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Research article

Periodic solutions in reversible systems in second order systems with distributed delays*

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Abstract: In this paper, we study the existence and multiplicity of periodic solutions to a class of second-order nonlinear differential equations with distributed delay. Under assumptions that the nonlinearity is odd, differentiable at zero and satisfies the Nagumo condition, by applying the equivariant degree method, we prove that the delay equation admits multiple periodic solutions. The results are then illustrated by an example.

Keywords: delay differential system; distributed delay; equivariant degree; periodic solutions
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1. Introduction

The investigation of periodic solutions in DDEs(Delay Differential Equations) with discrete delay can be traced back to 1962 [1], when Jones explored the existence of periodic solutions in a delayed Wright equation by using fixed-point theorems. Over the years, a multitude of methodologies have been adopted to analyze periodic solutions in DDEs with discrete delay, including fixed-point theory, Kaplan-Yorke's method [2], the center manifold theorem [3, 4], variational theory [5–10], coincidence degree theory [11], Hopf bifurcation methods [12–14], analysis techniques [15, 16], and equivariant degree theory [17–20].

More recently, DDEs with distributed delay have found extensive applications in population models, industrial models, and economic models [21, 22]. Various methods have been developed to explore periodic solutions in DDEs with distributed delay, including fixed-point theory [23], the ejective fixed-point principle [24], Jacobi elliptic functions [25, 26], and the variational method [27–29].

In this paper, we address the problem of the existence of periodic solutions in DDEs with distributed delays by introducing a novel approach rooted in the Brouwer equivariant degree theory.

We commence our study by considering a system of autonomous DDEs with distributed delays and no friction term. The absence of a friction term enables the application of the Nagumo condition to obtain an a priori bound. Additionally, the commensurability of delays, combined with the autonomous nature of the systems, leads to the emergence of the symmetry group $O(2)$.

To be more precise, let a be such that $\pi > a > 0$, and define $b := 2\pi - a$. Assume $f : \mathbf{V}^2 \to \mathbf{V}$ to be a continuous function, where $\mathbf{V} := \mathbb{R}^N$. We study the following periodic problem for DDEs:

$$\begin{cases} \ddot{x}(t) = f\left(x(t), \int_a^b x(t-s)ds\right), \\ x(t) = x(t+2\pi),\ \dot{x}(t) = \dot{x}(t+2\pi). \end{cases} \tag{1.1}$$

Here, $t \in \mathbb{R}$, $x(t) \in \mathbf{V}$, and a and b are given numbers.

We introduce the notation $\boldsymbol{x} := (x, y) \in \mathbf{V}^2$ and

$$\boldsymbol{x}_t := (x(t), \int_a^b x(t-s)ds) \in \mathbf{V}^2, \quad t \in \mathbb{R}.$$

Then, problem (1.1) can be rewritten as follows:

$$\begin{cases} \ddot{x}(t) = f(\boldsymbol{x}_t), \\ x(t) = x(t+2\pi),\ \dot{x}(t) = \dot{x}(t+2\pi). \end{cases} \tag{1.2}$$

We make the following assumptions:

($\mathbf{A_1}$) f is odd, i.e., $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbf{V}^2$.

($\mathbf{A_2}$) There exists $R \geq 0$ such that for all $\boldsymbol{x} = (x, y) \in \mathbf{V}^2$, the following holds:

$$|x| > R \quad \text{and} \quad |y| \leq (b-a)|x| \quad \Rightarrow x \cdot f(x, y) > 0.$$

($\mathbf{A_3}$) f is differentiable at $0 \in \mathbf{V}^2$, and $A := Df(0) = [A_0, A_1]$.

Notice that system (1.2) is autonomous, so it automatically exhibits $SO(2)$-symmetries (where the group $SO(2)$ acts by shifting the argument of the functions denoted by $x(t)$). In addition, by the assumption that $2\pi - a = b$, with the time inversion acting as reflection, (1.2) becomes $O(2)$-symmetric. By assumption ($\mathbf{A_1}$), one can leverage the oddness property of the function f (i.e., $\mathbb{Z}_2$-equivariance) to distinguish constant solutions from non-constant periodic solutions. A modified Nagumo condition ($\mathbf{A_2}$) lead to a priori bounds for (1.2). Notice that problem (1.2) lacks a variational structure. In fact f is not even required to be differentiable except at zero (see condition ($\mathbf{A_3}$)).

Our primary objective is to study the existence of periodic solutions to (1.1) under the aforementioned conditions. To address this problem, we employ the equivariant Brouwer/Leray-Schauder degree method. Notice that (1.2), in conjunction with the reversibility and oddness properties, leads to an $O(2) \times \mathbb{Z}_2$-equivariant operator equation in the appropriate functional space. Our goal is to demonstrate how the $O(2) \times \mathbb{Z}_2$-equivariant degree can be applied to establish the existence and multiplicity of periodic solutions for (1.2).

The remainder of this paper is organized as follows: In Section 2, we establish a priori bounds for periodic solutions of (1.1). In Section 3, we reformulate the problem (1.1) as a non-linear $O(2) \times \mathbb{Z}_2$-equivariant equation in appropriate functional spaces (see Section 3). In Sections 4 and 5, we recall some basic properties of the equivariant degree and use them to compute $O(2) \times \mathbb{Z}_2$-deg$(\mathscr{A}, B(\mathscr{E}))$ for the linear isomorphism $\mathscr{A}$. In Section 6, we state our main results and give an example to illustrate how the abstract results can be applied to the concrete system (1.1).

