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Research article

On a binary Diophantine inequality involving prime numbers

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Abstract: Let N denote a sufficiently large real number. In this paper, we prove that for $1 < c < \frac{104349}{77419}$, $c \neq \frac{4}{3}$, for almost all real numbers $T \in (N, 2N]$ (in the sense of Lebesgue measure), the Diophantine inequality $|p_1^c + p_2^c - T| < T^{-\frac{9}{10c}\left(\frac{104349}{77419}-c\right)}$ is solvable in primes p_1, p_2. In addition, it is proved that the Diophantine inequality $|p_1^c + p_2^c + p_3^c + p_4^c - N| < N^{-\frac{9}{10c}\left(\frac{104349}{77419}-c\right)}$ is solvable in primes p_1, p_2, p_3, p_4. This result constitutes a refinement upon that of Li and Cai.

Keywords: Diophantine inequality; prime; exponential sum
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1. Introduction

The Waring-Goldbach problem is that every natural number n can be represented as the type

$$n = p_1^k + \cdots + p_s^k,$$

where s and k are given positive integers and $p_1, \cdots, p_s$ are prime variables. This well-known problem has spawned many analogues, which has attracted a large number of scholars to investigate and obtain many celebrated results. For instance, given $c > 1$ is not an integer and $\varepsilon > 0$. For every sufficiently large real N, suppose that $h(c)$ is the smallest natural number s satisfying the inequality

$$|p_1^c + p_2^c + \cdots + p_s^c - N| < \varepsilon \tag{1.1}$$

with solutions in primes $p_1, \cdots, p_s$. Piatetski-Shapiro showed in [13] that

$$\limsup_{c\to\infty} \frac{h(c)}{c \log c} \leq 4,$$

and $h(c) \leq 5$ holds for $1 < c < \frac{3}{2}$. Based on this result and Vinogradov's three prime theorem, we can conjecture that $h(c) \leq 3$ for c near one. Tolev [14] first proved this conjecture. More precisely, Tolev [15] showed that, if $1 < c < \frac{15}{14}$, the inequality has solutions in primes p_1, p_2, p_3:

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon,$$

where $\varepsilon = N^{-\frac{1}{c}(\frac{15}{14}-c)} \log^9 N$. Subsequently, this result was constantly improved by several authors (see [1–4, 7, 8]).

In 1999, Laporta [10] considered the corresponding binary problem. If $1 < c < \frac{15}{14}$ fixed and $\varepsilon = T^{1-\frac{15}{14c}} \log^8 T$, the following inequality

$$|p_1^c + p_2^c - T| < \varepsilon \tag{1.2}$$

has a solution for all real numbers $T \in (N, 2N] \setminus \mathfrak{T}$ with $|\mathfrak{T}| = O\left(N \exp\left(-\frac{1}{3}\left(\frac{\log N}{c}\right)^{\frac{1}{5}}\right)\right)$. Later, the exponent of c was improved to

$$\frac{43}{36} = 1.944444\cdots, \quad \frac{6}{5} = 1.2, \quad \frac{59}{44} = 1.340909\cdots$$

by Zhai and Cao [16], Kumchev and Laporta [9], and Li and Cai [11] successively.

In 2003, Zhai and Cao [17] first proved that the inequality

$$|p_1^c + p_2^c + \cdots + p_4^c - N| < \varepsilon \tag{1.3}$$

is solvable, where $1 < c < \frac{81}{68}$. Afterwards, the exponent of c was improved to

$$\frac{97}{81} = 1.197530\cdots, \quad \frac{6}{5} = 1.2, \quad \frac{1193}{889} = 1.341957\cdots$$

by Mu [12], Zhang and Li [18], and Li and Zhang [19] successively.

Here, we consider the cases $s = 2$ and $s = 4$ in inequality (1.1), and enlarge the exponent of c. Our results are as follows:

Theorem 1. *Assume $1 < c < \frac{104349}{77419}$, $c \neq \frac{4}{3}$, and N is a sufficiently large real number. Suppose $\varepsilon = T^{-\frac{9}{10c}(\frac{104349}{77419}-c)}$, and $B_0(T)$ is the number of solutions of inequality* (1.2). *Then, for all real numbers $T \in (N, 2N] \setminus \mathfrak{T}$ with $|\mathfrak{T}| = O\left(N \exp\left(-\frac{1}{3}\left(\frac{\log N}{c}\right)^{\frac{1}{5}}\right)\right)$, we obtain*

$$B_0(T) \gg \frac{\varepsilon T^{\frac{2}{c}-1}}{\log^2 T}.$$

Theorem 2. *Assume $1 < c < \frac{104349}{77419}$, $c \neq \frac{4}{3}$, and N is a sufficiently large real number. Suppose $\varepsilon = T^{-\frac{9}{10c}(\frac{104349}{77419}-c)}$, and $\mathcal{B}_0(N)$ is the number of solutions of inequality* (1.3). *Then, we obtain*

$$\mathcal{B}_0(N) \gg \frac{\varepsilon N^{\frac{4}{c}-1}}{\log^4 N}.$$

In this paper, our improvement mainly comes from the estimates of exponential sums. We transform the exponential sum into Type I and Type II sums by using Heath-Brown's identity. As a result, we improve the previous results by enlarging the upper bound of c. In addition, by using Lemma 2.8, the value of $c = \frac{4}{3}$ can be excluded.

2. Notation and preliminaries

Now, we will give some notations which are required throughout the paper.

