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Research article

Existence theory and stability analysis of neutral ψ–Hilfer fractional stochastic differential system with fractional noises and non-instantaneous impulses

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Abstract: In this article, with the help of Laplace transform, the existence of solution was established in a finite dimensional setting for nonlinear ψ-Hilfer fractional stochastic equation with both retarded and advanced arguments driven by multiplicative and fractional noises, with Hurst index $H \in (\frac{1}{2}, 1)$. At first, we obtained the existence and uniqueness results by using the Banach fixed point theorem (FPT). Second, the existence result was also obtained by applying Schaefer's fixed point theorem with less conservative conditions. Furthermore, we investigated the Hyers Ulam Rasisas stability for the aforementioned system. At the end, an example was illustrated to validate the obtained theoretical results.

Keywords: neutral; existence and uniqueness; stochastic equation; retarded and advanced arguments; Ulam's stability
Mathematics Subject Classification: 34A37, 34N05, 35F30, 35L05

1. Introduction

Fractional differential equations (FDEs) are an effective mathematical tool to model and analyze many real life problems; it has been used by researchers and scientists to get better results than the integer order differential equations. Fractional order differential equations offer a superior framework for capturing the intricate dynamics of real-world phenomena compared to their integer-order counterparts. This superiority stems from the unique ability of fractional integrals and derivatives to account for the inherent hereditary and memory characteristics present in diverse processes and materials. By harnessing these fractional operators, models can more accurately depict the nuanced behaviors observed in nature, thereby enhancing our understanding and predictive capabilities across a wide range of disciplines and applications. Many fractional derivatives, including Caputo derivative, Atangana-Baleanu derivative, Coimbra derivative, and Riemann-Liouville (R-L) derivative, are frequently used to examine FDEs and fractional order stochastic differential equations (SDEs). In a similar vein, the Hilfer fractional derivative (HFD), which was just recently used to do so, was developed by Hilfer [21], which is a generalized version of R-L and Caputo derivatives. In actuality, fractional derivative and integrals indicate greater accuracy than integral models and also depict broader physical applications in seepage, flow in porous media, nanotechnology, fluid dynamics and traffic models [6, 7, 12, 22, 24, 28, 31].

SDEs are the natural extension of deterministic systems. SDEs with impulses arise from many mathematical models of physical phenomena in different scientific fields for example, technology, physics, biology, economics, etc. They are important from the viewpoint of applications since they incorporate randomness into the mathematical description of the phenomena and provide a more accurate description of it. Certainly, in various fields like economics, bioengineering, chemistry, medicine, and biology, we often encounter situations where things change suddenly at specific points in time [5, 23, 29, 32]. These abrupt changes can be explained by what we call "impulsive effects." These impulsive effects are like sudden pushes that happen at certain moments and have a big impact on the system, being studied. These pushes play a crucial role in understanding and modeling how things change in the aforementioned diverse fields. For the mathematical models of such phenomena, finding their solution is a challenging task. As a result, Boundani et al. [8, 9] presented some specific conditions that help us to determine whether certain mathematical equations, involving randomness, can have solutions. These equations involve functional differential equations and a type of random behavior, called fractional Brownian motion.

In [20], Hernandez and O'Regan introduced non-instantaneous (NI) impulses. Many researchers have utilized these impulses and studied the corresponding dynamical systems [3, 4, 27, 36, 39]. To better understand NI impulses, we can think about human blood sugar levels. When they have too much or too little, glucose they get insulin medication through the bloodstream, in fact it doesn't work instantly but takes some time to be absorbed [37]. This gradual effect is like NI, where the impact lasts for a while in many real situations. Sudden changes don't explain things well. For instance, in the treatment of diseases with medication, we need to describe how things change over time more smoothly. That's where NI impulsive differential equations come in handy. They help us to model these gradual changes, like how drugs affect the body in pharmacotherapy.

Among the qualitative behaviors of different physical systems, different types of stabilities are the essential ones. One of these types of stabilities is Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR) stability [11, 16, 26, 30, 33].