2. A priori bound

In order to establish the a priori bounds that are used later to construct admissible homotopies, consider the following modification of problem (1.2):

$$\begin{cases} \ddot{x}(t) = \lambda(f(\boldsymbol{x}_t) - x(t)) + x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbf{V}, \quad \lambda \in [0,1], \\ x(t) = x(t+2\pi), \;\; \dot{x}(t) = \dot{x}(t+2\pi). \end{cases} \tag{2.1}$$

One has the following lemma:

Lemma 2.1. *Assume that* $f : \mathbf{V^2} \to \mathbf{V}$ *satisfies* $(\mathbf{A_2})$. *If a* C^2*-differentiable* 2π*-periodic function* $x : \mathbb{R} \to \mathbf{V}$ *such that* $\max_{t\in\mathbb{R}} |x(t)| \geq R$, *then* $x(t)$ *is not a solution to* (2.1) *for* $\lambda \in [0,1]$.

Proof. Let us argue by contradiction: Assume that $x(t)$ is a solution to (2.1) with $|x(t_o)| := \max_{t\in\mathbb{R}} |x(t)| \geq R$, and consider the function $\phi(t) := \frac{1}{2}|x(t)|^2$. Then, $\phi(t_0) = \max_{t\in\mathbb{R}} \phi(t), \phi'(t_0) = x(t_0) \bullet \dot{x}(t_0) = 0$ and $\phi''(t_0) = \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) \leq 0$. However, by condition $(\mathbf{A_2})$, one has the following for $1 \geq \lambda > 0$:

$$\begin{aligned} \phi''(t_0) &= \dot{x}(t_0) \bullet \dot{x}(t_0) + \ddot{x}(t_0) \bullet x(t_0) \\ &\geq x(t_0) \bullet f\left(x(t_0), \int_a^b x(t-s)ds\right) > 0, \end{aligned}$$

which contradicts the assumption that $\phi(t_0)$ is the maximum of $\phi(t)$, i.e., $\phi''(t_0) \leq 0$. The lemma follows immediately.

The required priori bound is provided by the following lemma.

Lemma 2.2. *Assume that* $f : \mathbf{V^2} \to \mathbf{V}$ *is continuous and satisfies* $(\mathbf{A_2})$. *Then, there exists a constant C>0 such that for every solution x(t) to (2.1) (for some* $\lambda \in [0,1]$), *one has:*

$$\forall_{t\in\mathbb{R}} \quad |x(t)| < C, \;\; |\dot{x}(t)| < C, \;\; |\ddot{x}(t)| < C.$$

Proof. By Lemma 2.1, there exists R>0 such that any 2π-periodic solution $x(t)$ to (2.1) satisfies the $|x(t)| < R$.

Put

$$K_R := \left\{(\lambda, \boldsymbol{x}); \in [0,1] \times \mathbf{V^2} : \boldsymbol{x} = (x,y), \; |x| \leq R, |y| \leq (b-a)R\right\}.$$

Clearly, the set K_R is compact. Since the map

$$\tilde{f}(\lambda, \boldsymbol{x}) := \lambda(f(\boldsymbol{x}) - x(t)) + x(t), \quad \boldsymbol{x} := (x,y), \quad \lambda \in [0,1],$$

is continuous, it follows that $\tilde{f}(K_R)$ is bounded, i.e., there exists $M_1 \geq 0$ such that $\left|\tilde{f}(\lambda, \boldsymbol{x})\right| \leq M_1$ for all $(\lambda, \boldsymbol{x}) \in K_R$. Therefore, every solution $x(t)$ to (2.1) satisfies $|\ddot{x}(t)| \leq M_1$.

Take $v \in \mathbf{V}$ with $|v| = 1$ and consider the scalar function $\psi_v(t) := \dot{x}(t) \bullet v$. Since $x(t)$ is 2π-periodic, there exists t_0 such that $\psi_v(t_0) = 0$ for some $t_0 \in \mathbb{R}$, and for $t_0 + 2\pi \geq t \geq t_0$ one has:

$$\begin{aligned}
|\dot{x}(t) \bullet v| = |\psi_v(t)| &= \left|\psi_v(t_0) + \int_{t_0}^{t} \psi_v'(s)ds\right| = \left|\int_{t_0}^{t} \psi'(s)ds\right| \\
&= \left|\int_{t_0}^{t} \ddot{x}(s) \bullet v ds\right| \leq \int_{t_0}^{t} |\ddot{x}(s) \bullet v|\, ds \leq \int_{t_0}^{t} |\ddot{x}(s)|\, |v|\, ds \\
&= \int_{t_0}^{t} |\ddot{x}(s)|\, ds \leq \int_{0}^{2\pi} |\ddot{x}(s)|\, ds \leq 2\pi M_1 =: M_2.
\end{aligned}$$

Therefore,

$$|\dot{x}(t)| = \sup_{|v| \leq 1} |\dot{x}(t) \bullet v| \leq M_2.$$

Summing up, $C := \max\{R, M_1, M_2\} + 1$ is as required.

