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Research article

Results on generalized neutral fractional impulsive dynamic equation over time scales using nonlocal initial condition

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Abstract: This paper explored the existence and uniqueness of a neutral fractional impulsive dynamic equation over time scales that included nonlocal initial conditions and employed the Caputo-nabla derivative (C∇D). The establishment of existence and uniqueness relies on the fine fixed point theorem. Furthermore, a comparison was conducted between the fractional order C∇D and the Riemann-Liouville nabla derivative (RL∇D) over time scales. Theoretical findings were substantiated through a numerical methodology, and an illustrative graph using MATLAB was presented for the provided example.

Keywords: neutral differential equations; C∇D; RL∇D; time scales
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1. Introduction

Similar to how fractional exponents evolve from integer exponents, classical calculus concepts like integral and derivative operators serve as the foundation for fractional calculus [1,2]. Many people are aware that depending on the geometrical and physical factors, integer-order derivatives and integrals have different meanings. This assumption, however, is disproved when dealing with fractional-order integration and differentiation, which covers a constantly expanding domain in both theory and practical applications to real-life challenges [3]. The study of fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory, probability, and research on viscoelastic materials are just a few of the scientific and engineering domains where it has lately been employed [4–6]. Fractional differential equations (FDEs) have drawn attention from several researches as a result of its frequent occurrence in disciplines such as physics, chemistry, and engineering. The commonly used Laplace transform approach, the iterative method, the Fourier transform technique, and the operational method are just a few of the strategies developed to deal with FDEs [7–9]. The majority of these techniques, however, are only relevant to a few types of FDEs, particularly those that are linear and have constant coefficients. The existence of solutions for fractional semilinear differential or integrodifferential equations is one of the theoretical fields being investigated by many authors. There has been a significant development in nonlocal problems for FDEs or inclusions [10, 11]. Reimann-Liouvile fractional derivative-based linear FDEs with variable coefficients have been solved using the decomposition approach. FDEs have recently received a lot of attention from academics. This is because FDEs are frequently used in engineering and science, including in the study of diffusion in porous media, nonlinear earth oscillation, fractional biological neurons, traffic flow, polymer rheology, modeling of neural networks, and viscoelastic panels in supersonic gas flow [12–16].

In the real world, there may be situations that cannot be fully captured by either wholly continuous or entirely discrete phenomena. In these cases, we require a shared domain to adequately support both conditions. In order to unify continuous and discrete calculus, Stefan Hilger created a usual state known as time scale T [17–19]. This domain is based on the unification of these requirements [20–23]. To solve this type of model, which combines differential and difference equations, dynamic equations on a time scale were developed [24, 25]. Many scholars worked on dynamic equations that involve local beginning and boundary conditions and might be either linear or nonlinear. Because fractional calculus is accurate and has an advantage in the physical interpretation, several authors have studied the dynamic equation using this method [26–28].

We have seen a number of equations in the real world where the systems are permitted to experience a brief disturbance, the length of which may be insignificant compared to the overall process duration. In this situation, jump discontinuities may appear in the solution of these equations at time $\varsigma_1 < \varsigma_2 < \varsigma_3 < \cdots$, given in the form

$$p(\varsigma_k^+) - p(\varsigma_k^-) = \mathcal{I}_k(\varsigma_k, p(\varsigma_k^-)).$$

Impulsive dynamic equations are dynamic equations with jump discontinuities as solutions [29–31]. Dynamic impulsive equations on time scales have caught the attention of many academics recently. However, there are very few publications that investigate impulsive dynamic equations using fractional calculus on time scales with nonlocal initial conditions [32, 33].

Neutral differential equations [34, 35] appear when $\max\{n_1, n_2, \cdots, n_k\} = n$. The past and present values of the function are determined by Neutral differential equations, which differs from retarded differential equations in that they depend on derivatives with delays. Elastic networks are simulated by neutral type differential equations in high speed computers for the express purpose of joining switching circuits [36]. Due to their extensive use in practical mathematics, neutral differential equations have recently attracted a lot of attention [37, 38]. Many scientists have sought to create neutral differential systems, taking note of varied fixed point strategies, mild solutions, and nonlocal situations. Also, in recent years, neural networks have been extensively studied and have been applied in many fields, such as combinatorial optimization, multiagent systems, fault diagnosis, and industrial automation. However, the practical applications of neural networks have been limited due to some inherent dynamic properties, such as the information latching phenomenon [39–41].

While the authors of [24, 25] used the tools of the delta (Hilger) derivative to investigate the fractional dynamic equation with local initial condition and instantaneous and non-instantaneous impulses, authors of [42] investigated the nonlocal initial condition's impulsive dynamic equation. In [43], the authors have discussed the impulsive fractional dynamic equation on time scales with a nonlocal initial condition. The exploration and elucidation of results pertaining to a generalized neutral fractional impulsive dynamic equation over time scales, specifically incorporating nonlocal initial conditions, is the main contribution of the this study.

As a result of the work described above, we assert that it's important to investigate the impulsive neutral fractional dynamic equation with nonlocal initial condition of the type:

$$\begin{cases} {}^{C}D^{\varpi}[u(\varsigma) - g(\varsigma, u)] = \mathfrak{L}(\varsigma, u(\varsigma), {}^{C}D^{\varpi}u(\varsigma)), & \varsigma \in \mathscr{I}, \ \ \varsigma \neq \varsigma_k \\ u(\varsigma_k^+) - u(\varsigma_k^-) = \mathcal{I}_k(\varsigma_k, u(\varsigma_k^-)), & k = 1, 2, \cdots, m \\ u(0) = \varphi(u), \end{cases} \tag{1.1}$$

where $k \in \mathbb{N} \cup \{0\}$ and

$$\mathscr{I} = [0, \mathscr{T}] \cap \mathfrak{T}$$

for $\mathscr{T} \in \mathfrak{T}$. Let the left dense(ld) continuous function be

$$\mathfrak{L} : \mathscr{I} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R}$$

and ${}^{C}D^{\varpi}$ is Caputo nabla derivative (C∇D). We assume that

$$0 < \varsigma_o < \varsigma_1 < \varsigma_2 < \varsigma_3 < \cdots < \varsigma_n < \varsigma_{n+1} = \mathscr{T},$$

which indicates the impulse at specific time, and the terms

$$u(\varsigma_k^+) = lim_{d \to 0^+} u(\varsigma_k + d)$$

and

$$u(\varsigma_k^-) = lim_{d \to 0^+} u(\varsigma_k - d)$$

represent the function's right and left limits u at $\varsigma = \varsigma_k$ in relation to time scales. $\mathcal{I}_k$ are continuous real valued functions on $\mathscr{R}$ $\forall$ $k = 1, 2, \cdots, m$, and $\mathcal{I}_k(\varsigma_k, u(\varsigma_k^-))$ are the action of impulses on the time scale interval $\mathscr{I}$.

2. Preliminaries

Definition 2.1. *[44] A function ρ: $\mathfrak{T} \to \mathscr{R}$ defined by*

$$\rho(\varsigma) = inf\{\theta \in \mathfrak{T} : \theta < \varsigma\}$$

is said to be a backward jump operator. Any $\varsigma \in \mathfrak{T}$ is said to be ld if $\rho(\varsigma) = \varsigma$ and if $\rho(\varsigma) = \varsigma - 1$, then ς is said to be a left scattered point on $\mathfrak{T}$.

