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Research article

Multiple positive solutions to the fractional Kirchhoff-type problems involving sign-changing weight functions

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Abstract: This paper was concerned with the following Kirchhoff type equation involving the fractional Laplace operator $(-\Delta)^s$

$$\begin{cases} \left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+\mu K(x)u=g(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3), \end{cases}$$

where $\alpha,\ \mu>0$, $s\in[\frac{3}{4},1)$, $2<p<4$. By filtration of the Nehari manifold and variational techniques, we obtained the existence of one and two positive solutions under some conditions imposed on K and g.

Keywords: fractional Laplace operator; Kirchhoff type problems; sign-changing; Nehari manifold; variational methods
Mathematics Subject Classification: 49J35

1. Introduction and statement of results

We are concerned with the existence of positive solutions to the fractional Kirchhoff type equation

$$\begin{cases} \left(1+\alpha\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^s u+\mu K(x)u=g(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $\alpha,\ \mu>0$, $s\in[\frac{3}{4},1)$, $2<p<4$, and $(-\Delta)^s$ represents commonly used fractional Laplacian operators defined (up to normalization factors) as follows (for more detailed information on fractional

Laplacian operators, please refer to [1] and the references therein)

$$(-\Delta)^s u(x) = C_{3,s} P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy = C_{3,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^3 \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy, \tag{1.2}$$

for $u \in \mathcal{S}(\mathbb{R}^3)$, where $C_{3,s} = \left(\int_{\mathbb{R}^3} \frac{1-cos\zeta_1}{|\zeta|^{3+2s}} d\zeta \right)^{-1}$, $P.V.$ stands for the Cauchy principle value on the integral, $B_\epsilon(x)$ denotes an open ball of radius ϵ centered at x, and $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of the rapidly decaying C^∞ function. The fractional Laplacian operators $(-\Delta)^s$ have been widely used to establish a variety of mathematical models in practical applications such as physics, chemistry, optimization, finance, fractional quantum mechanics, and so on. When waves propagate in porous spaces or spaces with non- smooth boundaries, classical models cannot reveal the influence of non-smooth geometric shapes on wave characteristics and a fractal model must be used. Recently, He's fractal derivative [2] has been widely adopted to establish a mathematical model for a very complex physical problem, which is defined by

$$\frac{Du(x_0)}{Dx^\sigma} = \Gamma(1+\sigma) \lim_{\substack{x-x_0 \to \Delta x \\ \Delta x \neq 0}} \frac{u(x)-u(x_0)}{(x-x_0)^\sigma},$$

where Γ is the gamma function and Δx is the smallest scale for measuring a porous structure or an unsmooth medium. The future research frontier might be in the fractal-fractional Kirchhoff-type equations, where the fractional order is determined by the fractal dimensions.

Throughout the paper, we assume that potential K satisfies the following conditions:

(K_1) $K \in C(\mathbb{R}^3, \mathbb{R})$ and $K \geq 0$ in $\mathbb{R}^3$.

(K_2) There exists $c > 0$ such that $\mathcal{K}_c := \{x \in \mathbb{R}^3 | K(x) < c\}$ is nonempty and has finite measure.

(K_3) $\Omega = \text{int} K^{-1}(0)$ in a nonempty open set with locally Liptschitz boundary and $\overline{\Omega} = K^{-1}(0)$.

Meanwhile, the weight function g fulfills the following assumptions:

(g_1) $g \in L^\infty(\mathbb{R}^3)$ and $g_{\Omega,\min} = \inf_{x\in\overline{\Omega}} g(x) > 0$.

(g_2) $g_{\max} \leq \frac{g_{\Omega,\min} S_s^{p/2}}{|\mathcal{K}_c|^{1/p_s^*} S_p^{p/2}(\Omega)}$, where $g_{\max} = \sup_{\mathbb{R}^3} g(x)$, $p_s^* = 2_s^*/(2_s^* - p)$ $(2_s^* = \frac{6}{3-2s})$, $S_p(\Omega)$, and S_s are the best constants for the embeddings of $H_0^1(\Omega)$ in $L^p(\Omega)$ and $D^{s,2}(\mathbb{R}^3)$ in $L^{2_s^*}(\mathbb{R}^3)$, respectively.

(g_3) There exist C_0, $R_0 > 0$ such that

$$|x|^{s\left[\frac{2}{\theta}-(4-p)\right]} g(x) \leq C_0 [K(x)]^{\frac{4-p}{2}},$$

for all $|x| > R_0$, where $\theta = \frac{2_s^*-2}{2_s^*-p}$.

Conditions (K_1)–(K_3), first introduced by Bartsch and Wang [3], have attracted attention of several researchers; see [4–10] and the references therein. To be more precise, in [7], Sun and Wu studied the existence and concentration of solutions for a class of Kirchhoff type problems with steep potential well:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \mu K(x) u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where $N \geq 3$, a, b, $\mu > 0$ are parameters and $f(x,t)$ is asymptotically $k-$ linear $(k = 1, 3, 4)$ with respect to t at infinity. By applying variational approach, under some suitable conditions, they showed the existence and the nonexistence of the nontrivial solution to (1.3). Very recently, Zhang and Du [10] studied (1.3) when $N = 3$ and $f(x,u) = |u|^{p-2}u$ with $p \in (2,4)$. By combining truncation techniques and the parameter dependent compactness lemma, the existence of positive solutions for b small and μ large is obtained. Meanwhile, Sun and Wu [8] considered (1.3) when $f(x,u) := g(x)|u|^{p-2}u$ with $p \in (2,4)$. They introduced a novel constraint method to prove the existence of one or two positive solutions to (1.3).

In the last decade, considerable attention has focused on the variational problems involving fractional operators. We point out that, when $\alpha = 0$ and $f(x,u)$ instead of $g(x)|u|^{p-2}u$, problem (1.1) is equivalent to the fractional Schrödinger equation of the type

$$(-\Delta)^s u + \mu K(x)u = f(x,u), \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

which has been extensively investigated by many authors; see [11–17]. For instance, Niu and Tang [13] were concerned with the existence of least energy solutions of (1.4) with $f(x,u) = u^{p-1},\ u \geq\ 0$. Yang and Liu [14] proved that such a class of equations with sublinear perturbation possess at least two nontrivial solutions by using variational methods. Liu, Luo, and Zhang [12] obtained the existence of nontrivial solutions for (1.4) with nonresonant nonlinearity.

Compared to (1.3) and (1.4), dealing with (1.1) involving fractional and nonlocal operators is more difficult. As far as we know, there is no existence result of positive solutions for problem (1.1) with steep potential well. Motivated by the above works, we intend to consider the fractional Kirchhoff type problem (1.1) with steep potential well and sign-changing weight functions.

We now summarize our main results.

Theorem 1.1. *Assume that conditions* (K_1)–(K_3) *and* (g_1)–(g_2) *hold, then there are* $\mu_*,\ \alpha_* > 0$ *such that for any* $\mu \in (\mu_*, \infty),\ \alpha \in (0, \alpha_*)$*, problem (1.1) has at least one positive solution* $u^-_{\mu,\alpha} \in H^s(\mathbb{R}^3)$ *satisfying*

$$0 < \|u^-_{\mu,\alpha}\|_{H^s_{\mu K}(\mathbb{R}^3)} < \left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{1/(p-2)}$$

and

$$0 < I_{\mu,\alpha}(u^-_{\mu,\alpha}) < \frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)},$$

where $I_{\mu,\alpha}$ *is defined in* (2.4).