In the literature, most of the results related to fractional stochastic differential equations (FSDEs) are given over infinite dimensional spaces [1,2,13,18,19,25,32,35,38], and very few have looked at similar results in finite spaces. There is no prior work on the specific topic of non-integers order impulses in finite-dimensional equations to the problem of fractional neutral SDEs including both noises. In FSDEs incorporating both retarded and advanced arguments, a significant characteristic emerges: The rate of change of the system at the present moment is influenced not only by its past history but also by its anticipated future states. This feature underscores the intricate interplay between past, present, and future dynamics, where the system's behavior is shaped by a combination of its memory effects and anticipatory responses.

In the current manuscript, we investigate the following ψ-Hilfer fractional stochastic equation (HFSE):

$$
\begin{aligned}
D^{\gamma,\beta}_{\psi(\varepsilon)}[\Gamma(\varepsilon) - h(\varepsilon,\Gamma(\varepsilon))] =& A\Gamma(\varepsilon) + \Delta(\varepsilon,\Gamma(\varepsilon), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\varepsilon)) \\
&+ \int_0^{\varepsilon} g(s,\Gamma(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\varepsilon))dB(s) \\
&+ \int_0^{\varepsilon} \lambda(s,\Gamma(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\varepsilon))dB^H(s), \\
\varepsilon \ &\in (s_k,\varepsilon_{k+1}] \subset \mathscr{J}' := (0,b], \quad k = 0,1,2,\dots,m, \\
\Gamma(\varepsilon) =& h_k(\varepsilon,\Gamma(\varepsilon)), \quad \varepsilon \in (\varepsilon_k, s_k], \quad k = 1,2,\dots,m, \\
& I^{1-\nu}_{0^+}\Gamma(\varepsilon)/_{\varepsilon=0} = \Gamma_0, \quad \nu = \gamma + \beta - \gamma\beta, \\
& \Gamma(\varepsilon) = \phi(\varepsilon), \varepsilon \in [0-r,0], \\
& \Gamma(\varepsilon) = \varphi(\varepsilon), \varepsilon \in [b,b+h],
\end{aligned} \tag{1.1}
$$

where $D^{\gamma,\beta}_{\psi(\varepsilon)}$ is the ψ– Hilfer FD of order $0 < \gamma < 1$ and of type $0 < \beta \leq 1$. Let $\mathscr{J} := [0,b]$, $b > 0$. The state vector $\Gamma \in R^n$, $A \in R^{n\times n}$ and nonlinear functions h : $\mathscr{J} \times R^n \longrightarrow R^n$, $\Delta : \mathscr{J} \times R^n \longrightarrow R^n$, $g : \mathscr{J} \times R^n \longrightarrow R^{n\times n}$, $\lambda : \mathscr{J} \times R^n \longrightarrow R^{n\times n}$, and $h_k : \mathscr{J} \times R^n \longrightarrow R^n$ are measurable and bounded functions. Also, Γ_0 is $\mathscr{F}_0$ measurable R^n-valued stochastic variable and B is an n-dimensional Wiener process.

Based on the value of H, the following kinds of the fractional Brownian motion (fBm) process exist:

(1) if $H = \frac{1}{2}$, then the process is a Brownian motion or a Wiener process exists;
(2) if $H > \frac{1}{2}$, then the increment of the process is positively correlated;
(3) if $H < \frac{1}{2}$, then the increment of the process is negatively correlated.

The contributions of this paper are described as below:

- Nonlinear ψ-HFSE is considered in R^n.
- The existence and uniqueness results are established by using the standard Banach contraction principle.
- The weaker sufficient conditions are derived by using the generalized Schaefer FPT for the system with measure of non-compactness (MNC).
- UHR stability results are derived for ψ-HFSE with NI impulse.
- An example is provided for the theoretical results.

