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Research article

The simplified modulus-based matrix splitting iteration method for the nonlinear complementarity problem

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Abstract: In this paper, the simplified modulus-based matrix splitting iteration method was extended to solve the nonlinear complementarity problem, and the convergence conditions were presented from the spectral radius and the matrix norm. Then, for the special cases of this method, we provided some concrete convergence conditions as well as the quasi-optimal parameter matrix. Moreover, some numerical examples were illustrated to show the validity of the convergence results.

Keywords: nonlinear complementarity; numerical solution; iteration method; matrix spitting; convergence
Mathematics Subject Classification: 65F10, 65N12

1. Introduction

We are concerned with the following nonlinear complementarity problem (NCP(A, φ)),

$$z^{\mathrm{T}}w = 0, \quad z \geq 0, \quad w = Az + \varphi(z) \geq 0, \tag{1.1}$$

here, $z \in R^n$ is to be found, and $A = (a_{ij}) \in R^{n\times n}$ is given with $\varphi(z) = (\varphi_1(z_1), \varphi_2(z_2), \ldots, \varphi_n(z_n))^{\mathrm{T}}$ satisfying $|\frac{d\varphi_i}{dz_i}| \leq \bar{\psi}_i$, where $\bar{\psi}_i$ is the upper boundary of $|\frac{d\varphi_i}{dz_i}|$, for $i = 1, 2, ..., n$. For such a problem and its general cases, some researchers have studied this (see [1–8] and the cited references). Other types of complementary problems (CP) have also been explored, such as, the linear complementarity problem (LCP), the implicit complementarity problem (ICP), the horizontal linear complementarity problem (HLCP), the vertical linear complementarity problems (VLCP), and the horizontal nonlinear complementarity problem (HNCP), etc. (see [9–16] for details).

To calculate the numerical solution of the NCP, many researchers have proposed many practical methods, such as, the projection-filter method [17], the interior proximal point algorithm [18], and the smoothing least square method [19]. If $\varphi(z) = q \in R^n$ in (1.1), the NCP(A, φ) is transformed into the simplest form, i.e., the linear complementarity problem LCP(A, q), which has attracted many researchers to study this numerically, and various kinds of popular numerical solving approaches have been proposed in recent decades ([20–22]). Among these numerical approaches, the modulus-based type matrix splitting iteration methods are very effective for solving the LCP with some particular system matrices (see [9, 23–26]). For the high effectiveness of such modulus-based type iteration methods, some researchers have generalized these methods to deal with other complementarity problems, such as, the nonlinear complementarity problems [3, 4, 6–8], the implicit complementarity problems [5, 10, 27], the horizontal complementarity problems [11, 28], and the vertical linear complementarity problems [12, 14, 15].

Motivated by the recent works in the LCPs and the developments of the NCPs, we continue to study the modulus-based type iteration methods for the NCP in this paper. Wu and Li proposed a new modulus-based matrix splitting iteration method to solve the LCP in [22], which differs to the existing modulus-based type matrix splitting iteration methods, appears simple in form, and shows high efficiency in the given experiments. We extend such methods to solve a kind of NCP and further discuss the convergence. In our discussion, the general convergence conditions of this method are first presented. Then, in order to facilitate practical applications, for the special matrix splitting of A and the special parameter matrix Φ, we propose some concrete convergence regions under certain conditions. Moreover, the quasi-optimal parameter matrix of the method is discussed and provided with respect to the spectral radius. The major results are shown in Section 3. Besides, some numerical examples are given to verify the effectiveness and the convergence of the method.

2. The SMMS iteration method

Initially, we propose the simplified modulus-based matrix splitting (SMMS) iteration method for solving (1.1), and then give some preliminaries for later discussion. Some related definitions and notations, such as, M-matrix, H_+-matrix, the comparison matrix $\langle A\rangle$, and H-splitting, readers refer to [9, 23] and the cited references.

For a positive diagonal matrix Φ, the NCP (1.1) is equivalent to

$$(\Phi z)^{\mathrm{T}} w = 0, \quad z \geq 0, \quad w = Az + \varphi(z) \geq 0, \tag{2.1}$$

thus, based on Lemma 3.1 in [22], the NCP (1.1) can be reformulated as an equivalent fixed-point equation

$$\Phi z + (Az + \varphi(z)) = |(Az + \varphi(z)) - \Phi z|. \tag{2.2}$$

It follows that if $A = F - G$ is a splitting of A, Eq (2.2) becomes

$$(\Phi + F)z = Gz + |(A - \Phi)z + \varphi(z)| - \varphi(z). \tag{2.3}$$

So, if $\Phi + F$ is invertible, we obtain the SMMS iteration method below for solving the NCP (1.1).

The simplified modulus-based matrix splitting (SMMS) iteration method

- Given any initial victor $z^{(0)} \in R^n$, for k=0,1,2,..., compute $z^{(k+1)}$ by solving

$$(\Phi + F)z^{(k+1)} = Gz^{(k)} + |(A - \Phi)z^{(k)} + \varphi(z^{(k)})| - \varphi(z^{(k)}). \tag{2.4}$$

- If $\|\min(z^{(k)}, Az^{(k)} + \varphi(z^{(k)}))\|_2 < \varepsilon$, the iteration stops, where $\|\cdot\|_2$ denotes the 2-norm of vectors and ε is a given positive constant.

The SMMS iteration method differs from the modulus-based matrix splitting (MMS) iteration method [3, 4], and one difference is that the former does not involve new vector x. Obviously, the iterative form of the SMMS iteration method is relatively simple. For the SMMS iteration method dealing with the linear complementarity problem (LCP), as well as the numerical experiments compared with other methods, readers refer to [22] for details.

