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Research article

A new characterization of the dual space of the HK-integrable functions

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Abstract: We construct the Henstock-Kurzweil (HK) integral as an extension of a linear form initially defined on L^1, but which is not continuous in this space. This gives us an alternative way to prove existing results. In particular, we give a new characterization of the dual space of Henstock-Kurzweil integrable functions in terms of a quotient space.

Keywords: Henstock-Kurzweil integral; Banach dual space
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1. Introduction

It is well known that λ is a continuous linear functional on the space of Henstock-Kurzweil integrable functions on $[a, b]$ if there exists a function $\vartheta : [a, b] \to \mathbb{R}$ of essentially bounded variation such that

$$\lambda(f) = \text{(HK)-}\int_a^b \vartheta f, \tag{1.1}$$

(see [1]). For example, in [2, Theorem 12.7] it is proved that

$$\inf_{\vartheta_1=\vartheta\ a.e} Var[\vartheta_1; [a, b]] \leq 2\|\lambda\|$$

where $Var[\vartheta_1; [a, b]]$ denotes the total variation of ϑ_1.

We present a new method to show that the space of essentially bounded variation functions is in fact isometric to the dual space of Henstock-Kurzweil integrable functions:

$$\inf_{\vartheta_1=\vartheta\ a.e} Var[\vartheta_1; [a, b]] = \|\ [\vartheta_1]\ \|_{BV_m} = \|\lambda\|,$$

where $[\vartheta_1]$ is an element of the quotient space, BV_m, of BV with given equivalence relation. It comes out that

$$\mathcal{A}' = BV_m$$

is an isometric isomorphism. $\mathcal{A}'$ denotes the dual space of the Henstock-Kurzweil integrable functions normed with the Alexiewicz norm. For details, see Theorem 3 of this article. In fact, Theorem 1 of [3] shows that the functionals in the space of Henstock-Kurzweil integrable functions are of the form

$$\text{(HK)-}\int_a^b f\vartheta$$

where ϑ is a bounded variation function. Modification of the function ϑ on a null set gives rise to the same continuous linear functional. So, the correspondence between linear functionals and bounded variation functions is not a bijection. In this paper we show that the dual space of the Henstock-Kurzweil integrable functions is isometrically isomorphic to the space of essentially bounded variation functions seen as a quotient space.

We mention also that this result is related to James' theorem, (see [4, 5]), which assures that in a non-reflexive space like the one in question not every continuous functional reaches its supremum. But rather, there must always be values arbitrarily close to the norm of the functional.

Furthermore, in this paper we build the Henstock-Kurzweil integral (abbreviated HK-integral), via the limit of integrals of functions that are Lebesgue integrable, thanks to the fact that $L^1[0, 2\pi]$ is dense in the space of Henstock-Kurzweil integrable functions. In fact, on $C[0, 2\pi]$ we can define the positive linear form

$$\lambda(f) = \lim_{n\to\infty} \int_0^{2\pi} f_n(x)dx,$$

with the integral in the sense of Riemann and (f_n) converging to f in L^1-sense. This leads to the Daniell-Stone integral and the space $\mathcal{L}^1(\lambda)$ of integrable functions in this sense. It contains the Riemann integral and coincides with the usual Lebesgue space $L^1[0, 2\pi]$, (see [6, Theorem 1.1]).

Following a similar idea we can extend the notion of integration, inheriting naturally the properties of the previous integration theory on which it is based but adding new features. Our proofs of important results in this theory as the multiplier Theorem are elementary and easy.