This paper is structured as follows: In Section 2, we present a number of lemmas and some fundamental definitions for fractional calculus. Section 3 derives the solution representation of ψ-Hilfer fractional SDEs with NI impulses. To demonstrate the key results, the generalized Schaefer's and contraction mapping principles are used in Section 4. In Section 5, UHR stability of a ψ-HFSE is discussed. Example is demonstrated for the validity of theoretical results in Section 6.

Notations:

- $(\Omega, \mathscr{F}, \mathscr{P})$ represents the complete probability space along a probability measure $\mathscr{P}$ on Ω.
- $B(\varepsilon)$ and $B^H(\varepsilon)$ denote, respectively, the n-dimensional Brownian motion and fBm with Hurst index $\frac{1}{2} < H < 1$.
- $\{\mathscr{F}_\varepsilon | \varepsilon \in \mathscr{J}\}$ represents the filtration generated by $\{B(\varepsilon) : 0 \leq s \leq \varepsilon\}$.
- $L_2(\Omega, \mathscr{F}_\varepsilon, \mathscr{P}, R^n) := L_2(\Omega, R^n)$ is the space of all $\mathscr{F}_\varepsilon$-measurable square integrable random variables with values in R^n.

Let $\mathscr{J}_k = (\varepsilon_k, \varepsilon_{k+1}]$, $k = 1, 2, \ldots, m$ be such that impulse times satisfy $0 = \varepsilon_0 = s_0 < \varepsilon_1 < s_1 < \varepsilon_2 < \cdots < \varepsilon_m \leq s_m < \varepsilon_{m+1} = \mathrm{b}$. Let $C(\mathscr{J}, L_2(\Omega, R^n))$ denote $C_n(\mathscr{J})$ and be the Banach space of all continuous maps from $\mathscr{J}$ into $L_2(\Omega, R^n)$ of $\mathscr{F}_\varepsilon$-adapted square integrable functions $\Gamma(\varepsilon)$ and for its norm $\sup E\|\Gamma(\varepsilon)\|^2 < \infty$.

Define the space

$$\begin{aligned}\mathbb{Y} = PC_n^v(\mathscr{J}) &= PC_n^v(\mathscr{J}, L_2(\Omega, R^n)) \\ &= \Big\{\Gamma : \mathscr{J} \to L_2(\Omega, R^n), \quad \Gamma/\mathscr{J}_k \in C_n(\mathscr{J}_k, L_2(\Omega, R^n)),\end{aligned}$$

and there exist $\Gamma(\varepsilon_k^-)$ and $\Gamma(\varepsilon_k^+)$ with $\Gamma(\varepsilon_k) = \Gamma(\varepsilon_k^-)$, $k = 1, \ldots, m$, endowed with the norm

$$\|\Gamma\|_{\mathbb{Y}}^2 = \max_{k=0,1,\ldots,m} \sup_{\varepsilon \in \mathscr{J}_k} \{E\|(\varepsilon - \varepsilon_k)^{(1-v)}\Gamma(\varepsilon)\|\}\Big\}.$$

Clearly, $\mathbb{Y}$ is a Banach space.

2. Preliminaries

Lemma 2.1. *[39] Let $p \geq 2$ and $\mathfrak{f} \in L_p(\mathscr{J}, R^{n\times n})$ such that $E|\int_0^b \mathfrak{f}(\mathfrak{s})dB(s)|^p < \infty$, then*

$$E\left|\int_0^b \mathfrak{f}(s)dB(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} b^{\frac{p-2}{2}} E\int_0^b |\mathfrak{f}(s)|^p ds.$$

Lemma 2.2. *[10] Let $\varphi : \mathscr{J} \to L_2^0$ satisfy $\int_0^b \|\varphi(s)\|_{L_2^0}^2 ds < \infty$, then we get*

$$E\left\|\int_0^\varepsilon \varphi(s)dB^H(s)\right\|^2 \leq 2H\varepsilon^{2H-1}\int_0^\varepsilon E\,\|\varphi(s)\|\,|_{L_2^0}^2 ds.$$

