}$$

is compact.

Proof. Define an operator $\varphi : E \to L^{p'(x)}(\mathbb{R}^N)$ as

$$\varphi\upsilon(x) = h(x, \upsilon, \nabla\upsilon) \text{ for } x \in \mathbb{R}^N \text{ and } \upsilon \in E.$$

We prove that φ is bounded and continuous.

For every $\upsilon \in E$, using the embedding $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$ and condition (H), we obtain

$$|\varphi\upsilon|_{p'(x)} \leq \varrho_{p'(x)}(\varphi\upsilon) + 1 = \int_{\mathbb{R}^N} |h(x, \upsilon(x), \nabla\upsilon(x))|^{p'(x)}\, dx + 1$$

$$
\begin{aligned}
&\leq C\left(\varrho_{p'(x)}(\gamma)+\varrho_{p(x)}(\upsilon)+\varrho_{p(x)}(\nabla\upsilon)\right)+1\\
&\leq C\left(|\gamma|_{p'(x)}^{p'^{+}}+|\gamma|_{p'(x)}^{p'^{-}}+|\upsilon|_{p(x)}^{p^{+}}+|\upsilon|_{p(x)}^{p^{-}}+\varrho(\upsilon)\right)+1\\
&\leq C\left(|\gamma|_{p'(x)}^{p'^{+}}+|\gamma|_{p'(x)}^{p'^{-}}+\|\upsilon\|^{p^{+}}+\|\upsilon\|^{p^{-}}+\|\upsilon\|^{q^{+}}\right)+1,
\end{aligned}
$$

where $C>0$ stands for arbitrary constant, which means that φ is bounded on E.

Let $\upsilon_n \to \upsilon$ in E, then $\upsilon_n \to \upsilon$ in $L^{p(x)}(\mathbb{R}^N)$ and $\nabla\upsilon_n \to \nabla\upsilon$ in $\left(L^{p(x)}(\mathbb{R}^N)\right)^N$. Thus, there exist a subsequence $\{\upsilon_k\}_{k\in\mathbb{N}}$ of $\{\upsilon_n\}_{n\in\mathbb{N}}$, and measurable functions $\vartheta \in L^{p(x)}(\mathbb{R}^N)$ and $\zeta \in \left(L^{p(x)}(\mathbb{R}^N)\right)^N$ satisfy

$$\upsilon_k(x) \to \upsilon(x) \text{ and } \nabla\upsilon_k(x) \to \nabla\upsilon(x),$$

$$|\upsilon_k(x)| \leq \vartheta(x) \text{ and } |\nabla\upsilon_k(x)| \leq \zeta(x),$$

for any $k\in\mathbb{N}$ and a.e. $x\in\mathbb{R}^N$. Since h is a Carathéodory function, we get

$$h\left(x,\upsilon_k(x),\nabla\upsilon_k(x)\right) \to h\left(x,\upsilon(x),\nabla\upsilon(x)\right), \text{ as } k\to\infty. \tag{4.2}$$

It follows from (H) that

$$h\left(x,\upsilon_k(x),\nabla\upsilon_k(x)\right) \leq \gamma(x)+d_1|\vartheta(x)|^{p(x)-1}+d_2|\zeta(x)|^{p(x)-1}, \tag{4.3}$$

for any $k\in\mathbb{N}$ and a.e. $x\in\mathbb{R}^N$. Note that

$$\gamma(x)+d_1|\vartheta(x)|^{p(x)-1}+d_2|\zeta(x)|^{p(x)-1} \in L^{p'(x)}(\mathbb{R}^N).$$

According to (4.2), (4.3), and the dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} \left|h\left(x,\upsilon_k(x),\nabla\upsilon_k(x)\right)-h\left(x,\upsilon(x),\nabla\upsilon(x)\right)\right|^{p'(x)}dx \to 0, \text{ as } k\to\infty,$$

that is,

$$\varphi\upsilon_k \to \varphi\upsilon \text{ in } L^{p'(x)}(\mathbb{R}^N).$$

Therefore, the entire sequence $\varphi\upsilon_n$ converges to $\varphi\upsilon$ in $L^{p'(x)}(\mathbb{R}^N)$. Thus, φ is continuous.

Recall that the embeding $I: E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$. It is known that the adjoint $I^*: L^{p'(x)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow E^*$. Hence, we conclude that the composition $K = I^*\circ\varphi$ is compact.