2. Extension of the Lebesgue integral

For a given $f \in L^1[0, 2\pi]$ one can define the positive linear form

$$\mathfrak{m}(f) := \int_0^{2\pi} f,$$

which easily leads to

$$\left| \mathfrak{m}(f) \right| \leq \sup_{0\leq x\leq 2\pi} \left| \int_0^x f \right|. \tag{2.1}$$

The supremum is known as the Alexiewicz norm of f, denoted as $\|f\|_{\mathcal{A}}$. The linear space $L^1[0, 2\pi]$ with the Alexiewicz norm is normed but not complete. A basic result is the following:

Lemma 1. *The space* $(L^1[0, 2\pi], \|\cdot\|_{\mathcal{A}})$ *is a normed space.*

Definition 1. *The completion of the space $L^1[0, 2\pi]$ under the norm $\|\cdot\|_{\mathcal{A}}$ is denoted by $\mathcal{A}$.*

The map $\mathfrak{m}$ can be extended over functions out of $L^1[0, 2\pi]$. The extension of the linear form $\mathfrak{m}$ on $\mathcal{A}$ is denoted by the same symbol.

Definition 2. *The Henstock-Kurzweil integral is defined on $\mathcal{A}$ by*

$$\text{(HK)-}\int_0^{2\pi} f(t)dt := \mathfrak{m}(f) = \lim_{n\to\infty} \mathfrak{m}(f_n),$$

where $f_n \to f$ in the $\mathcal{A}$-norm.

The prefix (HK) makes it clear that it is not a Lebesgue integral.

Remark 1. *Definition 2 agrees with the classical integral of Henstock-Kurzweil (see [7]).*

The following results are well known, (see [8, Theorems 1 and 3] and [9, Theorems 2.5.1 to 2.5.12]).

Theorem 1.

(a) The Henstock-Kurzweil integral contains the Lebesgue integral in the sense that

$$\text{(HK)-}\int_0^{2\pi} f = \int_0^{2\pi} f \quad (\forall\, f \in L^1[0, 2\pi]).$$

(b) For given $f_1,\ f_2 \in \mathcal{A}$, and $\beta \in \mathbb{R}$, the equality

$$\text{(HK)-}\int_0^{2\pi} (f_1 + \beta f_2) = \text{(HK)-}\int_0^{2\pi} f_1 + \beta\text{(HK)-}\int_0^{2\pi} f_2$$

is valid.

(c) If $f \in \mathcal{A}$ and $(f_n) \subset L^1[0, 2\pi]$ converges to f in $\mathcal{A}$-norm, then

$$f\chi_{[0,t]} := \lim_{n\to\infty} f_n\chi_{[0,t]} \in \mathcal{A},$$

and $F(t) :=$ (HK)-$\int_0^t f :=$ (HK)-$\int_0^{2\pi} f\chi_{[0,t]}$ is continuous on $[0, 2\pi]$.

Definition 3. *Let v be a real valued function over $\mathbb{R}$. It is said v is a bounded variation function over $[0, 2\pi]$ if*

$$Var[v, [0, 2\pi]] := \sup \sum_{i=1}^{n} |v(x_i) - v(x_{i-1})| < \infty$$

where the supremum is taken over all partitions on $[0, 2\pi]$.

The space of functions that have finite variation on $[0, 2\pi]$ is denoted by $BV[0, 2\pi]$. In the space $BV[0, 2\pi]$ we introduce a norm by

$$\|v\|_{BV} := |v(2\pi)| + Var[v, [0, 2\pi]].$$

Note that this norm is equivalent to the one in [10, Theorems 2.2.1 and 2.2.2]. Thus $BV[0, 2\pi]$ with the given norm is a Banach space.

In order to give a characterization of the dual space $(\mathcal{A})'$, we use

$$L^1[0, 2\pi] \subseteq \mathcal{A} \;\Rightarrow\; (\mathcal{A})' \subseteq L^1[0, 2\pi]' = L^\infty[0, 2\pi].$$

This allows us to precisely see not only an element of $(\mathcal{A})'$ as an element in $L^\infty[0, 2\pi]$, but also as an element in the dual space of $C[0, 2\pi]$. In fact, $C[0, 2\pi]'$ is characterized by Borel measures, which in turn are related to the space of functions of bounded variation $BV[0, 2\pi]$, (see [11, Theorem 7.1.1] and [12, Theorem 6.3.3]).