Definition 2.1. *[17] The generalized ψ-Hilfer FD of order $0 < \gamma < 1$ and of type $0 \leq \beta \leq 1$ is represented by*

$$D_{\psi(t)}^{\gamma,\beta} f(t) = I_{\psi(t)}^{\beta(n-\gamma)}\left(\frac{1}{\psi'(t)}.\frac{d}{dt}\right)^m I_{\psi(t)}^{1-\beta(n-\gamma)} f(t).$$

Definition 2.2. *[10] Let z be a bounded linear operator. The two parameter Mittage-Leffler (M-L) function is defined by*

$$M_{\gamma,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(r\gamma + \beta)}, \quad \gamma, \beta > 0, \quad z \in C.$$

One of the interesting properties of the M-L function, related with their Laplace integral, is given by

$$\int_0^{\infty} e^{-s\varepsilon}\varepsilon^{\beta-1}M_{\gamma,\beta}(\pm a\varepsilon^{\gamma})d\varepsilon = \frac{s^{\gamma-\beta}}{(s^{\gamma}\mathrm{I} \mp a)}.$$

That is

$$\mathcal{L}\{\varepsilon^{\beta-1}M_{\gamma,\beta}(\pm a\varepsilon^{\gamma})\}(s) = \frac{s^{\gamma-\beta}}{(s^{\gamma}\mathrm{I} \mp a)}.$$

Lemma 2.3. *[17] For $\gamma \in (n-1, n]$, $\beta \in [0, 1]$, the following Laplace formula for the ψ-Hilfer derivative is valid:*

$$\mathscr{L}_{\psi}\left\{{}_0\mathcal{D}^{\gamma,\beta}_{\psi(t)}f(t)\right\} = s^{\gamma}\mathcal{L}_{\psi}\{f(t)\} - \sum_{n=0}^{m-1} s^{m(1-\nu)+\gamma\beta-n-1}\left({}_0\mathcal{I}^{(1-\nu)(m-\beta)-n}_{\psi(t)}f\right)(0).$$

Corollary 2.1. *[17] If f is a function whose classical Laplace transform is $F(s)$, then the generalized Laplace transform of the function $f \circ \psi = f(\psi(t))$ is also $F(s)$:*

$$\mathscr{L}\{F(s)\} = F(s) \quad \Rightarrow \quad \mathscr{L}\{f(\psi(\varepsilon))\} = F(s).$$

Example 2.1. *[17]*

$$(a) \quad \mathscr{L}_{\psi}\{(\psi(\varepsilon)^{\mu})\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}, \quad for \quad s > 0.$$

$$(b) \quad \mathscr{L}_{\psi}\{e^{a(\psi(\varepsilon))}\} = \frac{1}{s-a}, \quad for \quad s > a.$$

$$(c) \quad \mathscr{L}_{\psi}\{M_{\mu}(A(\psi(\varepsilon))^{\mu})\} = \frac{s^{\mu-1}}{s^{\mu}-A}.$$

$$(d) \quad \mathscr{L}_{\psi}\{(\psi(\varepsilon))^{\mu-1}M_{\mu,\mu}(A(\psi(\varepsilon))^{\mu})\} = \frac{1}{s^{\mu}-A}, \quad for \quad Re(\mu) > 0 \quad and \; |\frac{A}{s^{\mu}}| < 1.$$

Example 2.2. *Assume that $Re(\mu) > 0$ and $|\frac{A}{s^{\mu}}| < 1$. If $M^{\gamma}_{\mu,\nu}$ denotes the Prabhakar function, then we have*

$$\mathscr{L}_{\psi}\{(\psi(\varepsilon))^{\nu-1}M^{\gamma}_{\mu,\nu}(A(\psi(\varepsilon))^{\mu})\} = \mathscr{L}\{\psi(\varepsilon)^{\nu-1}M^{\gamma}_{\mu,\nu}(A(\psi(\varepsilon)^{\mu}))\} = \frac{s^{\mu\gamma-\nu}}{(s^{\mu}-A)^{\gamma}}.$$