Remark 2.2. *If $\mathfrak{T}$ is a minimum right scattered point y, then set $\mathfrak{T}_\upsilon = \mathfrak{T} \backslash \{y\}$, otherwise $\mathfrak{T}_\upsilon = \mathfrak{T}$.*

Definition 2.3. *[42] A function*

$$x : \mathfrak{T} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R}$$

is said to be an ld continuous function, if $x(\cdot, p, q)$ is ld continuous on $\mathfrak{T}$ for each ordered pair $(\varsigma, \theta) \in \mathscr{R} \times \mathscr{R}$ and $x(\varsigma, \cdot, \cdot)$ is continuous on $\mathscr{R} \times \mathscr{R}$ for fixed point $\varsigma \in \mathfrak{T}$.

Proposition 2.4. *[17] Assume g is an increasing continuous function on* $[0, \mathscr{T}] \cap \mathfrak{T}$*. If* $\mathbb{G}$ *is an addition to g in* $[0, \mathscr{T}]$*,* $\mathscr{T} \in \mathscr{R}$*, one can obtain*

$$\mathbb{G}(\varsigma) = \begin{cases} g(\varsigma), & if \varsigma \in \mathfrak{T}, \\ g(\theta), & if \varsigma \in (\varsigma, \rho(\varsigma)) \notin \mathscr{R}, \end{cases}$$

then,

$$\int_s^t g(\varsigma)\nabla\varsigma \leq \int_s^t g(\varsigma)d\varsigma, \tag{2.1}$$

for $s, t \in [0, \mathscr{T}] \cap \mathfrak{T}$*, such that* $s < t$*.*

Definition 2.5. *([44], Higher order nabla derivative) Consider an ld continuous function* $\mathfrak{H}\colon \mathfrak{T}_\upsilon \to \mathscr{R}$ *over* $\mathfrak{T}$*. Here,* $\mathfrak{H}_\nabla$ *is differentiable over* $\mathfrak{T}_\upsilon^{(2)} = \mathfrak{T}_{\upsilon\upsilon}$ *along*

$$\mathfrak{H}_\nabla^{(2)} = (\mathfrak{H})_\nabla : \mathfrak{T}_\upsilon^{(2)} \to \mathscr{R},$$

where $\mathfrak{H}_{\nabla\nabla} = \mathfrak{H}_\nabla^{(2)}$ *is the second order nabla derivative. Also, proceeding upto* n^{th} *order, one can get* $\mathfrak{H}_\nabla^{(n)}\colon \mathfrak{T}_\upsilon^{(n)} \to \mathscr{R}$*.*

Definition 2.6. *[44] Consider ld continuous function* $\mathfrak{H}\colon \mathfrak{T}_\upsilon^{(n)} \to \mathscr{R}$*, such that* $\mathfrak{H}_\nabla^{(n)}(\varsigma)$ *(*n^{th} *order of nabla derivative) appears, then the C∇D is*

$${}^C D_a^{\mathfrak{w}} \mathfrak{H}(\varsigma) = \frac{1}{\Gamma(n-\mathfrak{w})} \int_a^\varsigma (\varsigma - \rho(\theta))^{n-\mathfrak{w}-1} \mathfrak{H}_\nabla^n(\theta)\nabla\theta.$$

If $\mathfrak{w} \in (0, 1)$*, we obtain*

$${}^C D_a^{\mathfrak{w}} \mathfrak{H}(\varsigma) = \frac{1}{\Gamma(1-\mathfrak{w})} \int_a^\varsigma (\varsigma - \rho(\theta))^{-\mathfrak{w}} \mathfrak{H}_\nabla \nabla\theta.$$

Definition 2.7. *[44] On the set* $\mathfrak{T}_\upsilon$*, consider* $\mathfrak{H}$ *to be any ld continuous function, then the Riemann-Liouville nabla derivative (RL∇D) is*

$$D_{\varsigma_o}^{\mathfrak{w}} x(t) = \frac{1}{\Gamma(1-\mathfrak{w})} \Big(\int_{\varsigma_o}^\varsigma (\varsigma - \rho(\theta))^{-\mathfrak{w}} x(\theta)\nabla\theta \Big)^\nabla.$$

Definition 2.8. *[17] Assume* $\mathfrak{H}\colon \mathscr{I}_{\mathcal{J}} \to \mathscr{R}$*, then the RL∇D fractional integral of* $\mathfrak{H}$ *is*

$$D_{\varsigma_o}^{-\mathfrak{w}} \mathfrak{H}(\varsigma) = \mathbb{I}_{\varsigma_o}^{\mathfrak{w}} \mathfrak{H}(\varsigma) = \frac{1}{\Gamma(\mathfrak{w})} \int_{\varsigma_o}^\varsigma (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{H}(\theta)\nabla\theta.$$

The RL∇D integral always satisfies the condition

$$\mathbb{I}_{\varsigma_o}^{\mathfrak{w}} \mathbb{I}_{\varsigma_o}^{u} \mathfrak{H}(\varsigma) = \mathbb{I}_{\varsigma_o}^{w+u} \mathfrak{H}(\varsigma).$$

Lemma 2.9. *[17] Assume the ld continuous function is* $u(\varsigma)$*, then*

$$\begin{cases} D^u \mathbb{I}^{\mathfrak{w}} \imath p(\varsigma) = u(\varsigma), \\ D^u \mathbb{I}^{\mathfrak{w}} \imath p(\varsigma) = \mathbb{I}^{w-u} u(\varsigma). \end{cases}$$

Theorem 2.10. *[25] Assume $D \subset \mathfrak{C}(\mathfrak{T}, \mathscr{R})$. Let D be bounded and equicontinuous simultaneously, then it is relatively compact.*

Theorem 2.11. *[42] A function $\mathfrak{H}(\mathcal{B})$ is relatively compact in $\mathcal{A}$ for $\mathfrak{H}$: $\mathcal{A} \to \mathcal{B}$, which is completely continuous.*

Theorem 2.12. *([24], Nonlinear alternatives Leray-Schauder's type) Let $C \subset X$ be closed and convex and X be as Banach space. Let G: $\mathcal{U} \to C$ be a compact map and $\mathcal{U}$ be a relatively open subset of C with $0 \in \mathcal{U}$, then*

(i) G has a fixed point in $\mathcal{U}$; or

(ii) there is a point $\mathfrak{u} \in \delta U$ and $\gamma \in (0, 1)$ with $u = \gamma G(\mathfrak{u})$.

