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Review

Wiener Tauberian theorem and half-space problems for parabolic and elliptic equations

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Abstract: For various kinds of parabolic and elliptic partial differential and differential-difference equations, results on the stabilization of solutions are presented. For the Cauchy problem for parabolic equations, the stabilization is treated as the existence of a limit as the time unboundedly increases. For the half-space Dirichlet problem for parabolic equations, the stabilization is treated as the existence of a limit as the independent variable orthogonal to the boundary half-plane unboundedly increases. In the classical case of the heat equation, the necessary and sufficient condition of the stabilization consists of the existence of the limit of mean values of the initial-value (boundary-value) function over balls as the ball radius tends to infinity. For all linear problems considered in the present paper, this property is preserved (including elliptic equations and differential-difference equations). The Wiener Tauberian theorem is used to establish this property. To investigate the differential-difference case, we use the fact that translation operators are Fourier multipliers (as well as differential operators), which allows one to use a standard Gel'fand-Shilov operational scheme. For all quasilinear problems considered in the present paper, the mean value from the stabilization criterion is changed: It undergoes a monotonic map, which is explicitly constructed for each investigated nonlinear boundary-value problem.

Keywords: stabilization; asymptotical closeness; elliptic equations; parabolic equations; differential-difference equations; quasilinear equations
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1. Introduction

The stabilization property of solutions of the Cauchy problem for the heat equation has been well known since the middle of the previous century: The long-time behavior of the solution is determined by the limit properties of means of the initial-value function.

The following pioneering result in this research direction was obtained in [1]: If $u(x,t)$ is the classical bounded solution of the Cauchy problem for the heat equation with a bounded initial-value function, then, for each x, its limit as $t \to \infty$ exists (and we say that the stabilization of the solution takes place) if and only if the mean of the initial-value function over balls centered at the origin has a limit as the radius of the balls tends to infinity; moreover, if these limits exist, then they are equal to each other. In particular, this means that, though solutions of such problems are functions of $n + 1$ variables, they cannot have nonconstant limits as $t \to \infty$.

During the last half of century, this remarkable fundamental result was substantially generalized in various directions. The consideration is extended to various equations with variable coefficients (see, e.g., [2–5] and references therein). Proximity (asymptotical closeness) theorems were obtained in [6–9] (see references therein as well). Theorems of such kind establish the decay of the difference $u(x, t) - v(x, t)$, where $u(x, t)$ is the investigated solution, while $v(x, t)$ is the so-called "etalon function" with known qualitative properties (such as the solution of the heat equation with a prescribed trace on the initial hyperplane). The advantage of proximity theorems compared with stabilization ones is as follows: The knowledge about the behavior of the solution is provided even in the case where there is no stabilization. The case of unbounded initial-value functions is investigated in [2, 3, 10] (see references therein as well). The fundamental novelty of this case is that the stabilization of solutions to nonconstant limits is possible now.

Several important extensions are the concern of the present review.

Section 3 is devoted to the Cauchy problem for differential-difference equations (i.e., equations containing translation operators apart from differential ones) of the parabolic type. The worldwide interest in differential-difference equations (and to the more general object called *functional-differential* equations) is caused both by their numerous applications not covered by classical models of mathematical physics and by purely theoretical reasons: The nonlocal problem of such equations generates qualitatively new phenomena not arising in the classical case of differential equations, while various research methods proved to be efficient for the theory of differential equations turn out to be inapplicable to functional-differential ones (e.g., this refers to all methods based on the maximum principle because, unlike differential equations, the investigated equation links values of the desired function at different points); hence, qualitatively new methods are to be developed (see [11–14] and references therein).

In Section 4, the stabilization phenomenon is considered for elliptic equations. Though all independent variables are spatial in the elliptic case, the term "stabilization" is reasonable for the Dirichlet problem in half-spaces for elliptic equations: Once we violate the domain isotropy, leaving only one half of $\mathbb{R}^n$, the independent variable selected this way (i.e., the only independent variable varying along the semiaxis) becomes qualitatively different from all other ones: The resolving operator of the specified problem possesses the semigroup property with respect to the *spatial* variable y, and the solution of the specified problem is represented by the convolution of the boundary-value function with a function qualitatively similar to the Poisson kernel of the Cauchy problem for the heat equation (it vanishes at infinity sufficiently fast to ensure the convergence of the specified convolution for each bounded boundary-value function). This causes the stabilization of solutions of elliptic problems (as $y \to \infty$), and this holds not only for differential parabolic equations but for differential-difference ones as well.

Section 5 is devoted to nonlinear generalizations of the stabilization theory. We consider quasilinear equations with nonlinearities of the Kardar-Parisi-Zhang type (KPZ-type), i.e., equations containing the term $|\nabla u|^2$. Equations of this kind, studied since the pioneering paper [15], arise in numerous applications not covered by the linear theory (see, e.g., [16] and references therein). On the other hand, such nonlinearities attract the attention of worldwide researchers from the purely theoretical viewpoint: It is known (see, e.g., [17–19]) that the second power is the greatest one such that Bernstein-type conditions for the corresponding boundary-value problem guarantee the validity of a priori L^∞-estimates of first-order derivatives of the solution via the L^∞-estimate of the solution itself. Stabilization theorems are valid for both parabolic and elliptic equations with KPZ-nonlinearities.

2. Cauchy problem for heat equation

The following fact is well known as a corollary of the Wiener Tauberian theorem (see, e.g., [20, p. 1003–1004]):

Let $f \in L^\infty(0,\infty)$ and there exist ψ_0 from $L_1(0,\infty)$ such that its Mellin transform $\int\limits_0^\infty \tau^{ix}\psi_0(\tau)d\tau$ has no real zeroes, and

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^\infty \psi_0\left(\frac{\tau}{t}\right)f(\tau)d\tau = 0.$$

Then, $\lim\limits_{t\to\infty}\frac{1}{t}\int\limits_0^\infty \psi\left(\frac{\tau}{t}\right)f(\tau)d\tau = 0$ *for each ψ from $L_1(0,\infty)$.*

To explain (on a simple visual sample) how to use it, consider the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \Delta u \tag{2.1}$$

with an initial-value function $u_0(x,t) =: u_0(x_1,\dots,x_n,t)$ belonging to $L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$.