3. Solution representation

Consider the linear deterministic system, which is represented in the following form:

$$D^{\gamma,\beta}_{\psi(\varepsilon)}[\Gamma(\varepsilon) - h(\varepsilon, \Gamma(\varepsilon))] = A\Gamma(\varepsilon) + \Delta(\varepsilon, \Gamma(\varepsilon)),$$
$$I^{1-\nu}_{0^+}\Gamma(\varepsilon)/_{\varepsilon=0} = \Gamma_0, \quad \nu = \gamma + \beta - \gamma\beta.$$

By the Laplace transformation, we get

$$s^{\gamma}\hat{\Gamma}(s) - \hat{h}(s) - s^{-\beta(1-\gamma)}(\Gamma(0) - h(\varepsilon, 0)) = A\hat{\Gamma}(s) + \hat{\Delta}(s),$$

$$(s^{\gamma}I - A)\left(\hat{\Gamma}(s)\right) = s^{-\beta(1-\gamma)}\left(\Gamma(0) - h(\varepsilon, 0)\right) + \hat{h}(s) + \hat{\Delta}(s),$$
$$\hat{\Gamma}(s) = \frac{s^{-\beta(1-\gamma)}}{(s^{\gamma}I - A)}\left[\Gamma(0) - h(\varepsilon, 0)\right] + \frac{1}{(s^{\gamma}I - A)}\hat{h}(s) + \frac{1}{(s^{\gamma}I - A)}\hat{\Delta}(s),$$

where I is the identity matrix.

$$\Gamma(s) = \mathscr{L}_{\psi}^{-1}\frac{s^{-\beta(1-\gamma)}}{(s^{\gamma}I - A)}\left[\Gamma(0) - h(\varepsilon, 0)\right] + \mathscr{L}_{\psi}^{-1}\frac{1}{(s^{\gamma}I - A)}\hat{h}(s) + \mathscr{L}_{\psi}^{-1}\frac{1}{(s^{\gamma}I - A)}\hat{\Delta}(s).$$

Substituting the $L_{\psi}T$ of the M-L function, one can obtain that

$$\begin{aligned}\Gamma(\varepsilon) =&\psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)\left[\Gamma_0 - h(\varepsilon, 0)\right] + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})h(s, \Gamma(s))ds\\ &+ \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})\Delta(s, \Gamma(s))ds.\end{aligned}$$

Therefore, the solution of (1.1) is given as follows:

$$\Gamma(\varepsilon) = \begin{cases}\Gamma(\varepsilon) = \phi(\varepsilon), \quad \varepsilon \in [0 - r, 0],\\ \Gamma(0), \quad \varepsilon = 0,\\ \psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)\left[\Gamma_0 - h(\varepsilon, 0)\right] + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})h(s, \Gamma(s))ds\\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})\Delta(s, \Gamma(s))ds, D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)\\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})(\int_0^s g(\eta, \Gamma(\eta)), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)dB(\eta))ds\\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})(\int_0^s \lambda(\eta, \Gamma(\eta)), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)dB^H)ds, \quad \varepsilon\epsilon(0, \varepsilon_1],\\ h_k(\varepsilon, \Gamma(\varepsilon)), \quad \varepsilon\epsilon(\varepsilon_k, s_k], \quad k = 1, 2, 3, \ldots, m,\\ (\psi(\varepsilon) - s_k)^{\nu-1}M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right)h_k(s_k, \gamma(s_k)) + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})h(s, \Gamma(s))ds\\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})\Delta(s, \Gamma(s))ds, D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)\\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})(\int_0^s g(\eta, \Gamma(\eta)), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)dB(\eta))ds\\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma})(\int_0^s \lambda(\eta, \Gamma((\eta, \gamma(\eta))), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(\varepsilon)dB^H)ds, \quad \varepsilon\epsilon(s_k, \varepsilon_{k+1}],\\ \Gamma(\varepsilon) = \phi(\varepsilon), \quad \varepsilon \in [b, b + h].\end{cases}$$