Theorem 1.2. *Assume that conditions* (K_1)–(K_3) *and* (g_1)–(g_3) *hold, then there are* $\mu^*,\ \alpha_* > 0$ *such that for any* $\mu \in (\mu^*, \infty)$, $\alpha \in (0, \alpha_*)$*, problem (1.1) has at least two positive solutions* $u^{\pm}_{\mu,\alpha} \in H^s(\mathbb{R}^3)$ *satisfying*

$$0 < \|u^-_{\mu,\alpha}\|_{H^s_{\mu K}(\mathbb{R}^3)} < \left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{1/(p-2)} < \|u^+_{\mu,\alpha}\|_{H^s_{\mu K}(\mathbb{R}^3)}$$

and

$$I_{\mu,\alpha}(u^+_{\mu,\alpha}) < 0 < I_{\mu,\alpha}(u^-_{\mu,\alpha}) < \frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)},$$

and $u^+_{\mu,\alpha}$ *is a ground state solution.*

Remark 1.1. *It is worth noting that letting $s = 1$, our Theorems can improve existing results in [8, 10]. Theorems 1.1 and 1.2 seem to be the first existence and multiplicity results of positive solutions to (1.1) respectively.*

Remark 1.2. *From condition (g_1), it's not difficult to see that the weight function g is sign-changing. There indeed exist functions K and g, which satisfy conditions (K_1)–(K_3) and (g_1)–(g_3). For instance, let $g(x) \equiv C_0 > 0$ and*

$$K(x) = \begin{cases} 0, & \text{if } |x| \leq r_0, \\ \dfrac{R_0^{\frac{2s\left[\frac{2}{\theta}-(4-p)\right]}{4-p}}}{R_0 - r_0}(|x| - r_0), & \text{if } r_0 < |x| \leq R_0, \\ |x|^{\frac{2s\left[\frac{2}{\theta}-(4-p)\right]}{4-p}}, & \text{if } |x| > R_0, \end{cases}$$

where $0 < r_0 < R_0$. We can easily see that conditions (K_1)–(K_2) and (g_3) hold. Moreover, from [18], by simple calculations we conclude that

$$S_p(B_{r_0}(0)) = (4\pi)^s \left(\frac{3}{4\pi}\right)^{\frac{2}{pp_s^*}} \frac{\Gamma\left(\frac{3+2s}{2}\right)}{\Gamma\left(\frac{3-2s}{2}\right)} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\right)^{\frac{2s}{3}} r_0^{-\frac{6}{pp_s^*}},$$

where Γ is the gamma function, $B_{r_0}(0)$ denotes the ball of radius r_0 centered at 0, and p_s^* is given in condition (g_2), then, we have the equality in (g_2):

$$\frac{4}{3}\pi r_0^3 S_p^{pp_s^*/2}(B_{r_0}(0)) = (4\pi)^{spp_s^*/2} \left(\frac{\Gamma\left(\frac{3+2s}{2}\right)}{\Gamma\left(\frac{3-2s}{2}\right)}\right)^{pp_s^*/2} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\right)^{\frac{spp_s^*}{3}} = S_s^{pp_s^*/2},$$

which implies that there exists $c > 0$ such that condition (g_2) holds.

Remark 1.3. *As far as we know, there was no result for problem (1.1) with steep potential and sign-changing weight function for $p \in (2, 4)$. We point out that in the process of estimating the energy functional, we must restrict $s \in [\frac{3}{4}, 1)$. More precisely, we shall prove $I_{\mu,\alpha}$ is bounded below on $\mathcal{N}_{\mu,\alpha}^-$ in such restriction on s. However, if $s \in (0, \frac{3}{4})$, the exponent $p_s^* = 2_s^*/(2_s^* - p)$ given in Lemma 2.2 will not be well defined due to $2_s^*, p \in (2, 4)$.*

The rest of this article is organized as follows. In Section 2, we introduce some symbols and technical lemmas. We prove the main results in Section 3.

2. Preliminaries

To prove our main Theorems, we need the following notation and useful results. The Hilbert space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) := \left\{u \in L^2(\mathbb{R}^3) | \, (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^3)\right\},$$

endowed with the inner product

$$(u, v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} uv dx$$

and the norm

$$\|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (u^2 + |(-\Delta)^{\frac{s}{2}} u|^2) dx\right)^{\frac{1}{2}}.$$

According to (1.2) and [1, Proposition 3.6], we have

$$\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^N)} = \frac{2}{C_s} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy,$$

where C_s is a positive constant that depends on s.

As in [18, Theorem 1.1], let $S_s > 0$ be the best constant of the fractional Sobolev embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_s}(\mathbb{R}^3)$ defined by

$$S_s = \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{L^{2^*_s}(\mathbb{R}^3)}}, \tag{2.1}$$

where $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is defined by

$$\mathcal{D}^{s,2}(\mathbb{R}^3) := \left\{u \in L^{2^*_s}(\mathbb{R}^3) |\, (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^3)\right\},$$

endowed with the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx\right)^{\frac{1}{2}}.$$

The best constant for the embedding $H^s_0(\Omega) \hookrightarrow L^p(\Omega)$ is denoted by

$$S_p(\Omega) = \inf_{u\in H^s_0(\Omega)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{L^p(\mathbb{R}^3)}}. \tag{2.2}$$

Throughout this paper, we define

$$H^s_K(\mathbb{R}^3) = \left\{u \in H^s(\mathbb{R}^3) |\int_{\mathbb{R}^3} K(x)|u|^2 dx < +\infty\right\},$$

which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{H^s_K(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} K(x) uv dx$$

and the norm

$$\|u\|_{H^s_K(\mathbb{R}^3)} = \left(\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} K(x)|u|^2 dx\right)^{\frac{1}{2}}.$$

For $\mu > 0$, define the next inner product

$$\langle u, v\rangle_{H^s_{\mu K}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} \mu K(x) uv dx,$$

and the norm

$$\|u\|_{H^s_{\mu K}(\mathbb{R}^3)} = \left(\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \mu K(x)|u|^2 dx\right)^{\frac{1}{2}}.$$

Obviously, for $\mu \geq 1$, $\|u\|_{H^s_K(\mathbb{R}^3)} \leq \|u\|_{H^s_{\mu K}(\mathbb{R}^3)}$, then, by (K_1)–(K_2), the Hölder inequality, and (2.1), we derive that

$$\begin{aligned}\int_{\mathbb{R}^3}(u^2+|(-\Delta)^{\frac{s}{2}}u|^2)dx &\leq \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx+|\mathcal{K}_c|^{\frac{2s}{3}}\left(\int_{\mathcal{K}_c}|u|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}+\frac{1}{\mu c}\int_{\mathbb{R}^3\setminus\mathcal{K}_c}\mu K(x)u^2dx\\ &\leq \max\left\{1+|\mathcal{K}_c|^{\frac{2s}{3}}S_s^{-1},(\mu c)^{-1}\right\}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}.\end{aligned}$$