Its (unique classical bounded) solution $u(x,t)$ obeys the famous Repnikov-Eidel'man alternative, the most essential part of which reads as follows (see [1]):

$$\lim_{t\to\infty} u(x,t) = 0 \quad \Longleftrightarrow \quad \lim_{R\to\infty}\frac{1}{|\{|y|<R\}|}\int\limits_{\{|y|<R\}} u_0(x+y)dy = 0$$

for each x from $\mathbb{R}^n$.

The hardest (at least, historically) part of the proof is the necessity.

To prove this, fix an arbitrary x and assume that $\lim\limits_{t\to\infty} u(x,t) = 0$. Then, $\lim\limits_{t\to\infty} u\left(x,\frac{t^2}{4}\right) = 0$, i.e.,

$$0 \overset{\infty\leftarrow t}{\longleftarrow} \frac{1}{t^n}\int\limits_{\mathbb{R}^n} e^{-\frac{|\xi-x|^2}{t^2}}u_0(\xi)d\xi = \int\limits_{\mathbb{R}^n} e^{-|\eta|^2}u_0(x+t\eta)d\eta$$

$$= \int\limits_0^\infty \int\limits_{\{|\eta|=\rho\}} e^{-|\eta|^2}u_0(x+t\eta)d\sigma_\eta d\rho = \int\limits_0^\infty e^{-\rho^2}\int\limits_{\{|\eta|=\rho\}} u_0(x+t\eta)d\sigma_\eta d\rho.$$

Change the variable as follows: $t\eta := y$. Then, $d\sigma_y = t^{n-1} d\sigma_\eta$, and, therefore,

$$0 \overset{\infty \leftarrow t}{\longleftarrow} \frac{1}{t^{n-1}} \int_0^\infty e^{-\rho^2} \int_{\{|y|=t\rho\}} u_0(x+y) d\sigma_y d\rho = \text{const} \int_0^\infty e^{-\rho^2} \rho^{n-1} F(x; t\rho) d\rho,$$

where $F(x; r)$ is the mean value of u_0 over the sphere of radius r centered at x.
Denoting $t\rho$ by τ, obtain that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^\infty e^{-\left(\frac{\tau}{t}\right)^2} \left(\frac{\tau}{t}\right)^{n-1} F(x; \tau) d\tau = 0.$$

Assign $f(r) := F(x; r) \in L^\infty(0, \infty)$ and $\psi_0(r) := e^{-r^2} r^{n-1} \in L_1(0, \infty)$.

Now, let us check the Mellin transform of the function $e^{-r^2} r^{n-1}$ for zeroes:

$$\int_0^\infty \tau^{ix} e^{-\tau^2} \tau^{n-1} d\tau = \frac{1}{2} \int_0^\infty z^{\frac{n+ix}{2}-1} e^{-z} dz = \frac{1}{2} \Gamma\left(\frac{n+ix}{2}\right);$$

indeed, if x is real, then there are no real zeroes.
It remains to represent $\frac{1}{|\{|y| < R\}|} \int_{\{|y|<R\}} u_0(x+y) dy$ as

$$\frac{\text{const}}{R^n} \int_0^R \int_{\{|y|=r\}} u_0(x+y) d\sigma_y dr = \frac{\text{const}}{R^n} \int_0^R r^{n-1} F(x; r) dr.$$

This is equal to $\frac{\text{const}}{R} \int_0^\infty \psi\left(\frac{r}{R}\right) f(r) dr$,

where $\psi(r) = \begin{cases} r^{n-1} \text{ if } r \le 1, \\ 0 \text{ otherwise} \end{cases}$, i.e., $\psi \in L_1(0, \infty)$.

Thus, the above corollary of the Wiener Tauberian theorem is applicable, and, therefore, the necessity is proved.

The next step is to prove that

$$\lim_{R\to\infty} \left[\frac{1}{|\{|y| < R\}|} \int_{\{|y|<R\}} u_0(x+y) dy - \frac{1}{|\{|y| < R\}|} \int_{\{|y|<R\}} u_0(y) dy \right] = 0$$

for each x.
To do that, it suffices to note that the measure of the geometrical difference between the sets $\{|y| < R\} \cup \{|y - x| < R\}$ and $\{|y| < R\} \cap \{|y - x| < R\}$ is estimated from above by $\frac{\text{const}}{R^{n-1}}$.

It remains to change the zero limit for an arbitrary real constant l. To do that, the initial-value function $u_0(x)$ is changed for $u_0(x) - l$; this changes the solution for the function $u(x, t) - l$ (because the problem is linear and the solution is unique).

This yields the Repnikov-Eidel'man alternative (see [1]):
either the solution tends to a constant as $t \to \infty$, or its limit as $t \to \infty$ exists for no x from $\mathbb{R}^n$.

Numerous extensions of this fundamental result to cases where the elliptic operator in Eq (2.1) is generalized in various directions are obtained though the restriction for the initial-value function to be bounded is, in a way, unimprovable (see, e.g., [3] and references therein).

3. Parabolic differential-difference equations

The above phenomenon takes place for equations of the kind

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{j=1}^{n} \sum_{k=1}^{m_j} a_{jk} u(x + b_{jk} h_j, t), \tag{3.1}$$

where $h_j = (h_{j1}, \dots, h_{jn})$ are mutually orthogonal (for $j = \overline{1, n}$) in $\mathbb{R}^n$ unit vectors, while a_{jk} and b_{jk} are real constants and of the kind

$$\frac{\partial u}{\partial t} = L_{(n)} u := \Delta u + \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \frac{\partial^2 u}{\partial x_i^2}(x_1, \dots, x_{i-1}, x_i + b_{ij}, x_{i+1}, \dots, x_n, t), \tag{3.2}$$

where a_{ij} and b_{ij} are real constants.