Theorem 2.13. *[42] For $\mathfrak{w} \in (0, 1)$, u is a solution for $\mathfrak{L}$: $\mathscr{I}_{\mathcal{J}} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R}$, then*

$$^{C}D^{\mathfrak{w}}u(\varsigma) = \mathfrak{L}(\varsigma, u(\varsigma), {}^{C}D^{\mathfrak{w}}u(\varsigma)), \quad u(\varsigma)|_{\varsigma=0} = \varphi(u),$$

if u is the solution of equation

$$u(\varsigma) = \varphi(u) + \frac{1}{\Gamma(\mathfrak{w})} \int_{\varsigma_o}^{\varsigma} (\varsigma - \rho(x))^{\mathfrak{w}-1} \mathfrak{L}(x, u(x), {}^{C}D^{\mathfrak{w}}u(x)) \nabla x. \tag{2.2}$$

Definition 2.14. *[45] Let $\mathscr{X}$ be a Banach space and*

$$\mathscr{A} = \{\mathcal{L}(t) \in \mathscr{L}(\mathscr{X}) : t \geq 0\},$$

where $\mathscr{L}(\mathscr{X})$, a family of linear, bounded operators $\mathscr{L}(\mathscr{X})$: $\mathscr{X} \to \mathscr{X}$ for all $t \geq 0$. A is called a semigroup if, and only if,

$$\mathcal{L}(0) = \mathscr{I} \quad \text{and} \quad \mathcal{L}(s + t) = \mathcal{L}(s)\mathcal{L}(t), \quad \forall t, s \geq 0.$$

3. Analysis between RL∇D and C∇D

Proposition 3.1. *Let $m - 1 < \mathfrak{w} < m, m \in \mathbb{N}$ for any $\mathfrak{w} \in \mathscr{R}$, such that ${}^{C}D^{\mathfrak{w}}_{\varsigma_o}\mathbb{G}(\varsigma)$ exists over time scale $\mathfrak{T}$, then*

$$^{C}D^{\mathfrak{w}}_{\varsigma_o}\mathbb{G}(\varsigma) = \mathbb{I}^{m-\mathfrak{w}}_{\varsigma_o}\mathbb{G}^{(m)}_{\nabla}(\varsigma).$$

Proof. The proof is evident from Theorems 2.12 and 2.13. □

Theorem 3.2. *For $m = [\mathfrak{w} + 1]$ and for any $\varsigma \in \mathfrak{T}_{\nu^n}$, the C∇D and RL∇D satisfies:*

$$^{C}D^{\mathfrak{w}}_{\alpha}\mathbb{G}(\varsigma) = D^{\mathfrak{w}}_{\alpha}(\mathbb{G}(\varsigma) - \sum_{\nu=0}^{m-1} \frac{(\varsigma - \alpha)^{\nu}}{\Gamma(\nu + 1)} \mathbb{G}^{(\nu)}_{\nabla}(\alpha)),$$

for a fixed point $\alpha \in \mathfrak{T}$. Taylor's theorem defined in [27] proves this theorem.

Proof. Assume ld continuous function $\mathbb{G}$, then $\forall$ fixed $\alpha \in \mathfrak{T}$ and $m \in \mathbb{N} \cup \{0\}, m < n$. One can obtain

$$\begin{aligned}\mathbb{G}(\varsigma) &= \sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) + \frac{1}{\Gamma(m)}\int_{\alpha}^{\varsigma}(\varsigma-\rho(\theta))\nabla\theta \\ &= \sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) + \mathbb{I}_{\alpha}^{m}\mathbb{G}_{\nabla}^{(m)}(\varsigma). \end{aligned} \tag{3.1}$$

Taking the Riemann-Liouville derivative $D_{\alpha}^{\mathfrak{w}}$ in each side of Eq (3.1), Lemma 2.9 and Proposition 3.1 are used below:

$$\begin{aligned}D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma) &= D_{\alpha}^{\mathfrak{w}}\sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) + D_{\alpha}^{\mathfrak{w}}\mathbb{I}_{\alpha}^{m}\mathbb{G}_{\nabla}^{(m)}(\varsigma) \\ &= D_{\alpha}^{\mathfrak{w}}\sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) + \mathbb{I}_{\alpha}^{m-\mathfrak{w}}\mathbb{G}_{\nabla}^{(m)}(\varsigma) \\ &= D_{\alpha}^{\mathfrak{w}}\sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) + {}^{C}D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma). \end{aligned} \tag{3.2}$$

From the above equation, we obtain

$$\begin{aligned}{}^{C}D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma) &= D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma) - D_{\alpha}^{\mathfrak{w}}\sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha) \\ &= D_{\alpha}^{\mathfrak{w}}\Big(\mathbb{G}(\varsigma) - \sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha)\Big). \end{aligned} \tag{3.3}$$

□

Proposition 3.3. *If* $\mathfrak{w} \in (0,1)$, *then* $m = 1$. *Hence, from the Eq (3.3),*

$${}^{C}D_{\alpha}^{\mathfrak{w}}(\varsigma) = D_{\alpha}^{\mathfrak{w}}(\mathbb{G}(\varsigma) - \mathbb{G}(\alpha)).$$

Case 1. *If* $\mathbb{G}(\alpha) \to 0$, *as* $\alpha \to 0$, *then*

$${}^{C}D_{\alpha}^{\mathfrak{w}}(\varsigma) = D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma). \tag{3.4}$$

Hence, C∇D and the Riemann-Liouville derivative coincide with each other.

Case 2. *If* $\mathfrak{w} \in \mathbb{N}$, *by applying Eq (3.1) in Eq (3.2) and applying Lemma 2.9, one can get*

$$\begin{aligned}{}^{C}D_{\alpha}^{\mathfrak{w}}\mathbb{G}(\varsigma) &= D_{\alpha}^{\mathfrak{w}}\Big(\mathbb{G}(\varsigma) - \sum_{v=0}^{m-1} \frac{(\varsigma-\alpha)^v}{\Gamma(v+1)}\mathbb{G}_{\nabla}^{(v)}(\alpha)\Big) \\ &= D_{\alpha}^{\mathfrak{w}}\mathbb{I}_{\alpha}^{m}\mathbb{G}_{\nabla}^{(m)}(\varsigma) \\ &= \mathbb{G}^{(m)}(\varsigma). \end{aligned}$$

∴ *C∇D coincides with the nabla derivative.*

4. Existence and uniqueness of impulsive neutral fractional dynamic equation

A population dynamics model featuring a stop-start phenomenon can be used to compare the dynamic Eq (1.1) to that model. If we take into account a negative impact on that particular species, we can observe the population change, that the C∇D ${}^{C}D^{\mathfrak{w}}u(\varsigma)$ presents (at the initial stage of time), with respect to ς on

$$\mathscr{I} = [0, \mathscr{T}] \cap \mathfrak{T}.$$

We investigate a scenario in specific times $\varsigma_1, \varsigma_2, \varsigma_3, \cdots$ such that

$$0 < \varsigma_1 < \varsigma_2 < \varsigma_3, \cdots, \varsigma_m < \varsigma_{m+1} = \mathscr{T}, \quad lim_k = \infty.$$

Impulse effects have an impact on people " momentarily," so there is a surge in the population $u(\varsigma)$, and $u(\varsigma_k^+)$ and $u(\varsigma_k^-)$ show the species population at the time ς_k before and after the impulsive effect.