Thus, for $\mu \geq c^{-1}(1+|\mathcal{K}_c|^{\frac{2s}{3}}S_s^{-1})^{-1}$, there holds

$$\|u\|^2_{H^s(\mathbb{R}^3)} \leq (1+|\mathcal{K}_c|^{\frac{2s}{3}}S_s^{-1})\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}.$$

It is easy to see the embedding $H^s_{\mu K}(\mathbb{R}^3) \hookrightarrow H^s(\mathbb{R}^3)$ is continuous. Analogously, for any $r \in [2, 2_s^*]$, there is $\mu_0 = c^{-1}S_s|\mathcal{K}_c|^{-\frac{2s}{3}}$ such that for $\mu \geq \mu_0$, we have

$$\int_{\mathbb{R}^3}|u|^rdx \leq \left(\int_{\mathbb{R}^3\setminus\mathcal{K}_c}u^2dx+\int_{\mathcal{K}_c}u^2dx\right)^{\frac{2_s^*-r}{2_s^*-2}}\left(S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}_{\mathcal{D}^{s,2}(\mathbb{R}^3)}\right)^{\frac{r-2}{2_s^*-2}} \leq |\mathcal{K}_c|^{\frac{2_s^*-r}{2_s^*}}S_s^{-r/2}\|u\|^r_{H^s_{\mu K}(\mathbb{R}^3)}. \tag{2.3}$$

We consider the variational functional associated to problem (1.1) defined by

$$I_{\mu,\alpha}(u)=\frac{1}{2}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-\frac{1}{p}\int_{\mathbb{R}^3}g(x)|u|^pdx. \tag{2.4}$$

We can see $I_{\mu,\alpha} \in C^1(H^s_{\mu K}(\mathbb{R}^3),\mathbb{R})$, and for any $u,\ \varphi \in H^s_{\mu K}(\mathbb{R}^3)$,

$$\langle I'_{\mu,\alpha}(u),\varphi\rangle=(1+\alpha\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)})\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx+\int_{\mathbb{R}^3}\mu K(x)u\varphi dx-\int_{\mathbb{R}^3}g(x)|u|^{p-2}u\varphi dx.$$

u is a solution to problem (1.1) if u is a critical point of $I_{\mu,\alpha}$. We introduce the following Nehari manifold

$$\mathcal{N}_{\mu,\alpha}=\left\{u\in H^s_{\mu K}(\mathbb{R}^3)\setminus\{0\}:\langle I'_{\mu,\alpha}(u),u\rangle=0\right\},$$

and the fibering map $\phi_u : t \in \mathbb{R}^+ \to I_{\mu,\alpha}(tu)$,

$$\phi_u(t)=\frac{t^2}{2}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\frac{\alpha t^4}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-\frac{t^p}{p}\int_{\mathbb{R}^3}g(x)|u|^pdx.$$

Thus, we get

$$\phi'_u(t)=t\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\alpha t^3\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-t^{p-1}\int_{\mathbb{R}^3}g(x)|u|^pdx$$

and

$$\phi''_u(t)=\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+3\alpha t^2\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-(p-1)t^{p-2}\int_{\mathbb{R}^3}g(x)|u|^pdx.$$

Therefore $u \in \mathcal{N}_{\mu,\alpha}$ is equivalent to $\phi'_u(1)=0$. Furthermore, for $u \in \mathcal{N}_{\mu,\alpha}$, we obtain

$$\phi''_u(1)=-(p-2)\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\alpha(4-p)\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}, \tag{2.5}$$

or

$$\phi_u''(1) = -2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + (4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx. \tag{2.6}$$

As [19], we devide $\mathcal{N}_{\mu,\alpha}$ into three parts as

$$\mathcal{N}^+_{\mu,\alpha} = \left\{u \in \mathcal{N}_{\mu,\alpha} | \phi_u''(1) > 0\right\},$$
$$\mathcal{N}^-_{\mu,\alpha} = \left\{u \in \mathcal{N}_{\mu,\alpha} | \phi_u''(1) < 0\right\},$$
$$\mathcal{N}^0_{\mu,\alpha} = \left\{u \in \mathcal{N}_{\mu,\alpha} | \phi_u''(1) = 0\right\}.$$

In view of [20, Lemma 3.2], we can derive the following lemma immediately.

Lemma 2.1. *If u is a minimizer of $I_{\mu,\alpha}$ on $\mathcal{N}_{\mu,\alpha}$ such that $u \notin N^0_{\mu,\alpha}$, then $I'_{\mu,\alpha}(u) = 0$ in $H^{-s}_{\mu K}(\mathbb{R}^3)$.*

Lemma 2.2. *Assume that conditions (K_1)–(K_2) hold, then for any $\mu,\ \alpha > 0$, the functional $I_{\mu,\alpha}$ is coercive and bounded below on $\mathcal{N}^-_{\mu,\alpha}$. Moreover, for all $u \in \mathcal{N}^-_{\mu,\alpha}$, there holds*

$$I_{\mu,\alpha}(u) \geq \frac{p-2}{4p}\left(g^{-1}_{\max}|\mathcal{K}_c|^{-\frac{1}{p_s^*}} S_s^{p/2}\right)^{\frac{2}{p-2}}.$$

Proof. For $u \in \mathcal{N}^-_{\mu,\alpha}$, it follows from (2.3) that

$$\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} \leq \|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + \alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} g(x)|u|^p dx \leq g_{\max}|\mathcal{K}_c|^{\frac{1}{p_s^*}} S_s^{-p/2}\|u\|^p_{H^s_{\mu K}(\mathbb{R}^3)}.$$

Thus, we obtain

$$\int_{\mathbb{R}^3} g(x)|u|^p dx \geq \|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} \geq \left(g^{-1}_{\max}|\mathcal{K}_c|^{-\frac{1}{p_s^*}} S_s^{p/2}\right)^{\frac{2}{p-2}}. \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$\begin{aligned}
I_{\mu,\alpha}(u) &= I_{\mu,\alpha}(u) - \frac{1}{4}\left\langle I'_{\mu,\alpha}(u), u\right\rangle \\
&= \frac{1}{4}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{4-p}{4p}\int_{\mathbb{R}^3} g(x)|u|^p dx \\
&\geq \frac{p-2}{4p}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} \\
&\geq \frac{p-2}{4p}\left(g^{-1}_{\max}|\mathcal{K}_c|^{-\frac{1}{p_s^*}} S_s^{p/2}\right)^{\frac{2}{p-2}}.
\end{aligned}$$

Hence, we deduce that $I_{\mu,\alpha}$ is coercive and bounded below on $\mathcal{N}^-_{\mu,\alpha}$. □

Define

$$T = \left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)}. \tag{2.8}$$

For any $u \in \mathcal{N}_{\mu,\alpha}$ with $I_{\mu,\alpha}(u) < \frac{p-2}{4p}T$, we obtain that

$$\frac{p-2}{4p}T > I_{\mu,\alpha}(u) = \frac{p-2}{2p}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{\alpha(4-p)}{4p}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \geq \frac{p-2}{2p}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{\alpha(4-p)}{4p}\|u\|^4_{H^s_{\mu K}(\mathbb{R}^3)}.$$