Suppose that $z^{(*)}$ is the solution of the NCP (1.1), then from (2.3) and (2.4), there is a relationship if $\varphi_i(z_i)$ is differentiable for $i = 1, 2, ..., n$, that is,

$$\begin{aligned}
&(\Phi + F)(z^{(k+1)} - z^{(*)})\\
=&G(z^{(k)} - z^{(*)}) + |(A - \Phi)z^{(k)} + \varphi(z^{(k)})| - |(A - \Phi)z^{(*)} + \varphi(z^{(*)})| - \varphi(z^{(k)}) + \varphi(z^{(*)})\\
=&(G - \Psi^{(k)})(z^{(k)} - z^{(*)}) + |(A - \Phi)z^{(k)} + \varphi(z^{(k)})| - |(A - \Phi)z^{(*)} + \varphi(z^{(*)})|,
\end{aligned} \tag{2.5}$$

where $\Psi^{(k)} = \text{diag}((\frac{d\varphi_1}{d_{z_1}}|_{\xi_1^{(k)}}, \frac{d\varphi_2}{d_{z_2}}|_{\xi_2^{(k)}}, \dots, \frac{d\varphi_n}{d_{z_n}}|_{\xi_n^{(k)}}))$ is a diagonal matrix, and $\xi^{(k)} = (\xi_1^{(k)}, \xi_2^{(k)}, \dots, \xi_n^{(k)})^{\mathrm{T}}$ with $\xi_i^{(k)} \in [z_i^{(k)}, z_i^{(*)}]$ or $\xi_i^{(k)} \in [z_i^{(*)}, z_i^{(k)}]$ for $i = 1, 2, \dots, n$. In our later discussion, without special statements, we always assume that

$$|\Psi^{(k)}| \le \bar{\Psi} = \text{diag}((\bar{\psi}_1, \bar{\psi}_2, \dots, \bar{\psi}_n)) \text{ with } k = 1, 2, ... \tag{2.6}$$

3. Convergence analysis

We discuss the convergence of the SMMS iteration method (2.4) for solving the NCP (1.1) with $\varphi(z)$ satisfying (2.6). In our discussion, we assume that the NCP (1.1) has a unique solution. We first give the convergence conclusions based on (2.5) from the spectral radius and the matrix norm, and then study some concrete convergence conditions in terms of the special matrix splittings.

Theorem 3.1. *Let $A = F - G$ be a splitting of A with $\Phi + F$ being nonsingular. If either of the conditions below holds:*

$$\begin{aligned}
\delta_1 &= \rho\Big(\sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{|(\Phi + F)^{-1}| \, (|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\}\Big) < 1 \text{ and}\\
\delta_2 &= \rho\Big(\sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{|(\Phi + F)^{-1}| \, (|F - \Phi| + 2|G - \Psi^{(k)}|)\}\Big) < 1,
\end{aligned} \tag{3.1}$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$.

Proof. Let $z^{(*)}$ be the unique solution of (1.1), for $\Phi + F$ is invertible, from Eq (2.5),

$$z^{(k+1)} - z^{(*)} = (\Phi + F)^{-1}((G - \Psi^{(k)})(z^{(k)} - z^{(*)}) + |(A - \Phi)z^{(k)} + \varphi(z^{(k)})| - |(A - \Phi)z^{(*)} + \varphi(z^{(*)})|). \tag{3.2}$$

We take the absolute value function on both sides of Eq (3.2), then

$$\begin{aligned}
|z^{(k+1)} - z^{(*)}| \le& |(\Phi + F)^{-1}| \, (|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|) \, |z^{(k)} - z^{(*)}|\\
\le& |(\Phi + F)^{-1}| \, (|F - \Phi| + 2|G - \Psi^{(k)}|) \, |z^{(k)} - z^{(*)}|.
\end{aligned} \tag{3.3}$$

Since $\Psi^{(k)}$ is a diagonal matrix and satisfies $|\Psi^{(k)}| \le \bar{\Psi}$, we know that both $\sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{|(\Phi + F)^{-1}| \, (|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\}$ and $\sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{|(\Phi + F)^{-1}| \, (|F - \Phi| + 2|G - \Psi^{(k)}|)\}$ exist. Thus, if either of the inequalities

$$\delta_1 = \rho\Big(\sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{|(\Phi + F)^{-1}|\,(|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\}\Big) < 1 \text{ and}$$
$$\delta_2 = \rho\Big(\sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{|(\Phi + F)^{-1}|\,(|F - \Phi| + 2|G - \Psi^{(k)}|)\}\Big) < 1$$

holds, from (3.3), it is easy to know that $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$. □

Similarly, if we take the 2-norm $\|\cdot\|_2$ on both sides of Eq (3.2) in Theorem 3.1, the relationship below holds,

$$\begin{aligned} \|z^{(k+1)} - z^{(*)}\|_2 &\le \|(\Phi + F)^{-1}\|_2\,(\|G - \Psi^{(k)}\|_2 + \|A - \Phi + \Psi^{(k)}\|_2)\,\|z^{(k)} - z^{(*)}\|_2 \\ &\le \|(\Phi + F)^{-1}\|_2\,(\|F - \Phi\|_2 + 2\|G - \Psi^{(k)}\|_2)\,\|z^{(k)} - z^{(*)}\|_2. \end{aligned} \tag{3.4}$$

Based on (3.4), we obtain the convergence conclusion below from the 2-norm $\|\cdot\|_2$.

Theorem 3.2. *Let $A = F - G$ be a splitting of A with $\Phi + F$ being nonsingular. If either of the conditions below holds:*

$$\begin{aligned} \sigma_1 &= \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{\|(\Phi + F)^{-1}\|_2\,(\|G - \Psi^{(k)}\|_2 + \|A - \Phi + \Psi^{(k)}\|_2)\} < 1 \text{ and} \\ \sigma_2 &= \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{\|(\Phi + F)^{-1}\|_2\,(\|F - \Phi\|_2 + 2\|G - \Psi^{(k)}\|_2)\} < 1, \end{aligned} \tag{3.5}$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$.

We remark here that for δ_1, δ_2 in Theorem 3.1, and σ_1, σ_2 in Theorem 3.2, the symbol '$\sup_{|\Psi^{(k)}|\le\bar{\Psi}}$' can be put inside, that is

$$\begin{aligned} \delta_1 &= \rho\Big(|(\Phi + F)^{-1}| \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|\}\Big) < 1, \\ \delta_2 &= \rho\Big(|(\Phi + F)^{-1}|\,(|F - \Phi| + 2 \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{|G - \Psi^{(k)}|\})\Big) < 1, \\ \sigma_1 &= \|(\Phi + F)^{-1}\|_2 \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{\|G - \Psi^{(k)}\|_2 + \|A - \Phi + \Psi^{(k)}\|_2\} < 1, \\ \sigma_2 &= \|(\Phi + F)^{-1}\|_2\,(\|F - \Phi\|_2 + 2 \sup_{|\Psi^{(k)}|\le\bar{\Psi}} \{\|G - \Psi^{(k)}\|_2\}) < 1. \end{aligned}$$