Throughout this paper, suppose that N is a sufficiently large integer, $\Lambda(n)$ stands for the von Mangoldt function, and $|\mathfrak{T}|$ is the cardinality of the set $\mathfrak{T}$. $a \sim A$ means $A < a \le 2A$. As usual, the constants in the $\ll$-symbols and O-terms are absolute or depend only on c.

In addition, we write

$$\begin{aligned}
&1 < c < \frac{104349}{77419},\ c \neq \frac{4}{3}; \quad X = \frac{N^{\frac{1}{c}}}{4}, \quad \iota = X^{\frac{50489}{154838}-\frac{c}{2}},\\
&\mathcal{K} = X^{\frac{104349}{77419}-c}, \quad e(t) = e^{2\pi i t}, \quad \varepsilon = N^{-\frac{9}{10c}\left(\frac{104349}{77419}-c\right)},\\
&\delta = \frac{1}{1000}\left(\frac{104349}{77419}-c\right), \quad E = \exp\left(-\left(\frac{\log N}{c}\right)^{\frac{1}{5}}\right),\\
&P = \left(\frac{2}{E^2}\right)^{\frac{1}{3}} \log N, \quad \mathfrak{S}(t) = \sum_{p\sim X} e(p^c t)\log p,\\
&\mathcal{I}(t) = \int_X^{2X} e\left(u^c t\right) du.
\end{aligned}$$

Next, we shall recall some preliminary lemmas that are necessary in this paper.

Lemma 2.1 ([15, Lemma 1]). *Assume that $\xi(y)$ is a function which is $\omega = [\log X]$ times continuously differentiable and satisfies*

$$\begin{cases} \xi(y) = 1, \ |y| \le \frac{4}{5}\varepsilon,\\ 0 < \xi(y) < 1, \ \frac{4}{5}\varepsilon < |y| \le \varepsilon,\\ \xi(y) = 0, \ |y| \ge \varepsilon. \end{cases}$$

For its Fourier transformation

$$\Xi(t) = \int_{-\infty}^{\infty} e(-ty)\xi(y)dy,$$

then

$$|\Xi(t)| \le \min\left(\frac{9}{5}\varepsilon, \frac{1}{\pi|t|}, \frac{1}{\pi|t|}\left(\frac{5\omega}{\pi|t|\varepsilon}\right)^{\omega}\right).$$

Lemma 2.2 ([15, Lemma 14]). *If $|t| \le \iota$, then*

$$\mathfrak{S}(t) = \mathcal{I}(t) + O\left(X\exp\left(-\log^{\frac{1}{5}} X\right)\right).$$

Lemma 2.3 ([5, Lemma 3.1]). *There is a trivial bound that*

$$\mathcal{I}(t) \ll X^{1-c}|t|^{-1}. \tag{2.1}$$

Lemma 2.4 ([15, Lemma 7]). *There exist the following inequalities*

$$(i) \int_{|t|<\iota} |\mathfrak{S}(t)|^2 dt \ll X^{2-c}\log^3 X; \tag{2.2}$$

$$(ii)\int_{|t|<\iota}|\mathcal{I}(t)|^2dt \ll X^{2-c}\log^3 X; \tag{2.3}$$

$$(iii)\int_{n}^{n+1}|\mathfrak{S}(t)|^2dt \ll X\log^3 X \text{ uniformly in } n. \tag{2.4}$$

Lemma 2.5. *We have*

$$(i)\int_{-\infty}^{\infty}\mathcal{I}^2(t)\Xi(t)e(-Tt)dt \gg \varepsilon T^{\frac{2}{c}-1}, \tag{2.5}$$

$$(ii)\int_{-\infty}^{\infty}\mathcal{I}^4(t)\Xi(t)e(-Nt)dt \gg \varepsilon X^{4-c}. \tag{2.6}$$

Proof. The above two inequalities (2.5) and (2.6) can be found in [9] and [17] independently. □

Lemma 2.6 ([6, Lemma 5]). *For any complex number α_n, we have the inequality*

$$\left|\sum_{H\le n\le 2H}\alpha_n\right|^2 \le \left(1+\frac{H}{\mathcal{U}}\right)\sum_{0\le|u|\le\mathcal{U}}\left(1-\frac{|u|}{\mathcal{U}}\right)\left(\sum_{H\le n\le 2H-u}\alpha_{n+u}\overline{\alpha_n}\right),$$

where $H \geqslant 1$, $\mathcal{U} \le H$, and $\overline{\alpha_n}$ stands for the conjugate of α_n.

Lemma 2.7 ([11, Lemma 2.3]). *Suppose $|D| > 0$ and $A \le A' \le 10A$, then the inequality*

$$\sum_{A\le a\le A'} e(Da^c) \ll (|D|A^c)^{\kappa}A^{\lambda-\kappa} + \frac{A}{|D|A^c}$$

holds for any exponent pair (κ,λ) with $0 \le \kappa \le \frac{1}{2} \le \lambda \le 1$.