Theorem 4.1. Assume that condition (H) hold. Then problem (H_V) has a weak solution in E.

Proof. Due to the Lemma 4.1 and the definition of the operator $\mathcal{L}$, we have that $\upsilon\in E$ is a weak solution of problem (H_V) when, and only when,

$$\mathcal{L}\upsilon = -K\upsilon. \tag{4.4}$$

By the proof of Lemmas 2.3 and 4.1, we known that the inverse operator $T = \mathcal{L}^{-1}$ is continuous, bounded, and of type (S_+), and the operator K is continuous, bounded, and quasimonotone.

Therefore, Eq (4.4) is equivalent to

$$\upsilon = T\xi \ \text{ and } \ \xi + K \circ T\xi = 0. \tag{4.5}$$

Next, we solve Eq (4.5) with degree theory. First, we prove that the set

$$A := \{\xi \in E^* | \xi + \eta K \circ T\xi = 0 \text{ for some } \eta \in [0, 1]\}$$

is bounded. Indeed, let $\xi \in A$. Set $\upsilon = T\xi$, then $\|\upsilon\| = \|T\xi\|$.

(i)If $\|\upsilon\| \leq 1$, then $\|T\xi\|$ is bounded.

(ii)If $\|\upsilon\| > 1$, then

$$\begin{aligned}
\|T\xi\|^{p^-} = \|\upsilon\|^{p^-} \leq & \varrho(\upsilon) = \langle \mathcal{L}\upsilon, \upsilon\rangle = \langle \xi, T\xi\rangle \\
= & -\eta\langle K \circ T\xi, T\xi\rangle = \eta \int_{\mathbb{R}^N} h(x, \upsilon, \nabla\upsilon)\upsilon dx \\
\leq & \int_{\mathbb{R}^N} |\gamma||\upsilon| dx + d_1 \int_{\mathbb{R}^N} |\upsilon|^{p(x)} dx + d_2 \int_{\mathbb{R}^N} |\nabla\upsilon|^{p(x)-1}|\upsilon| dx \\
\leq & 2|\gamma|_{p'(x)}|\upsilon|_{p(x)} + d_1\varrho_{p(x)}(\upsilon) + \frac{d_2}{p'(x)}\varrho_{p(x)}(\nabla\upsilon) + \frac{d_2}{p(x)}\varrho_{p(x)}(\upsilon) \\
\leq & \max\left\{d_2 - \frac{d_2}{p^+}, \frac{d_1 p^- + d_2}{V_0 p^-}\right\}\varrho(\upsilon) + 2|\gamma|_{p'(x)}|\upsilon|_{p(x)}.
\end{aligned} \tag{4.6}$$

Now, we choose $\varsigma = 1 - \max\left\{d_2 - \frac{d_2}{p^+}, \frac{d_1 p^- + d_2}{V_0 p^-}\right\} > 0$, then by embedding $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$, we obtain

$$\|T\xi\|_E^{p^-} \leq C\|T\xi\| + \frac{1}{\varsigma}.$$

Thanks to the assumption $p^- > 1$, $\|T\xi\|$ is bounded, which means $\{T\xi | \xi \in A\}$ is bounded.

Moreover, the boundedness of operator K and (4.5) implies the set A is bounded in E. Therefore, there exists $a > 0$ such that

$$|\xi|_{E^*} < a, \ \text{for any } \xi \in A.$$

This means that

$$\xi + \eta K \circ T\xi \neq 0, \ \text{for each } \xi \in \partial S_a(0) \text{ and each } \eta \in [0, 1].$$

From Lemma 3.1, we conclude that

$$I + K \circ T\xi \in \mathcal{L}_T(\overline{S_a(0)}), \ \text{and } I = \mathcal{L} \circ T \in \mathcal{L}_T(\overline{S_a(0)}),$$

and $I + K \circ T$ is also bounded due to that the operators I, K, and T are bounded. It follows that

$$I + K \circ T\xi \in \mathcal{L}_{T,B}(\overline{S_a(0)}), \ \text{and } I = \mathcal{L} \circ T \in \mathcal{L}_{T,B}(\overline{S_a(0)}).$$

Next, discuss a homotopy $\mathcal{H} : [0, 1] \times \overline{S_a(0)} \to E^*$ as

$$\mathcal{H}(\eta, \xi) = \xi + \eta K \circ T\xi, \ \text{for } (\eta, \xi) \in [0, 1] \times \overline{S_a(0)}.$$

Based on the normalization property and homotopy invariance of the degree d in Theorem 3.1, we have

$$d(I + K \circ T, S_a(0), 0) = d(I, S_a(0), 0) = 1.$$

Thus, there exists a point $\xi \in S_a(0)$ that satisfies equation

$$\xi + K \circ T\xi = 0,$$

which means that $\upsilon = T\xi$ is a weak solution of problem (H_V).