Obviously, these formulas are more convenient in practice. In addition, if the detailed range of $\Psi^{(k)}$ is known, these convergence conditions in these two theorems can be more refined. For example, if we know that $\bar{\Psi}_1 \le \Psi^{(k)} \le \bar{\Psi}_2$ for $k = 1, 2, ..., n$, and both bounds $\bar{\Psi}_1$ and $\bar{\Psi}_2$ can be reached, then the symbol '$\sup_{|\Psi^{(k)}|\le\bar{\Psi}}$' can be replaced by '$\sup_{\bar{\Psi}_1\le\Psi^{(k)}\le\bar{\Psi}_2}$' in the formulas. Moreover, for this case, δ_1 can be refined as

$$\iota = \rho\Big(|(\Phi + F)^{-1}|\ \max(|G - \bar{\Psi}_1| + |A - \Phi + \bar{\Psi}_1|, |G - \bar{\Psi}_2| + |A - \Phi + \bar{\Psi}_2|)\Big) < 1,$$

where 'max' is a function in matlab software. In addition, we can see that all the conditions are only sufficient conditions to ensure that the SMMS iteration method converges, not necessary conditions.

When these conditions are not satisfied, the iteration method may also converge. It is apparent that convergence conditions $\delta_1 < 1$ and $\sigma_1 < 1$ are more precise than the other two convergence conditions.

Next, we consider some special convergence conditions. We assume that F is symmetric positive definite in the matrix splitting $A = F - G$, and $\Phi = \phi I$ is a given positive scalar matrix. Then, we obtain the convergence conclusion.

Theorem 3.3. *Let $A = F - G$ be a matrix splitting of A with F being symmetric positive definite. Denote the smallest and the largest eigenvalues of F by λ_1 and λ_n, respectively, and $\tau = \sup_{|\Psi^{(k)}|\leq\bar{\Psi}}\{\|G - \Psi^{(k)}\|_2\}$. Set $\Phi = \phi I(\phi > 0)$, if $\tau < \lambda_1$, then when*

$$\phi \in (\frac{\lambda_n - \lambda_1}{2} + \tau, +\infty), \tag{3.6}$$

$\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$. In addition, $\phi = \frac{\lambda_1 + \lambda_n}{2}$ is a quasi-optimal parameter.

Proof. Since F is symmetric positive definite, the expression of σ_2 in Theorem 3.2 can be refined as

$$\begin{aligned}
\sigma_2 &= \sup_{|\Psi^{(k)}|\leq\bar{\Psi}} \{\|(\Phi + F)^{-1}\|_2 \, (\|F - \Phi\|_2 + 2\|G - \Psi^{(k)}\|_2)\} \\
&= \sup_{|\Psi^{(k)}|\leq\bar{\Psi}} \{\|(\phi I + G)^{-1}\|_2 \, (\|F - \phi I\|_2 + 2\|G - \Psi^{(k)}\|_2)\} \\
&= \frac{\max_\lambda \left\{|\phi - \lambda|\right\} + 2\tau}{\phi + \lambda_1} = \begin{cases} \dfrac{\lambda_n - \phi + 2\tau}{\phi + \lambda_1}, \text{if } \phi \leq \dfrac{\lambda_1 + \lambda_n}{2}, \\ \dfrac{\phi - \lambda_1 + 2\tau}{\phi + \lambda_1}, \text{if } \phi > \dfrac{\lambda_1 + \lambda_n}{2}, \end{cases}
\end{aligned} \tag{3.7}$$

where λ represents the eigenvalue of F. Therefor, based on (ii) in Theorem 3.2, solving

$$\text{(I)} \begin{cases} \dfrac{\lambda_n - \phi + 2\tau}{\phi + \lambda_1} < 1, \\ \phi \leq \dfrac{\lambda_1 + \lambda_n}{2}, \end{cases} \quad \text{and} \quad \text{(II)} \begin{cases} \dfrac{\phi - \lambda_1 + 2\tau}{\phi + \lambda_1} < 1, \\ \phi > \dfrac{\lambda_1 + \lambda_n}{2}, \end{cases} \tag{3.8}$$

in turn, we obtain the convergence region of parameter ϕ as follows.

From (I), if $\tau < \lambda_1$, the parameter ϕ satisfies

$$\phi \in (\frac{\lambda_n - \lambda_1}{2} + \tau, \frac{\lambda_1 + \lambda_n}{2}].$$

From (II), if $\tau < \lambda_1$, the parameter ϕ satisfies

$$\phi \in (\frac{\lambda_1 + \lambda_n}{2}, +\infty).$$

Combining (I) with (II), the convergence region of ϕ is

$$\phi \in (\frac{\lambda_n - \lambda_1}{2} + \tau, +\infty).$$

Then, the first part of this theorem is proved.

For the second part of Theorem 3.3, we consider the functions appeared in (3.8), that is

$$f_1(\phi) = \frac{\lambda_n - \phi + 2\tau}{\phi + \lambda_1} \text{ and } f_2(\phi) = \frac{\phi - \lambda_1 + 2\tau}{\phi + \lambda_1},$$

respectively. It is easy to know that if $\tau < \lambda_1$, $f_1(\phi)$ is decreasing when

$$\phi \in (\frac{\lambda_n - \lambda_1}{2} + \tau, \frac{\lambda_1 + \lambda_n}{2}],$$

and $f_2(\phi)$ is increasing when

$$\phi \in [\frac{\lambda_1 + \lambda_n}{2}, +\infty).$$

So, $\phi = \frac{\lambda_1 + \lambda_n}{2}$ is the minimum value point of the above two functions. According to that $\sigma_2 < 1$ is a convergence condition of the SMMS iteration method (2.4), and the smaller the value of σ_2, the better the convergence in general, we know that $\phi = \frac{\lambda_1 + \lambda_n}{2}$ is a quasi-optimal parameter when $\Phi = \phi I$ ($\phi > 0$). Then the second part of this theorem is verified. □

Corollary 3.1. *Let $A = F - G$ be a matrix splitting with F satisfying $F = tI$ ($t > 0$). Denote $\tau = \sup_{|\Psi^{(k)}| \le \bar{\Psi}}\{\|N - \Psi^{(k)}\|_2\}$. Set $\Phi = \phi I (\phi > 0)$, if $\tau < t$, then when*

$$\phi \in (\tau, +\infty), \tag{3.9}$$

$\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$. In addition, $\phi = t$ is a quasi-optimal parameter.