Lemma 2.8 ([1, Therorem 2]). *Suppose τ, υ are real numbers such that*

$$\tau\upsilon(\tau-1)(\upsilon-1)(\tau-2)(\tau+\upsilon-2)(\tau+\upsilon-3)(\tau+2\upsilon-3)(2\tau+\upsilon-4) \neq 0.$$

Let

$$\sum\nolimits_I = \sum_{a\le A}\alpha(a)\sum_{b\in I_a}e\left(Da^{\tau}b^{\upsilon}\right),$$

where $D > 0$, $A \ge 1$, $B \ge 1$, $|\alpha(a)| \le 1$, and I_a is a subinterval of $[B, 2B]$. Assume $G = DA^{\tau}B^{\upsilon}$. For any $\eta > 0$, then

$$\sum\nolimits_I \ll \left(G^{\frac{3}{14}}A^{\frac{41}{56}}B^{\frac{29}{56}} + G^{\frac{1}{5}}A^{\frac{3}{4}}B^{\frac{11}{20}} + G^{\frac{1}{8}}A^{\frac{13}{16}}B^{\frac{11}{16}} + A^{\frac{3}{4}}B + AB^{\frac{3}{4}} + G^{-1}AB\right)(AB)^{\eta}.$$

Lemma 2.9 ([10, Lemma 1]).

$$\mathbb{V} = \max_{N<v_2\le 2N}\int_{N}^{2N}\left|\int_{\iota<|t|\le\mathcal{K}}e((v_1-v_2)t)dt\right|dv_1 \ll \log N.$$

Lemma 2.10 ([10, Lemma 2]). *Let the letters $\mathbb{C}$ and $\mathbb{R}$ be the sets of complex and real numbers, respectively. Suppose that F_1 and F_2 are measurable subsets of $\mathbb{R}^n$, and*

$$||g||_i = \left(\int_{\mathsf{F}_i}|g(t)|^2dt\right)^{\frac{1}{2}},\ \langle g, f\rangle_i = \int_{\mathsf{F}_i}g(t)\overline{f}(t)dt$$

stand for the usual norm and inner product in $L^2(F_i,\mathbb{C})$ $(i=1,2)$, respectively.

Assume ϖ is a measurable complex-valued function defined on $F_1 \times F_2$, then

$$\sup_{y\in F_1}\int_{F_2}|\varpi(y,t)|dt < +\infty,\ \sup_{t\in F_2}\int_{F_1}|\varpi(y,t)|dt < +\infty.$$

Hence,

$$\left|\int_{F_1}\phi(y)\langle\psi,\varpi(y,\cdot)\rangle_2 dy\right| \ll \|\psi\|_2\|\phi\|_1\left(\sup_{y'\in F_1}\int_{F_1}|\langle\varpi(y,\cdot),\varpi(y',\cdot)\rangle_2|\,dy\right)^{\frac{1}{2}},$$

where $\phi\in L^2(F_1,\mathbb{C})$, $\psi\in L^2(F_2,\mathbb{C})$.

3. The estimation of $\mathfrak{S}(t)$

In this section, we transform the exponential sum into Type I and Type II sums by taking advantage of Lemma 3.1.

Lemma 3.1 ([6, Lemma 3]). *For $3<L<M<Q<X$, $\{Q\}=\frac{1}{2}$, $X\geq 64Q^2L$, $Q\geq 4L^2$, and $M^3\geq 32X$, if $G(n)$ is a complex valued function satisfying $|G(n)|\leq 1$, then the sum*

$$\sum_{X\leq n\leq 2X}\Lambda(n)G(n)$$

may be decomposed into $O(\log^{10}X)$ sums, each either of type I:

$$\sum_{A\leq a\leq 2A}\alpha(a)\sum_{B\leq b\leq 2B}G(ab)$$

with $B>Q$, $AB\asymp X$, $|\alpha(a)|\ll a^{\eta}$, or of type II:

$$\sum_{A\leq a\leq 2A}\alpha(a)\sum_{B\leq b\leq 2B}\beta(b)G(ab)$$

with $L\ll B\ll M$, $AB\asymp X$, $|\alpha(a)|\ll a^{\eta}$, $|\beta(b)|\ll b^{\eta}$.

Lemma 3.2. *For any $\eta>0$, $\alpha(a)$ is a sequence of complex numbers satisfying $|\alpha(a)|\ll a^{\eta}$. If $\iota\leq|t|\leq\mathcal{K}$ and $A\ll X^{\frac{3685827905}{6017314356}}$, then*

$$\mathfrak{S}_I=\sum_{a\leq A}\alpha(a)\sum_{X\leq ab\leq 2X}e\left(t(ab)^c\right)\ll X^{\frac{1417193203}{1504328589}+2\eta},$$

where $c\in(1,\frac{104349}{77419}]$ and $c\neq\frac{4}{3}$.

Proof. Suppose $A\ll X^{\frac{3685827905}{10530300123}}$, and we use Lemma 2.7 by choosing the exponent pair $\left(\frac{2}{7},\frac{4}{7}\right)$, then

$$\begin{aligned}\mathfrak{S}_I \ll& A^{\eta}\sum_{a\leq A}\left|\sum_{X\leq ab\leq 2X}e\left(t(ab)^c\right)\right|\\ \ll& A^{\eta}\sum_{a\leq A}\left((|t|X^c)^{\frac{2}{7}}X^{\frac{2}{7}}+\frac{1}{|t|X^{c-1}a}\right)\end{aligned}$$

$$\ll X^{\frac{1417193203}{1504328589}+2\eta}.$$

In addition, suppose $X^{\frac{3685827905}{10530300123}} \ll A \ll X^{\frac{3685827905}{6017314356}}$, then from Lemma 2.8 we derive that

$$\begin{aligned}\mathfrak{S}_I =& A^\eta \sum_{a\le A}\frac{a(m)}{A^\eta}\sum_{X\le ab\le 2X} e\left(t(ab)^c\right)\\ \ll& \Big((\mathcal{K}X^c)^{\frac{3}{14}}A^{\frac{3}{14}}X^{\frac{29}{56}} + (\mathcal{K}X^c)^{\frac{1}{5}}A^{\frac{1}{5}}X^{\frac{11}{20}} + (\mathcal{K}X^c)^{\frac{1}{8}}A^{\frac{1}{8}}X^{\frac{11}{16}}\\ &+A^{-\frac{1}{4}}X + A^{\frac{1}{4}}X^{\frac{3}{4}} + (\iota X^c)^{-1}X\Big)X^{2\eta}\\ \ll& X^{\frac{1417193203}{1504328589}+2\eta}.\end{aligned}$$