3. Operator reformulation in functional spaces

Let us introduce the spaces of interest. First, consider the space $C_{2\pi}(\mathbb{R}; \mathbf{V})$ of continuous 2π-periodic functions equipped with the norm

$$\|x\|_\infty = \sup_{t \in \mathbb{R}} |x(t)|, \quad x \in C_{2\pi}(\mathbb{R}; \mathbf{V}). \tag{3.1}$$

Then, the space $\mathscr{E}$ from $\mathbb{R}$ to $\mathbf{V}$ is denoted by $C^2_{2\pi}(\mathbb{R}, \mathbf{V})$ for twice continuously differentiable 2π-periodic functions equipped with the norm

$$\|x\| := \max\left\{\|x\|_\infty, \|\dot{x}\|_\infty, \|\ddot{x}\|_\infty\right\}. \tag{3.2}$$

Put $G := O(2) \times \mathbb{Z}_2$ with the G-action on $\mathscr{E}$ defined by

$$(e^{i\theta}, \pm 1)x(t) := \pm x(t + \theta), \tag{3.3}$$

$$(e^{i\theta}\kappa, \pm 1)x(t) := \pm x(-t + \theta), \tag{3.4}$$

where $x \in \mathscr{E}$, $\kappa \in O(2)$. Clearly, $\mathscr{E}$ is an isometric Banach G-representation. According to formulas (3.3) and (3.4), the isometric G-representations are defined on the space of periodic functions $C_{2\pi}(\mathbb{R}, \mathbf{V})$ and $C_{2\pi}(\mathbb{R}, \mathbf{V^2})$. The G-isotypic decomposition of $\mathscr{E}$ is simple to define. Consider the subspaces of $\mathscr{E}$ corresponding to its Fourier models as G-subrepresentations

$$\mathscr{E} = \overline{\bigoplus_{k=0}^{\infty} \mathscr{E}_k}, \qquad \mathscr{E}_k := \{\cos(kt)u + \sin(kt)v : u, v \in \mathbf{V}\}. \tag{3.5}$$

Clearly, each $\mathscr{E}_k$, for k=1,2,$\cdots$, identifies with the complexification $\mathbf{V}^c := \mathbf{V} \oplus i\mathbf{V}$ (as a real $O(2)\times\mathbb{Z}_2$-representation) of $\mathbf{V}$, where the rotation $e^{i\theta} \in SO(2)$ acts on vectors $\mathbf{z} \in \mathbf{V}^c$ by $e^{i\theta}(\mathbf{z}) := e^{-ik\theta} \cdot \mathbf{z}$

and $\kappa\mathbf{z} := \bar{\mathbf{z}}$ a space (here '·' stands for complex multiplication). In fact, the linear isomorphism $\varphi_k : \mathbf{V}^c \to \mathscr{E}_k$ given by

$$\varphi_k(u + iv) := \cos(kt)u + \sin(kt)v, \quad u,\ v \in \mathbf{V}, \tag{3.6}$$

is $O(2) \times \mathbb{Z}_2$-equivariant. Also, with the trivial action of $O(2)$ and the antipodal action of $\mathbb{Z}_2$, $\mathscr{E}_0$ and $\mathbf{V}$ can be identified, while $\mathscr{E}_k$, $k = 1, 2, ...$, is modeled on the irreducible $O(2)$-representation $\mathscr{W}_k \simeq \mathbb{R}^{2N}$ with the antipodal $\mathbb{Z}_2$-action.

Let us introduce the subsequent operators:

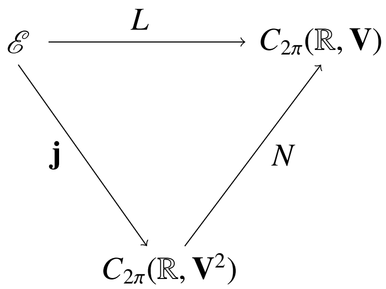
$$L : \mathscr{E} \to C_{2\pi}(\mathbb{R}, \mathbf{V}), \ \ Lx := \ddot{x} - x,$$

$$\mathbf{j} : \mathscr{E} \to C_{2\pi}(\mathbb{R}, \mathbf{V}^2), \ \ \mathbf{j}(x)(t) := (x(t), \int_a^b x(t - s)ds),$$

and $N : C_{2\pi}(\mathbb{R}, \mathbf{V}^2) \to C_{2\pi}(\mathbb{R}, \mathbf{V})$, which is defined by

$$N(x, y) := f(x(t), y(t)) - x(t), \quad (x, y) \in C_{2\pi}(\mathbb{R}, \mathbf{V}^2),$$

and which the following (non-commutative) figure serves to illustrate:



Problem (2.1) is equivalent to the following:

$$Lx = \lambda(N(\mathbf{j}x)), \ \ x \in \mathscr{E}, \ \ \lambda \in [0, 1], \tag{3.7}$$

which, for $\lambda = 1$, is equivalent to (1.2). Equation (3.7) can be recast as follows since L is an isomorphism:

$$\mathscr{F}_\lambda(x) := x - \lambda L^{-1}N(\mathbf{j}(x)) = 0, \ \ x \in \mathscr{E}, \ \ \lambda \in [0, 1]. \tag{3.8}$$

Proposition 3.1. *Assume that f satisfies assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *and* (3.8) *provides the nonlinear operator $\mathscr{F}_\lambda : \mathscr{E} \to \mathscr{E}$. Then, $\mathscr{F}_\lambda$ is a G-equivariant completely continuous field for every* $\lambda \in [0, 1]$.