5. Existence of strong generalized solutions

Since the Banach space E is separable, we can find a Galerkin basis of E, which means a sequence $\{E_n\}_{n\in\mathbb{N}}$ of vector subspaces of E with

$$dim(E_n) < \infty, E_n \subset E_{n+1} \text{ for all } n \in \mathbb{N} \text{ and } \overline{\bigcup_{n=1}^{\infty} E_n} = E.$$

First, we introduce the notion of the strong generalized solution, then we derive the existence of strong generalized solutions for the problem (H_V) based on the Galerkin method. Our approach is largely inspired by [16].

Definition 5.1. A function $\upsilon \in E$ is a strong generalized solution to equation (H_V), if there exists a sequence $\{\upsilon_n\}_{n\in\mathbb{N}} \subseteq E$ satisfying the following statements
(i) $\upsilon_n \rightharpoonup \upsilon$ in E, as $n \to \infty$;
(ii) $-\Delta_\lambda^V \upsilon_n - h(\cdot, \upsilon_n(\cdot), \nabla\upsilon_n(\cdot)) \rightharpoonup 0$ in E^*, as $n \to \infty$;
(iii) $\lim_{n\to\infty}\langle -\Delta_\lambda^V \upsilon_n, \upsilon_n - \upsilon\rangle = 0$.

Lemma 5.1. Assume that assumption (H) holds. One has the inequality

$$\left|\int_{\mathbb{R}^N} h(x,\upsilon,\nabla\upsilon)\xi dx\right| \le \left(2|\gamma|_{p'(x)} + C_1 + C_2\right)|\xi|_{p(x)},$$

for any $\upsilon, \xi \in E$, where C_1, C_2 is shown below.

Proof. Using condition (H), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N} h(x,\upsilon,\nabla\upsilon)\xi dx\right| \le& \left|\int_{\mathbb{R}^N} \left(\gamma(x) + d_1|\upsilon|^{p(x)-1} + d_2|\nabla\upsilon|^{p(x)-1}\right)\xi dx\right| \\ \le& \int_{\mathbb{R}^N} |\gamma||\xi| dx + d_1\int_{\mathbb{R}^N} |\upsilon|^{p(x)-1}|\xi| dx + d_2\int_{\mathbb{R}^N} |\nabla\upsilon|^{p(x)-1}|\xi| dx \\ \le& 2|\gamma|_{p'(x)}|\xi|_{p(x)} + 2d_1|\upsilon|_{p(x)}^{p(x)-1}|\xi|_{p(x)} + 2d_2|\nabla\upsilon|_{p(x)}^{p(x)-1}|\xi|_{p(x)} \\ \le& 2|\gamma|_{p'(x))}|\xi|_{p(x)} + C_1|\xi|_{p(x)} + C_2|\xi|_{p(x)},\end{aligned}$$

where

$$C_1 = 2d_1\left(|\upsilon|_{p(x)}^{p^+-1} + |\upsilon|_{p(x)}^{p^--1}\right), \quad C_2 = 2d_2\left(|\nabla\upsilon|_{p(x)}^{p^+-1} + |\nabla\upsilon|_{p(x)}^{p^--1}\right).$$

Lemma 5.2. Let $(E, \|\cdot\|)$ be a normed finite dimensional space and $B : E \to E^*$ be a continuous map. Suppose that there exists some $\delta > 0$, which satisfies

$$\langle B(\upsilon), \upsilon\rangle \ge 0, \text{ for any } \upsilon \in E \text{ with } \|\upsilon\| = \delta,$$

then $B(\upsilon) = 0$ has a solution $\upsilon \in E$ with $\|\upsilon\| \le \delta$.