Combining the above two cases, we complete the proof of this lemma. □

Lemma 3.3. *For any $\eta > 0$, $\alpha(a)$ and $\beta(b)$ are sequences of complex numbers satisfying $|\alpha(a)| \ll a^\eta$, $|\beta(b)| \ll b^\eta$. If $\iota \le |t| \le \mathcal{K}$ and $X^{\frac{174270772}{1504328589}} \ll A \ll X^{\frac{427164751875025}{1190813782635796}}$, then*

$$\mathfrak{S}_{II} = \sum_{A<a\le 2A}\alpha(a)\sum_{X<ab\le 2X}\beta(b)e\left(t(ab)^c\right) \ll X^{\frac{1417193203}{1504328589}+3\eta}.$$

Proof. Let $\mathcal{U} = X^{\frac{174270772}{1504328589}-\eta}$. Cauchy's inequality and Lemma 2.6 gives us

$$\begin{aligned}|\mathfrak{S}_{II}| &\ll \left(\sum_{b\in I_a}|\beta(b)|^2\right)^{1/2}\left(\sum_{b\in I_a}\left|\sum_{A<a\le 2A}\alpha(a)e(t(ab)^c)\right|^2\right)^{1/2}\\ &\ll X^{2\eta}\left(\frac{X^2}{\mathcal{U}} + \frac{X}{\mathcal{U}}\sum_{1\le u\le\mathcal{U}}\sum_{A\le a\le 2A-u}|E_u|\right)^{1/2},\end{aligned}\tag{3.1}$$

where I_a stands for a subinterval of $\left[\frac{X}{2A}, \frac{2X}{A}\right]$, and $E_u = \sum_{b\in I_a} e\left(tb^c((a+u)^c - a^c)\right)$.

Following from Lemma 2.7 and choosing the exponent pair $\left(\frac{13}{31}, \frac{16}{31}\right)$, we obtain the estimate of E_u:

$$E_u \ll \left(|t|X^{c-1}u\right)^{\frac{13}{31}} X^{\frac{16}{31}}A^{-\frac{16}{31}} + \frac{1}{|t|X^{c-1}u}.$$

Then, taking the estimate of E_u into (3.1), we get

$$\begin{aligned}|\mathfrak{S}_{II}| \ll& X^{2\eta}\left(\frac{X^2}{\mathcal{U}} + \frac{X}{\mathcal{U}}\left(\mathcal{U}^{\frac{44}{31}}|\mathcal{K}|^{\frac{13}{31}}X^{\frac{13}{31}c+\frac{3}{31}}A^{\frac{15}{31}} + X^{1-c}|\iota|^{-1}A\log\mathcal{U}\right)\right)^{1/2}\\ \ll& X^{\frac{1417193203}{1504328589}+3\eta}.\end{aligned}$$

□

Lemma 3.4. *Assume that η is any arbitrarily small positive number and $\iota \le |t| \le \mathcal{K}$, then*

$$\mathfrak{S}(t) \ll X^{\frac{1417193203}{1504328589}+5\eta},$$

where $c \in (1, \frac{104349}{77419}]$ and $c \ne \frac{4}{3}$.

Proof. First of all, we define

$$\mathfrak{M}(t)=\sum_{n\sim X}\Lambda(n)e(n^c t).$$

Obviously, we can deduce that

$$\mathfrak{S}(t)=\mathfrak{M}(t)+O\left(X^{\frac{1}{2}}\right). \tag{3.2}$$

Suppose

$$L=X^{\frac{174270772}{1504328589}},\ M=X^{\frac{427164751875025}{1190813782635796}},\ Q=\left[X^{\frac{2331486451}{6017314356}}\right]+\frac{1}{2}.$$

Following from Lemma 3.1 with $G(n)=e(tn^c)$, we reduce the sum $\mathfrak{M}(t)$ of type I:

$$\sum_{A\le a\le 2A}\alpha(a)\sum_{B\le b\le 2B}G(ab),\ B>Q$$

or of type II:

$$\sum_{A\le a\le 2A}\alpha(a)\sum_{B\le b\le 2B}\beta(b)G(ab), L\ll B\ll M.$$

From this combined with Lemmas 3.2 and 3.3, we deduce

$$\mathfrak{M}(t)\ll X^{\frac{1417193203}{1504328589}+5\eta}.$$

Inserting the bound of $\mathfrak{M}(t)$ into (3.2), we finish the proof of Lemma 3.4. □

4. Proof of Theorem 1

In this section, we shall give the details of the proof of Theorem 1.