The multiplier theorem for functions on $\mathcal{A}$ is well known, but for the convenience of the reader, we give a new proof in our context.

Theorem 2. *If $f \in \mathcal{A}$ and $\vartheta \in BV[0, 2\pi]$, then $f\vartheta \in \mathcal{A}$. Its integral is given by*

$$\text{(HK)-}\int_0^{2\pi} f\,\vartheta = \vartheta(2\pi)F(2\pi) - \int_0^{2\pi} F\,d\vartheta \tag{2.2}$$

where $F(t) = \text{(HK)-}\int_0^t f$ and the integral on the right side of the equation is in the Riemann-Stieltjes sense.

Proof. First suppose that $f \in L^1[0, 2\pi]$. Since ϑ is a bounded measurable function, then $f\vartheta \in L^1[0, 2\pi]$. Due to [13, Theorem 3.135], for given $f \in L^1[0, 2\pi]$, we can choose a sequence of functions $(f_n) \in C[0, 2\pi]$ converging to f in L^1-norm. If $F_n(t) := \int_0^t f_n$, then

$$\|F_n - F\|_\infty = \sup_{0\le x\le 2\pi} \left| \int_0^x f_n - f \right| \le \|f_n - f\|_{L^1} \to 0. \tag{2.3}$$

The integration by parts formula for the Riemann-Stieltjes integral and [12, Theorem 6.2.8] imply that

$$\int_0^{2\pi} \vartheta dF_n \quad \text{exists due to} \quad \int_0^{2\pi} F_n d\vartheta \quad \text{existing.}$$

Therefore, by the fundamental theorem of calculus and [14, Theorem 7.8] we get

$$\int_0^{2\pi} f_n\vartheta \;=\; \int_0^{2\pi} F_n'\vartheta \;=\; \int_0^{2\pi} \vartheta dF_n \;=\; \vartheta(2\pi)F_n(2\pi) - \int_0^{2\pi} F_n d\vartheta. \tag{2.4}$$

From [15, Corollary H.4] we get

$$\left| \int_0^{2\pi} F_n d\vartheta - \int_0^{2\pi} F d\vartheta \right| \le \|F_n - F\|_\infty \, \|\vartheta\|_{BV} \to 0.$$

This inequality and (2.3) yield, after taking limits on both sides of formula (2.4),

$$\int_0^{2\pi} f\,\vartheta = \vartheta(2\pi)F(2\pi) - \int_0^{2\pi} F\,d\vartheta \quad (\forall f \in L^1[0, 2\pi]). \tag{2.5}$$

On the other hand, we can define for $\vartheta \in BV[0, 2\pi]$ the multiplication operator

$$T_\vartheta : L^1[0, 2\pi] \to \mathcal{A}; \qquad T_\vartheta(f) := \vartheta f.$$

From (2.5) we get that T_ϑ is a bounded operator:

$$\|T_\vartheta(f)\|_{\mathcal{A}} \leq c\|f\|_{\mathcal{A}} \quad (\forall\, f \in L^1[0,2\pi]), \tag{2.6}$$

where c is a positive constant. It follows from the Bounded Linear Transformation Theorem [16, Theorem 1.7] that T_ϑ extends uniquely to a bounded operator from the completion of $(L^1[0,2\pi], \|\cdot\|_{\mathcal{A}})$ to $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. Denoting by the same symbol the extension of the operator on $\mathcal{A}$, one has

$$T_\vartheta(f) := \vartheta f \equiv f\vartheta = \lim_{n\to\infty} T_\vartheta(f_n) \in \mathcal{A}$$

whenever (f_n) converges to $f \in \mathcal{A}$. Now, using (2.5) and (2.6) and a sequence $L^1[0,2\pi] \ni f_n \to f$ in the $\mathcal{A}$-norm, we get the validity of the formula for all $f \in \mathcal{A}$. This proves the theorem. □