Theorem 5.1. Suppose that condition (H) is satisfied, then for all $n \in \mathbb{N}$ and $\psi \in E_n$, there exists $\upsilon_n \in E_n$ such that

$$\langle -\Delta_\lambda^V \upsilon_n, \psi \rangle = \int_{\mathbb{R}^N} h(x, \upsilon_n(x), \nabla \upsilon_n(x))\psi(x)dx. \tag{5.1}$$

Proof. For every $n \in \mathbb{N}$, we define the operator $B_n : E_n \to E_n^*$ by

$$\langle B_n(\upsilon), \psi \rangle = \langle -\Delta_\lambda^V \upsilon, \psi \rangle - \int_{\mathbb{R}^N} h(x, \upsilon(x), \nabla \upsilon(x))\psi(x)dx,$$

for every $\upsilon, \psi \in E_n$. From (H) and (4.6), we have the following estimate

$$\begin{aligned}
\langle B_n(\upsilon), \upsilon \rangle = &\int_{\mathbb{R}^N} \left(|\nabla \upsilon|^{p(x)} + \lambda(x)|\nabla \upsilon|^{q(x)}\right) dx + \int_{\mathbb{R}^N} V(x)\left(|\upsilon|^{p(x)} + \lambda(x)|\upsilon|^{q(x)}\right) dx - \int_{\mathbb{R}^N} h(x, \upsilon, \nabla \upsilon)\upsilon dx \\
\geq &\varrho(\upsilon) - \int_{\mathbb{R}^N} |\gamma(x)|dx - d_1 \int_{\mathbb{R}^N} |\upsilon|^{p(x)}dx - d_2 \int_{\mathbb{R}^N} |\nabla \upsilon|^{p(x)-1}|\upsilon|dx \\
\geq &\left(1 - \max\left\{d_2 - \frac{d_2}{p^+}, \frac{d_1 p^- + d_2}{V_0 p^-}\right\}\right)\varrho(\upsilon) - \int_{\mathbb{R}^N} |\gamma(x)|dx.
\end{aligned}$$

If $\|\upsilon\| > 1$, then there exists $\delta > 0$ large enough. Whenever $\upsilon \in E_n$ with $\|\upsilon\| = \delta$, such that

$$\langle B_n(\upsilon), \upsilon \rangle \geq \varsigma \|\upsilon\|^{p^-} - |\gamma|_{L^1(\mathbb{R}^N)} \geq 0,$$

since $p^- > 1$ and $\varsigma > 0$. In view of Lemma 5.2, the equation $B_n(\upsilon) = 0$ has an approximate solution $\upsilon_n \in E_n$, which is (5.1). The proof is complete.

Lemma 5.3. If condition (H) holds, then the sequence $\{\upsilon_n\}_{n\in\mathbb{N}}$ with $\upsilon_n \in E_n$ constructed in Theorem 5.1 is bounded in E.

Proof. If $\|\upsilon_n\| \leq 1$ for any $n \in \mathbb{N}$, then $\{\upsilon_n\}_{n\in\mathbb{N}}$ is bounded in E. So, when $\|\upsilon_n\| > 1$ for any $n \in \mathbb{N}$, insert $\psi = \upsilon_n$ in (5.1), then we have

$$\langle -\Delta_\lambda^V \upsilon_n, \upsilon_n \rangle = \int_{\mathbb{R}^N} h(x, \upsilon_n(x), \nabla \upsilon_n(x))\upsilon_n dx.$$

Based on hypotheses (H) and (4.6), we obtain

$$\begin{aligned}
\|\upsilon_n\|^{p^-} \leq &\varrho(\upsilon_n) = \int_{\mathbb{R}^N} h(x, \upsilon_n, \nabla \upsilon_n)\upsilon_n dx \\
\leq &\int_{\mathbb{R}^N} |\gamma(x)|dx + d_1 \int_{\mathbb{R}^N} |\upsilon_n|^{p(x)}dx + d_2 \int_{\mathbb{R}^N} |\nabla \upsilon_n|^{p(x)-1}|\upsilon_n|dx \\
\leq &\max\left\{d_2 - \frac{d_2}{p^+}, \frac{d_1 p^- + d_2}{V_0 p^-}\right\}\varrho(\upsilon_n) + |\gamma|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

Recalling that $p^- > 1$ and $\varsigma > 0$, we conclude that the desired conclusion.

Theorem 5.2. If conditions (H) holds, then equation (H_V) has a strong generalized solution in E.