The following property is used: Translation operators are Fourier multipliers. Using this property, one can apply the Gel'fand-Shilov operational scheme of [21] to construct solutions of the Cauchy problem (again, with continuous and bounded initial-value functions) for the specified equations in the form

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{E}(x - \xi, t) u_0(\xi) d\xi, \tag{3.3}$$

where

$$\mathcal{E}(x, t) = \int_{\mathbb{R}^n} e^{-tG_1(\xi)} \cos\Big[x \cdot \xi - tG_2(\xi)\Big] d\xi, \tag{3.4}$$

$G_1(\xi) = |\xi|^2 - \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{m_j} a_{jk} \cos b_{jk} h_j \cdot \xi$ and $G_2(\xi) = \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{m_j} a_{jk} \sin b_{jk} h_j \cdot \xi$ for the case of Eq (3.1), and $G_1(\xi) = \sum\limits_{k,j=1}^{n} a_{kj} \xi_k^2 \cos b_{kj} \xi_j$ and $G_2(\xi) = \sum\limits_{k,j=1}^{n} a_{kj} \xi_k^2 \sin b_{kj} \xi_j$ for the case of Eq (3.2).

For the said problem for Eq (3.1), the function $u(x, t)$ defined by (3.3) is the unique classical solution bounded in the layer $\{0 \leq t \leq T\}$ for each positive T, and this solution is infinitely smooth outside the initial-value hyperplane without any additional restriction for a_{jk} and b_{jk}. To establish its long-time behavior, we have to impose such restrictions. To do that, we have to introduce the notion of differential-difference operators strongly elliptic in the whole space (cf. [11, § 8] for bounded domains).

Definition 3.1. A Fourier multiplier is said to be *strongly elliptic* if the real part of its symbol is bounded from above by $-C|\xi|^2$, where C is a positive constant.

Considering the case of Eq (3.1), assume (without loss of generality) that the (finite) number sequence $\{a_{jk}\}_{k=1}^{m_j}$, $j=\overline{1,n}$, does not decrease. For any $j\in\overline{1,n}$, denote $\min\limits_{a_{jk}>0} k$ by m_j^0; if j is such that $a_{jk}<0$ for any $k\in\overline{1,m_j}$, then denote m_j+1 by m_j^0. Denote the differential-difference operator at the right-hand part of Eq (3.1) by L. Also, introduce the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u=\Delta u+\sum_{j=1}^{n}\sum_{k<m_j^0}a_{jk}u(x+b_{jk}h_j,t).$$

The following assertion is valid for the case of Eq (3.1).

Theorem 3.1. *Let the operator* $\mathcal{L}-\sum\limits_{j=1}^{n}\sum\limits_{k<m_j^0}a_{jk}I$ *be strongly elliptic. Then,*

$$\lim_{t\to+\infty}\left[e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j}a_{jk}}u(x,t)-w\left(\frac{x_1+q_1t}{p_1},\dots,\frac{x_n+q_nt}{p_n},t\right)\right]=0 \tag{3.5}$$

for each x *from* $\mathbb{R}^n$, *where* $w(x,t)$ *is the bounded solution of the Cauchy problem for Eq* (2.1) *with the initial-value function* $u_0(p_1x_1,\dots,p_nx_n)$,

$$p_j=\sqrt{1+\frac{1}{2}\sum_{k=1}^{m_j}a_{jk}b_{jk}^2},\qquad \text{and}\qquad q_j=\sum_{k=1}^{m_j}a_{jk}b_{jk},\ j=\overline{1,n}.$$

Remark 3.1. The positivity of the last radicand is guaranteed by the strong ellipticity of the operator $\mathcal{L}-\sum\limits_{j=1}^{n}\sum\limits_{k<m_j^0}a_{jk}I$.

Note that Theorem 3.1 is an asymptotical closeness theorem. In general, assertions of such a kind are stronger than stabilization ones: Unlike stabilization theorems, they provide the information even in the case where the etalon function w has no limit. In particular, Theorem 3.1 implies the following stabilization result:

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied,* $l\in\mathbb{R}^1$, *and* $a_j\perp b_j$ *for any* $j\in\overline{1,n}$, *where* $a_j=(a_{j1},\dots,a_{jm_j})$ *and* $b_j=(b_{j1},\dots,b_{jm_j})$. *Then,*

$$\lim_{t\to\infty}e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j}a_{jk}}u(x,t)=l\ \textit{for any}\ x\in\mathbb{R}^n$$

if and only if

$$\lim_{t\to\infty}\frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}t^n\prod\limits_{j=1}^{n}p_j}\int\limits_{\left\{\sum\limits_{j=1}^{n}\frac{x_j^2}{p_j^2}<t^2\right\}}u_0(x)dx=l. \tag{3.6}$$

For the Cauchy problem for Eq (3.2), under the assumption that the operator $-L_{(n)}$ is strongly elliptic in $\mathbb{R}^n$, the function $u(x,t)$ defined by (3.3) is the unique solution in the sense of generalized functions, and it is infinitely smooth outside the initial-value hyperplane without any additional restriction for a_{jk} and b_{jk}.