Let $\xi(y)$ and $\Xi(t)$ stand for the functions that appear in Lemma 2.1. For $T\in[N,2N]$, we write

$$\mathfrak{B}(T)=\sum_{\substack{p_1,p_2\sim X\\ |p_1^c+p_2^c-T|<\varepsilon}}(\log p_1)(\log p_2).$$

It suffices to show that $\mathfrak{B}(T)\ge\mathfrak{B}_1(T)$, where

$$\begin{aligned}\mathfrak{B}_1(T)&=\sum_{X<p_1,p_2\le 2X}(\log p_1)(\log p_2)\int_{-\infty}^{\infty}e((p_1^c+p_2^c-T)t)\Xi(t)dt\\&=\int_{-\infty}^{\infty}\mathfrak{S}^2(t)\Xi(t)e(-Tt)dt\\&=\left(\int_{|t|\le\iota}+\int_{\iota<|t|\le\mathcal{K}}+\int_{|t|>\mathcal{K}}\right)\mathfrak{S}^2(t)\Xi(t)e(-Tt)dt\\&=Q_1(T)+Q_2(T)+Q_3(T).\end{aligned}$$

In addition, we write

$$P(T)=\int_{-\infty}^{\infty}\mathcal{I}^2(t)e(-Tt)\Xi(t)dt,\ P_1(T)=\int_{-\iota}^{\iota}\mathcal{I}^2(t)\Xi(t)e(-Tt)dt.$$

Lemma 4.1. *We have*

$$\int_{\iota<|t|\leq\mathcal{K}}|\mathfrak{S}(t)|^2|\Xi(t)|dt \ll X\log^4 X.$$

Proof. It follows from Lemma 2.1 and (2.4) that

$$\begin{aligned}&\int_{\iota<|t|\leq\mathcal{K}}|\mathfrak{S}(t)|^2|\Xi(t)|dt\\ &\ll \varepsilon\sum_{0\leq n\leq\frac{1}{\varepsilon}}\int_n^{n+1}|\mathfrak{S}(t)|^2dt+\sum_{\frac{1}{\varepsilon}-1\leq n\leq\mathcal{K}}\frac{1}{n}\int_n^{n+1}|\mathfrak{S}(t)|^2dt\\ &\ll X\log^4 X.\end{aligned}$$

□

Lemma 4.2. *We have*

$$\int_{\iota<|t|\leq\mathcal{K}}|\mathfrak{S}(t)|^4|\Xi(t)|dt \ll \varepsilon^{\frac{1}{2}}X^{\frac{10007332969}{3008657178}-\frac{c}{2}+12\delta}.$$

Proof. If $V(t)$ is a continuous function defined for $-\mathcal{K}\leq t\leq\mathcal{K}$, then we find that

$$\begin{aligned}\left|\int_{\iota<|t|\leq\mathcal{K}}\mathfrak{S}(t)V(t)dt\right| &= \left|\sum_{X<p\leq 2X}(\log p)\int_{\iota<|t|\leq\mathcal{K}}V(t)e(p^ct)dt\right|\\ &\leq \sum_{X<p\leq 2X}(\log p)\left|\int_{\iota<|t|\leq\mathcal{K}}V(t)e(p^ct)dt\right|\\ &\leq (\log X)\sum_{X<m\leq 2X}\left|\int_{\iota<|t|\leq\mathcal{K}}V(t)e(m^ct)dt\right|. \end{aligned}\tag{4.1}$$

Suppose $H(t)=\sum_{m\sim X}e(m^ct)$. Using Cauchy's inequality, we deduce that

$$\begin{aligned}\left|\int_{\iota<|t|\leq\mathcal{K}}\mathfrak{S}(t)V(t)dt\right| &\leq X^{\frac{1}{2}}(\log X)\left|\sum_{X<m\leq 2X}\left|\int_{\iota<|t|\leq\mathcal{K}}V(t)e(m^ct)dt\right|^2\right|^{\frac{1}{2}}\\ &= X^{\frac{1}{2}}(\log X)\left|\sum_{X<m\leq 2X}\int_{\iota<|t|\leq\mathcal{K}}V(t)e(m^ct)dt\int_{\iota<|y|\leq\mathcal{K}}\overline{V(y)e(m^cy)}dy\right|^{\frac{1}{2}}\\ &= X^{\frac{1}{2}}(\log X)\left|\int_{\iota<|y|\leq\mathcal{K}}\overline{V(y)}dy\int_{\iota<|t|\leq\mathcal{K}}V(t)H(t-y)dt\right|^{\frac{1}{2}}\\ &\leq X^{\frac{1}{2}}(\log X)\left|\int_{\iota<|y|\leq\mathcal{K}}|V(y)|dy\int_{\iota<|t|\leq\mathcal{K}}|V(t)||H(t-y)|dt\right|^{\frac{1}{2}}. \end{aligned}\tag{4.2}$$

Then, we estimate the inner integral in (4.2).

First, we need to consider $H(t-y)$. Following from Lemma 2.7 and choosing the exponent pair $\left(\frac{19126}{58293},\frac{31369}{58293}\right)$, we obtain that the inequality

$$H(t)\ll(|t|X^c)^{\frac{19126}{58293}}X^{\frac{12243}{58293}}+\frac{X}{|t|X^c}$$

holds for $X^{-c} < |t| \leq 2\mathcal{K}$.