Proof. We know that $\{\upsilon_n\}$ is bounded in E by Lemma 5.2. Since E is reflexive, then

$$\upsilon_n \rightharpoonup \upsilon \text{ in } E, \text{ for some } \upsilon \in E. \tag{5.2}$$

Lemma 5.1 indicates that the Nemytskii operator $N_h : E \to E^*$ given by

$$N_h(\upsilon) = h(x, \upsilon, \nabla\upsilon), \text{ for any } \upsilon \in E,$$

is well defined, and we can find that constants $C_1, C_2 > 0$ satisfy

$$\|N_h(\upsilon_n)\|_{E^*} \leq \left(|\gamma|_{p'(x)} + C_1 + C_2\right), \ \upsilon_n \in E.$$

Thus, the Nemytskii operator N_h is bounded. In association with (5.2), then

$$\{N_h(\upsilon_n)\}_{n\in\mathbb{N}} \text{ is bounded in } E^*. \tag{5.3}$$

The boundedness of the operator $-\Delta_\lambda^V : E \to E^*$ implies that

$$\{-\Delta_\lambda^V \upsilon_n - N_h(\upsilon_n)\}_{n\in\mathbb{N}} \text{ is also bounded in } E^*. \tag{5.4}$$

Again, by the reflexivity of E^*, we obtain

$$-\Delta_\lambda^V \upsilon_n - N_h(\upsilon_n) \rightharpoonup \kappa \text{ in } E^*, \tag{5.5}$$

for some $\kappa \in E^*$.

Let $\zeta \in \cup_{i=1}^\infty E_n$, then we can find $m \in \mathbb{N}$ such that $\zeta \in E_m$. So, Theorem 5.1 implies that equality (5.1) is ture for any $n \geq m$. As $n \to \infty$ in (5.1), then

$$\langle \kappa, \zeta \rangle = 0 \text{ for each } \zeta \in \cup_{i=1}^\infty E_n.$$

Since $\zeta \in \cup_{i=1}^\infty E_n$ is dense in E^*, then we deduce that $\kappa = 0$. Therefore, (5.5) becomes

$$-\Delta_\lambda^V \upsilon_n - N_h(\upsilon_n) \rightharpoonup 0 \text{ in } E^*. \tag{5.6}$$

Next, choose $\psi = \upsilon_n$ in (5.1), that is,

$$\langle -\Delta_\lambda^V \upsilon_n, \upsilon_n \rangle = h(x, \upsilon_n, \nabla\upsilon_n), \text{ for any } n \in \mathbb{N}. \tag{5.7}$$

In addition to this, from (5.6) we have

$$\langle -\Delta_\lambda^V \upsilon_n - N_h(\upsilon_n), \upsilon \rangle \to 0, \text{ as } n \to \infty. \tag{5.8}$$

Combining (5.7) and (5.8), we get

$$\lim_{n\to\infty} \langle -\Delta_\lambda^V \upsilon_n - N_h(\upsilon_n), \upsilon_n - \upsilon \rangle = 0. \tag{5.9}$$

From Lemma 5.1, choosing the test function $\xi = \upsilon_n - \upsilon$, we get

$$\left| \int_{\mathbb{R}^N} h(x, \upsilon, \nabla\upsilon)(\upsilon_n - \upsilon) dx \right| \leq \left(2|\gamma|_{p'(x)} + C_1 + C_2\right) |\upsilon_n - \upsilon|_{p(x)},$$

Due to the compact embeddings $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$, we deduct that $\upsilon_n \to \upsilon$ in $L^{p(x)}(\mathbb{R}^N)$. The $\{\upsilon_n\}_{n\in\mathbb{N}}$ is bounded in E and, hence, in $L^{p(x)}(\mathbb{R}^N)$. Also, $\{\nabla\upsilon_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p(x)}(\mathbb{R}^N)$. This implies that

$$\int_{\mathbb{R}^N} h(x, \upsilon_n, \nabla\upsilon_n)(\upsilon_n - \upsilon)dx \to 0 \text{ as } n \to \infty. \tag{5.10}$$

Consequently, (5.4) gives us

$$\lim_{n\to\infty}\langle -\Delta_\lambda^V \upsilon_n, \upsilon_n - \upsilon\rangle = 0 \tag{5.11}$$

Obviously, (5.2), (5.6) and (5.11) show that $\upsilon \in E$ is a strong generalized solution to equation (H_V). This completes the proof.

Corollary 5.1. Assume that the equation (H_V) has a strong generalized solution $\upsilon \in E$ stated in Theorem 5.2, then $\upsilon \in E$ is a weak solution to equation (H_V). The same holds in the opposite sense.