Proof. Based on (ii) in Theorem 3.2, by the similar proof way of Theorem 3.3, we can have

$$\begin{aligned} \sigma_2 &= \sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{\|(\Phi + F)^{-1}\|_2 \,(\|F - \Phi\|_2 + 2\|G - \Psi^{(k)}\|_2)\} \\ &= \sup_{|\Psi^{(k)}| \le \bar{\Psi}} \{\|(\phi I + tI)^{-1}\|_2 \,(\|tI - \phi I\|_2 + 2\|G - \Psi^{(k)}\|_2)\} \\ &= \frac{|\phi - t| + 2\tau}{\phi + t} \\ &= \begin{cases} \frac{t - \phi + 2\tau}{\phi + t}, \text{if } \phi \le t, \\ \frac{\phi - t + 2\tau}{\phi + t}, \text{if } \phi > t. \end{cases} \end{aligned} \tag{3.10}$$

Then, solving

$$\text{(I)} \begin{cases} \frac{t - \phi + 2\tau}{\phi + t} < 1, \\ \phi \le t, \end{cases} \text{ and (II)} \begin{cases} \frac{2\tau + \phi - t}{\phi + t} < 1, \\ \phi > t, \end{cases} \tag{3.11}$$

in turn, for (I), if $\tau < t$, ϕ satisfies $\phi \in (\tau, t]$, and for (II), if $\tau < t$, ϕ satisfies $\phi \in (t, +\infty)$. So, combining these two cases, the convergence region of ϕ is $(\tau, +\infty)$ if $\tau < t$. Then, the first part of Corollary 3.1 is verified.

Similar to the proof of Theorem 3.3, for the function $f_1(\phi) = \dfrac{t-\phi+2\tau}{\phi+t}$ in (3.11) is decreasing when $\phi \in (\tau, t]$, and the function $f_2(\phi) = \dfrac{2\tau+\phi-t}{\phi+t}$ in (3.11) is increasing when $\phi \in [t, +\infty)$, we know that $\phi = t$ is a quasi-optimal parameter when $\Phi = \phi I$ with $\phi \in (\tau, +\infty)$. Therefore, the second part of Corollary 3.1 is proved. □

Next, we assume that the system matrix A is an H_+-matrix, and consider the convergence region of Φ in the SMMS iteration method when the matrix splitting $A = F - G$ satisfies certain conditions.

Theorem 3.4. *Assume that A is an H_+-matrix and $A = F - G$ is a splitting of A with*

$$\langle F\rangle - |G| - |\Psi^{(k)}| + \Psi^{(k)}$$

being an M-matrix for any $|\Psi^{(k)}| \le \bar{\Psi}$. If Φ satisfies

$$\Phi \ge D + \bar{\Psi}, \tag{3.12}$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$, where D represents the diagonal part of A.

Proof. From the first relationship of (3.3) in the proof of Theorem 3.1, that is

$$|z^{(k+1)} - z^{(*)}| \le |(\Phi + F)^{-1}|\,(|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\,|z^{(k)} - z^{(*)}|,$$

we will verify that

$$\rho\Big(|(\Phi + F)^{-1}|\,(|G - J^{(k)}| + |\Phi - A - \Psi^{(k)}|)\Big) < 1 \tag{3.13}$$

for any $|\Psi^{(k)}| \le \bar{\Psi}$ under the conditions given in this theorem.

Since $\langle F\rangle - |G| + \Psi^{(k)} - |\Psi^{(k)}|$ is an F-matrix for any $|\Psi^{(k)}| \le \bar{\Psi}$, according to $\langle F\rangle - |G| \ge \langle F\rangle - |G| + \Psi^{(k)} - |\Psi^{(k)}|$ and $\langle F\rangle \ge \langle F\rangle - |G|$, we know that both $\langle F\rangle - |G|$ and $\langle F\rangle$ are M-matrices. Thus $\Phi + \langle F\rangle$ is an M-matrix. Then, these two inequalities

$$\begin{gathered}|(\Phi + F)^{-1}| \le (\Phi + \langle F\rangle)^{-1} \text{ and}\\ |(\Phi + F)^{-1}|\,(|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|) \le (\Phi + \langle F\rangle)^{-1}\,(|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|)\end{gathered} \tag{3.14}$$

hold. For the nonnegative matrix $(\Phi + \langle F\rangle)^{-1}\,(|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|)$ in the second inequality of (3.14), we consider the matrix splitting

$$\begin{aligned}(\Phi + \langle F\rangle) - (|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|) =&(\Phi + \langle F\rangle) - (|G - \Psi^{(k)}| + |\Phi - D - \Psi^{(k)}| + |B|)\\ \ge&(\Phi + \langle F\rangle) - (|G| + 2|J^{(k)}| + \Phi - D - 2\Psi^{(k)} + |B|)\\ \ge&2(\langle F\rangle - |G| - |\Psi^{(k)}| + \Psi^{(k)}),\end{aligned} \tag{3.15}$$

here, we use the inequality relationship $D - |B| \ge \langle F\rangle - |G|$ appeared in [8]. Therefor, according to the condition that $\langle F\rangle - |G| - |\Psi^{(k)}| + \Psi^{(k)}$ is an M-matrix for any $|\Psi^{(k)}| \le \bar{\Psi}$, we know that $(\Phi + \langle F\rangle) - (|G -$

$\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|)$ is an M-matrix too. Thus, the matrix splitting $(\Phi+\langle F\rangle)-(|G-\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|)$ in (3.15) is an M-splitting, so

$$\rho((\Phi+\langle F\rangle)^{-1}(|G-\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|))<1$$