Proof. Adding the definition of L yields (3.5) and (3.6):

$$L|_{\mathscr{E}_k} = -(k^2 + 1)\text{Id} \ : \mathbf{V}^c \to \mathbf{V}^c \quad \textit{and} \quad L|_{\mathscr{E}_0} = -\text{Id} \quad (k > 0). \tag{3.9}$$

Specifically, L^{-1} and, by extension, L is G-equivariant. Since $\mathbf{j}$ is an embedding, it also satisfies the conditions of G-equivariance. It follows that $\mathscr{F}_\lambda$ is a fully continuous field since L and N are continuous and $\mathbf{j}$ is a compact operator. Furthermore, by conditions ($\mathbf{A_1}$), the operator N is $\mathbb{Z}_2$-equivariant. Since problem (1.1) is autonomous, it follows that $N \circ \mathbf{j}$ is $SO(2)$-equivariant. The proof just has to demonstrate that $N \circ \mathbf{j}$ commutes with the κ-action. For every t, one obtains:

$$\begin{aligned} N(\mathbf{j}(\kappa x))(t) &= f\left(x(-t), \int_a^b x(-t+s)ds\right) - x(-t) \\ &= f\left(\kappa x(t), -\int_{2\pi-a}^{2\pi-b} \kappa x(-t+(2\pi-s))ds\right) - x(-t) \\ &= f\left(\kappa x(t), \int_a^b \kappa x(-t-s))ds\right) - x(-t) \\ &= \kappa N(\mathbf{j}(x))(t). \end{aligned}$$

On the other hand, $x \equiv 0$ is a solution to (3.8) for any $\lambda \in [0,1]$. Assuming that condition ($\mathbf{A_3}$) is satisfied, put

$$\mathscr{A} := D\mathscr{F}_1(0) : \mathscr{E} \to \mathscr{E}. \tag{3.10}$$

Then,

$$\mathscr{A} = \mathrm{Id} - L^{-1}(DN(0)) \circ \mathbf{j} : \mathscr{E} \to \mathscr{E}. \tag{3.11}$$

One can easily check that $\mathscr{A} \in L_c(\mathscr{E})$ and, as such, is a Fredholm operator of index zero[1*]. Therefore, $\mathscr{A}$ is an isomorphism if and only if $0 \notin \sigma(\mathscr{A})$ (here $\sigma(\mathscr{A})$ stands for the spectrum of $\mathscr{A}$). Also, since $G_{\{0\}} = G$, it follows that $\mathscr{A} \in L_c^G(\mathscr{E})$.

Lemma 3.1. *Furthermore, assume that* $0 \notin \sigma(\mathscr{A})$ *under the assumptions* ($\mathbf{A_1}$) *and* ($\mathbf{A_3}$). *The map* $\mathscr{F} := \mathscr{F}_1$ *(cf.* (3.8)*) is then* Ω_ε*-admissibly G-equivariantly homotopic to* $\mathscr{A}$*, as given by* (3.10) *and* (3.11) *for a sufficiently small* $\varepsilon > 0$ *(here* $\Omega_\varepsilon := \{x \in \mathscr{E} : \|x\| < \varepsilon\}$*).*

Proof. We set $\mathscr{H}_\lambda(x) := (1-\lambda)\mathscr{A}(x) + \lambda\mathscr{F}(x)$, $x \in \mathscr{E}$, $\lambda \in [0,1]$, and show that there exists a sufficiently small $\varepsilon > 0$ such that $\mathscr{H}_\lambda(\cdot)$ is an Ω_ε-admissible homotopy. Indeed, assume for contraction, that there exist sequences $x_n \subset \mathscr{E}$ and $\lambda_n \subset [0,1]$ such that $x_n \to 0, \lambda_n \to \lambda_0$ and

$$\mathscr{H}_{\lambda_n}(x_n) = \mathscr{A}(x_n) - \lambda_n(\mathscr{A}(x_n) - \mathscr{F}(x_n)) = 0 \quad for\ \ all\ \ n \in \mathbb{N}.$$

Then, by linearity and differentiability, one has:

$$\frac{\mathscr{A}(x_n)}{\|x_n\|} = \mathscr{A}(\frac{x_n}{\|x_n\|}) = \frac{\lambda_n(\mathscr{A}(x_n) - \mathscr{F}(x_n))}{\|x_n\|} \to 0 \quad as\ \ n \to \infty. \tag{3.12}$$

Set $v_n := \frac{x_n}{\|x_n\|}$. Combining (3.12) with (3.11) yields:

$$\mathscr{A}(v_n) = v_n - L^{-1}(DN(0)(\mathbf{j}(v_n))) \to 0 \quad as\ \ n \to \infty. \tag{3.13}$$

Since $\mathbf{j}$ is a compact operator, there exist y_0 and a subsequence $\{v_{n_k}\}$ such that $L^{-1}(DN(0)(\mathbf{j}(v_{n_k}))) \to y_0$. Hence, by the continuity of $\mathscr{A}$ combined with (3.13), one has that $v_{n_k} \to y_0$ and $\|y_0\| = 1$. Therefore, $\mathscr{A}(y_0) = 0$ which is impossible since $\mathscr{A}$ is an isomorphism.