Proof. If $\omega \in E$ is a strong generalized solution to equation (H_V), then

$$\lim_{n\to\infty}\langle -\Delta_\lambda^V \upsilon_n, \upsilon_n - \upsilon\rangle = 0,$$

which means $\upsilon_n \to \upsilon$ in E, since $-\Delta_\lambda^V$ fulfills the (S_+)-property. Using again Definition 5.1, we have

$$-\Delta_\lambda^V \upsilon_n - h(\cdot, \upsilon_n(\cdot), \nabla\upsilon_n(\cdot)) \rightharpoonup 0 \text{ in } E^* \text{ as } n \to \infty,$$

and it follows that

$$-\Delta_\lambda^V \upsilon - h(\cdot, \upsilon(\cdot), \nabla\upsilon(\cdot)) = 0.$$

Thus, $\upsilon \in E$ is a weak solution of equation (H_V) (see (2.3)). If posing $\upsilon = \upsilon_n$, it is clear that any weak solution is a strong generalized solution for equation (H_V).

Remark 5.1. Note that for the problem (H_V), each weak solution is a generalized solution. However, a generalized solution does not necessarily derive the notion of weak solution. For the definition of a generalized solution, one can refer to [16].

6. Conclusions

In this article, we study a class of Schrödinger equations in $\mathbb{R}^N$. One of the main features of the paper is the presence of a new double phase operator with variable exponents. We give the corresponding Musielak-Orlicz Sobolev spaces and compact embedding result. Another significant characteristic of the paper is the presence of convection term. Based on the topological degree theory and Galerkin method, we not only obtain the existence of strong generalized solutions, but also the existence of weak solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Postgraduate Research Practice Innovation Program of Jiangsu Province under Grant (KYCX23-0669) and the Doctoral Foundation of Fuyang Normal University under Grant (2023KYQD0044).

Conflict of interest

The authors declare no conflicts of interest.

References

1. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR-Izvestiya*, **29** (1987), 33–66. https://doi.org/10.1070/im1987v029n01abeh000958
2. F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pur. Appl.*, **195** (2016), 1917–1959. https://doi.org/10.1007/s10231-015-0542-7
3. W. Liu, G. Dai, Existence and multiplicity results for double phase problem, *J. Differ. Equ.*, **265** (2018), 4311–4334. https://doi.org/10.1016/j.jde.2018.06.006
4. K. Perera, M. Squassina, Existence results for double phase problems via Morse theory, *Commun. Contemp. Math.*, **20** (2018), 1750023. https://doi.org/10.1142/S0219199717500237
5. W. Liu, G. Dai, Multiplicity results for double phase problems in $\mathbb{R}^N$, *J. Math. Phys.*, **61** (2020), 091508. https://doi.org/10.1063/5.0020702
6. R. Steglinski, Infinitely many solutions for double phase problem with unbounded potential in $\mathbb{R}^N$, *Nonlinear Anal.*, **214** (2022), 112580. https://doi.org/10.1016/j.na.2021.112580
7. J. Shen, L. Wang, K. Chi, B. Ge, Existence and multiplicity of solutions for a quasilinear double phase problem on the whole space, *Complex Var. Elliptic*, **68** (2023), 206–316. https://doi.org/10.1080/17476933.2021.1988585
8. C. Farkas, P. Winkert, An existence result for singular Finsler double phase problems, *J. Differ. Equ.*, **286** (2021), 455–473. https://doi.org/10.1016/j.jde.2021.03.036
9. N. Cui, H. Sun, Existence and multiplicity results for double phase problem with nonlinear boundary condition, *Nonlinear Anal. Real*, **60** (2021), 103307. https://doi.org/10.1016/j.nonrwa.2021.103307
10. K. Wang, Q. Zhou, On a double phase problem with sublinear and superlinear nonlinearities, *Complex Var. Elliptic*, **66** (2021), 1182–1193. https://doi.org/10.1080/17476933.2021.1885383
11. Z. Liu, N.S. Papageorgiou, A double phase equation with convection, *Electron. J. Qual. Theory Differ. Equ.*, **91** (2021), 1–11. https://doi.org/10.14232/ejqtde.2021.1.91
12. B. Ge, X. Cao, W. Yuan, Existence of two solutions for double-phase problems with a small perturbation, *Appl. Anal.*, 2021. https://doi.org/10.1080/00036811.2021.1909725
13. L. Gasinski, P. Winkert, Existence and uniqueness results for double phase problems with convection term, *J. Differ. Equ.*, **268** (2020), 4183–4193. https://doi.org/10.1016/j.jde.2019.10.022