Unlike the case of Eq (3.1), no additional restrictions are imposed in assertions on asymptotical properties. The main result on asymptotical properties is as follows:

Theorem 3.3. *If the operator $-L_{(n)}$ is strongly elliptic, then*

$$\lim_{t\to+\infty}[u(x,t)-w(x,t)]=0 \tag{3.7}$$

for each x from $\mathbb{R}^n$, where $w(x,t)$ is the bounded solution of the Cauchy problem for the parabolic differential equation

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}p_i\frac{\partial^2 u}{\partial x_i^2}, \tag{3.8}$$

where $p_i=1+\sum_{j=1}^{m_i}a_{ij}$, $i=\overline{1,n}$, with the initial-value function $u_0(x_1,\dots,x_n)$.

Remark 3.2. The positivity of each coefficient p_i is guaranteed by the strong ellipticity of the operator $-L_{(n)}$. Thus, Eq (3.8) is parabolic and, therefore, the specified function $w(x,t)$ is well defined.

As above, the stronger result on the asymptotical closeness implies the stabilization criterion:

Theorem 3.4. *If the operator $-L_{(n)}$ is strongly elliptic, and $l\in\mathbb{R}^1$, then $\lim_{t\to\infty}u(x,t)=l$ for any $x\in\mathbb{R}^n$ if and only if* (3.6) *is satisfied.*

The principal technical distinction of the differential-difference case is that the Wiener Tauberian theorem is applied to prove the sufficiently fast decay of the Poissonian kernel $\mathcal{E}(x,t)$ (and, therefore, the well-definiteness of its convolutions with bounded functions) as well (more exactly, the author is not aware of any proof not using the Wiener Tauberian theorem). This is visibly illustrated on the sample case of the prototype equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\sum_{k=1}^{m}a_k u(x-h_k,t), \tag{3.9}$$

i.e., Eq (3.1) with one spatial variable.
In this case, the function $\mathcal{E}(x,t)$ can be decomposed into its even and odd (with respect to x) terms $\mathcal{E}_1(x,t)$ and $\mathcal{E}_2(x,t)$:

$$\mathcal{E}_1(x,t)=\int_0^\infty e^{-t(\xi^2-\sum_{k=1}^{m}a_k\cos h_k\xi)}\cos x\xi\,\cos\left(t\sum_{k=1}^{m}a_k\sin h_k\xi\right)d\xi$$

and

$$\mathcal{E}_2(x,t)=\int_0^\infty e^{-t(\xi^2-\sum_{k=1}^{m}a_k\cos h_k\xi)}\sin x\xi\,\sin\left(t\sum_{k=1}^{m}a_k\sin h_k\xi\right)d\xi.$$

Change the variable as follows: $\eta = x\xi$. This yields the relation

$$\mathcal{E}_1(x,t) = \frac{1}{x}\int\limits_0^\infty e^{-t(\frac{\eta}{x})^2} e^{t\sum\limits_{k=1}^m a_k \cos\frac{h_k\eta}{x}} \cos\left(t\sum_{k=1}^m a_k \sin\frac{h_k\eta}{x}\right)\cos\eta\, d\eta = \frac{1}{x}\int\limits_0^\infty \psi\left(\frac{\eta}{x}\right) f(\eta)d\eta,$$

where

$$f(\tau) = \cos\tau \in L^\infty(\mathbb{R}^1_+),$$
$$\psi(\tau) = e^{-t\tau^2} e^{t\sum\limits_{k=1}^m a_k\cos h_k\tau} \cos\left(t\sum_{k=1}^m a_k \sin h_k\tau\right) \in L_1(\mathbb{R}^1_+).$$

Denoting $e^{-t\tau^2}$ by $\psi_0(\tau)$, we see that $\psi_0(\tau) \in L_1(\mathbb{R}^1_+)$. Further, the Mellin transform of the function $\psi_0(\tau)$ is defined on the real axis, and it has no real zeros; indeed,

$$\int\limits_0^\infty \tau^{ix}\psi_0(\tau)d\tau = \frac{1}{2t^{\frac{1+ix}{2}}}\int\limits_0^\infty z^{\frac{ix-1}{2}} e^{-z}dz = \frac{\Gamma(\frac{1+ix}{2})}{2t^{\frac{1+ix}{2}}}.$$

Further,

$$\frac{1}{r}\int\limits_0^\infty \psi_0\left(\frac{\tau}{r}\right) f(\tau)d\tau = \frac{\sqrt{\pi}}{2\sqrt{t}}e^{-\frac{r^2}{4t}} \xrightarrow{r\to\infty} 0\ .$$

Then,

$$\frac{1}{r}\int\limits_0^\infty \psi\left(\frac{\tau}{r}\right) f(\tau)d\tau \xrightarrow{r\to\infty} 0$$

due to the above corollary of the Wiener Tauberian theorem, i.e., $\mathcal{E}_1(x,t)$ tends to zero as $x \to \infty$ for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$.

Now, consider $\mathcal{E}_2(x,t)$.

Denote the function $e^{-t\tau^2} e^{t\sum\limits_{k=1}^m a_k\cos h_k\tau} \sin\left(t\sum\limits_{k=1}^m a_k\sin h_k\tau\right)$ by $\psi(\tau) \in L_1(\mathbb{R}^1_+)$. Denote the function $\sin\tau$ by $f(\tau) \in L^\infty(\mathbb{R}^1_+)$. Then,

$$\frac{1}{r}\int\limits_0^\infty \psi_0\left(\frac{\tau}{r}\right) f(\tau)d\tau = \frac{r}{2t}F\left(1, \frac{3}{2}, -\frac{r^2}{4t}\right) \xrightarrow{r\to\infty} 0,$$

where F denotes the second-kind degenerate hypergeometric function.

Thus, the assumptions of the Wiener Tauberian theorem are satisfied. Hence, for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$, we have

$$\mathcal{E}_2(x,t) = \frac{1}{x}\int\limits_0^\infty \psi\left(\frac{\tau}{x}\right) f(\tau)d\tau \xrightarrow{r\to\infty} 0\ .$$

Thus,

$$\lim_{x\to\infty}\mathcal{E}(x,t) = 0$$

for any positive t and any $a, h \in \mathbb{R}^m$.