Definition 3.1. *[40] The set* S *is called a quasi-equicontinuous in* $\mathscr{T}$ *if, for* $\varepsilon > 0$*, there exists a* $\delta > 0$*, such that if* $\Gamma \in \mathrm{S}, K \in \mathscr{N}, \tau_1, \tau_2 \in \mathscr{T}_{\mathscr{K}} \bigcap \mathscr{T}$*, and* $|\tau_2 - \tau_1| < \delta$*, then* $|\Gamma(\tau_2) - \Gamma(\tau_1)| < \varepsilon$*.*

Lemma 3.1. *[34] The set* $\mathrm{S} \subset PC(\mathscr{J}, R^n)$ *is relatively compact if*

(i) S *is uniformly bounded, i.e.,* $\|\Gamma\|_{PC} \leq \mathrm{k}$ *for each* $\Gamma \in \mathrm{S}$ *and some* $\mathrm{k} > 0$*;*
(ii) S *is quasi-equicontinous in* $\mathscr{T}$*.*

Definition 3.2. *[14] An operator* $\mathrm{T} : \mathcal{Z} \to \mathcal{Z}$ *is said to be* χ*-condensing of any bounded set* $\mathscr{B}$ *of* Z *with* χ*-condensing(*$\mathscr{B}$*)* > 0*,* $\chi(\mathrm{T}(\mathscr{B})) < \chi(\mathscr{B})$*, where*

$$\chi(\mathscr{B}) = \inf\{\mathrm{k} > 0, \mathscr{B} \text{ is covered by a finite number of sets of diameter} \leq \mathrm{k}\}$$

is Kuratowskii MNC of a bounded set $\mathscr{B}$ *of* Z*.*

Now, we state the generalized Schaefer's type FPT with χ-condensing operators.

Theorem 3.1. *[15] Let* $\mathrm{T} : \mathrm{Z} \to \mathrm{Z}$ *be an operator and* Z *be a separable Banach space satisfying*
*(*A1*)* T *is* χ*-condensing and continuous.*
*(*A2*) The set* S=$\{\Gamma \in \mathrm{Z} : \Gamma = \delta\mathrm{T}(\Gamma)$ *for some* $0 < \delta < 1\}$ *is bounded, then* T *has a fixed point.*

For convenience, define the following:

$$M_1 = \sup_{\xi\in\mathcal{J}} \|M_{\gamma,\nu}(A(\psi(\varepsilon))^{\gamma})\|^2;$$
$$M_2 = \sup_{\xi\in\mathcal{J}} \|M_{\gamma,\gamma}(A(b-\psi(\varepsilon))^{\gamma})\|^2.$$

To derive the existence result, we imposed the following assumptions:
(H1) The functions Δ, h, g, λ, and h_k $k = 1, 2, \ldots, m$ are Lipschitz continuous.

(1) $E\|h(\varepsilon, u_1) - h(\varepsilon, u_2)\|^2 \leq M_h\|u_1 - u_2\|^2_{\mathbb{Y}}$;
(2) $E\|\Delta(\varepsilon, u_1, v_1) - \Delta(\varepsilon, u_2, v_2)\|^2 \leq M_{f_1}\|u_1 - u_2\|^2_{\mathbb{Y}} + M_{f_2}\|v_1 - v_2\|^2_{\mathbb{Y}}$;
(3) $E\|g((\varepsilon, u_1, v_1)) - g((\varepsilon, u_2, v_2))\|^2 \leq M_{g_1}\|u_1 - u_2\|^2_{\mathbb{Y}} + M_{g_2}\|v_1 - v_2\|^2_{\mathbb{Y}}$;
(4) $E\|\lambda(\varepsilon, u_1, v_1) - \lambda(\varepsilon, u_2, v_2)\|^2 \leq M_{\lambda_1}\|u_1 - u_2\|^2_{\mathbb{Y}} + M_{\lambda_2}\|v_1 - v_2\|^2_{\mathbb{Y}}$;
(5) $E|h_k(\varepsilon, u_1) - h_k(\varepsilon, v_1)\|^2 \leq M_{hk}\|u_1 - v_1\|^2_{\mathbb{Y}}$,