Combining with the trivial upper bound $H(t-y) \ll X$, we get

$$H(t-y) \ll \min\left((|t-y|X^c)^{\frac{19126}{58293}} X^{\frac{12243}{58293}} + \frac{X}{|t-y|X^c}, X \right). \tag{4.3}$$

Next, inserting the estimate of $H(t-y)$ into the inner integral in (4.2), we find that

$$\begin{aligned}
&\int_{\iota<|t|\leq\mathcal{K}} |V(t)||H(t-y)|dt \\
\ll &\int_{\substack{\iota<|t|\leq\mathcal{K} \\ |t-y|\leq X^{-c}}} |V(t)||H(t-y)|dt + \int_{\substack{\iota<|t|\leq\mathcal{K} \\ X^{-c}<|t-y|\leq 2\mathcal{K}}} |V(t)||H(t-y)|dt \\
\ll &X \int_{\substack{\iota<|t|\leq\mathcal{K} \\ |t-y|\leq X^{-c}}} |V(X)|dt + \int_{\substack{\iota<|t|\leq\mathcal{K} \\ X^{-c}<|t-y|\leq 2\mathcal{K}}} |V(t)| \left((|t-y|X^c)^{\frac{19126}{58293}} X^{\frac{12243}{58293}} + \frac{1}{|t-y|X^{c-1}} \right) dt \\
\ll &X \max_{\iota<|t|\leq\mathcal{K}} |V(t)| \int_{|t-y|\leq X^{-c}} dt + X^{\frac{981206597}{1504328589}} \int_{\iota<|t|\leq\mathcal{K}} |V(t)|dt \qquad (4.4) \\
&+ X^{1-c} \max_{\iota<|t|\leq\mathcal{K}} |V(t)| \int_{X^{-c}<|t-y|\leq 2\mathcal{K}} \frac{1}{|t-y|} dt \\
\ll &X^{\frac{981206597}{1504328589}} \int_{\iota<|t|\leq\mathcal{K}} |V(t)|dt + X^{1-c}(\log X) \max_{\iota<|t|\leq\mathcal{K}} |V(t)|. \qquad (4.5)
\end{aligned}$$

From (4.2) and (4.4), we have

$$\begin{aligned}
\left| \int_{\iota<|t|\leq\mathcal{K}} \mathfrak{S}(t)V(t)dt \right| \ll &X^{\frac{2485535186}{3008657178}} (\log X) \int_{\iota<|t|\leq\mathcal{K}} |V(t)|dt \\
&+ X^{1-\frac{c}{2}} (\log^{\frac{3}{2}} X) \left| \max_{\iota<|t|\leq\mathcal{K}} |V(t)| \int_{\iota<|t|\leq\mathcal{K}} |V(t)|dt \right|^{\frac{1}{2}}. \qquad (4.6)
\end{aligned}$$

Taking $V(t) = |\Xi(t)|\overline{\mathfrak{S}(t)}|\mathfrak{S}(t)|$, from Lemmas 3.4 and 4.1 and (4.6), we get

$$\begin{aligned}
&\int_{\iota<|t|\leq\mathcal{K}} |\Xi(t)||\mathfrak{S}(t)|^3 dt \\
= &\int_{\iota<|t|\leq\mathcal{K}} V(t)\mathfrak{S}(t)dt \\
\ll &X^{\frac{1242767598}{1504328589}} (\log X) \int_{\iota<|t|\leq\mathcal{K}} |\Xi(t)||\mathfrak{S}(t)|^2 dt + \varepsilon^{\frac{1}{2}} X^{\frac{2921521792}{1504328589}-\frac{c}{2}+6\delta} \left| \int_{\iota<|t|\leq\mathcal{K}} |\Xi(t)||\mathfrak{S}(t)|^2 dt \right|^{\frac{1}{2}} \\
\ll &X^{\frac{2747096182}{1504328589}} \log^5 X + \varepsilon^{\frac{1}{2}} X^{\frac{7347372173}{3008657178}-\frac{c}{2}+7\delta} \\
\ll &X^{\frac{2747096182}{1504328589}+7\delta}. \qquad (4.7)
\end{aligned}$$

Next, taking $V(t) = |\Xi(t)|\overline{\mathfrak{S}(t)}|\mathfrak{S}(t)|^2$, by Lemma 3.4, (4.6), and (4.7), we obtain

$$\int_{\iota<|t|\leq\mathcal{K}} |\mathfrak{S}(t)|^4 |\Xi(t)| dt$$

$$
\begin{aligned}
&= \int_{\iota<|t|\le\mathcal{K}} V(t)\mathfrak{S}(t)dt\\
&\ll X^{\frac{1242767598}{1504328589}}(\log X)\int_{\iota<|t|\le\mathcal{K}}|\mathfrak{S}(t)|^3|\Xi(t)|dt+\varepsilon^{\frac12}X^{\frac{2420078929}{1002885726}-\frac{c}{2}+8\delta}\left|\int_{\iota<|t|\le\mathcal{K}}|\mathfrak{S}(t)|^3|\Xi(t)|dt\right|^{\frac12}\\
&\ll X^{\frac{3989863780}{1504328589}+8\delta}+\varepsilon^{\frac12}X^{\frac{10007332969}{3008657178}-\frac{c}{2}+12\delta}\\
&\ll \varepsilon^{\frac12}X^{\frac{10007332969}{3008657178}-\frac{c}{2}+12\delta}.
\end{aligned}
$$