However, the obtained limit relation is not sufficient to prove the convergence of the convolution of the Poissonian kernel with bounded initial-value functions. We have to estimate the rate of the proved decay. To do that, we integrate the term $\mathcal{E}_1(x,t)$ by parts:

$$\int_0^\infty e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \cos(t\sum_{k=1}^m a_k \sin h_k\xi) \cos x\xi d\xi$$
$$= \frac{1}{x}\left[e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \cos(t\sum_{k=1}^m a_k \sin h_k\xi) \sin x\xi \Big|_{\xi=0}^{\xi=\infty} + t\int_0^\infty e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \right.$$
$$\left. \times \left((2\xi + \sum_{k=1}^m h_k a_k \sin h_k\xi) \cos(t\sum_{k=1}^m a_k \sin h_k\xi) + \sin(t\sum_{k=1}^m a_k \sin h_k\xi) \sum_{k=1}^m h_k a_k \cos h_k\xi \right) \sin x\xi d\xi \right]$$
$$= \frac{t}{x}\int_0^\infty e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \left(2\xi\cos(t\sum_{k=1}^m a_k \sin h_k\xi) + \sum_{k=1}^m a_k h_k \sin(h_k\xi + t\sum_{k=1}^m a_k \sin h_k\xi) \right) \sin x\xi d\xi.$$

Denote the derivative (with respect to ξ) of

$$e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \left(2\xi\cos(t\sum_{k=1}^m a_k \sin h_k\xi) + \sum_{k=1}^m a_k h_k \sin(h_k\xi + t\sum_{k=1}^m a_k \sin h_k\xi) \right)$$

by $\psi(\xi)$ and integrate by parts again. We see that

$$\mathcal{E}_1(x,t) = \frac{t}{x^2}\left[e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \left(2\xi\cos(t\sum_{k=1}^m a_k \sin h_k\xi) \right.\right.$$
$$\left.\left. + \sum_{k=1}^m a_k h_k \sin(h_k\xi + t\sum_{k=1}^m a_k \sin h_k\xi) \right) \cos x\xi \Big|_{\xi=\infty}^{\xi=0} + \int_0^\infty \psi(\xi)\cos x\xi d\xi \right] = \frac{t}{x^2}\int_0^\infty \psi(\xi)\cos x\xi d\xi,$$

i.e., $x^2\mathcal{E}_1(x,t) = \frac{t}{x}\int_0^\infty \psi\left(\frac{\eta}{x}\right)\cos\eta d\eta.$

Since $\psi(\xi) \in L_1(\mathbb{R}^1_+)$, it follows that the assumptions of the Wiener Tauberian theorem are satisfied. Hence, $x^2\mathcal{E}_1(x,t) \overset{x\to\infty}{\longrightarrow} 0$ for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$.

In the same way, consider the second term of the Poissonian kernel.

$$\mathcal{E}_2(x,t) = \frac{1}{x}\left[e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \sin(t\sum_{k=1}^m a_k \sin h_k\xi) \cos x\xi \Big|_{\xi=\infty}^{\xi=0} \right.$$
$$- \int_0^\infty e^{t(\sum_{k=1}^m a_k \cos h_k\xi - \xi^2)} \left(t(\sum_{k=1}^m a_k h_k \sin h_k\xi + 2\xi) \sin(t\sum_{k=1}^m a_k \sin h_k\xi) \right.$$
$$\left.\left. - t\cos(t\sum_{k=1}^m a_k \sin h_k\xi) \sum_{k=1}^m a_k h_k \cos h_k\xi \right) \cos x\xi d\xi \right].$$

Thus, the second term of the Poissonian kernel is equal to

$$-\frac{t}{x}\int\limits_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\left(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi)-\sum_{k=1}^m a_kh_k\cos(h_k\xi+t\sum_{k=1}^m a_k\sin h_k\xi)\right)\cos x\xi d\xi$$

$$=-\frac{t}{x^2}\left[\sin x\xi\, e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\left(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi)-\sum_{k=1}^m a_kh_k\cos(h_k\xi+t\sum_{k=1}^m a_k\sin h_k\xi)\right)\Bigg|_{\xi=0}^{\xi=\infty}\right.$$

$$\left.-\int\limits_0^\infty \psi(\xi)\sin x\xi d\xi\right]=\frac{t}{x^3}\int\limits_0^\infty \psi(\frac{\eta}{x})\sin\eta d\eta,$$

where

$$\psi(\xi)=\left[e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\left(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi)-\sum_{k=1}^m a_kh_k\cos(h_k\xi+t\sum_{k=1}^m a_k\sin h_k\xi)\right)\right]'\in L_1(\mathbb{R}^1_+).$$

By virtue of the Wiener Tauberian theorem, this implies that $x^2\mathcal{E}_2(x,t)\overset{x\to\infty}{\longrightarrow}0$ for all fixed $t>0$ and $a,h\in\mathbb{R}^m$.

Thus, $x^2\mathcal{E}(x,t)\overset{x\to\infty}{\longrightarrow}0$ for all fixed $t>0$ and $a,h\in\mathbb{R}^m$. Therefore, the Poissonian kernel decays at infinity sufficiently fast to ensure that convolution (3.3) is well defined.

Complete proofs of the results of this section are provided in [22].