Consequently, for $\alpha \in (0, \frac{p-2}{(4-p)T})$, there exist $A_1,\ A_2 > 0$ such that

$$\|u\|_{H^s_{\mu K}(\mathbb{R}^3)} < A_1 \quad \text{or } \|u\|_{H^s_{\mu K}(\mathbb{R}^3)} > A_2,$$

with

$$\sqrt{\frac{T}{2}} < A_1 < \sqrt{T} < A_2. \tag{2.9}$$

Moreover, it follows from (2.8) and (2.9) that $A_1 \to \infty$ as $p \to 4^-$, then, we consider the next two sets

$$\mathcal{N}^{(1)}_{\mu,\alpha} := \left\{ u \in \mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) \Big| \|u\|_{H^s_{\mu K}(\mathbb{R}^3)} < A_1 \right\}$$

and

$$\mathcal{N}^{(2)}_{\mu,\alpha} := \left\{ u \in \mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) \Big| \|u\|_{H^s_{\mu K}(\mathbb{R}^3)} > A_2 \right\}.$$

Therefore, we obtain

$$\mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) = \left\{ u \in \mathcal{N}_{\mu,\alpha} | I_{\mu,\alpha}(u) < \frac{p-2}{4p}T \right\} = \mathcal{N}^{(1)}_{\mu,\alpha} \cup \mathcal{N}^{(2)}_{\mu,\alpha}.$$

For all $u \in \mathcal{N}^{(1)}_{\mu,\alpha}$, applying (2.9), we deduce that

$$\|u\|_{H^s_{\mu K}(\mathbb{R}^3)} < A_1 < \sqrt{T}. \tag{2.10}$$

Similarly, for all $u \in \mathcal{N}^{(2)}_{\mu,\alpha}$, we obtain that

$$\|u\|_{H^s_{\mu K}(\mathbb{R}^3)} > A_2 > \sqrt{T}. \tag{2.11}$$

The lemma below shows that $\mathcal{N}^{(1)}_{\mu,\alpha}$ and $\mathcal{N}^{(2)}_{\mu,\alpha}$ are sub-manifolds of $\mathcal{N}^-_{\mu,\alpha}$ and $\mathcal{N}^+_{\mu,\alpha}$, respectively.

Lemma 2.3. *Assume that $\alpha \in (0, \frac{p-2}{(4-p)T})$ and conditions (K_1)–(K_2), (g_1)–(g_2) hold, then $\mathcal{N}^{(1)}_{\mu,\alpha} \subset \mathcal{N}^-_{\mu,\alpha}$ and $\mathcal{N}^{(2)}_{\mu,\alpha} \subset \mathcal{N}^+_{\mu,\alpha}$ are C^1 sub-manifolds. Moreover, every local minimizer of the functional $I_{\mu,\alpha}$ in $\mathcal{N}^{(1)}_{\mu,\alpha}$ and $\mathcal{N}^{(2)}_{\mu,\alpha}$ is a critical point of $I_{\mu,\alpha}$ in $H^s_{\mu K}(\mathbb{R}^3)$.*

Proof. For $u \in \mathcal{N}^{(1)}_{\mu,\alpha}$, it follows from (2.6), (2.3), condition (g_2), and (2.10) that

$$\begin{aligned}
\phi''_u(1) &= -2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + (4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx \\
&\leq -2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + (4-p)g_{\max}|\mathcal{K}_c|^{\frac{1}{p^*_s}} S_s^{-p/2}\|u\|^p_{H^s_{\mu K}(\mathbb{R}^3)} \\
&< -2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}\left(1 - \frac{(4-p)g_{\Omega,\min}}{2S_p^{p/2}(\Omega)}\|u\|^{p-2}_{H^s_{\mu K}(\mathbb{R}^3)}\right) \\
&< 0,
\end{aligned}$$

which implies that $u \in \mathcal{N}^-_{\mu,\alpha}$.

For $u \in \mathcal{N}^{(2)}_{\mu,\alpha}$, it follows from (2.11) that

$$\frac{1}{4}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{4-p}{4p}\int_{\mathbb{R}^3} g(x)|u|^p dx = I_{\mu,\alpha}(u) < \frac{p-2}{4p}T < \frac{p-2}{4p}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)},$$

which yields that

$$2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} < (4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx.$$

Combining with (2.6), we obtain

$$\phi''_u(1) = -2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + (4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx > 0.$$

□

For $u \in H^s_{\mu K}(\mathbb{R}^3) \setminus \{0\}$, let

$$M_f(u) = \left(\frac{\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}}{\int_{\mathbb{R}^3} g(x)|u|^p dx}\right)^{\frac{1}{p-2}}, \quad \hat{t}(u) = \left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} M_f(u).$$

Lemma 2.4. *Assume that condition (K_1) holds, then for any $\alpha > 0$ and $u \in H^s_{\mu K}(\mathbb{R}^3) \setminus \{0\}$ satisfying*

$$\int_{\mathbb{R}^3} g(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\alpha(4-p)}{p-2}\right)^{(p-2)/2} \|u\|^p_{H^s_{\mu K}(\mathbb{R}^3)}, \tag{2.12}$$

there exists $t_1 > \hat{t}(u)$ such that

$$\inf_{t\geq 0} I_{\mu,\alpha}(tu) = \inf_{\hat{t}(u)<t<t_1} I_{\mu,\alpha}(tu) < 0.$$

Proof. Fix $u \in H^s_{\mu K}(\mathbb{R}^3) \setminus \{0\}$ satisfying (2.12) and define $\psi_u(t) : \mathbb{R}^+ \to \mathbb{R}$ as

$$\psi_u(t) = \frac{t^{-2}}{2}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{t^{p-4}}{p}\int_{\mathbb{R}^3} g(x)|u|^p dx. \tag{2.13}$$

We remark that $I_{\mu,\alpha}(tu) = 0$ if, and only if, $\psi_u(t) + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = 0$, then, it is easy to show that there exists a unique $t_0 > 0$ such that

$$\psi_u(t_0) = 0, \ \lim_{t\to 0^+}\psi_u(t) = \infty \text{ and } \lim_{t\to\infty}\psi_u(t) = 0, \tag{2.14}$$

where $t_0 = (\frac{p}{2})^{\frac{1}{p-2}} M_f(u)$. From (2.13), we can see $\psi_u(t)$ is decreasing when $t \in (0, \hat{t}(u))$ and is increasing when $t \in (\hat{t}(u), \infty)$, and

$$\inf_{t>0}\psi_u(t) = \psi_u(\hat{t}(u)) = -\frac{p-2}{2(4-p)}\left(\frac{p\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}}{(4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx}\right)^{-\frac{2}{p-2}} \|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}.$$

According to (2.12), we can deduce that

$$\inf_{t>0}\psi_u(t) = \psi_u(\hat{t}(u)) < -\alpha\|u\|^4_{H^s_{\mu K}(\mathbb{R}^3)} < -\alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}. \tag{2.15}$$

Together with (2.14), we obtain that there exists $0 < t_2 < \hat{t}(u) < t_1$ such that

$$\psi_u(t_1) + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = 0 \text{ and } \psi_u(t_2) + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = 0.$$

Thus, it follows from (2.15) that

$$I_{\mu,\alpha}(\hat{t}(u)u) = (\hat{t}(u))^4\left[\psi_u(\hat{t}(u)) + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}\right] < 0,$$

which implies that $\inf_{t\geq 0} I_{\mu,\alpha}(tu) < 0$. We recall that

$$\phi_u^{'}(t) = 4t^3\left[\psi_u(t) + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}\right] + t^4\psi_u'(t),$$

which implies that $\phi_u^{'}(t) < 0$ for any $t \in (t_2, \hat{t}(u))$ and $\phi_u^{'}(t_1) > 0$. □

Define

$$\tilde{t}(u) = \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} M_f(u).$$

We note that $\tilde{t}(u) < \hat{t}(u)$ for $2 < p < 4$. The following lemma indicates that $\mathcal{N}^-_{\mu,\alpha}$ and $\mathcal{N}^+_{\mu,\alpha}$ are nonempty.