14. C. O. Alves, A. Moussaoui, Existence of solutions for a class of singular elliptic systems with convection term, *Asymptotic Anal.*, **90** (2014), 237–248. https://doi.org/10.3233/ASY-141245

15. C. Vetro, Variable exponent $p(x)$-Kirchhoff type problem with convection, *J. Math. Anal. Appl.*, **506** (2022), 125721. https://doi.org/10.1016/j.jmaa.2021.125721

16. D. Motreanu, Quasilinear Dirichlet problems with competing operators and convection, *Open Math.*, **18** (2020), 1510–1517. https://doi.org/10.1515/math-2020-0112

17. W. Bu, T. An, Y. Li, J. He, Kirchhoff-type problems involving logarithmic nonlinearity with variable exponent and convection term, *Mediterr. J. Math.*, **20** (2023), 77. https://doi.org/10.1007/s00009-023-02273-w

18. K. Ho, I. Sim, A-priori bounds and existence for solutions of weighted elliptic equations with a convection term, *Adv. Nonlinear Anal.*, **6** (2017), 427–445. https://doi.org/10.1515/anona-2015-0177

19. D. Averna, N. S. Papageorgiou, E. Tornatore, Positive solutions for nonlinear Robin problems with convection, *Math. Method. Appl. Sci.*, **42** (2019), 1907–1920. https://doi.org/10.1002/mma.5484

20. N. S. Papageorgiou, V. D. Radulescu, D. D. Repovs, Positive solutions for nonlinear Neumann problems with singular terms and convection, *J. Math. Pur. Appl.*, **136** (2020), 1–21. https://doi.org/10.1016/j.matpur.2020.02.004

21. A. Crespo-Blanco, L. Gasinski, P. Harjulehto, P, Winkert, A new class of double phase variable exponent problems: Existence and uniqueness, *J. Differ. Equ.*, **323** (2022), 182–228. https://doi.org/10.1016/j.jde.2022.03.029

22. F. Vetro, P. Winkert, Constant sign solutions for double phase problems with variable exponents, *Appl. Math. Lett.*, **135** (2023), 108404. https://doi.org/10.1016/j.aml.2022.108404

23. A. Aberqi. J. Bennouna, O. Benslimane, M. A, Ragusa, Existence results for double phase problem in Sobolev-COrlicz spaces with variable exponents in complete manifold, *Mediterr. J. Math.*, **19** (2022), 158. https://doi.org/10.1007/s00009-022-02097-0

24. I. H. Kim, Y. H. Kim, M. W. Oh, S. Zeng, Existence and multiplicity of solutions to concave-convex-type double-phase problems with variable exponent, *Nonlinear. Anal. Real*, **67** (2022), 103627.

25. A. Bahrouni, V. D. Radulescu, P. Winkert, Double phase problems with variable growth and convection for the Baouendi-Grushin operator, *Z. Angew. Math. Phys.*, **71** (2020), 183. https://doi.org/10.1007/s00033-020-01412-7

26. V. Benci, D. Fortunato, Discreteness conditions of the spectrum of Schröinger operators, *J. Math. Anal. Appl.*, **64** (1978), 695–700. https://doi.org/10.1016/0022-247x(78)90013-6

27. C. O. Alves, S. Liu, On superlinear p(x)-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.*, **73** (2010), 2566–2579. https://doi.org/10.1016/j.na.2010.06.033

28. A. Salvatore, Multiple solutions for perturbed elliptic equations in unbounded domains, *Adv. Nonlinear Stud.*, **3** (2003), 1–23. https://doi.org/10.1515/ans-2003-0101

29. L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Springer, 2011.

30. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1983.
31. I. S. Kim, S. J. Hong, A topological degree for operators of generalized (S_+) type, *Fixed Point Theory Appl.*, **2015** (2015), 194. https://doi.org/10.1186/s13663-015-0445-8
32. J. Berkovits, Extension of the Leray-Schauder degree for abstract Hammerstein type mappings, *J. Differ. Equ.*, **234** (2007), 289–310. https://doi.org/10.1016/j.jde.2006.11.012

AIMS Press