4. Elliptic case

Half-space boundary-value problems are traditionally treated to be typical for parabolic and hyperbolic equations: The only independent variable varying on the semiaxis is naturally treated as time, while all other independent variables are considered to be spatial ones. Correspondingly, the boundary of the domain, i.e., the boundary hyperplane, is treated as the initial (zero-time) hyperplane. However, well-posed half-space problems for *stationary* equations are known as well (see, e.g., the case of the Laplace equation in [23, 24]). In such cases, the only spatial variable varying on the semiaxis acquires so-called *timelike* properties. Indeed, the problem

$$\sum_{j=1}^n u_{x_jx_j}+u_{yy}=0,\ x\in\mathbb{R}^n,\ y>0,\tag{4.1}$$

$$u\Big|_{y=0}=u_0(x),\ x\in\mathbb{R}^n,\tag{4.2}$$

i.e., the half-space Dirichlet problem for the Laplace equation, is well defined in the class of bounded (classical) solutions, its resolving operator possesses the semigroup property with respect to the *spatial* variable y, and its solution $u(x,y)$ is represented by the convolution of the boundary-value function u_0 with the Poissonian kernel $\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}$ (cf. $\frac{e^{-\frac{|x|^2}{4y}}}{(2\sqrt{\pi y})^n}$ for the Cauchy problem for the heat equation $u_y=\sum\limits_{j=1}^n u_{x_jx_j}$ in the same domain with the same initial-value function). This allows one to investigate the long-time behavior of the solution of problems (4.1) and (4.2): It turns out that the Repnikov-Eidel'man stabilization criterion from [1] holds for the elliptic case as well. To prove this, one can apply the same method as in Section 2, assigning $\psi_0(r) := \frac{r^{n-1}}{\left(1+r^2\right)^{\frac{n+1}{2}}}$. This function belongs to $L_1(0,+\infty)$, and its Mellin transform is equal to $\frac{\Gamma\left(\frac{n}{2}+\frac{x}{2}i\right)\Gamma\left(\frac{1}{2}-\frac{x}{2}i\right)}{2e^{\pi m x}\Gamma\left(\frac{n+1}{2}\right)}$, $m \in \mathbb{Z}$, which has no real zeroes, i.e., the Wiener Tauberian theorem works again (see, e.g., [25] with the zero value of the parameter k).

For elliptic *differential-difference* equations, the Wiener Tauberian theorem is applicable as well. Up to now, this is shown only for the planar case. Similarly to parabolic differential-difference problems, equations with sums of differential operators and translation operators (in other words, equations with nonlocal potentials) and equations with their superpositions are considered separately. For the former kind of equations, the following assertion is proved.

Theorem 4.1. *Let $u_0 \in L^\infty(\mathbb{R}^1) \cap C(\mathbb{R}^1)$, let there exist a positive constant a_0 such that the inequality*

$$\xi^2 + \sum_{j=1}^{m} a_j \cos h_j\xi \geq a_0 \tag{4.3}$$

holds on the real line, and let the function $\mathcal{E}(x,y)$ be defined as follows:

$$\mathcal{E}(x,y) = \int_0^\infty e^{-yG_1(\xi)} \cos\left[x\xi - yG_2(\xi)\right] d\xi, \tag{4.4}$$

where $G_1(\xi) = \rho(\xi)\cos\theta(\xi)$, $G_2(\xi) = \rho(\xi)\sin\theta(\xi)$,

$$\theta(\xi) = \frac{1}{2}\arctan\frac{\sum_{j=1}^{m} a_j \sin h_j\xi}{\xi^2 + \sum_{j=1}^{m} a_j \cos h_j\xi}, \tag{4.5}$$

and

$$\rho(\xi) = \left[\left(\xi^2 + \sum_{j=1}^{m} a_j \cos h_j\xi\right)^2 + \left(\sum_{j=1}^{m} a_j \sin h_j\xi\right)^2\right]^{\frac{1}{4}}. \tag{4.6}$$

If $\sum_{j=1}^{m} a_j \geq 0$, then the function

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathcal{E}(x-\xi,y)u_0(\xi)d\xi \tag{4.7}$$

satisfies the problem

$$u_{xx} + u_{yy} - \sum_{j=1}^{m} a_j u(x+h_j,y) = 0,\ x \in (-\infty,+\infty),\ y \in (0,+\infty), \tag{4.8}$$

$$u\Big|_{y=0} = u_0(x),\ x \in (-\infty, +\infty), \tag{4.9}$$

in the classical sense and is infinitely smooth in the open half-plane $\mathbb{R}^1 \times (0, +\infty)$.

As in the parabolic case, this theorem is proved by means of the Gel'fand-Shilov operational scheme. The transformation with respect to the variable x is formally applied to the investigated problem. As a result, instead of a boundary-value problem for a *partial differential-difference* equation, we obtain an initial-value problem for an *ordinary differential* equation, depending on the scalar parameter ξ:

$$\frac{d^2\widehat{u}}{dy^2} = \left(\xi^2 + \sum_{j=1}^{m} a_j e^{ih_j\xi}\right)\widehat{u},\ x \in (-\infty, +\infty), \tag{4.10}$$

$$\widehat{u}(0;\xi) = \widehat{u_0}(\xi). \tag{4.11}$$

The characteristic equation of Eq (4.10) has the two roots $\pm\rho(\xi)[\cos\theta(\xi) + i\sin\theta(\xi)]$, which yields two linearly independent solutions (depending on the parameter ξ) of Eq (4.10). Now, it remains to take their linear combination, formally apply the inverse Fourier transformation (with respect to ξ) to it, annihilate the terms with odd integrands, and select the arbitrary constants depending on the parameter ξ to nullify all imaginary terms (such a choice is possible because Eq (4.10) has the second order, while the initial-value condition is unique).

This formal procedure leads to convolution (4.7), but we have to justify it, proving that the constructed function (4.4) and all its derivatives contained in Eq (4.8) vanish sufficiently fast as $x \to \infty$. Again, this is done by means of the Wiener Tauberian theorem: We assign $\psi_0(r) := e^{-yr^2}$, $f(r) := \cos r$, and $\psi(r) := e^{-yG_1(r)}\cos[yG_2(r)]$.

For elliptic equations with superpositions of differential operators and translation operators, the following results are obtained.