Assume a collection of every ld continuous function $\mathfrak{C}(\mathscr{I}, \mathscr{R})$. Put $\mathscr{I}_o = [0, \varsigma_1]$ and $\mathscr{I}_k = [\varsigma_k, \varsigma_{k+1}]$ for each $k = 1, 2, \cdots, m$. Let

$$\mathfrak{PC}(\mathscr{I}, \mathscr{R}) = \{u : \mathscr{I}_k \to \mathscr{R}, u \in \mathfrak{C}(\mathscr{I}, \mathscr{R}) \text{ and } u(\varsigma_k^+) \text{ and } u(\varsigma_k^-) \text{ exist with } u(\varsigma_k^-) = u(\varsigma_k), k = 1, 2, \cdots, m\}$$

and

$$\mathfrak{PC}^1(\mathscr{I}, \mathscr{R}) = \{u : \mathscr{I}_k \to \mathscr{R}, u \in \mathfrak{C}^1(\mathscr{I}, \mathscr{R}), k = 1, 2, \cdots, m\},$$

where $\mathfrak{PC}^1(\mathscr{I}, \mathscr{R})$ is collection of every function from $\mathscr{I}_k$ to $\mathscr{R}$, i.e., ld continuously ∇ differentiable function.

The set $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$ is a Banach space

$$\|u\|_{\mathfrak{PC}} = sup_{\varsigma \in \mathscr{I}} |u(\varsigma)|.$$

Definition 4.1. *A function $u \in \mathfrak{PC}^1(\mathscr{I}, \mathscr{R})$ is a solution of the Eq (1.1), if u satisfies the Eq (1.1) on $\mathscr{I}$ having*

$$u(\varsigma_k^+) - u(\varsigma_k^-) = \mathcal{I}_k(\varsigma_k, u(\varsigma_k^-)) \text{ and } u(0) = \varphi(\mathscr{T}).$$

Lemma 4.2. *Assume an ld continuous function $\mathscr{H}: \mathscr{I} \to \mathscr{R}$, such that solution (1.1) is*

$$\begin{cases} {}^{C}D^{\mathfrak{w}}[u(\varsigma) - g(\varsigma, u)] = \mathscr{H}(\varsigma), & \varsigma \in \mathscr{I}, \ \varsigma \neq \varsigma_k \\ u(\varsigma_k^+) - u(\varsigma_k^-) = \mathcal{I}_k(\varsigma_k, u(\varsigma_k^-)), & k = 1, 2, 3, \cdots, m \\ u(0) = \varphi(u), \end{cases} \tag{4.1}$$

where the integral equation specifies

$$u(\varsigma) \begin{cases} \varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_0^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta, & \varsigma \in \mathscr{I}_o, \\ \varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \sum_{i=1}^{k} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta & \\ + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_k}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta + \sum_{i=1}^{k} \mathcal{I}_i(\varsigma_i, u(\varsigma_i^-)), & \varsigma \in \mathscr{I}_k. \end{cases} \tag{4.2}$$

Proof. If $\varsigma \in \mathscr{I}_o$, then the solution of the Eq (4.1) is given by

$$u(\varsigma) = \varphi(p) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_0^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta. \tag{4.3}$$

For $\varsigma \in \mathscr{I}_1$, the problem

$$\begin{cases} {}^{C}D^{\mathfrak{w}}[u(\varsigma) - g(\varsigma, u)] = \mathscr{H}(\varsigma), \\ u(\varsigma_1^+) - u(\varsigma_1^-) = \mathcal{I}_1(\varsigma_1, u(\varsigma_1^-)), \end{cases}$$

holds the solution

$$u(\varsigma) = u(\varsigma_1^+) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_1}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta. \tag{4.4}$$

Again,

$$u(\varsigma_1^+) - u(\varsigma_1^-) = \mathcal{I}_1(\varsigma_1, u(\varsigma_1^-)). \tag{4.5}$$

Applying Eq (4.5) in Eq (4.4), then

$$u(\varsigma) = u(\varsigma_1^-) + \mathcal{I}_1(\varsigma_1, u(\varsigma_1^-)) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_1}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta,$$

which follows that

$$\begin{aligned} u(\varsigma) =& \varphi(p) + \mathcal{I}_1(\varsigma_1, u(\varsigma_1^-)) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_1}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta \\ &+ \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{0}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta, \qquad \varsigma \in \mathscr{I}_1. \end{aligned}$$

Using the idea of mathematical induction and generalizing in this way for $\varsigma \in \mathscr{I}_k$, $k = 1, 2, \cdots, m$, one can say,

$$\begin{aligned} u(\varsigma) =& \varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{0}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta \\ &+ \frac{g(0)}{\Gamma(\mathfrak{w})} \sum_{i=1}^{k} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i - \rho(\theta))^{\mathfrak{w}-1} \mathscr{H}(\theta) \nabla\theta + \sum_{i=1}^{k} \mathcal{I}_{\mathrm{i}}(\varsigma_{\mathrm{i}}, u(\varsigma_{\mathrm{i}})), \qquad k = 1, 2, 3, \cdots, m. \end{aligned}$$

□

The following hypotheses are necessary in order to prove the existence and uniqueness of the solution to Eq (1.1):

(A1) $\mathfrak{L}$: $\mathscr{I} \times \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is ld continuous and there should be a constant $\mathcal{K} > 0$ and $0 < \mathscr{G} < 1$, which contents

$$|\mathfrak{L}(\varsigma, \theta_1, \theta_2) - \mathfrak{L}(\varsigma, \zeta_1, \zeta_2)| \leq \mathcal{K}|\theta_1 - \zeta_1| + \mathscr{G}|\theta_2 - \zeta_2|, \qquad \forall \varsigma \in \mathbb{I},$$

$\theta_i, \zeta_i \in \mathscr{R}$ for $\mathrm{i} = 1, 2$.

(A2) There exist constants $\mathcal{A} > 0, \mathcal{F} > 0$, and $0 < \mathcal{E} < 1$, such that

$$|\mathfrak{L}(\varsigma, \theta, \zeta)| \leq \mathcal{A} + \mathcal{F}|\theta| + \mathcal{E}|\zeta|, \qquad \forall \theta, \zeta \in \mathscr{R}.$$

(A3) $\mathcal{I}_k(\varsigma, u)$ is continuous $\forall\ k = 1, 2, \cdots m$ and contents:

(i) There exists a "+" ve constant $\mathscr{M}_k$ for $k = 1, 2, \cdots, m$ such that

$$|\mathcal{I}_k(\varsigma, u)| \leq \mathscr{M}_k, \qquad \forall \varsigma \in \mathscr{I}_k, u \in \mathscr{R}.$$

(ii) There exists a "+" ve constatnt $\mathcal{L}_k$, for $k = 1, 2, 3, \cdots, m$ such that

$$|\mathcal{I}_k(\varsigma, u) - \mathcal{I}_k(\varsigma, \mathfrak{h})| \leq \mathcal{L}_k|u - \mathfrak{h}|, \qquad \forall \varsigma \in \mathscr{I}_k, u, \mathfrak{h} \in \mathscr{R}.$$

(A4) There must be a non "-" ve increasing function μ: $\mathscr{R}^+ \to \mathscr{R}^+$ such that

$$|\varphi(\varsigma) - \varphi(\theta)| \leq \mathscr{H}|\varsigma - \theta|, \qquad \forall \varsigma \in \mathscr{I},$$

and a "+" ve constant $\mathscr{H}$ such that

$$|\varphi(\varsigma) - \varphi(\theta)| \leq \mathscr{H}|\varsigma - \theta|, \qquad \forall \varsigma, \theta \in \mathscr{I}.$$

(A5) For $\varsigma \in \mathscr{I}_o$ in a time scale interval, let the function $u(\varsigma)$ be

$$u(\varsigma) = \varphi(p) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_0^\varsigma (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{L}(\varsigma, u(\varsigma), {}^C D^{\mathfrak{w}} u(\varsigma)) \nabla\theta.$$

The Banach contraction theorem forms the basis of the following theorem.