and $h_k \in C((\varepsilon_k, s_k], L_2(\Omega, R^n))$, where M_h, M_f, M_g, M_λ, and M_{hk} are positive constants, $u_1, u_2, v_1, v_2 \in R^n$, and $\varepsilon \in \mathscr{T}$.
(H2) There exist $l_h, m_h, n_h \in \mathbb{Y}$ with $l^*_h = \sup_{\xi\in\mathcal{J}} l_f(\varepsilon)$, $m^*_h = \sup_{\xi\in\mathcal{J}} m_h(\varepsilon)$, and $n^*_h = \sup_{\xi\in\mathcal{J}} n_h(\varepsilon)$ such that

$$E\|h(\varepsilon, u_1, v_1)\|^2 \leq l_h(\varepsilon) + m_h(\varepsilon)\|u_1\|^2 + n_h(\varepsilon)\|v_1\|^2 \text{ for } \varepsilon \in \mathscr{T}, u_1, v_1 \in R^n.$$

(H3) There exist $l_f, m_f, n_f \in \mathbb{Y}$ with $l^*_f = \sup_{\xi\in\mathcal{J}} l_f(\varepsilon)$, $m^*_f = \sup_{\xi\in\mathcal{J}} m_f(\varepsilon)$, and $n^*_f = \sup_{\xi\in\mathcal{J}} n_f(\varepsilon)$ such that

$$E\|\Delta(\varepsilon, u_1, v_1)\|^2 \leq l_f(\varepsilon) + m_f(\varepsilon)\|u_1\|^2 + n_f(\varepsilon)\|v_1\|^2 \text{ for } \varepsilon \in \mathscr{T}, u_1, v_1 \in R^n.$$

(H4) There exist $l_g, m_g \in \mathbb{Y}$ with $l^*_g = \sup_{\xi\in\mathcal{J}} l_g(\varepsilon)$, $m^*_g = \sup_{\xi\in\mathcal{J}} m_g(\varepsilon)$, and $n^*_g = \sup_{\xi\in\mathcal{J}} n_g(\varepsilon)$ such that

$$E\|g(\varepsilon, u_1, v_1)\|^2 \leq l_g(\varepsilon) + m_g(\varepsilon)\|u_1\|^2 + n_g(\varepsilon)\|v_1\|^2 \text{ for } \varepsilon \in \mathscr{T}, u_1, v_1 \in R^n.$$

(H5) There exist $l_\lambda, m_h, n_h \in \mathbb{Y}$ with $l^*_\lambda = \sup_{\xi\in\mathcal{J}} l_\lambda(\varepsilon)$, $m^*_\lambda = \sup_{\xi\in\mathcal{J}} m_\lambda(\varepsilon)$, and $n^*_\lambda = \sup_{\xi\in\mathcal{J}} n_\lambda(\varepsilon)$ such that

$$E\|\lambda(\varepsilon, u_1, u_2\|^2 \leq l_\lambda(\varepsilon) + m_\lambda(\varepsilon)\|u_1\|^2 + n_\lambda(\varepsilon)\|v_1\|^2 \text{ for } \varepsilon \in \mathscr{T}, u_1, v_1 \in R^n.$$

(H6) There exist $M_{hk} > 0$, for all $u \in R^n$, such that

$$E\|h_k(\varepsilon, u)\|^2 \leq M_{hk}(1 + \|u\|^2_{\mathbb{Y}}).$$

4. Existence and uniqueness of solution

To prove the existence and uniqueness of solution first, we need to transform the problem (1.1) into a fixed point problem and define an operator $\mathscr{H} : \mathbb{Y} \to \mathbb{Y}$ by