□

Lemma 4.3 ([11, Lemma 3.3]). *We have*

$$\int_N^{2N}|Q_1(T)-P_1(T)|^2dT \ll \varepsilon^2N^{\frac4c-1}E^{\frac13}.$$

Lemma 4.4. *We have*

$$\int_N^{2N}|Q_2(T)|^2dT \ll \varepsilon^2N^{\frac4c-1}E^{\frac13}.$$

Proof. Suppose

$$F_1=\{T:T\sim N\},\ F_2=\{t:\iota<|t|\le\mathcal{K}\}.$$

Taking advantage of Lemma 2.10 with $\psi(t)=\Xi(t)\mathfrak{S}^2(t), \varpi(t,T)=e(Tt), \phi(T)=\overline{Q_2(T)}$, we obtain

$$
\begin{aligned}
\int_N^{2N}|Q_2(T)|^2dT &= \int_{F_1}\overline{Q_2(T)}\langle\Xi(t)\mathfrak{S}^2(t),e(tT)\rangle_2dT\\
&\ll\left(\int_{F_2}|\Xi(t)\mathfrak{S}^2(t)|^2dt\right)^{\frac12}\left(\int_{F_1}|\overline{Q_2(T)}|^2dT\right)^{\frac12}\left(\sup_{t'\in F_1}\int_{F_1}|\langle e(tT),e(t'T)\rangle_2|\,dt\right)^{\frac12}. \qquad (4.8)
\end{aligned}
$$

According to Lemmas 2.1, 2.9, and 4.2, we get

$$
\begin{aligned}
\int_N^{2N}|Q_2(T)|^2dT &\ll \mathbb{V}\int_{\iota<|t|\le\mathcal{K}}|\mathfrak{S}(t)|^4|\Xi(t)|^2dt\\
&\ll \varepsilon(\log N)\int_{\iota<|t|\le\mathcal{K}}|\mathfrak{S}(t)|^4|\Xi(t)|dt\\
&\ll \varepsilon^{\frac32}X^{\frac{10007332969}{3008657178}-\frac{c}{2}+12\delta}\log N\\
&\ll \varepsilon^2N^{\frac4c-1}E^{\frac13}.
\end{aligned}
$$

Hence, we finish the proof of this lemma. □

Lemma 4.5. *We have*

$$\int_N^{2N}|Q_3(T)|^2dT \ll N.$$

Proof. It follows from the estimate of $\Xi(t)$ in Lemma 2.1 that

$$\int_N^{2N}|Q_3(T)|^2dT \ll N\left|\int_{t\ge\mathcal{K}}|\mathfrak{S}^2(t)||\Xi(t)|dt\right|^2$$

$$\ll NX^4 \left| \int_{t \geq \mathcal{K}} \left(\frac{5\omega}{\pi t \varepsilon} \right)^{\omega} \frac{dt}{t} \right|^2$$
$$\ll NX^4 \left(\frac{5\omega}{\pi \mathcal{K} \varepsilon} \right)^{2\omega}$$
$$\ll N.$$

□

Lemma 4.6. *We have*

$$\int_N^{2N} |\mathfrak{B}_1(T) - P(T)|^2 dT \ll \varepsilon^2 N^{\frac{4}{c}-1} E^{\frac{1}{3}}.$$

Proof. We obtain

$$\int_N^{2N} |\mathfrak{B}_1(T) - P(T)|^2 = \int_N^{2N} |Q_1(t) - P_1(T) + Q_2(t) + Q_3(t) + P_1(T) - P(T)|^2 \tag{4.9}$$
$$\leq \int_N^{2N} |Q_1(T) - P_1(T)|^2 dT + \int_N^{2N} |Q_2(T)|^2 dT$$
$$+ \int_N^{2N} |Q_3(T)|^2 dT + \int_N^{2N} |P(T) - P_1(T)|^2 dT. \tag{4.10}$$

For the last integral, we will use Lemmas 2.1 and 2.3 to estimate and obtain

$$\int_N^{2N} |P(T) - P_1(T)|^2 dT \ll \int_N^{2N} \int_{|t|>\iota} |\mathcal{I}(t)|^4 |\Xi(t)|^2 dt dT$$
$$\ll NX^{4-4c} \int_{|t|>\iota} |\Xi(t)|^2 |t|^{-4} dt$$
$$\ll \varepsilon^2 NX^{4-4c} \iota^{-3}. \tag{4.11}$$

From (4.9) and (4.11), we have

$$\int_N^{2N} |\mathfrak{B}_1(T) - P(T)|^2 dT \ll \int_N^{2N} |Q_1(T) - P_1(T)|^2 dT + \int_N^{2N} |Q_2(T)|^2 dT$$
$$+ \int_N^{2N} |Q_3(T)|^2 dT + \varepsilon^2 NX^{4-4c} \iota^{-3}$$
$$\ll \varepsilon^2 N^{\frac{4}{c}-1} E^{\frac{1}{3}}, \tag{4.12}$$

where Lemmas 4.3, 4.4, and 4.5 are employed. □

Lemma 4.6 implies

$$\mathfrak{B}_1(T) = P(T) + O\left(\varepsilon N^{\frac{2}{c}-1} E^{\frac{1}{9}}\right), \tag{4.13}$$

where $T \in (N, 2N] \backslash \mathfrak{T}$ with $|\mathfrak{T}| = O\left(NE^{1/3}\right)$.

Then, it follows from (4.13) and (2.5) that

$$B_0(T) \geq \frac{\mathfrak{B}(T)}{\log^2 2X} \geq \frac{\mathfrak{B}_1(T)}{\log^2 2X} \geq \frac{P(T)}{\log^2 2X} \gg \frac{\varepsilon T^{\frac{2}{c}-1}}{\log^2 T}. \tag{4.14}$$

Thus, we finish the proof of Theorem 1.