Theorem 4.2. *Let* $u_0 \in L^\infty(\mathbb{R}^1) \cap C(\mathbb{R}^1)$, *let there exist a positive constant* a_0 *such that the inequality*

$$1 + \sum_{k=1}^{m} a_k \cos h_k\xi \geq a_0 \tag{4.12}$$

holds on the real line, and let the function $\mathcal{E}(x, y)$ *be defined by relation* (4.4)*, where*

$$G_{\left\{ {1 \atop 2} \right\}}(\xi) = \xi\sqrt{\frac{\varphi(\xi) \pm a(\xi) \pm 1}{2}},$$

$\varphi(\xi) = \left[a^2(\xi) + b^2(\xi) + 2a(\xi) + 1\right]^{\frac{1}{2}}$, $a(\xi) = \sum\limits_{k=1}^{m} a_k \cos h_k\xi$, *and* $b(\xi) = \sum\limits_{k=1}^{m} a_k \sin h_k\xi$. *Then, function* (4.7) *satisfies the Dirichlet problem with Condition* (4.9) *for the equation*

$$u_{xx} + \sum_{k=1}^{m} a_k u_{xx}(x + h_k, y) + u_{yy} = 0,\ x \in (-\infty, +\infty),\ y > 0, \tag{4.13}$$

in the sense of generalized functions and is infinitely smooth in the open half-plane $\mathbb{R}^1 \times (0, +\infty)$.

The scheme of the proof is the same as above. The dual ordinary differential equation obtained after the formal applying of the Fourier transformation has the form

$$\frac{d^2\widehat{u}}{dy^2} = \xi^2\left(1 + \sum_{k=1}^{m} a_k e^{ih_k\xi}\right)\widehat{u},$$

which causes the distinction in the definition of the functions G_1 and G_2 but does not affect the choice of the functions f, ψ_0, and ψ for the corollary from the Wiener Tauberian theorem.

Remark 4.1. In both cases, no restrictions about the commensurability are imposed on the real parameters $a_1, \dots, a_m$ and $h_1, \dots, h_m$; they are linked only by Conditions (4.3) and (4.12) for Theorems 4.1 and 4.2, respectively.

If the nonlocal term of Eq (4.13) is unique, then Theorem 4.2 is complemented by the following result about the asymptotical behavior of the constructed solution:

Theorem 4.3. *If $m = 1$ and $|a| < 1$, then, under the assumptions of Theorem 4.2, the function $u(x, y)$ obeys the relation $\lim\limits_{y\to+\infty} [u(x, y) - v(x, y)] = 0$ for each real x, where $v(x, y)$ is the classical bounded solution of the equation*

$$(a + 1)u_{xx} + u_{yy} = 0 \tag{4.14}$$

satisfying Condition (4.9).

Remark 4.2. The assumptions of the theorem guarantee the ellipticity of Eq (4.14). Hence, the function $v(x, y)$ is well defined.

Complete proofs of the results of this section are provided in [26–28].

5. Nonlinear case

For quasilinear (both parabolic and elliptic) equations, the stabilization phenomenon takes place as well, though no direct usage of the Wiener Tauberian theorem is possible. The technique of monotone maps, proposed in [29], is applied.

To demonstrate this technique, consider the problem

$$\frac{\partial u}{\partial t} = \Delta u + g(u)|\nabla u|^2, \quad x \in \mathbb{R}^n,\ t > 0, \tag{5.1}$$

$$u\Big|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n, \tag{5.2}$$

where g is continuous, while u_0 is continuous and bounded.
The existence and uniqueness of its classical bounded solution $u(x, t)$ is known, e.g., from [32].
The specified solution obeys the following assertion.

Theorem 5.1. *If $x \in \mathbb{R}^n$, then $\lim\limits_{t\to\infty} u(x, t)$ exists if and only if $\lim\limits_{t\to\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^n} \int\limits_{\{|x|<t\}} f[u_0(x)]dx$ exists, where*

$$f(s) = \int_0^s e^{\int_0^x g(\tau)d\tau} dx. \tag{5.3}$$

If those limits exist, then the latter one is equal to

$$\lim_{t\to\infty} u(x,t) = f^{-1}\left(\lim_{t\to\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^n} \int\limits_{\{|x|<t\}} f[u_0(x)]dx\right).$$

To prove this, we compute $f'(s) = e^{\int_0^s g(\tau)d\tau}$ and see this is strictly positive everywhere, i.e., f is strictly monotone. Further, we note that $f''(s) = g(s)e^{\int_0^s g(\tau)d\tau}$ and, therefore, $g(s) = \dfrac{f''(s)}{f'(s)}$.

Defining $v(x,t) := f[u(x,t)]$, we find that it satisfies Eq (2.1). Further, taking into account that f is continuous and u is bounded, we conclude that $v(x,t)$ is bounded as well (as a continuous function f on the segment $[\inf u, \sup u]$). Thus, being a bounded solution of the heat equation, the function $v(x,t)$ obeys the stabilization criterion from [1], i.e., for any $x \in \mathbb{R}^n$, $\lim\limits_{t\to\infty} v(x,t)$ exists if and only if $\lim\limits_{t\to\infty} \dfrac{n\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}} t^n} \int\limits_{\{|x|<t\}} v(x,0)dx$ exists, and if those limits exist, then they are equal to each other.

Now, it remains to take into account that f is invertible due its strong monotonicity, and f^{-1} is continuous due the smoothness of f.

The last theorem is generalized to the case of singular coefficients, but the generalization holds only for positive solutions. More exactly, the following assertion is valid:

Theorem 5.2. *Let $g(s) = \dfrac{\alpha}{s^\beta}$, where $\alpha \in \mathbb{R}^1$, while $\beta \in (0,1)$. Let u_0 be continuous, bounded, nonnegative, and nontrivial in $\mathbb{R}^n$. Then, there exists a unique positive bounded solution of problems* (5.1) *and* (5.2)*, and the assertion of Theorem 5.1 holds for it.*

The scheme of the proof, proposed above, still works (with the same anzatz), but we have to restrict the sign of the solution to guarantee the map monotonicity.