3. Duality and non-reflexivity in the space $\mathcal{A}$

In the case of the classical space $L^1(X; d\mu)$, with X an arbitrary measure space of positive measure μ, its dual space is the quotient space $L^\infty(X; d\mu) = \mathcal{L}^\infty(X; d\mu)\,/\mathcal{W}$, where $\mathcal{L}^\infty(X; d\mu)$ denotes the space of bounded measurable real functions defined in X and

$$\mathcal{W} = \{f : X \to \mathbb{R} : f = 0 \; a.e.\}\,.$$

The norm is defined on $L^\infty(X; d\mu)$ as:

$$\|f\|_\infty = \inf\{M \geq 0 : |f|\chi_{[M,\infty)} \in \mathcal{W}\}\,.$$

The map $g \longrightarrow F_g(f) = \int_X fg d\mu$ is an isometric isomorphism of $L^\infty(X; d\mu)$ onto $L^1(X; d\mu)'$ when $(X; d\mu)$ is σ-finite, (see [17, page 375]). We will show an analogous result for the space $\mathcal{A}$. We introduce the Banach space defined by the quotient space

$$BV_m := BV/Z,$$

where $Z := \{v \in BV[0,2\pi] : v(x) = 0 \;\; a.e.\}$.

Lemma 2. *BV_m is a Banach space of equivalence classes* $[v] := \{\, v' \in BV[0,2\pi] \; : \; v - v' \in Z \,\}$*, with the norm given by*

$$\|\,[v]\,\|_{BV_m} := \inf_{v' \in Z} \|v - v'\|_{BV}.$$

Proof. We will show that Z is a closed subspace of $BV[0,2\pi]$. Then the statement of the lemma is implied by a classical result in Functional Analysis [17, Theorem 4.2]. Given a Cauchy sequence $(v_n) \subset Z$ in the norm of $BV[0,2\pi]$ with limit v, we have that for each $n \in \mathbb{N}$, there exists a measurable set $\Upsilon_n \subset [0,2\pi]$ with

$$v_n \Big|_{\Upsilon_n} \equiv 0, \quad \text{such that } \gamma(\Upsilon_n^c) = 0.$$

Here γ is denoting the Lebesgue measure. Therefore, due to convergence in $BV[0,2\pi]$ implying in particular pointwise convergence, we get

$$v(x) = \lim_{n\to\infty} v_n(x) = 0 \qquad \Big(\forall\, x \in \bigcap_{m\in\mathbb{N}} \Upsilon_m\Big).$$

Furthermore,

$$\gamma\Big(\left(\cap_{m\in\mathbb{N}}\Upsilon_m\right)^c \Big) \leq \sum_m \gamma(\Upsilon_m^c) = 0.$$

This proves that the limit v belongs to Z. □

In [18, page 241] the normalized functions are defined. For our purposes we slightly modify that concept.

Definition 4. *Let $v \in BV[0, 2\pi]$. For $x \in [0, 2\pi)$ we define*

$$v^+(x) := \lim_{t\to x^+} v(t),$$

and $v^+(2\pi) = 0$. We will say that v^+ is the normalization in the Alexiewicz sense of the function v. The set of all such functions will be denoted by $\mathcal{NABV}[0, 2\pi]$.

Lemma 3. *Let $v \in BV[0, 2\pi]$. Then there exists a unique element $v^+ \in [v]$ belonging to $\mathcal{NABV}[0, 2\pi]$.*

Proof. Let $u, w \in [v]$ and $x \in [0, 2\pi)$. By [10, Corollary 2.1.23] the limits $u^+(x), w^+(x)$ exist. Let $\epsilon > 0$ be given. There exists $\delta_1(\epsilon) > 0$, such that if $s \in (x, x + \delta_1(\epsilon))$, then

$$|u^+(x) - u(s)| < \frac{\epsilon}{2}.$$

Also, there exists $\delta_2(\epsilon) > 0$ such that if $s \in (x, x + \delta_2(\epsilon))$, then

$$|w^+(x) - w(s)| < \frac{\epsilon}{2}.$$

On the other hand, there necessarily exists $y \in (x, x + \delta_\epsilon)$ such that