[*1] Here $L_c(\mathscr{E})$ stands for the space of linear operators $T : \mathscr{E} \to \mathscr{E}$ of the type $T : \mathrm{Id} - K$, where K is a compact operator.

4. Equivariant invariant and abstract existence result

To formulate a result by using the equivariant degree in relation to problem (1.2), we need additional concepts.

Definition 4.1. If there is $k > 0$ and $u \neq 0, u \in \mathscr{E}_k$, such that $H = G_u$ and (H) is a maximal orbit type in $\Phi(G, \mathscr{E}_k \setminus \{0\})$, i.e., $H = D^d_{2k} \leq O(2) \times \mathbb{Z}_2$, then an orbit type (H) in the space $\mathscr{E}$ is said to be of *maximal kind*, where

$$D^d_{2k} := \{(1,1), (\gamma, -1), \dots, (\gamma^{2k-1}, -1), (\kappa, 1)(\gamma\kappa, -1), \dots, (\gamma^{2k-1}\kappa, -1)\},$$

where $\gamma := e^{i\frac{\pi}{k}} \in SO(2)$.

Remark 4.1. The above concepts have a very transparent meaning. Without extra assumptions, typical equivariant degree based results provide minimal spatio-temporal symmetries of the corresponding periodic solutions for Definition 4.1.

Under the conditions $(\mathbf{A_1})$ and $(\mathbf{A_3})$, the G-equivariant degree $G\text{-deg}(\mathscr{A}, B(\mathscr{E})) \in A(G)$ is correctly defined provided that $0 \notin \sigma(\mathscr{A})$ (here $B(\mathscr{E})$ denotes the unit ball in $\mathscr{E}$). Set

$$\omega := (G) - G\text{-deg}(\mathscr{A}, B(\mathscr{E})). \tag{4.1}$$

Now we can formulate a result that is abstract.

Proposition 4.1. *Assume that $f : \mathbf{V^2} \to \mathbf{V}$ satisfies conditions $(\mathbf{A_1}) - (\mathbf{A_3})$. Furthermore, assume that $0 \notin \sigma(\mathscr{A})$ (cf. (3.8), (3.10), (3.11)). Finally, assume that*

$$\omega = n_1(H_1) + n_2(H_2) + \cdots + n_N(H_N), \quad n_j \neq 0, \quad (H_j) \in \Phi_0(G), \tag{4.2}$$

(cf. (4.1)). Then,

- *(a) there exists a G-orbit of 2π-periodic solutions $x \in \mathscr{E} \setminus \{0\}$ to (1.2) such that $(G_x) \geq (H_j)$, for every $j = 1, 2, ..., N$,*
- *(b) if (H_j) is of maximal kind, i.e., $H_j = D^d_{2k}$, then the solution x is non-constant and $G_x = D^d_{2m}$ when $m \in \mathbb{N}$ is a multiple of k.*

proof. (a) Consider Ω_ε as given in Lemma 3.1 and set $\mathscr{F} := \mathscr{F}_1$ (see (3.8)). Then, $\mathscr{F}$ is Ω_ε-admissible and, by equivariant homotopy invariance of the equivariant degree,

$$G\text{-deg}(\mathscr{F}, \Omega_\varepsilon) = G\text{-deg}(\mathscr{A}, B(\mathscr{E})). \tag{4.3}$$

Similarly, consider C as given in Lemma 2.2 and set $\Omega_c := \{x \in \mathscr{E} : \|x\| < C\}$. Then, $\mathscr{F}$ is Ω_c-admissible and equivariantly homotopic to $\mathscr{F}_0 = \text{Id}$. Hence,

$$G\text{-deg}(\mathscr{F}, \Omega_c) = (G). \tag{4.4}$$

Put

$$\Omega := \Omega_c \setminus \overline{\Omega_\varepsilon}. \tag{4.5}$$

Then, using the degree's additivity property,

$$G\text{-deg}\,(\mathscr{F},\Omega) = G\text{-deg}\,(\mathscr{F},\Omega_c) - G\text{-deg}\,(\mathscr{F},\Omega_\varepsilon) = (G) - G\text{-deg}\,(\mathscr{A}, B(\mathscr{E})).$$

Thus, combining (4.1)–(4.4) yields that $\omega = G\text{-deg}\,(\mathscr{F},\Omega)$, which, by the statement follows.

(b) Note that $G_x \geq O(2)$ if $x \in \mathscr{E}$ is a constant function. However, the following property holds for any (H) of maximum kind, $H = D^d_{2k}$; if $(K) \in \Phi_0(G, \mathscr{E} \setminus \{0\})$ and $(K) \geq (H)$, then there exists $s \in \mathbb{N}$ such that $(K) = (D^d_{2sk})$. Specifically, (K) is also of maximal kind. As a result, (K) is an orbit type for a 2π-periodic function that is not constant.

5. Computation of $G\text{-deg}\,(\mathscr{A}, B(\mathscr{E}))$: Reduction to basic degrees

By Proposition 4.1, problem (1.2) is reduced to a computation of $G\text{-deg}\,(\mathscr{A}, B(\mathscr{E}))$. One needs a workable formula for $G\text{-deg}\,(\mathscr{A}, B(\mathscr{E}))$ to analyze the non-triviality of some of the coefficients of ω.