5. Proof of Theorem 2

Suppose

$$\mathcal{B}(N)=\sum_{\substack{X<p_1,\cdots,p_4\le 2X\\ |p_1^c+\cdots+p_4^c-N|<\varepsilon}}(\log p_1)\cdots(\log p_4).$$

It follows from the definition of $\xi(y)$ and $\Xi(t)$ in Lemma 2.1 that

$$\mathcal{B}(N)\ge\mathcal{B}_1(N),\tag{5.1}$$

where

$$\begin{aligned}\mathcal{B}_1(N)&=\int_{-\infty}^{\infty}\mathfrak{S}^4(t)\Xi(t)e(-Nt)dt\\&=\left(\int_{-\iota}^{\iota}+\int_{\iota<|t|\le\mathcal{K}}+\int_{|t|>\mathcal{K}}\right)\mathfrak{S}^4(t)\Xi(t)e(-Nt)dt\\&=Q_1(N)+Q_2(N)+Q_3(N).\end{aligned}\tag{5.2}$$

First of all, we need to estimate $Q_1(N)$. Let

$$\begin{aligned}\mathcal{P}_1(N)&=\int_{-\iota}^{\iota}\mathcal{I}^4(t)\Xi(t)e(-tN)dt,\\ \mathcal{P}(N)&=\int_{-\infty}^{\infty}\mathcal{I}^4(t)\Xi(t)e(-tN)dt,\end{aligned}$$

then we find that

$$\begin{aligned}Q_1(N)&=\int_{-\iota}^{\iota}\mathfrak{S}^4(t)\Xi(t)e(-Nt)dt\\&=\mathcal{P}(N)+(\mathcal{P}_1(N)-\mathcal{P}(N))+(Q_1(N)-\mathcal{P}_1(N)).\end{aligned}\tag{5.3}$$

For the second integral in (5.3), we use Lemmas 2.1 and 2.3 to estimate and obtain

$$\begin{aligned}|\mathcal{P}_1(N)-\mathcal{P}(N)|&\ll\int_{-\iota}^{\iota}|\mathcal{I}(t)|^4|\Xi(t)|dt\\&\ll X^{4-4c}\int_{|t|>\iota}|\Xi(t)||t|^{-4}dt\\&\ll\varepsilon X^{4-4c}\iota^{-3}\\&\ll\frac{\varepsilon X^{4-c}}{\log X}.\end{aligned}\tag{5.4}$$

For the third integral in (5.3), we use Lemmas 2.1 and 2.2 to estimate and get

$$\begin{aligned}|Q_1(N)-\mathcal{P}_1(N)|&\ll\int_{|t|\le\iota}|\mathfrak{S}^4(t)-\mathcal{I}^4(t)||\Xi(t)|dt\\&\ll\varepsilon\max_{-\iota\le t\le\iota}|\mathfrak{S}(t)-\mathcal{I}(t)|\int_{|t|\le\iota}(|\mathfrak{S}(t)|+|\mathcal{I}(t)|)(|\mathfrak{S}(t)|^2+|\mathcal{I}(t)|^2)dt\end{aligned}$$

$$
\begin{aligned}
&\ll \varepsilon X \exp\left(-\log^{\frac{1}{5}} X\right) X \int_{|t|\le\iota} (|\mathfrak{S}(t)|^2 + |\mathcal{I}(t)|^2)dt \\
&\ll \varepsilon X^2 \exp\left(-\log^{\frac{1}{5}} X\right) X^{2-c}(\log^3 X) \\
&\ll \varepsilon X^{4-c} \exp\left(-\log^{\frac{1}{5}} X\right), \qquad (5.5)
\end{aligned}
$$

where (2.2) and (2.3) in Lemma 2.4 are utilized. Combining (2.6) and (5.3)–(5.5), we derive that

$$
Q_1(N) \gg \varepsilon X^{4-c}. \qquad (5.6)
$$

From Lemma 4.2, we have

$$
\begin{aligned}
Q_2(N) &\ll \int_{\iota<|t|\le\mathcal{K}} |\mathfrak{S}(t)|^4 |\Xi(t)| dt \\
&\ll \varepsilon^{\frac{1}{2}} X^{\frac{10007332969}{3008657178} - \frac{c}{2} + 12\delta} \\
&\ll \frac{\varepsilon X^{4-c}}{\log X}. \qquad (5.7)
\end{aligned}
$$

From Lemma 2.1, we obtain

$$
\begin{aligned}
Q_3(N) &\ll \int_{-\mathcal{K}}^{\mathcal{K}} |\mathfrak{S}^4(t)||\Xi(t)| dt \\
&\ll X^4 \int_{-\mathcal{K}}^{\mathcal{K}} \left(\frac{5\omega}{\pi|t|\varepsilon}\right)^{\omega} \frac{dt}{|t|} \\
&\ll X^4 \left(\frac{5\omega}{\pi\mathcal{K}\varepsilon}\right)^{\omega} \\
&\ll 1. \qquad (5.8)
\end{aligned}
$$

It follows from (5.2), (5.6)–(5.8) that

$$
\mathcal{B}_1(N) \gg \varepsilon X^{4-c}. \qquad (5.9)
$$

From (5.1) and (5.9), we have

$$
\mathcal{B}_0(N) \ge \frac{\mathcal{B}(N)}{\log^4 2X} \ge \frac{\mathcal{B}_1(N)}{\log^4 2X} \gg \frac{\varepsilon N^{\frac{4}{c}-1}}{\log^4 N}. \qquad (5.10)
$$

Hence, we finish the proof of Theorem 2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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