$$u(y) = w(y)$$

where $\delta_\epsilon = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$. So,

$$|u^+(x) - w^+(x)| \leq |u^+(x) - u(y)| + |w^+(x) - w(y)| < \epsilon$$

and since ϵ is arbitrary, we obtain

$$u^+(x) = w^+(x) \ \ \forall x \in [0, 2\pi].$$ □

Lemma 4. *Given $v \in BV[0, 2\pi]$ its normalization in the Alexiewicz sense v^+ belongs to $BV[0, 2\pi]$.*

Proof. Let $w \in [v]$. For an arbitrary partition $\{x_j\}_{j=0}^n$ of $[0, 2\pi]$, let $\{x'_j\}_{j=0}^n$ such that $x_j < x'_j < x_{j+1}$ for $0 \leq j \leq n-1$ and $x'_n = x_n$. Given $\epsilon > 0$, we choose x'_j sufficiently close to x_j such that

$$|v^+(x_j) - w(x'_j)| < \frac{\epsilon}{2^j}$$

for all $j = 0, 1, ..., n-1$. Therefore,

$$\sum_{j=1}^n |v^+(x_j) - v^+(x_{j-1})| \leq \sum_{j=1}^n |v^+(x_j) - w(x'_j)|$$

$$
\begin{aligned}
& + \sum_{j=1}^{n} |w(x'_j) - w(x'_{j-1})| + \sum_{j=1}^{n} |v^+(x_{j-1}) - w(x'_{j-1})| \\
\leq\ & |w(2\pi)| + \sum_{j=1}^{n-1} |v^+(x_j) - w(x'_j)| + \sum_{j=1}^{n} |w(x'_j) - w(x'_{j-1})| \\
& + \sum_{j=1}^{n} |v^+(x_{j-1}) - w(x'_{j-1})| \\
\leq\ & |w(2\pi)| + \sum_{j=1}^{n-1} \frac{\epsilon}{2^j} + \sum_{j=1}^{n} |w(x'_j) - w(x'_{j-1})| + \sum_{j=1}^{n} \frac{\epsilon}{2^{j-1}}.
\end{aligned}
$$

Now, taking the supremum over all partitions on $[0, 2\pi]$ we get

$$\|v^+\|_{BV} \leq 3\epsilon + \|w\|_{BV}.$$

Since ϵ is arbitrary, we conclude

$$\|v^+\|_{BV} \leq \|w\|_{BV} \quad (\forall\ w \in [v]).$$ □

Corollary 1. *If* $v \in BV[0, 2\pi]$*, then* $\|\ [v]\ \|_{BV_m} = \|v^+\|_{BV}$.

Theorem 3. *The spaces* $(\mathcal{A})'$ *and* BV_m *are isometrically isomorphic:*

(a) $(\mathcal{A})' = BV_m$.
(b) For every $\lambda \in (\mathcal{A})'$, *there exists a unique* $[v] \in BV_m$ *such that*

$$\|\lambda\|_{\mathcal{A}'} = \|\ [v]\ \|_{BV_m}.$$

Proof. (a) Let $\mathfrak{l} \in (\mathcal{A})'$. For given $f \in L^1[0, 2\pi]$, we take

$$F_0(t) := \int_0^t f \in C_0[0, 2\pi] \subset C[0, 2\pi], \tag{3.1}$$

where $C_0[0, 2\pi]$ consists of the continuous functions vanishing at zero. We define

$$\mathfrak{n}(F_0) = \mathfrak{l}(F'_0). \tag{3.2}$$

It follows that,

$$|\mathfrak{n}(F_0)| = |\mathfrak{l}(F'_0)| \leq c\|F'_0\|_{\mathcal{A}} = c\|F_0\|_\infty.$$