First, we get the equivariant spectral information associated with $\mathscr{A}$. Since (by G-equivariance) $\mathscr{A}$ respects the isotypic decomposition, establish

$$\xi := \int_a^b e^{-iks} ds = \frac{e^{-ika} - e^{-ikb}}{ik} = -\frac{2\sin ka}{k},$$

and define (see (3.9)) $\mathscr{A}_k := \mathscr{A}|_{\mathscr{E}_k}$ to satisfy

$$\mathscr{A}_k = \text{Id} + \frac{1}{k^2+1}(A_0 + \xi A_1 - \text{Id}). \tag{5.1}$$

To simplify the computations, assume the following:

($\mathbf{A_4}$) The matrices A_0 and A_1 are diagonalizable and $A_0A_1 = A_1A_0$, with $\sigma(A_0) = \{\mu_j : j = 1,2,\dots,r\}$, $\sigma(A_1) = \{\nu_j : j = 1,2,\dots,r\}$, $E(\mu_j) = E(\nu_j)$.

This gives one the following spectrum description of $\mathscr{A}$:

$$\sigma(\mathscr{A}) = \bigcup_{k=0}^{\infty} \sigma(\mathscr{A}_k), \tag{5.2}$$

where

$$\sigma(\mathscr{A}_0) = \sigma(A_0),$$
$$\sigma(\mathscr{A}_k) = \left\{1 + \frac{1}{k^2+1}\left(\mu_j - \frac{2\nu_j \sin ak}{k} - 1\right) : k = 1,2,\dots,\ \mu_j \in \sigma(A_0),\ \nu_j \in \sigma(A_1)\right\}.$$

Put

$$m_j := \dim E(\mu_j) = \dim E(\nu_j),$$

i.e., m_j stands for the multiplicity of the eigenvalues μ_j and ν_j. Denote

$$\lambda_{k,j} := 1 + \frac{1}{k^2+1}\left(\mu_j - \frac{2\nu_j \sin ak}{k} - 1\right), \quad j = 1,2,\dots,r,$$

for $k \in \mathbb{N}$. Notice that the G-isotypic multiplicity of $\lambda_{k,j}$ is equal to m_j. Moreover, one has

$$\lambda_{k,j} < 0 \quad \Leftrightarrow \quad k^3 + k\mu_j - 2\nu_j \sin ak < 0, \tag{5.3}$$

which allows us to describe the negative spectrum $\sigma_-(\mathscr{A})$

$$\sigma_-(\mathscr{A}) = \sigma_-(A_0) \cup \bigcup_{k=1}^{\infty} \{\lambda_{k,j} : k^3 + k\mu_j - 2\nu_j \sin ak < 0, \quad \text{and} \quad j = 1, 2, \dots, r\}.$$

Put

$$N_k := \left\{ j \in \{1, 2, \dots, r\} : k^3 + k\mu_j - 2\nu_j \sin ak < 0 \right\}, \quad k \in \mathbb{N}$$

and

$$\mathfrak{m}_k := \sum_{j \in N_k} m_j. \tag{5.4}$$

The degree $G\text{-deg}\,(\mathscr{A}, B(\mathscr{E}))$ can be calculated; by using the following formula:

$$G\text{-deg}\,(\mathscr{A}, B(\mathscr{E})) = G\text{-deg}\,(\mathscr{A}_0, B(\mathbf{V})) \cdot \prod_{k=1}^{\infty} (\deg_{\mathcal{V}_k})^{\mathfrak{m}_k}. \tag{5.5}$$

The basic degrees $\deg_{\mathcal{V}_k}$ are given by

$$\deg_{\mathcal{V}_k} = (O(2) \times \mathbb{Z}_2) - (D^d_{2k}).$$

6. Main results and examples

In this section, we present our primary findings and provide illustrative examples by using $G = O(2) \times \mathbb{Z}_2$. Since $\mathscr{A}$ is a Fredholm operator of index zero, its invertibility depends on whether zero belongs to $\sigma(\mathscr{A})$. Depending on that, we distinguish between non-degenerate and degenerate cases. The following statement is a non-degenerate version of the main result.

Theorem 6.1. *Assume that $f : \mathbf{V}^2 \to \mathbf{V}$ satisfies conditions $(\mathbf{A_1}) - (\mathbf{A_4})$ and, in addition, that $0 \notin \sigma(\mathscr{A})$, where $\sigma(\mathscr{A})$ is given by* (5.2). *If for some $k > 0$ the number $\mathfrak{m}_k$ (given by* (5.4)*) is odd, then problem* (1.2) *admits a non-constant 2π-periodic solution with symmetries denoted by D^d_{2m}, where m is a multiple of k.*

Proof. By Proposition 4.1 we need to prove that $\omega = G\text{-deg}\,(\mathscr{F}, \Omega)$ has a nonzero coefficient corresponding to the orbit type (D^d_{2k}). By (5.5),

$$G\text{-deg}\,(\mathscr{A}, B(\mathscr{E})) = G\text{-deg}\,(\mathscr{A}_0, B(\mathbf{V})) \cdot \prod_{l=1}^{\infty} (\deg_{\mathcal{V}_l})^{\mathfrak{m}_l},$$

and since for $l = k$, $\mathfrak{m}_k$ is odd, one has

$$(\deg_{\mathcal{V}_k})^{\mathfrak{m}_k} = \deg_{\mathcal{V}_k} = (G) - (D^d_{2k}).$$