Lemma 2.5. *Assume that condition (K_1) holds, then for any $\alpha > 0$ and $u \in H^s_{\mu K}(\mathbb{R}^3)\backslash\{0\}$ satisfying (2.12), there exists $M_f(u) < t^- = t^-(u) < \tilde{t}(u) < t^+ = t^+(u)$ such that $t^-u \in \mathcal{N}^-_{\mu,\alpha}$, $t^+u \in \mathcal{N}^+_{\mu,\alpha}$ and*

$$I_{\mu,\alpha}(t^-u) = \sup_{0\leq t\leq t^+} I_{\mu,\alpha}(tu), \quad I_{\mu,\alpha}(t^+u) = \inf_{t\geq t^-} I_{\mu,\alpha}(tu) = \inf_{t\geq 0} I_{\mu,\alpha}(tu) < 0.$$

Proof. Fix $u \in H^s_{\mu K}(\mathbb{R}^3) \setminus \{0\}$ satisfying (2.12) and define $h_u(t) : \mathbb{R}^+ \to \mathbb{R}$ as

$$h_u(t) = t^{-2}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - t^{p-4}\int_{\mathbb{R}^3} g(x)|u|^p dx.$$

As a consequence, $tu \in \mathcal{N}_{\mu,\alpha}$ if, and only if, $h_u(t) + \alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = 0$. Similar to (2.14) and (2.15), we can find that

$$h_u(M_f(u)) = 0, \ \lim_{t\to 0^+} h_u(t) = \infty \ \lim_{t\to\infty} h_u(t) = 0,$$

$h_u(t)$ is decreasing when $t \in (0, \tilde{t}(u))$ and is increasing when $t \in (\tilde{t}(u), \infty)$, and

$$\begin{aligned}\inf_{t>0} h_u(t) = h_u(\tilde{t}(u)) &= -\frac{p-2}{4-p}\left(\frac{2\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}}{(4-p)\int_{\mathbb{R}^3} g(x)|u|^p dx}\right)^{-\frac{2}{p-2}} \|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} \\ &< -2\alpha\left(\frac{p}{2}\right)^{\frac{2}{p-2}} \|u\|^4_{H^s_{\mu K}(\mathbb{R}^3)} < -\alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)},\end{aligned}$$

noting that $\left(\frac{p}{2}\right)^{\frac{2}{p-2}} > 1$. Therefore, there exists $M_f(u) < t^- = t^-(u) < \tilde{t}(u) < t^+ = t^+(u)$ such that

$$h_u(t^+) = -\alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = h_u(t^-), \quad h_u^{'}(t^+) > 0, \quad h_u^{'}(t^-) < 0,$$

which yields that $t^+u,\ t^-u \in \mathcal{N}_{\mu,\alpha}$. Due to $\phi''_{tu}(1) = t^5 h'_u(t)$, we conclude $t^+u \in \mathcal{N}^+_{\mu,\alpha}$ and $t^-u \in \mathcal{N}^-_{\mu,\alpha}$. From $\phi'_u(t) = t^3\left(h_u(t) + \alpha\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}\right)$, it follows that $\phi'_u(t) > 0$ for all $t \in (0,t^-) \cup (t^+,\infty)$ and $\phi'_u(t) < 0$ for all $t \in (t^-,t^+)$. This implies that

$$I_{\mu,\alpha}(t^-u) = \sup_{0\le t\le t^+} I_{\mu,\alpha}(tu), \quad I_{\mu,\alpha}(t^+u) = \inf_{t\ge t^-} I_{\mu,\alpha}(tu).$$

Obviously, $I_{\mu,\alpha}(t^+u) < I_{\mu,\alpha}(t^-u)$. Applying Lemma 2.4, we obtain that

$$I_{\mu,\alpha}(t^+u) = \inf_{t\ge 0} I_{\mu,\alpha}(tu),$$

which completes the proof. □

Notice the following fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u = |u|^{p-2}u, & \text{in } \Omega, \\ u \in H^s_0(\Omega), \end{cases} \tag{2.16}$$

where $2 < p < 4$ and Ω is given in condition (K_3), admits a ground state solution ω satisfying

$$\int_\Omega |(-\Delta)^{\frac{s}{2}}\omega|^2 dx = \int_\Omega |\omega|^p dx = S_p^{\frac{p}{p-2}}(\Omega). \tag{2.17}$$

Moreover, we have

$$\inf_{u\in\mathcal{N}_\Omega} J(u) = J(\omega) = \frac{p-2}{2p} S_p^{\frac{p}{p-2}}(\Omega),$$

where J is the variational functional associated to problem (2.16) defined by

$$J(u) = \frac{1}{2}\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} - \frac{1}{p}\int_\Omega |u|^p dx,$$

and

$$\mathcal{N}_\Omega = \left\{u \in H^s_0(\Omega)\backslash\{0\} | \langle J'(u), u\rangle = 0\right\}.$$

Note that for $\omega \in H^s_0(\Omega)$, $\|\omega\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \|\omega\|_{H^s_{\mu K}(\mathbb{R}^3)}$. Define

$$\alpha_* := \frac{p-2}{2(4-p)}\left(\frac{(4-p)g_{\Omega,\min}}{pS_p^{p/2}(\Omega)}\right)^{\frac{2}{p-2}}. \tag{2.18}$$

For any $\alpha \in (0,\alpha_*)$, by (2.17) and (2.18), we obtain

$$\int_{\mathbb{R}^3} g(x)|\omega|^p dx \ge g_{\Omega,\min}\int_\Omega |\omega|^p dx = g_{\Omega,\min} S_p^{-p/2}(\Omega)\|\omega\|^p_{H^s_{\mu K}(\mathbb{R}^3)} > \frac{p}{4-p}\left(\frac{2\alpha(4-p)}{p-2}\right)^{\frac{p-2}{2}} \|\omega\|^p_{H^s_{\mu K}(\mathbb{R}^3)}.$$