We get a continuous linear functional defined on a subspace of $C[0, 2\pi]$. From the Hahn-Banach Theorem there exists an extension on the space $C[0, 2\pi]$ with the same norm. We denote this extension by the same symbol $\mathfrak{n}$. Riesz Theorem [19, Theorem 2.14] assures that there exists a unique finite signed measure ρ so that

$$\mathfrak{n}(F_0) = \int_0^{2\pi} F_0\, d\rho \quad (\forall\ F_0 \in C_0[0, 2\pi]).$$

By the Jordan Decomposition [11, Theorem 5.1.8] each finite signed measure ρ is the difference of two measures μ_1 and μ_2, implying

$$\mathfrak{n}(F_0) = \int_0^{2\pi} F_0(t)d\mu_1 - \int_0^{2\pi} F_0(t)d\mu_2.$$

Substitution in this equation of the equality (3.1) and application of Fubini's Theorem yield

$$\begin{aligned} \mathfrak{n}(F_0) &= \int_0^{2\pi} f(s)ds \int_s^{2\pi} d\mu_1 - \int_0^{2\pi} f(s)ds \int_s^{2\pi} d\mu_2 \\ &= \int_0^{2\pi} f(s)(\mu_1([s, 2\pi] - \mu_2([s, 2\pi]))ds \\ &= \int_0^{2\pi} f(s)v^+(s)ds. \end{aligned} \tag{3.3}$$

Where, $v(s) = \mu_1([s, 2\pi]) - \mu_2([s, 2\pi])$ for all $s \in [0, 2\pi]$ is a bounded variation function [15, page 104], and $v^+ \in \mathcal{NABV}[0, 2\pi]$ is the normalization of v. It yields

$$\mathfrak{l}(f) = \mathfrak{l}(F_0') = \mathfrak{n}(F_0) = \int_0^{2\pi} f\, v^+, \tag{3.4}$$

where

$$F_0(s) = \int_0^s f, \qquad (\forall\, f \in L^1[0, 2\pi]).$$

For $f \in \mathcal{A}$, we can take a sequence $(f_n) \in L^1[0, 2\pi]$ such that (f_n) converges to f in the $\mathcal{A}$-norm. Continuity of $\mathfrak{l} \in \mathcal{A}'$ together with (3.4) give

$$\mathfrak{l}(f) = \lim_{n\to\infty} \int_0^{2\pi} v^+ f_n = \text{(HK)-}\int_0^{2\pi} v^+ f \quad (\forall\, f \in \mathcal{A}). \tag{3.5}$$

Due to $v^+ \in BV[0, 2\pi]$ and Lemma 3, we obtain $(\mathcal{A})' \subset BV_m$. The other contention is a consequence of item (a). In fact, if $v \in BV[0, 2\pi]$, then

$$\lambda(f) = \text{(HK)-}\int_0^{2\pi} fv^+$$

defines a continuous functional on $\mathcal{A}$, and is associated with a unique element of the quotient space BV_m by Lemma 3. (b) Let $\lambda \in (\mathcal{A})'$, and from (a) we know that there exists $v^+ \in [v]$ representing λ. Corollary 1 and (2.2) imply

$$\|\lambda\|_{\mathcal{A}'} \le \|v^+\|_{BV} = \|\,[v]\,\|_{BV_m}. \tag{3.6}$$

We now prove the reverse inequality. Let $\{x_i'\}_{i=1}^n$ be a partition such that

$$\sum_{i=1}^n |v^+(x_i') - v^+(x_{i-1}')| \ge \|\,[v]\,\|_{BVm} - \epsilon.$$

We pick up points $\{x_i\}_{i=0}^{n}$ of $[0, 2\pi]$ as follows: Let $x_0 = x_0^{'} = 0$ and $x_n = x_n^{'} = 2\pi$. Since v^+ has a countable number of discontinuities, we can find points $x_i^{'} < x_i < x_{i+1}^{'}$ such that v^+ is continuous at every x_i. Moreover, we choose each x_i close enough to $x_i^{'}$ so that

$$|v^+(x_i^{'}) - v^+(x_i)| \le \frac{\epsilon}{2^i}, \quad \text{for } i = 1, 2, 3, ..., n-1.$$