Theorem 4.3. *If all conditions (A1)–(A4) and*

$$\sum_{i=1}^{m} \mathcal{L}_i + \mathscr{H} + g(\varsigma) + \frac{g(0)\mathcal{K}\mathscr{T}^{\mathfrak{w}}(m + 1)}{(1 - \mathscr{G})(\mathfrak{w} + 1)} < 1$$

hold, then Eq (1.1) must contain a solution on $\mathscr{I}$.

Proof. Assume

$${}^C D^{\mathfrak{w}}[u(\varsigma) - g(\varsigma, u)] = \mathfrak{h}(\varsigma).$$

Let $\Pi \subseteq \mathfrak{PC}(\mathscr{I}_k, \mathscr{R})$ such that

$$\Pi = \{u \in \mathfrak{PC}^1(\mathscr{I}_k, \mathscr{R}) : ||u||_{\mathfrak{PC}} \leq \omega\}$$

and χ: $\Pi \to \Pi$ such that

$$(\chi u)(\varsigma) = \varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \int_0^\varsigma (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{L}(\varsigma, u(\varsigma), {}^C D^{\mathfrak{w}} u(\varsigma)) \nabla\theta,$$

for $\varsigma \in \mathscr{I}_o$, and

$$\begin{aligned}(\chi u)(\varsigma) =& \varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})} \sum_{i=1}^{k} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{L}(\varsigma, u(\varsigma), \mathfrak{h}(\varsigma)) \nabla\theta + \sum_{i=1}^{k} \mathcal{I}_i(\varsigma_i, u(\varsigma_i^-)) \\ &+ \frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_k}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{L}(\varsigma, u(\varsigma), {}^C D^{\mathfrak{w}} u(\varsigma)) \nabla\theta,\end{aligned}$$

for $\varsigma \in \mathscr{I}_k$, then $k = 1, 2, 3, \cdots, m$.

Case 1. Let $\varsigma \in \mathscr{I}_k$, then $u \in \Pi$,

$$|(\chi u)(\varsigma)| = |\varphi(u)| + |g(\varsigma)| + |\frac{g(0)}{\Gamma(\mathfrak{w})} \sum_{i=1}^{k} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{h}(\theta) \nabla\theta| + |\sum_{i=1}^{k} \mathcal{I}_i(\varsigma_i, u(\varsigma_i^-))|$$
$$+ |\frac{g(0)}{\Gamma(\mathfrak{w})} \int_{\varsigma_k}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1} \mathfrak{h}(\theta)) \nabla\theta|,$$

where $\mathfrak{h} \in \Pi$, $\varsigma \in \mathscr{I}$. By Eq (1.1), one can get

$$\mathfrak{h} = \mathfrak{L}(\varsigma, u, \mathfrak{h}),$$

and

$$\begin{aligned} |\mathfrak{h}| &= |\mathfrak{L}(\varsigma, u, \mathfrak{h})| \\ &\leq \mathcal{A} + \mathcal{F}|u(\varsigma)| + \mathcal{E}|\mathfrak{h}(\varsigma)| \\ &\leq \frac{\mathcal{A} + \mathcal{F}\omega}{1 - \mathcal{E}}. \end{aligned} \tag{4.6}$$

Again, taking the norm of $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$ in (4.6),

$$\|u\|_{\mathfrak{PC}} \leq \frac{\alpha + \mathcal{F}\omega}{1 - \mathcal{E}},$$

where

$$\|\mathcal{A}\|_{\mathfrak{PC}} = \alpha.$$

Using the condition of Case 1 and Proposition 2.4, we obtain

$$\begin{aligned} \|\chi\|_{\mathfrak{PC}} &= sup_{\varsigma \in \mathbb{I}} |\chi u(\varsigma)| \\ &\leq \mu|u| + g(\varsigma) + \sum_{i=1}^{m} \mathscr{M}_i + \frac{g(0)[\mathcal{A} + \mathcal{F}|u|]}{(1 - \mathcal{E})\Gamma(\mathfrak{w})} \Big[\sum_{i=1}^{m} \int_{\varsigma_{i}-1}^{\varsigma_i} (\varsigma - \theta)^{(\mathfrak{w}-1)} d\theta + \int_{\varsigma_k}^{\varsigma} (\varsigma - \theta)^{(\mathfrak{w}-1)} d\theta \Big] \\ &\leq \mu\omega + g(\varsigma) + \sum_{i=1}^{m} \mathscr{M}_i + \frac{g(0)\mathscr{T}^{\mathfrak{w}}(\alpha + \mathcal{F}\omega)(m+1)}{\Gamma(\mathfrak{w}+1)(1 - \mathcal{E})} \\ &\leq \omega, \end{aligned} \tag{4.7}$$

where

$$\omega = \frac{\sum_{i=1}^{m} \mathscr{M}_i + g(\varsigma) + \frac{g(0)(m+1)\mathscr{T}^{\mathfrak{w}}\alpha}{\Gamma(\mathfrak{w}+1)(1-\mathcal{E})}}{1 - \mu + \frac{(m+1)\mathscr{T}^{\mathfrak{w}}\mathcal{F}g(0)}{\Gamma(\mathfrak{w}+1)(1-\mathcal{E})}}.$$

Case 2. If $\varsigma \in \mathscr{I}_o$, by a similar way, one can obtain

$$\begin{aligned} \|\chi_u\|_{\mathfrak{PC}} &\leq \mu\omega + g(\varsigma) + \frac{g(0)\mathscr{T}^{\mathfrak{w}}(\alpha + \mathcal{F}\omega)}{\Gamma(\mathfrak{w}+1)} \\ &\leq \omega. \end{aligned} \tag{4.8}$$