From Lemma 2.5, there exists $M_f(\omega) < t^- < \tilde{t}(\omega) < t^+$ such that $t^-\omega \in \mathcal{N}^-_{\mu,\alpha},\ t^+\omega \in \mathcal{N}^+_{\mu,\alpha}$, and

$$I_{\mu,\alpha}(t^-\omega) = \sup_{0\le t\le t^+} I_{\mu,\alpha}(tu), \quad I_{\mu,\alpha}(t^+\omega) = \inf_{t\ge t^-} I_{\mu,\alpha}(tu) = \inf_{t\ge 0} I_{\mu,\alpha}(tu) < 0, \tag{2.19}$$

which implies that $t^+\omega \in \mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right)$. Together with Lemma 2.3, we have $t^+\omega \in \mathcal{N}_{\mu,\alpha}^{(2)}$. Thus, $\mathcal{N}_{\mu,\alpha}^{(2)}$ is nonempty. By (2.17), we obtain that

$$\begin{aligned} I_{\mu,\alpha}(t^-\omega) &= \frac{(t^-)^2}{4}\|\omega\|^2_{H^s_{\mu K}(\mathbb{R}^3)} - \frac{(4-p)(t^-)^p}{4p}\int_{\mathbb{R}^3} g(x)|\omega|^p dx \\ &\le \frac{1}{4}\left[(t^-)^2 - \frac{(4-p)g_{\Omega,\min}}{p}(t^-)^p\right]\|\omega\|^2_{H^s_{\mu K}(\mathbb{R}^3)} \\ &< \frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)}. \end{aligned} \tag{2.20}$$

By using a similar strategy of $t^+\omega$, we observe that $t^-\omega \in \mathcal{N}_{\mu,\alpha}^{(1)}$. Therefore, $\mathcal{N}_{\mu,\alpha}^{(1)}$ is nonempty.

3. Proof of Theorems 1.1 and 1.2

In this section, we deal with the existence and multiplicity of solutions to problem (1.1). To begin, we present the next compactness result.

Lemma 3.1. *Assume that conditions (K_1)–(K_2) and (g_1) hold, then there exists $\mu_* > \mu_0$ such that $I_{\mu,\alpha}$ satisfies Palais-Smale condition in $\mathcal{N}_{\mu,\alpha}^{(1)}$ at the level $c < \frac{p-2}{4p}T$ for any $\mu \in (\mu_*, \infty)$ and $\alpha \in (0, \alpha_*)$.*

Proof. Let $\{u_n\} \subset \mathcal{N}_{\mu,\alpha}^{(1)}$ be a $(PS)_c$ sequence satisfying

$$I_{\mu,\alpha}(u_n) \to c \quad \text{and} \quad \|I'_{\mu,\alpha}(u_n)\|_{H^{-s}_{\mu K}(\mathbb{R}^3)} \to 0 \text{ as } n \to \infty. \tag{3.1}$$

It follows from the definition of $\mathcal{N}_{\mu,\alpha}^{(1)}$ that $\{u_n\}$ is bounded in $H^s_{\mu K}(\mathbb{R}^3)$. Hence, we may assume that, up to a subsequence, there exists $u \in H^s_{\mu K}(\mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\to u, && \text{a.e. in } \mathbb{R}^3, \\ u_n &\rightharpoonup u, && \text{weakly in } H^s_{\mu K}(\mathbb{R}^3), \\ u_n &\to u, && \text{strongly in } L^r_{loc}(\mathbb{R}^3),\ 1 \le r < 2^*_s. \end{aligned} \tag{3.2}$$

Meanwhile, there exists $D \in \mathbb{R}$ such that $\|u_n\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \to D^2$ and $\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le D^2$. Let us define $v_n = u_n - u$, then, from (K_2) and (3.2), we see that

$$\|v_n\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3\setminus\mathcal{K}_c} v_n^2 dx + \int_{\mathcal{K}_c} v_n^2 dx \le \frac{1}{\mu c}\|v_n\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + o_n(1).$$

Together with the Hölder inequality and (2.1), we infer that

$$\|v_n\|_{L^p(\mathbb{R}^3)} \le \|v_n\|^{\sigma}_{L^2(\mathbb{R}^3)}\|v_n\|^{1-\sigma}_{L^{2^*_s}(\mathbb{R}^3)} \le S_s^{-\frac{1-\sigma}{2}}\|v_n\|^{\sigma}_{L^2(\mathbb{R}^3)}\|v_n\|^{1-\sigma}_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le S_s^{-\frac{1-\sigma}{2}}(\mu c)^{-\sigma/2}\|v_n\|_{H^s_{\mu K}(\mathbb{R}^3)}, \tag{3.3}$$

where $\sigma = \frac{2}{p}\left(\frac{2^*_s - p}{2^*_s - 2}\right) < 1$. It follows from Brezis-Lieb Lemma [21] that

$$\int_{\mathbb{R}^3} g(x)|v_n|^p dx = \int_{\mathbb{R}^3} g(x)|u_n|^p dx - \int_{\mathbb{R}^3} g(x)|u|^p dx + o_n(1). \tag{3.4}$$

Applying (3.1) and (3.2), we obtain that for any $\phi \in C_0^\infty(\mathbb{R}^3)$,

$$\left(1+\alpha D^2\right)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi dx+\int_{\mathbb{R}^3}\mu K(x)u\phi dx-\int_{\mathbb{R}^3}g(x)|u|^{p-2}u\phi dx=0,$$

which implies that

$$\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\alpha D^2\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-\int_{\mathbb{R}^3}g(x)|u|^p dx=0. \tag{3.5}$$

From (2.3), (2.10), and (3.3)–(3.5), we conclude that

$$\begin{aligned}
o_n(1) &= \left\langle I'_{\mu,\alpha}(u_n),u_n\right\rangle_{H^s_{\mu K}(\mathbb{R}^3)}\\
&= \|u_n\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\alpha\|u_n\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-\int_{\mathbb{R}^3}g(x)|u_n|^p dx-\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)}-\alpha D^2\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}+\int_{\mathbb{R}^3}g(x)|u|^p dx\\
&= \|v_n\|^2_{H^s_{\mu K}(\mathbb{R}^3)}+\alpha D^2\|v_n\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}-\int_{\mathbb{R}^3}g(x)|v_n|^p dx\\
&\geq \|v_n\|^2_{H^s_{\mu K}(\mathbb{R}^3)}-g_{\max}\left(\int_{\mathbb{R}^3}|v_n|^p dx\right)^{(p-2)/p}\left(\int_{\mathbb{R}^3}|v_n|^p dx\right)^{2/p}\\
&\geq \left[1-g_{\max}\left(|\mathcal{K}_c|^{\frac{1}{p_s^*}}S_s^{-p/2}(2A_1)^p\right)^{(p-2)/p}S_s^{\sigma-1}(\mu c)^{-\sigma}\right]\|v_n\|^2_{H^s_{\mu K}(\mathbb{R}^3)},
\end{aligned}$$

which implies that there exists $\mu_* > \mu_0$ such that $u_n \to u$ in $H^s_{\mu K}(\mathbb{R}^3)$ for all $\mu > \mu_*$. □

Now, we will give the estimation of the infimum of $I_{\mu,\alpha}$ on the $\mathcal{N}^{(1)}_{\mu,\alpha}$ and $\mathcal{N}^{(2)}_{\mu,\alpha}$. We define

$$c^-_{\mu,\alpha}=\inf_{u\in\mathcal{N}^{(1)}_{\mu,\alpha}}I_{\mu,\alpha}(u)=\inf_{u\in\mathcal{N}^-_{\mu,\alpha}}I_{\mu,\alpha}(u).$$