(see [9, 29]). Thus, based on the spectral theories of nonnegative matrices (see [30]) and (3.14), we obtain that

$$\rho\Big(|(\Phi+F)^{-1}|\,(|G-\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|)\Big)\le\rho\Big((\Phi+\langle F\rangle)^{-1}(|G-\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|)\Big)<1$$

for any $|\Psi^{(k)}|\le\bar{\Psi}$. So, the inequality

$$\delta_1=\rho\Big(\sup_{|\Psi^{(k)}|\le\bar{\Psi}}\{|(\Phi+F)^{-1}|\,(|G-\Psi^{(k)}|+|A-\Phi+\Psi^{(k)}|)\}\Big)<1$$

holds. Therefor, from (i) of Theorem 3.1, this theorem is established. □

Next, a special matrix splitting of A is considered for the SMMS iteration method (2.4), that is, the accelerated overrelaxation (AOR) splitting, which is defined as

$$A=F_{\nu\omega}-G_{\nu\omega},\ F_{\nu\omega}=\frac{1}{\nu}(D-\omega L),\ G_{\nu\omega}=\frac{1}{\nu}[(1-\nu)D+(\nu-\omega)L+\nu U],\ \nu>0,\ \omega\ge0,\tag{3.16}$$

and has been studied by many researchers in the complementarity literatures [3, 9, 21], where D is the the diagonal part of A, $-L$ and $-U$ are the strictly lower triangular and strictly upper triangular parts of A, respectively. For such matrix splitting, the iteration method (2.4) is accordingly called the simplified modulus-based accelerated overrelaxation (SMAOR) iteration method. When we let ν,ω be some special values in (3.16), the SMAOR iteration method turns to be some special cases, i.e., the simplified modulus-based successive overrelaxation (SMSOR) iteration method ($\nu=\omega$), the simplified modulus-based Gauss Seidel (SMGS) iteration method ($\nu=\omega=1$), and the simplified modulus-based Jacobian (SMJ) iteration method ($\nu=1,\omega=0$). We have the following conclusion for the SMAOR iteration method.

Theorem 3.5. *Assume that A is an H_+-matrix and $A=F-G$ is the AOR splitting with Φ satisfying $\Phi\ge D+\bar{\Psi}$. If any of the conditions below holds:*

$$\begin{aligned}&\text{(i) } D>\bar{\Psi},\ \rho((D-\bar{\Psi})^{-1}(|L|+|U|))<1,\ 0<\nu\le1,\ \omega\le\nu,\\&\text{(ii) } D>\bar{\Psi},\ \rho((D-\bar{\Psi})^{-1}(\frac{\omega}{\nu}|L|+|U|))<1,\ 0<\nu\le1,\ \omega\ge\nu,\\&\text{(iii) } \frac{1}{\nu}D>\bar{\Psi},\ \rho((\frac{1}{\nu}D-\bar{\Psi})^{-1}(|L|+|U|))<1,\ \nu>1,\ \omega\le\nu,\\&\text{(iv) } \frac{1}{\nu}D>\bar{\Psi},\ \rho((\frac{1}{\nu}D-\bar{\Psi})^{-1}(\frac{\omega}{\nu}|L|+|U|))<1,\ \nu>1,\ \omega\ge\nu,\end{aligned}\tag{3.17}$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMAOR iteration method converges for any $z^{(0)}\in R^n$.

Proof. Since $A=F-G$ is the AOR splitting and A is an H_+ matrix, we know that $\langle F\rangle$ is an M-matrix and $\langle F\rangle+\Phi$ is an M-matrix for any nonnegative diagonal matrix Φ. For $\Phi\ge D+\bar{\Psi}$, just as the proof of Theorem 3.4, for the expression (3.15), we have

$$
\begin{aligned}
&(\Phi + \langle F\rangle) - (|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|)\\
=&(\Phi + \langle F\rangle) - (|G - \Psi^{(k)}| + (\Phi - D - \Psi^{(k)}) + |B|)\\
\geq&\langle F\rangle - |G| - |\Psi^{(k)}| + \Psi^{(k)} + D - |B|\\
=&\frac{1 + \nu - |1 - \nu|}{\nu} D - |\Psi^{(k)}| + \Psi^{(k)} - \frac{\nu + \omega + |\nu - \omega|}{\nu}|L| - 2|U|\\
=&\frac{2\min\{1, \nu\}}{\nu} D - |\Psi^{(k)}| + \Psi^{(k)} - \frac{2\max\{\nu, \omega\}}{\nu}|L| - 2|U|\\
=&\begin{cases} 2D - |\Psi^{(k)}| + \Psi^{(k)} - \dfrac{2\max\{\nu, \omega\}}{\nu}|L| - 2|U| \text{ when } 0 < \nu \leq 1,\\ \dfrac{2}{\nu}D - |\Psi^{(k)}| + \Psi^{(k)} - \dfrac{2\max\{\nu, \omega\}}{\nu}|L| - 2|U| \text{ when } \nu > 1,\end{cases}\\
\geq&\begin{cases} 2D - 2\bar{\Psi} - 2|L| - 2|U| \text{ when } 0 < \nu \leq 1 \text{ and } \omega \leq \nu,\\ 2D - 2\bar{\Psi} - \dfrac{2\omega}{\nu}|L| - 2|U| \text{ when } 0 < \nu \leq 1 \text{ and } \omega \geq \nu,\\ \dfrac{2}{\nu}D - 2\bar{\Psi} - 2|L| - 2|U| \text{ when } \nu > 1 \text{ and } \omega \leq \nu,\\ \dfrac{2}{\nu}D - 2\bar{\Psi} - \dfrac{2\omega}{\nu}|L| - 2|U| \text{ when } \nu > 1 \text{ and } \omega \geq \nu.\end{cases}
\end{aligned}
\tag{3.18}
$$

Under the conditions (3.17), we know that each of four matrices in the last inequality of (3.18) is an M-matrix. Then, the splitting $(\Phi+\langle F\rangle)-(|G-\Psi^{(k)}|+|\Phi-A-\Psi^{(k)}|)$ appeared in (3.18) is an M-splitting. Then

$$\rho\Big(|(\Phi + F)^{-1}|\,(|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|)\Big) \leq \rho\Big((\Phi + \langle F\rangle)^{-1}(|G - \Psi^{(k)}| + |\Phi - A - \Psi^{(k)}|)\Big) < 1$$

for any $|\Psi^{(k)}| \leq \bar{\Psi}$. So the convergence condition

$$\delta_1 = \rho\Big(\sup_{|\Psi^{(k)}|\leq\bar{\Psi}} \{|(\Phi + F)^{-1}|\,(|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\}\Big) < 1$$

holds. Therefor, from Theorem 3.1, the theorem is established. □

According to the proofs of Theorems 3.4 and 3.5, we can find that if $\Psi^{(k)} \geq O$ for any nonnegative integer k, and then $-|\Psi^{(k)}| + \Psi^{(k)} = O$ holds. Thus, $-|\Psi^{(k)}| + \Psi^{(k)}$ appeared in (3.15) and (3.18) can be deleted. Thus, we can obtain the following two corollaries derived from these two Theorems, respectively, and the proofs are omitted.