Thus, from (4.8), $\|\chi_u\|_{\mathfrak{PC}} \le \omega$. Hence, $\chi(\Pi)$ is bounded. Also, for $u, v \in \Pi$,

$$\begin{aligned}
\|\chi_u - \chi_v\|_{\mathfrak{PC}} =& sup_{\varsigma\in\mathscr{I}_k}|(\chi_u)(\varsigma) - (\chi_v)(\varsigma)| \\
&\le \sum_{i=1}^{k} |\mathcal{I}_{\mathfrak{i}}(\varsigma_{\mathfrak{i}}, u(\varsigma_{\mathfrak{i}}^{-})) - \mathcal{I}_{\mathfrak{i}}(\varsigma_{\mathfrak{i}}, v(\varsigma_{\mathfrak{i}}^{-}))| + |g(\varsigma)| + \frac{g(0)}{\Gamma(\mathfrak{w})}\left| \int_{\varsigma_k}^{\varsigma} (\varsigma - \rho(\theta))^{\mathfrak{w}-1}(\mathfrak{h}(\theta) - \mathfrak{i}(\theta))\nabla\theta \right| \\
&+ \frac{g(0)}{\Gamma(\mathfrak{w})}\left| \sum_{i=1}^{k} \int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i - \rho(\theta))^{\mathfrak{w}-1}(\mathfrak{h}(\theta) - \mathfrak{i}(\theta))\nabla\theta \right| + |\varphi(u) - \varphi(v)|,
\end{aligned} \tag{4.9}$$

where $\mathfrak{i} \in \Pi$, then $\mathfrak{i}(\varsigma) = \mathfrak{L}(\varsigma, v(\varsigma), \mathfrak{i}(\varsigma))$. For $\varsigma \in \mathscr{I}$, one can get

$$\begin{aligned}
|\mathfrak{h}(\varsigma) - \mathfrak{i}(\varsigma)| &= |\mathfrak{L}(\varsigma, u(\varsigma), \mathfrak{h}(\varsigma)) - \mathfrak{L}(\varsigma, v(\varsigma), \mathfrak{i}(\varsigma))| \\
&\le \mathcal{K}|u(\varsigma) - v(\varsigma)| + \mathscr{G}|\mathfrak{h}(\varsigma) - \mathfrak{i}(\varsigma)| \\
&\le \frac{\mathcal{K}|u(\varsigma) - v(\varsigma)|}{1 - \mathscr{G}}.
\end{aligned} \tag{4.10}$$

Taking the norm of $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$, (4.10) becomes

$$\|\mathfrak{h} - \mathfrak{i}\|_{\mathfrak{PC}} \le \frac{\mathcal{K}\|u - v\|_{\mathfrak{PC}}}{1 - \mathscr{G}}. \tag{4.11}$$

Using (4.11) in (4.9) and applying the Proposition 2.4,

$$\begin{aligned}
\|\chi_u - \chi_v\|_{\mathfrak{PC}} \le& \sum_{i=1}^{m} \mathcal{L}_i|u(\varsigma_i^{-}) - v(\varsigma_i^{-})| + g(\varsigma) + \frac{\mathcal{K}g(0)|u(\theta) - v(\theta)|}{(1 - \mathscr{G})\Gamma(\mathfrak{w})} \int_{\varsigma_k}^{\varsigma} (\varsigma - \theta)^{\mathfrak{w}-1} d\theta \\
&+ \frac{\mathcal{K}g(0)|u(\theta) - v(\theta)|}{(1 - \mathscr{G})\Gamma(\mathfrak{w})} \sum_{i=1}^{m} \int_{\varsigma_i - 1}^{\varsigma_i} (\varsigma - \theta)^{\mathfrak{w}-1} d\theta + \mathscr{H}|u - v| \\
\le& \|u - v\|_{\mathfrak{PC}} \sum_{i=1}^{m} \mathcal{L}_i + g(\varsigma) + \frac{\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)\|u - v\|_{\mathfrak{PC}}}{(1 - \mathscr{G})\Gamma(\mathfrak{w} + 1)} \\
&+ \frac{m\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)\|u - v\|_{\mathfrak{PC}}}{(1 - \mathscr{G})\Gamma(\mathfrak{w} + 1)} + \mathscr{H}\|u - v\|_{\mathfrak{PC}} \\
\le& \left(\sum_{i=1}^{m} \mathcal{L}_i + g(\varsigma) + \frac{\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)(m + 1)}{(1 - \mathscr{G})\Gamma(\mathfrak{w} + 1)} + \mathscr{H} \right) \|u - v\|_{\mathfrak{PC}}.
\end{aligned} \tag{4.12}$$

Similarly, for $\varsigma \in \mathscr{I}_o$,

$$\|\chi_u - \chi_v\|_{\mathfrak{PC}} \le \left(\mathscr{H} + g(\varsigma) + \frac{\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)}{(1 - \mathscr{G})\Gamma(\mathfrak{w} + 1)} \right) \|u - v\|_{\mathfrak{PC}}. \tag{4.13}$$

Thus, from (4.12) and (4.13), we obtain

$$\|\chi_u - \chi_v\|_{\mathfrak{PC}} \le \mathcal{U}\|u - v\|_{\mathfrak{PC}},$$

where

$$\mathcal{U} = \sum_{i=1}^{m} \mathcal{L}_i + g(\varsigma) + \frac{\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)(m + 1)}{(1 - \mathscr{G})\Gamma(\mathfrak{w} + 1)} + \mathscr{H}.$$

Here, $\mathcal{U} < 1$, then χ: $\Pi \to \Pi$ is a contraction operator. According to the Banach contraction theorem, it has a fixed point, which is the solution to Eq (1.1). □

Equation (1.1)'s adequate condition for a solution is based on the nonlinear alternative to Leray-Schauder's fixed point theorem.

Theorem 4.4. *If (A1) through (A4) are true and there is a positive constant β, then*

$$\mu\beta + \sum_{i=1}^{m} \mathscr{M}_i + g(\varsigma) + \frac{(m+1)\mathscr{T}^{\mathfrak{w}} g(0)(\mathcal{A} + \mathcal{F}\beta)}{\Gamma(\mathfrak{w}+1)(1-\mathcal{E})} < \beta. \tag{4.14}$$

Therefore, there is at least one solution to Eq (1.1) in $\mathscr{I}$.

Proof. The following steps are used to demonstrate the theorem's proof.

Step 1. χ: $\Pi \to \Pi$ is continuous.

Assume $\{u_n\}$ is a sequence of Π such that $u_n \to u$, then $\varsigma \in \mathscr{I}_k, k = 1, 2, 3, \cdots, m$.

$$\begin{aligned} \|\chi_{u_n} - \chi_v\|_{\mathfrak{PC}} =& sup_{\varsigma \in \mathscr{I}_k} |(\chi_{u_n})(\varsigma) - (\chi_v)(\varsigma)| \\ \leq& \sum_{i=1}^{m} |\mathcal{I}_i(\varsigma_i, u_n(\varsigma_i^-)) - \mathcal{I}_i(\varsigma_i, u(\varsigma_i^-))| + |g(\varsigma)| + \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{\varsigma_k}^{\varsigma} (\varsigma-\theta)^{\mathfrak{w}-1}(\mathfrak{h}_n(\theta) - \mathfrak{h}(\theta))d\theta\right| \\ &+ \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\sum_{i=1}^{m}\int_{\varsigma_{i-1}}^{\varsigma_i} (\varsigma_i-\theta)^{\mathfrak{w}-1}(\mathfrak{h}_n(\theta) - \mathfrak{h}(\theta))d\theta\right| + |\varphi(u_n) - \varphi(u)|, \end{aligned} \tag{4.15}$$

where $\mathfrak{h}_n \in \Pi$, such that

$$\mathfrak{h}_n = \mathfrak{L}(\varsigma, u_n, \mathfrak{h}_n),$$

and for $\varsigma \in \mathscr{I}_k$, we obtain

$$\begin{aligned} |\mathfrak{h}_n - \mathfrak{h}| &= |\mathfrak{L}(\varsigma, u_n, \mathfrak{h}_n) - \mathfrak{L}(\varsigma, u, \mathfrak{h})| \\ &\leq \mathcal{K}|u_n - u| + \mathscr{G}|\mathfrak{h}_n - \mathfrak{h}| \\ &\leq \frac{\mathcal{K}|u_n - u|}{1-\mathscr{G}}. \end{aligned} \tag{4.16}$$