It follows from Lemma 2.2 and (2.20) that

$$0<c^-_{\mu,\alpha}<\frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)}. \tag{3.6}$$

Proof of Theorem 1.1. By Ekeland's variational principle [22], there exists a $(PS)_{c^-_\mu}$ sequence $\{u_n\} \subset \mathcal{N}^{(1)}_{\mu,\alpha}$ satisfying (3.1). From Lemma 3.1 and (3.6), we get that for any $\mu \in (\mu_*,\infty)$ and $\alpha \in (0,\alpha_*)$, there exists $u^-_{\mu,\alpha} \in \mathcal{N}^-_{\mu,\alpha}$ such that

$$I'_{\mu,\alpha}(u^-_{\mu,\alpha})=0,\quad I_{\mu,\alpha}(u^-_{\mu,\alpha})=c^-_{\mu,\alpha}.$$

Furthermore, we note that

$$I_{\mu,\alpha}(u^-_{\mu,\alpha})=c^-_{\mu,\alpha}\leq I_{\mu,\alpha}(t^-\omega)<\frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)},$$

which implies that $u^-_{\mu,\alpha} \in \mathcal{N}^{(1)}_{\mu,\alpha}$. By Lemma 2.1, as in [23, Proposition 3.5], we can obtain that $u^-_{\mu,\alpha}$ is a positive solution to (1.1). □

Define

$$c_{\mu,\alpha}^{+} = \inf_{u\in\mathcal{N}_{\mu,\alpha}^{(2)}} I_{\mu,\alpha}(u) = \inf_{u\in\mathcal{N}_{\mu,\alpha}^{+}} I_{\mu,\alpha}(u)$$

and let

$$\mu_1 := \frac{4-p}{p-2}\left(\frac{\bar{C}_0^{2/\theta}}{\alpha^{p-2}S_s^{2_s^*(\theta-1)/\theta}}\right)^{1/(4-p)},$$

where $\bar{C}_0 > 0$. We observe the following estimate.

Lemma 3.2. *Assume that conditions* (K_1)*–*(K_2)*,* (g_1)*, and* (g_3) *hold, then*
(i) There exists $\mu_1 > 0$ *such that for any* $\mu \in (\mu_1, \infty)$ *and* $\alpha \in (0, \infty)$*,* $\mathcal{N}_{\mu,\alpha}^{+}$ *is a bounded set.*
(ii) There exist $\beta,\ \mu_2 > 0$ *such that for any* $\mu \in (\mu_2, \infty)$ *and* $\beta \in (0, \infty)$*,* $-\beta < c_{\mu,\alpha}^{+} < 0$*.*

Proof. (i) For $u \in \mathcal{N}_{\mu,\alpha}^{+}$, it follows from the Hölder inequality, (2.1), condition (g_3), Caffarelli-Kohn-Nirenberg type inequality of fractional order [24], and (2.5) that

$$\begin{aligned}
1 &= \frac{\int_{\mathbb{R}^3} g(x)|u|^p dx}{\|u\|_{H_{\mu K}^s(\mathbb{R}^3)}^2 + \alpha\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^4} < \frac{\int_{\mathbb{R}^3} g(x)|u|^p dx}{\alpha\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^4} \\
&\le \frac{\left(\int_{|x|>R_0} |g(x)|^\theta u^2 dx + g_{\max}^\theta |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^2\right)^{\frac{1}{\theta}} \left(S_s^{-\frac{2_s^*}{2}}\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{2_s^*}\right)^{1-\frac{1}{\theta}}}{\alpha\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^4} \qquad (3.7)\\
&= \frac{1}{\alpha S_s^{\frac{2_s^*(\theta-1)}{2\theta}}}\left(\frac{\int_{|x|>R_0} |g(x)|^\theta u^2 dx + g_{\max}^\theta |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^2}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{(4-2_s^*)\theta+2_s^*}}\right)^{1/\theta}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\alpha S_s^{\frac{2_s^*(\theta-1)}{2\theta}}}\left[\frac{C_0^\theta \int_{|x|>R_0}\left(K(x)u^2\right)^{\frac{(4-p)\theta}{2}}\left(\frac{|u|^2}{|x|^{2s}}\right)^{\frac{2-(4-p)\theta}{2}} dx}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{(4-2_s^*)\theta+2_s^*}} + \frac{g_{\max}^\theta |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{\theta(4-p)}}\right]^{1/\theta} \\
&\le \frac{1}{\alpha S_s^{\frac{2_s^*(\theta-1)}{2\theta}}}\left[\frac{C_0^\theta \left(\int_{|x|>R_0} K(x)u^2 dx\right)^{\frac{(4-p)\theta}{2}}\left(\int_{|x|>R_0}\frac{u^2}{x^{2s}}dx\right)^{\frac{2-(4-p)\theta}{2}}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{(4-2_s^*)\theta+2_s^*}} + \frac{g_{\max}^\theta |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{\theta(4-p)}}\right]^{1/\theta} \\
&\le \frac{1}{\alpha S_s^{\frac{2_s^*(\theta-1)}{2\theta}}}\left[\frac{\bar{C}_0 \left(\int_{|x|>R_0} K(x)u^2 dx\right)^{\frac{(4-p)\theta}{2}}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{2\theta(4-p)}} + \frac{g_{\max}^\theta |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{\theta(4-p)}}\right]^{1/\theta} \\
&\le \frac{1}{\alpha S_s^{\frac{2_s^*(\theta-1)}{2\theta}}}\left[\bar{C}_0\left(\frac{\alpha(4-p)}{\mu(p-2)}\right)^{\frac{(4-p)\theta}{2}} + \frac{g_{\max}^\theta |B_{R_0}|^{\frac{2s}{3}} S_s^{-1}}{\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^{\theta(4-p)}}\right]^{1/\theta},
\end{aligned}$$

where $\bar{C}_0$ is the constant of C_0^θ and the constant of Caffarelli-Kohn-Nirenberg type inequality of fractional order. This implies that for any $\mu > \mu_1$, there exists a positive constant δ_1 dependent on α such that $\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^2 \le \delta_1$, which yields

$$\|u\|_{H_{\mu K}^s(\mathbb{R}^3)}^2 \le \frac{\alpha(4-p)}{p-2}\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)}^4 \le \frac{\alpha(4-p)}{p-2}\delta_1^2.$$