Corollary 3.2. *Assume that A is an H_+-matrix and $A = F - G$ is an H-splitting with $O \leq \Psi^{(k)} \leq \bar{\Psi}$ for any $k = 1, 2, \cdots$. If Φ satisfies*

$$\Phi \geq D + \bar{\Psi}, \tag{3.19}$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ produced by the SMMS iteration method (2.4) converges for any $z^{(0)} \in R^n$.

Corollary 3.3. *Assume that A is an H_+-matrix and $A = F - G$ is the AOR splitting with $O \leq \Psi^{(k)} \leq \bar{\Psi}$ for any $k = 1, 2, \cdots$. If Φ satisfies $\Phi \geq D + \bar{\Psi}$ and any of the following conditions holds:*

$$\text{(i) } \rho(D^{-1}(|L| + |U|)) < 1,\ 0 < \nu \leq 1,\ \omega \leq \nu,\ \text{(ii) } \rho(D^{-1}(\frac{\omega}{\nu}|L| + |U|)) < 1,\ 0 < \nu \leq 1,\ \omega \geq \nu,$$

$$\text{(iii) } \rho(D^{-1}(|L| + |U|)) < \frac{1}{\nu},\ \nu > 1,\ \omega \leq \nu, \qquad \text{(iv) } \rho(D^{-1}(\omega|L| + \nu|U|)) < 1,\ \nu > 1,\ \omega \geq \nu,$$

then $\{z^{(k)}\}_{k=0}^{+\infty}$ *produced by the SMAOR iteration method converges for any* $z^{(0)} \in R^n$.

We remark here that for an H_+-matrix A, the inequality $\rho(D^{-1}(|L| + |U|)) < 1$ holds, and it follows that the SMAOR iteration method always converges for any $z^{(0)} \in R^n$ when $0 < \nu \leq 1$ and $\omega \leq \nu$ according to (i) of Corollary 3.3. In addition, although some proposed convergence conditions are similar to that of [8], the SMMS iteration method discussed here differs to the MMS iteration method involved in [3, 4, 8].

4. Numerical examples

We illustrate some numerical examples in this section. We denote the number of iteration steps and the elapsed time by IT and CPU, respectively. The norm of residual vector of the NCP is denoted by RES(z), which is defined as follows

$$\mathrm{RES}(z) = \|\min(z, Az + \varphi(z))\|_2.$$

$z^{(k)}$ represents the kth numerical solution, and we set the iteration to cease if IT reaches 1000 or $\mathrm{RES}(z^{(k)}) < 1.0e-5$. In the first two experiments, we use $A(\eta, \mu) = M + \eta N + \mu S$ to generate the matrix A in the NCP (1.1), where η and μ are two constants, M, N and S are three given matrices. $M = \mathrm{Tridiag}(-I, T, -I) \in R^{n\times n}$ is a block-tridiagonal matrix, where S is a diagonal matrix, $N = \mathrm{tridiag}(0, 0, 1) \in R^{n\times n}$ and $T = \mathrm{tridiag}(-1, 4, -1) \in R^{m\times m}$ are two tridiagonal matrix with $n = m^2$. We set $z^{(*)} = (0, 1, 0, 1, \ldots)^{\mathrm{T}} \in R^n$ and $z^{(0)} = (0, 0, 0, 0, \ldots)^{\mathrm{T}} \in R^n$ to be the solution of (1.1) and the initial vector, respectively.

Example 4.1. *We test the convergence conditions given in Theorem 3.1 and compare the simplified modulus-based Gauss-Seidel(SMGS) iteration method with the modulus-based Gauss-Seidel(MGS) iteration method [3]. We set* $\varphi(z)$ *in the NCP (1.1) to be*

$$\varphi(z) = (z_1 - 2\sin(z_1) + q_1, z_2 - 2\sin(z_2) + q_2, \ldots, z_n - 2\sin(z_n) + q_n)^{\mathrm{T}} \in R^n$$

with $q = -(Az^{(*)} + \varphi(z^{(*)}))$*, then*

$$\frac{d\varphi_i}{dz_i} = 1 - 2\cos(z_i) \in [-1, 3], \text{ for } i = 1, 2, \ldots, n.$$

We set $S = \mathrm{diag}((1, 2, 3, 1, 2, 3, \ldots)) \in R^{n\times n}$*, and consider cases* $A(0, 2)$ *and* $A(1, 1)$*, which are a symmetric positive definite matrix and a nonsymmetric P-matrix, respectively. Let* Φ *be the diagonal part of A, i.e.,* $\Phi = D$ *in both the SMGS iteration method (2.4) and the MGS iteration method [3]. Then for* $A(0, 2)$, $\delta_1 = 0.9080$ *when* $n = 2500$ *and* $\delta_1 = 0.9609$ *when* $n = 3600$*, and for* $A(1, 1)$, $\delta_1 = 0.7024$ *when* $n = 2500$ *and* $\delta_1 = 0.8218$ *when* $n = 3600$*. So the the SMGS is convergent for any* $z^{(0)} \in R^n$ *based on Theorem 3.1. Table 1 below shows the numerical comparison.*

Table 1. Comparison of the SMGS iteration method and the MGS iteration method.

		$A(0,2)$			$A(1,1)$		
	n	IT	CPU	RES	IT	CPU	RES
SMGS	2500	11	0.001986	2.6220e-06	11	0.002040	8.4601e-06
	3600	13	0.002779	4.1634e-06	14	0.003468	6.8549e-06
MGS	2500	11	0.025748	3.6082e-06	13	0.031206	4.7946e-06
	3600	13	0.060169	9.7871e-06	14	0.062419	4.7127e-06

From Table 1, for cases $A(0,2)$ and $A(1,1)$, although the ITs are similar for the two iteration methods, the elapsed time is significantly different, i.e., the former SMGS iteration method costs less time than the latter MGS iteration method. This example shows that the SMMS iteration method (2.4) is usually more efficient than the MMS iteration method.