Taking the norm of $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$, (4.16) becomes

$$\|\mathfrak{h}_n - \mathfrak{h}\|_{\mathfrak{PC}} \leq \frac{\mathcal{K}\|u_n - u\|_{\mathfrak{PC}}}{1-\mathscr{G}}. \tag{4.17}$$

Using (4.17) in (4.15), we obtain

$$\|\chi_{u_n} - \chi_v\|_{\mathfrak{PC}} \leq \|u_n - u\|_{\mathfrak{PC}} \leq \left(\sum_{i=1}^{m} \mathcal{L}_i + g(\varsigma) + \frac{\mathcal{K}\mathscr{T}^{\mathfrak{w}} g(0)(m+1)}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)} + \mathscr{H}\right)$$

As $n \to \infty$, let $u_n \to u$ such that

$$\|\chi_{u_n} - \chi_v\|_{\mathfrak{PC}} \to 0.$$

As a result, χ is continuous.

Also, for $\varsigma \in \mathscr{I}_o$, the proof is similar.

Step 2. The operator χ map Π to $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$.

Assume $x_1, x_2 \in \mathscr{I}_k, k = 1, 2, \cdots, m$, such that $x_1 < x_2$, then

$$\begin{aligned}
\|\chi_u(x_2) - \chi_v(x_1)\|_{\mathfrak{PC}} =& sup_{\varsigma\in\mathscr{I}_k}|(\chi_u)(x_2) - (\chi_v)(x_1)| \\
\leq& \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{\varsigma_k}^{x_1}(x_2 - \rho(\theta))^{\mathfrak{w}-1} - (x_1 - \rho(\theta)^{\mathfrak{w}-1})\mathfrak{h}(\theta)\nabla\theta\right| + |g(\varsigma)| \\
&+ \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{x_1}^{x_2}(x_2 - \rho(\theta))^{\mathfrak{w}-1}\mathfrak{h}(\theta)\nabla\theta\right| + \sum_{0<\varsigma_k<x_2-x_1}|\mathcal{I}_{\varsigma_k}(\varsigma_k, u(\varsigma_k^-))| \\
\leq& \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{\varsigma_k}^{x_1}(x_2 - (\theta))^{\mathfrak{w}-1} - (x_1 - (\theta)^{\mathfrak{w}-1})\mathfrak{h}(\theta)\nabla\theta\right| + |g(\varsigma)| \\
&+ \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{x_1}^{x_2}(x_2 - (\theta))^{\mathfrak{w}-1}\mathfrak{h}(\theta)\nabla\theta\right| + \sum_{0<\varsigma_k<x_2-x_1}|\mathcal{I}_{\varsigma_k}(\varsigma_k, u(\varsigma_k^-))| \\
\leq& \frac{(\mathcal{A} + \mathcal{F}\omega)g(o)}{(1-\mathcal{E})\Gamma(\mathfrak{w})}\Bigg(\left|\int_{\varsigma_k}^{x_1}(x_2 - (\theta))^{\mathfrak{w}-1} - (x_1 - (\theta)^{\mathfrak{w}-1})\mathfrak{h}(\theta)\nabla\theta\right| \\
&+ \frac{g(0)}{\Gamma(\mathfrak{w})}\left|\int_{x_1}^{x_2}(x_2 - (\theta))^{\mathfrak{w}-1}\mathfrak{h}(\theta)\nabla\theta\right|\Bigg) + |g(\varsigma)| + \sum_{0<\varsigma_k<x_2-x_1}|\mathcal{I}_{\varsigma_k}(\varsigma_k, u(\varsigma_k^-))|.
\end{aligned}$$

Since $(x - (\theta))^{\mathfrak{w}-1}$ is continuous, if $x_1 \to x_2$, then

$$\|\chi_u(x_2) - \chi_v(x_1)\|_{\mathfrak{PC}} \to 0.$$

Thus, the operator χ is equicontinuous in $\mathscr{I}_k$. Since the result at $x_1, x_2 \in \mathscr{I}_o$ is comparable, the evidence is left out.

Step 3. Let χ map Π be a bounded set of $\mathfrak{PC}(\mathscr{I}, \mathscr{R})$.

From (4.7), it is clear that $\|\chi(p)\| \leq \omega$ for $\omega \in \mathscr{R}$. As a consequence of Steps 1–3, using the Arzela-Ascoli theorem, one can discover that χ is entirely continuous.

Step 4. Let $\gamma \in (0, 1)$,

$$k = \{u \in \mathfrak{PC}(\mathscr{I}_k, \mathscr{R}) : u = \gamma\chi(u), 0 < \gamma < 1\}$$

be bounded.

Also, by $\varsigma \in \mathscr{I}_k, k = 1, 2, 3, \cdots, m$, one can obtain

$$\begin{aligned}
|u(\varsigma)| = |\gamma\chi(u)\varsigma| =& \Bigg|\gamma\Bigg(\varphi(u) + g(\varsigma) + \frac{g(0)}{\Gamma(\mathfrak{w})}\sum_{i=1}^{k}\int_{\varsigma_{i-1}}^{\varsigma_i}(\varsigma - \rho(\theta))^{\mathfrak{w}-1}\mathfrak{h}(\theta)\nabla\theta \\
&+ \frac{g(0)}{\Gamma(\mathfrak{w})}\int_{\varsigma_k}^{\varsigma}(\varsigma - \rho(\theta))^{\mathfrak{w}-1}\mathfrak{h}(\theta)\nabla\theta + \sum_{i=1}^{k}\mathcal{I}_i(\varsigma_i, u(\varsigma_i^-))\Bigg)\Bigg| \\
\leq& \mu\|u\|_{\mathfrak{PC}} + \sum_{i=1}^{n}\mathscr{M}_{iti} + +g(\varsigma)\frac{(\mathcal{A} + \mathcal{F}\|u\|_{\mathfrak{PC}})g(0)\mathscr{T}^{\mathfrak{w}}(m+1)}{\Gamma(\mathfrak{w}+1)(1-\mathcal{E})}.
\end{aligned}$$

Thus,

$$\frac{\|u\|_{\mathfrak{PC}}}{\mu\|u\|_{\mathfrak{PC}} + \sum_{i=1}^{n}\mathscr{M}_{iti} + +g(\varsigma)\frac{(\mathcal{A} + \mathcal{F}\|u\|_{\mathfrak{PC}})g(0)\mathscr{T}^{\mathfrak{w}}(m+1)}{\Gamma(\mathfrak{w}+1)(1-\mathcal{E})}} \leq 1.$$

From Eq (4.14), we get a "+" ve constant β such that $||u||_{\mathfrak{PC}} \neq \beta$. Consider a set

$$\psi = \{u \in \mathfrak{PC}(\mathscr{I}, \mathscr{R}) : ||u||_{\mathfrak{PC}} < \beta\},$$

such that

$$\chi : \tilde{\psi} \to \mathfrak{PC}(\mathscr{I}, \mathscr{R})$$

is continuous and completely continuous.