So

$$G\text{-deg}(\mathscr{A}, B(\mathscr{E})) = G\text{-deg}(\mathscr{A}_0, B(\mathbf{V}))((G) - (D^d_{2k})) \cdot \prod_{l \neq k} (\deg_{\mathcal{V}_l})^{\mathfrak{m}_l}$$
$$= (G) \pm (D^d_{2k}) + \alpha,$$

where $\alpha \in A(G)$ is an element such that for $H := D^d_{2k}$ one has $\text{coeff}^H(\alpha) = 0$. Then,

$$\omega = (G) - G\text{-deg}(\mathscr{F}, B(\mathscr{E}))$$
$$= (G) - (G) \mp (D^d_{2k}) - \alpha,$$

i.e.,

$$\text{coeff}^H(\omega) = \mp 1 \neq 0,$$

and the conclusion follows from Proposition 4.1.

In Theorem 6.1, the obtained solution $x(t)$ to (1.2) admits the symmetries denoted by D^d_{2m} which can be simply written as the following condition

$$\forall_{t \in \mathbb{R}} \;\; x(t + \tau) = -x(t), \quad \text{for} \quad \tau := \frac{\pi}{m}. \tag{6.1}$$

Clearly, such a solution is non-constant.

The following "degenerate" version of the main conclusion can be established by using the same argument as that used in the demonstration of Theorem 6.1.

Theorem 6.2. *Assume that* $f : \mathbf{V}^2 \to \mathbf{V}$ *satisfies conditions* $(\mathbf{A_1}) - (\mathbf{A_4})$ *but* $0 \in \sigma(\mathscr{A})$*, i.e., the set*

$$\mathscr{C} := \left\{k \in \mathbb{N} \cup \{0\} : k^3 + k\mu_j - 2\nu_j \sin ak = 0\right\} \neq \emptyset.$$

Assume that $s \in \mathbb{N}$ *is such that*

$$\mathscr{C} \cap \{(2k-1)s : k \in \mathbb{N}\} = \emptyset, \tag{6.2}$$

and that there exists $k \in \mathbb{N}$ *such that* $\mathfrak{m}_{(2k-1)s}$ *is odd. Then, system* (1.2) *admits a non-constant* 2π*-periodic solution with the orbit type* (D^d_{2m})*, where m is a multiple of* $(2k-1)s$*.*

Proof. If $e^{i\pi/k} \in O(2)$ acts as $-\text{Id}$ on the representations denoted by $\mathscr{E}_m$, then the element $(e^{i\pi/k}, -1) \in O(2) \times \mathbb{Z}_2$ acts trivially on $\mathscr{E}_m$, i.e., the group formed by $(e^{i\pi/k}, -1)$ is contained in each isotropy group of the representation $\mathscr{E}_m$, i.e.,

$$\boldsymbol{K} = \mathbb{Z}^d_{2k} := \{e\} \times \{(1,1), (\gamma, -1), (\gamma^2, 1), \dots, (\gamma^{2k-1}, -1)\}, \qquad \gamma = e^{i\pi/k}. \tag{6.3}$$

$(\gamma, -1)$ acting on the function $\cos(lt)u + \sin(lt)v \in \mathscr{E}_l^{\mathbb{Z}^d_{2k}}$ is given by

$$(\gamma, -1)(\cos(lt)u + \sin(lt)v) = -\left(\cos\left(lt + \frac{l\pi}{k}\right)u + \sin\left(lt + \frac{l\pi}{k}\right)v\right).$$

Thus, if l is an odd multiple of k, then these functions are in the fixed point space $\mathscr{E}^{\boldsymbol{K}}$ and

$$\mathscr{E}^{\boldsymbol{K}} = \overline{\bigoplus_{l \in k(2\mathbb{N}-1)} \mathscr{E}_l}.$$

Notice that $\boldsymbol{K}$ is normal in G and $W(\boldsymbol{K}) = O(2)$. Suppose that any solution to the problem

$$\mathscr{F}^{\boldsymbol{K}}(x) = 0, \quad x \in \Omega^{\boldsymbol{K}}, \tag{6.4}$$

where $\mathscr{F}^{\boldsymbol{K}} := \mathscr{F}|_{\mathscr{E}^{\boldsymbol{K}}}$ is $O(2)$-equivariant, is also a solution to

$$\mathscr{F}(x) = 0, \quad x \in \Omega. \tag{6.5}$$

It is sufficient to show that $O(2)\text{-deg}(\mathscr{A}^{\boldsymbol{K}}, B(\mathscr{E}^{\boldsymbol{K}})$ is well-defined and

$$\text{coeff}^{D_{(2k-1)s}}(O(2)\text{-deg}(\mathscr{A}^{\boldsymbol{K}}, B(\mathscr{E}^{\boldsymbol{K}})) \neq 0.$$

By applying the same argument as before, we obtain that

$$\begin{aligned} O(2)\text{-deg}(\mathscr{A}^{\boldsymbol{K}}, B(\mathscr{E})) &= \prod_{l \in (2\mathbb{N}-1)s} (O(2) - (D_l))^{\mathfrak{m}_l} \\ &= (O(2)) \pm (D_{(2k-1)s}) + \alpha, \end{aligned}$$

where

$$\text{coeff}^{D_{(2k-1)s}}(\alpha) = 0.$$

Then, clearly, by the same argument as the one used in the proof of Theorem 6.1, one can show that there exists a solution x to (6.4) with symmetries denoted by D_m, where m is a multiple of $(2k-1)s$; thus, x is also a solution to (6.5) and has the orbit type (D^d_{2m}).