Example 4.2. *We test the convergence conditions given in Theorem 3.3. We set $\varphi(z)$ in (1.1) the NCP (1.1) to be*

$$\varphi(z) = (z_1 - 2\cos(z_1) + q_1, z_2 - 2\cos(z_2) + q_2, \dots, z_n - 2\cos(z_n) + q_n)^{\mathrm{T}} \in R^n$$

with $q = -(Az^{(*)} + \varphi(z^{(*)}))$, then

$$\frac{d\varphi_i}{dz_i} = 1 + 2\sin(z_i) \in [-1, 3], \text{ for } i = 1, 2, \dots, n.$$

Four cases are considered for A in the NCP (1.1), that is $A(0,4)$, $A(0,6)$, $A(2,3)$ and $A(1,3)$ with

$$S_1 = \mathrm{diag}((3,5,5,3,5,5,\dots)),\ S_2 = \mathrm{diag}((2,3,4,2,3,4,\dots)),$$
$$S_3 = \mathrm{diag}((4,3,5,4,3,5,\dots)),\ S_4 = \mathrm{diag}((4,5,4,4,5,4,\dots)),$$

respectively. Set $F = \mathrm{triu}(A,-1) - \mathrm{triu}(A,2)$ in $A = F - G$ for the first two cases, and set $F = \mathrm{triu}(A) - \mathrm{triu}(A,1) + \mathrm{triu}(A,-1) - \mathrm{triu}(A) + (\mathrm{triu}(A,-1) - \mathrm{triu}(A))^{\mathrm{T}}$ for the other two cases. We set $n = 1600$, then all cases satisfy the condition $\tau < \lambda_1$ given Theorem 3.3, and the SMMS iteration method converges for any $z^{(0)} \in R^n$ if we set $\Phi = \phi I$ with $\phi \in (\frac{\lambda_n - \lambda_1}{2} + \tau, +\infty)$. In order to see the numerical results clearly, we set

$$\phi = \frac{\lambda_n - \lambda_1}{2} + \tau : \delta : \frac{\lambda_n + \lambda_1}{2} + 5\delta$$

where $\delta = \frac{\lambda_1 - \tau}{5}$. Then $\frac{\lambda_n + \lambda_1}{2}$ is the fourth point in the 11 points. We test the convergence condition $\sigma_2 < 1$ given in Theorem 3.2. Table 2, Figures 1 and 2 below show the numerical results.

From Table 2, Figures 1 and 2, when $\Phi = \frac{\lambda_n + \lambda_1}{2}$, that is, the quasi-optimal parameter given in Theorem 3.3, IT is not very large, and the best parameter is sometimes near the quasi-optimal parameter in this example. In addition, we also can see that though the convergence condition is obtained by inequality reduction, the size of the boundary value is not exactly consistent with IT.

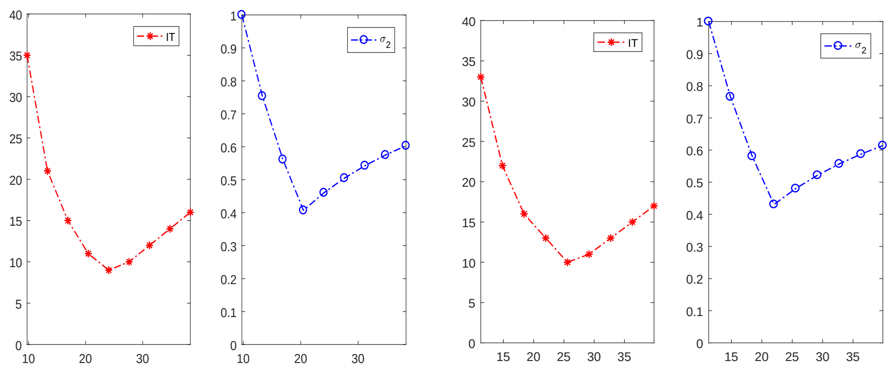

Figure 1. The numerical results for $A(0,3), A(0,6)$.

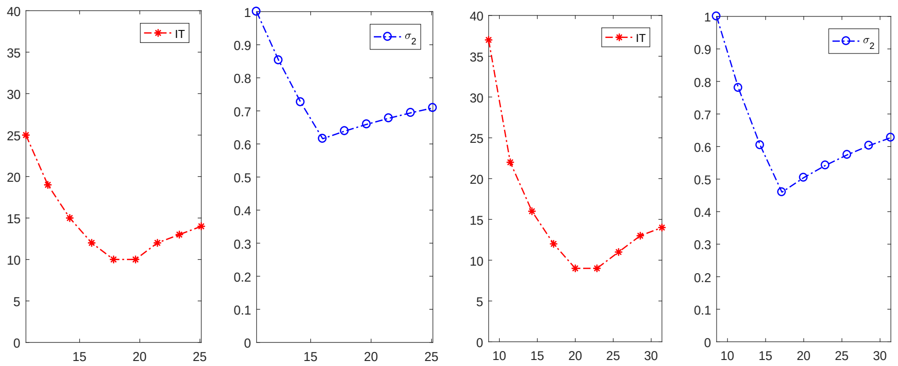

Figure 2. The numerical results for $A(2,3), A(1,3)$.

Table 2. The numerical results of the SMMS iteration method when $\Phi = \phi I$.

				$A(0,4),\ \tau = 4.9941,\ \lambda_1 = 15.7258$					
ϕ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
σ_2	1.0000	0.7536	0.5613	0.4070	0.4604	0.5049	0.5426	0.5750	0.6031
IT	35	21	15	11	9	10	12	14	16
				$A(0,6),\ \tau = 4.9941,\ \lambda_1 = 15.7302$					
ϕ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
σ_2	1.0000	0.7659	0.5808	0.4309	0.4802	0.5216	0.5570	0.5874	0.6140
IT	33	22	16	13	10	11	13	15	17
				$A(2,3),\ \tau = 6.9897,\ \lambda_1 = 12.4574$					
ϕ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
σ_2	1.0000	0.8531	0.7263	0.6157	0.6389	0.6594	0.6777	0.6941	0.7089
IT	25	19	15	12	10	10	12	13	14
				$A(1,3),\ \tau = 5.9916,\ \lambda_1 = 14.5548$					
ϕ	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
σ_2	1.0000	0.7803	0.6041	0.4597	0.5043	0.5421	0.5746	0.6028	0.6274
IT	37	22	16	12	9	9	11	13	14