Now we consider the partition $\{\xi_i\}_{i=0}^{2n-1} = \{x_i\}_{i=1}^{n-1} \cup \{x_i^{'}\}_{i=0}^{n}$. Note that $\xi_0 = x_0$, $\xi_1 = x_1^{'}$, $\xi_2 = x_1$, $\xi_3 = x_2^{'}$, $\xi_4 = x_2$, ... , $\xi_{2i-1} = x_i^{'}$, $\xi_{2i} = x_i$, ... , $\xi_{2n-3} = x_{n-1}^{'}$, $\xi_{2n-2} = x_{n-1}$, $\xi_{2n-1} = x_n$.

Let $\delta > 0$ such that $\delta < \min\{|\xi_i - \xi_{i-1}|, i = 1, ..., 2n-1\}$ and consider $F_{\delta,i}\colon [0, 2\pi] \to \mathbb{R}$, to be a continuous function defined by

$$F_{\delta,i}(x) = \begin{cases} 0 & if \quad x \le x_i \quad \text{or} \quad x \ge x_{i+1}^{'} + \delta \\ \frac{1}{\delta}(x - x_i) & if \qquad x_i < x < x_i + \delta \\ 1 & if \qquad x_i + \delta \le x \le x_{i+1}^{'} \\ 1 - \frac{1}{\delta}(x - x_{i+1}^{'}) & if \qquad x_{i+1}^{'} < x < x_{i+1}^{'} + \delta \end{cases}$$

for $i = 0, 1, 2, ..., n-2$. So,

$$\lambda(f_{\delta,i}) = \text{(HK)-}\int_0^{2\pi} f_{\delta,i} v^+,$$

where $f_{\delta,i} = F'_{\delta,i} \in \mathcal{A}$ is defined almost everywhere. Now, it is clear that

$$\begin{aligned} \text{(HK)-}\int_0^{2\pi} f_{\delta,i} v^+ &= -\int_0^{2\pi} F_{\delta,i} dv^+ \\ &= -\int_{x_i}^{x_i+\delta} \left(\frac{x - x_i}{\delta}\right) dv^+ - \int_{x_i+\delta}^{x_{i+1}^{'}} dv^+ \\ &\quad - \int_{x_{i+1}^{'}}^{x_{i+1}^{'}+\delta} \left(1 - \frac{x - x_{i+1}^{'}}{\delta}\right) dv^+. \end{aligned}$$

Using continuity in x_i and right continuity in $x_i^{'}$ of v^+, from [10, Corollary 2.3.4, Lemma 5.1.11] it yields

$$\lim_{\delta \to 0} \lambda(f_{\delta,i}) = -[v^+(x_{i+1}^{'}) - v^+(x_i)]$$

for $i = 0, 1, ..., n-2$. When $i = n-1$, we put

$$F_{\delta,n-1}(x) = \begin{cases} 0 & if \qquad x \le x_{n-1} \\ \frac{1}{\delta}(x - x_{n-1}) & if \quad x_{n-1} < x < x_{n-1} + \delta \\ 1 & if \quad x_{n-1} + \delta \le x \le 2\pi \end{cases}.$$

In any case the previous argument is valid. We can suppose that each $v^+(x_i^{'}) - v^+(x_{i-1})$ is negative; otherwise, we change the sign of each $F_{\delta,i}$ in the corresponding interval, so,

$$\lim_{\delta \to 0} \lambda(f_{\delta,i-1}) = |v^+(x_i^{'}) - v^+(x_{i-1})|$$

for $i = 1, ..., n$. On the other hand,

$$\sum_{i=1}^{2n-1} |v^+(\xi_i) - v^+(\xi_{i-1})| = \sum_{i=1}^{n} |v^+(x_i^{'}) - v^+(x_{i-1})| + \sum_{i=1}^{n-1} |v^+(x_i) - v^+(x_i^{'})|.$$