Example 6.1. We start by describing a class of maps that satisfy condition $(\mathbf{A_2})$. Take $\mathbf{V} := \mathbb{R}^N$ as equipped with the standard Euclidean norm, and consider a map $f : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ given by

$$f(x, y) = A_0 x + A_1 y + \|x\|^4 x - \|x\| \, \|y\| \, y, \qquad (x, y) \in \mathbf{V}, y = \int_a^b x(t-s)ds, \tag{6.6}$$

where $A_j : \mathbf{V} \to \mathbf{V}$, $j = 0, 1$, satisfies condition $(\mathbf{A_4})$ for $|y| \le (b-a)\,|x|$; then, one has

$$\begin{aligned} x \bullet f(x, y) &= x \bullet A_0 x + x \bullet A_1 y + x \bullet \|x\|^4 x - x \bullet \|x\| \, \|y\| \, y \\ &\ge \|x\|^6 - (b-a)^2 \|x\|^4 - \|A_0\| \, \|x\|^2 - (b-a) \|A_1\| \, \|x\|^2 > 0, \end{aligned}$$

which implies that the assumption $(\mathbf{A_2})$ is satisfied. Observe that the map $f(x, y)$ also satisfies conditions $(\mathbf{A_1})$ and $(\mathbf{A_3})$. Assume in addition that A_0 and A_1 are symmetric matrices such that $A_0 A_1 = A_1 A_0$. Then, there exists an orthonormal basis $\{v_1, v_2, \dots, v_N\}$ in $V = \mathbb{R}^N$ such that

$$A_0 v_k = \mu_k v_k, \;\; A_1 v_k = \nu_k v_k \qquad k = 1, 2, \dots, N,$$

and consequently the map f satisfies the assumption $(\mathbf{A_4})$.

To illustrate how Theorem 6.1 is applied to a concrete map $f : \mathbf{V^2} \to \mathbf{V}$, assume that $N = 5$, f is given by (6.6) and it satisfies conditions $(\mathbf{A_1}) - (\mathbf{A_4})$ with $A_0 v_j = \frac{1}{j} v_j$, $A_1 v_j = j v_j$, $j = 1, 2, \dots, 5$. Thus, we have the following system

$$\ddot{x}(t) = A_0 x(t) + A_1 \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} x(t-s)ds + \|x(t)\|^4 x(t)$$

$$- \|x(t)\| \left\| \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} x(t-s)ds \right\| \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} x(t-s)ds,$$

where $x(t) \in \mathbb{R}^5$.

Then, for $k \in \mathbb{N}$ one has the following:

$$\sin(\tfrac{k\pi}{3}) = \begin{cases} 0 & \text{if } \frac{k}{3} \in \mathbb{N}, \\ \frac{\sqrt{3}}{2}(-1)^{\lfloor \frac{3}{k} \rfloor} & \text{otherwise.} \end{cases}$$

Then, one can easily verify (by (5.3)) that

$$\sigma_-(\mathscr{A}) = \sigma_-(\mathscr{A}_1) \cup \sigma_-(\mathscr{A}_2),$$

where

$$\sigma_-(\mathscr{A}_1) = \{\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}, \lambda_{1,4}, \lambda_{1,5}\}, \quad \sigma_-(\mathscr{A}_2) = \{\lambda_{2,5}\}.$$

In our example for all $j = 1, 2, \dots, 5$, $m_j = 1$; thus, we obtain

$$\mathfrak{m}_1 = 5, \quad \mathfrak{m}_2 = 1,$$

which implies that

$$\begin{aligned} G\text{-deg}\,(\mathscr{A}) &= \deg_{\mathcal{V}_1}^5 \cdot \deg_{\mathcal{V}_2} = \deg_{\mathcal{V}_1} \cdot \deg_{\mathcal{V}_2} \\ &= (O(2) \times \mathbb{Z}_2) - (D_2^d) - (D_4^d) - 2(D_1^z) + (D_1), \end{aligned}$$

which implies that

$$\omega := (D_2^d) + (D_4^d) + 2(D_1^z) - (D_1),$$

where (D_2^d) and (D_4^d) are of maximal orbit kind.

7. Conclusions

The results presented in this paper show that the equivariant degree method is a viable and effective alternative to the main methods usually used to study periodic solutions of distributed delay differential equations, and, particularly, the fixed point methods and those of Kaplan-Yorke. Numerical simulations suggest that distributed delay differential equations may have multiple periodic solutions, and that there is clearly a question related to the topological and symmetric properties of these solutions. The equivariant degree theory is a topological tool that may provide some answers to this question. By converting the existence problem of periodic solutions for distributed delay differential equations into the existence problem of zeros of an equivariant operator, one can predict some equivariant topological properties of these periodic solutions.

It is important to mention that complex dynamical systems may not satisfy usual regularity conditions or admit variational structure, which does not constitute an issue for the equivariant degree method. Using these advantages of the method presented here, one can expect that it might be directly extended to a more general class of differential equations with multiple mixed delays (including distributed delays) and additional spatial symmetries.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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