Example 4.3. *We test the SMAOR iteration method for solving (1.1). We let A be the block tridialgonal matrix in [4], i.e.,* $A = \text{Tridiag}(T, K, T) \in R^{n\times n}$ *with* $K = \text{tridiag}(-4, 20, -4) \in R^{n_s\times n_s}$ *and* $T = \text{tridiag}(-1, 4, -1) \in R^{n_s\times n_s}$ *being two tridiagonal matrices with* $n = n_t \times n_s$, $n_t = 3 \cdot 2^t - 1$ *and* $n_s = 2 \cdot 2^t - 1$, *where t is a positive integer. We let $\varphi(z) \in R^n$ in (1.1) be*

$$\varphi(z) = (2z_1 + \sin(z_1) - \cos(z_1) + q_1, 2z_2 + \sin(z_2) - \cos(z_2) + q_2, \ldots, 2z_n + \sin(z_n) - \cos(z_n) + q_n)^{\mathrm{T}}.$$

Then,

$$\frac{d\varphi_i}{dz_i} = 2 + \cos(z_i) + \sin(z_i) \in [2 - \sqrt{2}, 2 + \sqrt{2}] \text{ for } i = 1, 2, \ldots, n.$$

We let $t = 4$ and Φ be $\Phi = D + \bar{\Psi}$ in the SMAOR iteration method. Then, A is an H_+-matrix and $0 \le \Psi^{(k)} \le \bar{\Psi}$ *for any* $k = 1, 2, \ldots$. *According to (i) in Corollary 3.3, we know that when* $0 < \nu \le 1$ and $\omega \le \nu$, *the SMAOR iteration method is convergent. In our experiment, we set the discrete values of ν and ω to be*

$$\nu = \frac{1}{7} : \frac{1}{7} : 1 + \frac{2}{7} \text{ and } \omega = \frac{1}{7} : \frac{1}{7} : 1 + \frac{2}{7}.$$

Table 3 and Figure 3 below show the numerical results.

Table 3. The numerical results of the SMAOR iteration method.

A = Tridiag(T, K, T), $t = 4$, $n = 1457$, $\rho(D^{-1}(\lvert L\rvert + \lvert U\rvert)) = 0.9958 < 1$									
$\rho_{\sup}$	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	ν_9
ω_1	0.9882	0.9825	0.9781	0.9746	0.9717	0.9693	0.9952	1.1210	1.2295
ω_2	1.0970	0.9741	0.9713	0.9689	0.9669	0.9651	0.9923	1.1223	1.2340
ω_3	1.1875	1.0781	0.9599	0.9596	0.9591	0.9585	0.9875	1.1221	1.2371
ω_4	1.2714	1.1712	1.0562	0.9463	0.9482	0.9493	0.9806	1.1202	1.2388
ω_5	1.3577	1.2574	1.1453	1.0352	0.9337	0.9373	0.9715	1.1165	1.2391
ω_6	1.4529	1.3388	1.2294	1.1190	1.0159	0.9220	0.9598	1.1110	1.2379
ω_7	1.5551	1.4204	1.3101	1.1990	1.0944	0.9984	0.9453	1.1034	1.2351
ω_8	1.6570	1.5059	1.3927	1.2777	1.1700	1.0720	1.0171	1.0935	1.2306
ω_9	1.7724	1.5953	1.4732	1.3576	1.2458	1.1431	1.0865	1.1628	1.2242
IT	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	ν_9
ω_1	72	38	27	21	18	26	39	63	99
ω_2	70	37	26	20	17	23	32	49	82
ω_3	68	36	25	20	16	20	28	40	61
ω_4	66	35	25	19	16	18	25	34	49
ω_5	65	34	24	19	15	16	22	30	41
ω_6	63	33	23	18	15	15	20	26	35
ω_7	62	33	23	18	15	14	18	24	31
ω_8	60	32	22	17	14	13	17	21	28
ω_9	59	31	22	17	14	12	15	20	25

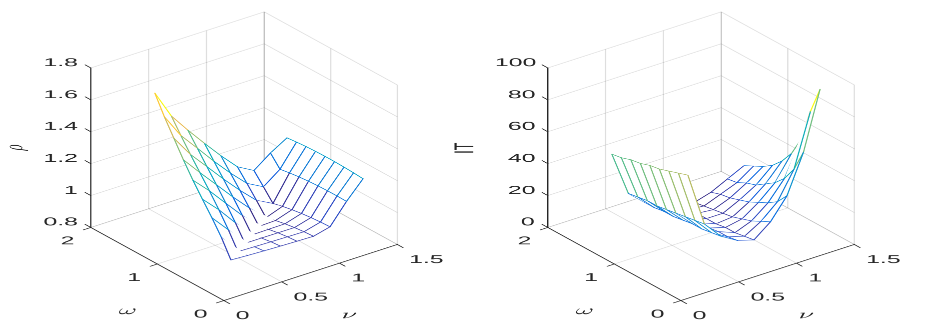

Figure 3. δ_1 and IT for the SMAOR iteration method.

From Table 3 and Figure 2, we can see that this example mainly verifies the first conclusion (i) given in Corollary 3.3, i.e., when $0 < \nu \leq 1$ and $\omega \leq \nu$. For other conclusions given in Corollary 3.3, such as, $\omega > \nu$, $\nu > 1$, $\omega > 1$, the numerical results are not obvious, and only when $\omega_7 > \nu_6$, $\delta_1 = \rho\Big(\sup_{|\Psi^{(k)}| \leq \bar{\Psi}} \{|(\Phi + F)^{-1}| \, (|G - \Psi^{(k)}| + |A - \Phi + \Psi^{(k)}|)\}\Big) = 0.9984 < 1$, which can ensure the convergence of the SMAOR iteration method (see Theorem 3.1). However, if we decrease the order of A, for instance, let $t = 2$, then the cases related to (ii)–(iv) given in Corollary 3.3 will be more, and the corresponding numerical results are omitted here. In addition, we can also see that the convergence conditions given in this paper are only sufficient, not necessary, and when $\nu = \omega$, i.e., the SMSOR iteration method is relatively better. Specially, the case $\nu = \omega = 1$, i.e., the SMGS iteration method is good although IT is not the smallest in this example.

5. Concluding remarks

The SMMS iteration method was extended to solve a kind of nonlinear complementarity problem in this paper. Both the general convergence conditions and the concrete convergence conditions were proposed. By comparing the SMGS iteration method with the MGS iteration method, the high efficiency of the SMMS iteration method was shown. The quasi-optimal parameter conclusion for the SMMS iteration method was also illustrated by the numerical experiments.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author has no conflict of interest.

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