Therefore,

$$\sum_{i=1}^{2n-1} |v^+(\xi_i) - v^+(\xi_{i-1})| \leq \sum_{i=1}^{n} \lim_{\delta\to 0} \lambda(f_{\delta,i-1}) + \sum_{i=1}^{n-1} \frac{\epsilon}{2^i} \leq \lim_{\delta\to 0} \lambda(f_\delta) + \epsilon$$

where $f_\delta = \sum_{i=1}^n f_{\delta,i-1}$. Thus, there exists $\rho > 0$ such that

$$\begin{aligned} |\lambda(f_\rho)| &\geq \sum_{i=1}^{2n-1} |v^+(\xi_i) - v^+(\xi_{i-1})| - 2\epsilon \geq \sum_{i=1}^{n} |v^+(x_i^{'}) - v^+(x_{i-1}^{'})| - 2\epsilon \\ &\geq \|\, [v] \,\|_{BV_m} - 3\epsilon. \end{aligned}$$

Yielding,

$$\|\lambda\|_{\mathcal{A}'} \cdot \|f_\rho\|_{\mathcal{A}} \geq |\lambda(f_\rho)| \geq \|\, [v] \,\|_{BVm} - 3\epsilon.$$

It is clear that

$$\|f_\rho\|_{\mathcal{A}} = 1.$$

From this and (3.6) we conclude

$$\|\lambda\|_{\mathcal{A}'} = \|\, [v] \,\|_{BV_m}. \qquad \square$$

The following example shows that $\mathcal{A}$ is not a reflexive space. We recall that for given $y \in \mathcal{A}$, $y^* \in BV'$ is defined by

$$y^*(v) := \int_0^{2\pi} yv \quad (\forall v \in BV).$$

$\mathcal{A}$ is called reflexive if every element $L \in BV_m'$ obeys $L = y^*$ for some $y \in \mathcal{A}$.

Example 1. *Let $L\colon BV_m \to \mathbb{R}$ be defined by*

$$L([v]) = \frac{1}{2}(\, v^+(2/3) - v^+(1/2)) + \frac{1}{2}(\, v^+((2/3)^-) - v^+((1/2)^-)\,).$$

Similar arguments as given previously show that

$$|L([v])| \leq \|\, [v] \,\|_{BV_m}.$$

So, this proves that $L \in BV_m'$. We will prove that there is no $y \in \mathcal{A}$ such that $y^* = L$. Let $y \in \mathcal{A}$ and

$$Y(t) = (HK) - \int_0^t y.$$

We note that $Y(0) = 0$, and suppose that $Y(t)$ is not the identically zero function. Continuity of Y implies that we can find $0 < c_1 \leq 2\pi$ such that $Y(c_1) \neq 0$. If $0 < c_1 < \frac{1}{2}$, we consider the function v^+ equal to -1 on $[0, c_1)$ and 0 otherwise. It follows that

$$y^*(v^+) = (HK) - \int_0^{2\pi} yv^+ = -\int_0^{2\pi} Y dv^+.$$

By [20, Theorems 6.1.1 and 6.1.6] we have

$$y^*(v^+) = -Y(c_1) \neq 0 = L([v]).$$

In the case $1/2 \leq c_1 \leq 2\pi$ *the proof is similar. We have proved that there is no* $y \in \mathcal{A}$ *such that* $y^* = L$.

Corollary 2. *The space* $\mathcal{A}$ *is not reflexive.*

Remark 2. *A Banach space X is reflexive if and only if every continuous linear functional on the space X attains its supremum on the closed unit ball, (see [21, 22]). In fact, in the proof of Theorem 3* (*b*) *we showed that, for a given functional* λ*, there exists a function* f *such that the absolute value* $|\lambda(f)|$ *is arbitrarily close to the norm of the functional, but does not necessarily attain this value.*

4. Conclusions

We have presented a new way of constructing the dual space of the space of integrable functions in HK, which may motivate a discussion about applying this method to similar studies in related spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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