The value $\beta = 1$ is reachable as well, but anzatz (5.3) is not applicable anymore, and it has to be changed for $f(s) := s^{\alpha+1}$. This yields the following result:

Theorem 5.3. *Let $g(s) = \dfrac{\alpha}{s}$, where $\alpha > -1$. Let u_0 be continuous, bounded, nonnegative, and nontrivial in $\mathbb{R}^n$. Then, there exists a unique positive bounded solution of problems* (5.1) *and* (5.2)*, and the following criterion is valid: If $x \in \mathbb{R}^n$, then $\lim\limits_{t\to\infty} u(x,t)$ exists if and only if $\lim\limits_{t\to\infty} \dfrac{1}{t^n} \int\limits_{\{|x|<t\}} u_0^{\alpha+1}(x)dx$ exists.*

If those limits exist, then $\lim\limits_{t\to\infty} u(x,t) = \left[\dfrac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \lim\limits_{t\to\infty} \dfrac{1}{t^n} \int\limits_{\{|x|<t\}} u_0^{\alpha+1}(x)dx\right]^{\frac{1}{\alpha+1}}$.

For the half-space Dirichlet problem for elliptic equations with nonlinearities of the above kind, the explained technique is applicable as well. Consider the problem

$$\Delta u + g(u)|\nabla u|^2, \quad x' \in \mathbb{R}^n,\ x_{n+1} > 0, \tag{5.4}$$

$$u\Big|_{x_{n+1}=0} = \varphi(x), \quad x' \in \mathbb{R}^n, \tag{5.5}$$

where $x = (x_1, \dots, x_n, x_{n+1}) := (x', x_{n+1})$, while φ is continuous and bounded in $\mathbb{R}^n$.

The following assertions are valid.

Theorem 5.4. *Let g be continuous in $\mathbb{R}^1$. Then, there exists a unique bounded solution $u(x,t)$ of problems* (5.4) *and* (5.5)*, and*

$$\lim_{x_{n+1}\to+\infty} u(x) = l \textit{ if and only if } \lim_{R\to+\infty} f^{-1}\left(\frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}R^n}\int\limits_{\{|y|<R\}} f[\varphi(y)]dy\right) = l$$

for each $x' \in \mathbb{R}^n$ and each $l \in \mathbb{R}^1$, where f is defined by relation (5.3).

Theorem 5.5. *Let $g(s) = \dfrac{\alpha}{s^\beta}$, where $0 < \beta < 1$, and $0 \le \varphi \not\equiv 0$. Then, there exists a unique bounded positive solution $u(x,t)$ of problems* (5.4) *and* (5.5)*, and the assertion of Theorem 5.4 is valid for it.*

Remark 5.1. In the last case, the function f can be represented as follows:

$$f(s) = \int\limits_0^s e^{\frac{\alpha}{\beta+1}\tau^{\beta+1}} d\tau.$$

Theorem 5.6. *If $g(s) = \dfrac{\alpha}{s}$, where $\alpha > -1$, and $0 \le \varphi \not\equiv 0$, then there exists a unique bounded positive solution $u(x,t)$ of problems* (5.4) *and* (5.5)*, and*

$$\lim_{x_{n+1}\to+\infty} u(x) = l \textit{ if and only if } \lim_{R\to+\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}R^n}\int\limits_{\{|y|<R\}} \varphi^{\alpha+1}(y)dy = l^{\alpha+1}$$

for each $x' \in \mathbb{R}^n$ and each nonnegative l.

The above results about elliptic problems are extended to equations with variable coefficients at linear terms (see [30]).

Complete proofs of the results of this section are provided in [30, 31].

6. Discussion

The most acute open question of this research area is whether the described phenomenon is preserved for elliptic differential-difference equations with several tangential (spacelike) independent variables. Currently, as far as the author is aware, no results in this direction are known. This might be a technical disadvantage because the Wiener Tauberian theorem is one-dimensional, while, comparing this case with the *elliptic differential* one and the *parabolic differential-difference* one, we see that their Poissonian kernels or integrands determining them possess certain symmetry properties: The function $\dfrac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}$ is radially symmetric with respect to the multidimensional *spacelike* variable x and the integrands in (3.4) can be reduced to factorable (with respect to the multidimensional *spatial* variable x) functions by means of the expansion with respect to plane waves (note that the specified procedure was originally invented to investigate *hyperbolic* problems).

For the *elliptic differential-difference* case, no such simplifying procedure has been found yet.

Another important open direction is uniqueness classes for problems with elliptic differential-difference equations. In general, the uniqueness study in the differential-difference case qualitatively differs from the classical case of partial *differential* equations because no maximum principle takes place for (parabolic and elliptic) differential-difference equations. To find uniqueness classes for parabolic problems, we pass to the dual ordinary *differential* equation. In the elliptic case, this task is harder because the dual equation has the second order, while the initial-value condition is unique.

As far as the author is aware, this problem is not solved yet.

7. Conclusions

Stabilization phenomena discovered for the heat equation in 1966 are typical for a much more broad class of equations than classical partial differential equations of the parabolic type. Stabilization theorems establishing the equivalence of the existence of a limit of the solution (as $t \to \infty$) and a limit of the mean value of the initial-value function over balls (as the ball radius tends to infinity) are generalized to the Cauchy problem for parabolic differential-difference equations and to the half-space Dirichlet problem for elliptic differential and differential-difference equations. In all the specified cases, the Wiener Tauberian theorem plays a key role for the proofs.

For parabolic and elliptic equations with KPZ-nonlinearities, this phenomenon takes place as well, but this is proved by other methods: Bitsadze's technique of monotone maps is applied.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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