(ii) From (2.19), we get that $c^+_{\mu,\alpha} < 0$. For $u \in \mathcal{N}^+_{\mu,\alpha}$, arguing similarly as (3.7), we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3} g(x)|u|^p dx \\
\leq& S_s^{-\frac{2_s^*(\theta-1)}{2\theta}} \|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \left[\bar{C}_0 \left(\frac{\alpha(4-p)}{\mu(p-2)} \right)^{\frac{(4-p)\theta}{2}} + \frac{g^\theta_{\max} |B_{R_0}(0)|^{\frac{2s}{3}} S_s^{-1}}{\|u\|^{\theta(4-p)}_{\mathcal{D}^{s,2}(\mathbb{R}^3)}} \right]^{1/\theta} \\
\leq& S_s^{-\frac{2_s^*(\theta-1)}{2\theta}} \|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \left[\bar{C}_0^{1/\theta} \left(\frac{\alpha(4-p)}{\mu(p-2)} \right)^{\frac{4-p}{2}} + \frac{g_{\max} |B_{R_0}(0)|^{\frac{2s}{3\theta}} S_s^{-1/\theta}}{\|u\|^{4-p}_{\mathcal{D}^{s,2}(\mathbb{R}^3)}} \right] \\
=& \bar{C}_0^{1/\theta} \left(\frac{\alpha(4-p)}{\mu(p-2)} \right)^{\frac{4-p}{2}} S_s^{-\frac{2_s^*(\theta-1)}{2\theta}} \|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} + g_{\max} |B_{R_0}(0)|^{\frac{2s}{3\theta}} S_s^{-p/2} \|u\|^p_{\mathcal{D}^{s,2}(\mathbb{R}^3)}.
\end{aligned}$$

Thus, we derive that

$$\begin{aligned}
I_{\mu,\alpha}(u) &= \frac{1}{2}\|u\|^2_{H^s_{\mu K}(\mathbb{R}^3)} + \frac{\alpha}{4}\|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} - \frac{1}{p}\int_{\mathbb{R}^3} g(x)|u|^p dx \\
&\geq \left[\frac{\alpha}{4} - \frac{\bar{C}_0^{1/\theta}}{p} \left(\frac{\alpha(4-p)}{\mu(p-2)} \right)^{\frac{4-p}{2}} S_s^{-\frac{2_s^*(\theta-1)}{2\theta}} \right] \|u\|^4_{\mathcal{D}^{s,2}(\mathbb{R}^3)} - p^{-1} g_{\max} |B_{R_0}(0)|^{\frac{2s}{3\theta}} S_s^{-p/2} \|u\|^p_{\mathcal{D}^{s,2}(\mathbb{R}^3)}.
\end{aligned}$$

Therefore, taking

$$\mu > \mu_2 := \left(\frac{4}{p}\right)^{\frac{2}{4-p}} \mu_1,$$

we can conclude that there exists $\beta > 0$ such that $I_{\mu,\alpha}(u) \geq -\beta$, noting $2 < p < 4$. □

Arguing as in Lemma 3.1, we can deduce the next compactness lemma for the functional $I_{\mu,\alpha}$ in $\mathcal{N}^{(2)}_{\mu,\alpha}$.

Lemma 3.3. *Assume that conditions* (K_1)*–*(K_2)*,* (g_1) *and* (g_3) *hold, then there exists* $\mu_3 > \max\{\mu_0,\ \mu_1\}$ *such that* $I_{\mu,\alpha}$ *satisfies* (PS) *condition in* $\mathcal{N}^{(2)}_{\mu,\alpha}$ *at the level* $c < \frac{p-2}{4p}T$ *for any* $\mu \in (\mu_3, \infty)$ *and* $\alpha \in (0, \alpha_*)$.

Proof of Theorem 1.2. Define $\mu^* := \max\{\mu_*,\ \mu_1,\ \mu_2,\ \mu_3\}$. It follows from the Ekeland's variational principle [22] and Lemma 3.2 that for any $\mu \in (\mu^*, \infty)$ and $\alpha \in (0, \alpha_*)$, there exists a bounded $(PS)_{c^+_\mu}$ sequence $\{u_n\} \subset \mathcal{N}^{(2)}_{\mu,\alpha}$ satisfying (3.1). From Lemma 3.3, we obtain that for any $\mu > \mu^*$ and $\alpha \in (0, \alpha_*)$, there exists $u^+_{\mu,\alpha} \in \mathcal{N}^+_{\mu,\alpha}$ such that

$$I'_{\mu,\alpha}(u^+_{\mu,\alpha}) = 0, \quad I_{\mu,\alpha}(u^+_{\mu,\alpha}) = c^+_{\mu,\alpha}.$$

Furthermore, we note that

$$I_{\mu,\alpha}(u^+_{\mu,\alpha}) = c^+_{\mu,\alpha} \leq I_{\mu,\alpha}(t^+\omega) < 0,$$

which implies that $u^+_{\mu,\alpha} \in \mathcal{N}^{(2)}_{\mu,\alpha}$. As in [23, Proposition 3.5], we can obtain that $u^+_{\mu,\alpha}$ is a positive solution to (1.1). Together with Theorem 1.1, we obtain that there are at least two positive solutions $u^\pm_{\mu,\alpha} \in H^s_{\mu K}(\mathbb{R}^3)$ to (1.1) satisfying

$$0 < \|u^-_{\mu,\alpha}\|_{H^s_{\mu K}(\mathbb{R}^3)} < \left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)} \right)^{1/(p-2)} < \|u^+_{\mu,\alpha}\|_{H^s_{\mu K}(\mathbb{R}^3)}$$

and

$$I_{\mu,\alpha}(u_{\mu,\alpha}^{+}) < 0 < I_{\mu,\alpha}(u_{\mu,\alpha}^{-}) < \frac{p-2}{4p}\left(\frac{2S_p^{p/2}(\Omega)}{g_{\Omega,\min}(4-p)}\right)^{2/(p-2)},$$

and $u_{\mu,\alpha}^{+}$ is a ground state solution. □

4. Conclusions

The main purpose of this paper is to study the existence of positive solutions for the fractional Kirchhoff type problem (1.1) with steep potential well and sign-changing weight functions. We find that the standard Nehari manifold method does not work, because the energy functional is not bounded below on the Nehari manifold. Moreover, the Nehari-Pohozaev manifold and monotonicity trick are also not applicable to solve this problem, due to the potential $K \in C^1$ is necessary.

To prove Theorem 1.1, as [8] we introduce the following sub-level set of Nehari manifold

$$\mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) = \left\{u \in \mathcal{N}_{\mu,\alpha} | I_{\mu,\alpha}(u) < \frac{p-2}{4p}T\right\} = \mathcal{N}_{\mu,\alpha}^{(1)} \cup \mathcal{N}_{\mu,\alpha}^{(2)},$$

where

$$\mathcal{N}_{\mu,\alpha}^{(1)} := \left\{u \in \mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) | \|u\|_{H_{\mu K}^{s}(\mathbb{R}^3)} < A_1\right\}$$

and

$$\mathcal{N}_{\mu,\alpha}^{(2)} := \left\{u \in \mathcal{N}_{\mu,\alpha}\left(\frac{p-2}{4p}T\right) | \|u\|_{H_{\mu K}^{s}(\mathbb{R}^3)} > A_2\right\}.$$

By some detailed estimates and analysis, we obtain that the energy functional $I_{\mu,\alpha}$ satisfies (PS) condition in $\mathcal{N}_{\mu,\alpha}^{(1)}$ at the level $c < \frac{p-2}{4p}T$. Since $\mathcal{N}_{\mu,\alpha}^{(1)}$ is bounded, combining Ekeland's variational principle [22] and the compactness result, we obtain one critical point. Furthermore, due to the assumption (g_3), we can show that $I_{\mu,\alpha}$ is bounded below on $\mathcal{N}_{\mu,\alpha}^{(2)}$ and $\mathcal{N}_{\mu,\alpha}^{(2)}$ is a bounded set. Then, similar to the above, we deserve the second critical point.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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