Thus, no $u \in \partial(\psi)$ can be found such that $u = \gamma\chi(u), \gamma \in (0,1)$. Hence, a nonlinear alternative of Leray Schauder's fixed point theorem gives that the answer to Eq (1.1) is a fixed point for χ.

The outcome for $\varsigma \in \mathscr{I}_o$ is almost identical, thus it is not included. □

5. Example

Example 5.1. *Take into account a nonlocal initial condition over time scale in a neutral impulsive fractional dynamic equation*

$$\mathfrak{T} = [0, \frac{1}{5}] \cup [\frac{1}{4}, 1],$$

and we get

$$\begin{cases} {}^{C}D^{\frac{1}{4}}[u(\varsigma) - g(\varsigma, u)] = \dfrac{e^{-5\varsigma}[4 + g(0)(|u(\varsigma)| + |{}^{C}D^{\mathfrak{w}}u(\varsigma)|) + g(\varsigma)]}{25e^{2\varsigma}(1 + |u(\varsigma)|)}, & \varsigma \in [0,1] \cap \mathscr{T}, \quad \varsigma \neq \frac{1}{5} \\ u(\frac{1}{5}^{+}) - u(\frac{1}{5}^{-}) = \dfrac{1 + u(\frac{1}{5})}{15}, & \varsigma_1 = \frac{1}{5} \\ u(0) = \dfrac{u}{10}. & \end{cases} \tag{5.1}$$

We set

$$\mathfrak{L}(\varsigma, u, v) = \frac{e^{-5\varsigma}[4 + g(0)(|u(\varsigma)| + |v(\varsigma)|) + g(\varsigma)]}{25e^{2\varsigma}(1 + |u(\varsigma)|)}. \tag{5.2}$$

It is evident that (5.2)'s right side is continuous for $u, v \in \mathscr{R}$ in relation to time scale. Again, ∀ $\varsigma \in [0, 1] \cap \mathfrak{T}$ and $\mathfrak{h}, \mathfrak{i} \in \mathscr{R}$. We obtain

$$\begin{aligned} \mathfrak{L}(\varsigma, u, v) &\leq \frac{4 + g(0)(|u(\varsigma)| + |v(\varsigma)|) + g(\varsigma)}{25e^2} \\ &\leq \frac{4}{25e^2} + \frac{1}{25e^2}|u(\varsigma)| + \frac{1}{25e^2}|v(\varsigma)| + \frac{2}{25e^2}, \end{aligned}$$

then, we get

$$\mathcal{A} = \frac{4}{25e^2}, \ \mathcal{F} = \frac{1}{25e^2}, \ \mathcal{E} = \frac{1}{25e^2}, \ g(0) = 1, \ g(\varsigma) = \frac{2}{25e^2}.$$

Next,

$$\begin{aligned} |\mathfrak{L}(\varsigma, u, v) - \mathfrak{L}(\varsigma, \mathfrak{h}, \mathfrak{i})| &\leq \frac{1}{25e^2}|u - \mathfrak{h}| + \frac{1}{25e^2}|v - \mathfrak{i}|, \\ |\mathcal{I}_1(\varsigma, u) - \mathcal{I}_1(\varsigma, v)| &\leq \frac{1}{15}|u - \mathfrak{h}|, \end{aligned}$$

$$|\varphi(u) - \varphi(\mathfrak{h})| \leq \frac{1}{10}|u - \mathfrak{h}|,$$
$$|\varphi(u)| \leq \frac{1}{10}.$$

Thus, one can obtain

$$\mathcal{K} = \frac{1}{25e^2}, \ \mathcal{G} = \frac{1}{25e^2}, \ \mathcal{L} = \frac{1}{15}, \ \mathcal{H} = \frac{1}{10}.$$

From the above data, we can say that the Eq (5.1) satisfies all the conditions of (A1)–(A4).

Again, for m = 1 we get

$$\mathcal{L} + g(\varsigma) + \frac{\mathcal{K}\mathcal{T}^{\mathfrak{w}}g(0)(m+1)}{(1-\mathcal{G})\Gamma(\mathfrak{w}+1)} + \mathcal{H} \leq \frac{1}{15} + \frac{1}{10} + \frac{1}{25e^2} + \frac{4\frac{1}{25e^2}}{(1-\frac{1}{25e^2})\Gamma(\frac{1}{4}+1)}$$
$$\leq 1.$$

As a result, the requirements of Theorem 4.3 are met. Consequently, we came to the conclusion that the solution to Eq (5.1) is unique.

Below Table 1 represents the numerical approach for the theoretical results.

Table 1. Variation of $u(\varsigma)$ value for different values of $\mathcal{L}$ and g.

$g\downarrow$	$\mathcal{L}$=1/15	$\mathcal{L}$=1/25	$\mathcal{L}$ = 1/35	$\mathcal{L}$=1/45	$\mathcal{L}$=1/55
1/50	0.6778	0.6511	0.6397	0.6333	0.6293
1/40	0.6828	0.6561	0.6447	0.6383	0.6343
1/30	0.6911	0.6645	0.6530	0.6467	0.6426
1/20	0.7078	0.6811	0.6697	0.6633	0.6593
1/10	0.7578	0.7311	0.7197	0.7133	0.7093

Figure 1 reveals a commendable correspondence between the numerical solution & exact solution across the entire interval.

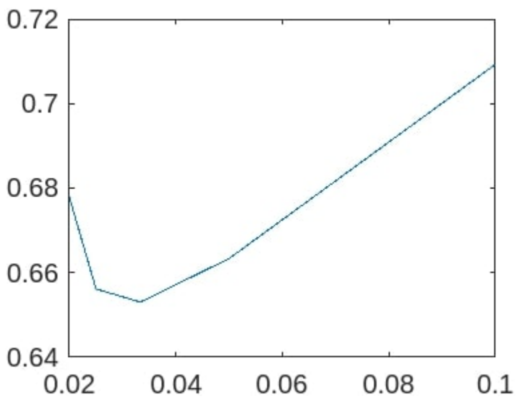

Figure 1. Graph of the approximate solution of $u(\varsigma)$.

6. Conclusions

In this article, we examine both operators in the setting of time scales and analyze the C∇D and RL∇D. Additionally, the C∇D of the fractional dynamic equation including instantaneous impulses and a nonlocal initial condition are also examined. Later, the numerical technique is followed by an example based on all theoretical findings on the existence and uniqueness of the solution. A graph using MATLAB is also represented for the example.

Futher, in modeling the spread of infectious diseases like COVID-19, a dynamic equation in time scales can be used to capture the various stages of infection, transmission rates, and the impact of interventions over time. Mathematical models, often expressed as differential equations or agent-based models, can be adapted to include time scales that represent different temporal aspects of the disease dynamics.

Similarly, in cancer modeling, incorporating a dynamic equation in time scales allows for the consideration of the progression of the disease, the growth of tumors, and the response to treatments over time. This can lead to more accurate predictions and insights into the evolution of the disease and the effectiveness of different therapeutic interventions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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