$$\mathscr{H}\Gamma(\varepsilon) = \begin{cases} \Gamma(\varepsilon) = \phi(\varepsilon), \quad \varepsilon \in [0-r, 0], \\ \Gamma(0), \quad \varepsilon = 0, \\ \psi(\varepsilon)^{\nu-1} M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)[\Gamma_0 - h(\varepsilon, 0)] + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) h(s, \Gamma(s)) ds \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \Delta(s, \Gamma(s)) ds, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s) \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s g(\eta, \Gamma(\eta)), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\eta) dB(\eta)) ds \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s \lambda(\eta, \Gamma(\eta)), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\beta) dB^H) ds, \quad \varepsilon \epsilon (0, \varepsilon_1], \\ h_k(\varepsilon, \Gamma(\varepsilon)), \quad \varepsilon \epsilon (\varepsilon_k, s_k], \quad k = 1, 2, 3, \dots, m, \\ (\psi(\varepsilon) - s_k)^{\nu-1} M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right) h_k(s_k, \gamma(s_k)) + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) h(s, \Gamma(s)) ds \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \Delta(s, \Gamma(s)) ds, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s) \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s g(\eta, \Gamma(\eta)), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\eta) dB(\eta)) ds \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s \lambda(\eta, \Gamma((\eta, \gamma(\eta))), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\beta) dB^H) ds, \ \varepsilon \epsilon (s_k, \varepsilon_{k+1}], \\ \Gamma(\varepsilon) = \varphi(\varepsilon), \quad \varepsilon \in [b, b+h]. \end{cases}$$

Let $x : [0-r, b+h] \to R$ be a function defined by

$$x(\varepsilon) = \begin{cases} \Gamma(\varepsilon) = \phi(\varepsilon), & if \quad \varepsilon \in [0-r, 0], \\ 0, & if \quad \varepsilon \in (0, b], \\ \Gamma(\varepsilon) = \varphi(\varepsilon), & if \quad \varepsilon \in [b, b+h]. \end{cases}$$

For each $z \in C([0, b], R)$ with $z(0) = 0$, we denote by u the function defined by

$$u(\varepsilon) = \begin{cases} \Gamma(\varepsilon) = \phi(\varepsilon), & if \quad \varepsilon \in [0-r, 0], \\ z(\varepsilon), & if \quad \varepsilon \in (0, b], \\ \Gamma(\varepsilon) = \varphi(\varepsilon), & if \quad \varepsilon \in [b, b+h]. \end{cases}$$

Let us set $\Gamma(\varepsilon) = z(\varepsilon) + x(\varepsilon)$ such that $y_\varepsilon = z_\varepsilon + x_\varepsilon$ for each $\varepsilon \in (0, b]$, where

$$\Gamma(\varepsilon) = \begin{cases} \Gamma(0), \quad \varepsilon = 0, \\ \psi(\varepsilon)^{\nu-1} M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)[\Gamma_0 - h(\varepsilon, 0)] + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) h(s, \Gamma_s) ds \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \Delta(s, \Gamma_s) ds, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s) \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s g(\eta, \Gamma_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\eta) dB(\eta)) ds \\ + \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s \lambda(\eta, \Gamma_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma_\beta dB^H) ds, \quad \varepsilon \epsilon (0, \varepsilon_1], \\ h_k(\varepsilon, \Gamma(\varepsilon)), \quad \varepsilon \epsilon (\varepsilon_k, s_k], \quad k = 1, 2, 3, \dots, m, \\ (\psi(\varepsilon) - s_k)^{\nu-1} M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right) h_k(s_k, \gamma(s_k)) + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) h(s, \Gamma_s) ds \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \Delta(s, \Gamma_s) ds, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s) \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s g(\eta, \Gamma_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\eta) dB(\eta)) ds \\ + \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^s \lambda(\eta, \Gamma((\eta, \gamma_\eta)), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(\eta) dB^H) ds, \quad \varepsilon \epsilon (s_k, \varepsilon_{k+1}]. \end{cases}$$

$$z(\varepsilon)=\begin{cases}\Gamma(0), \quad \varepsilon=0,\\ \psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma})\left[\Gamma_0-h(\varepsilon,0)\right]+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,z_s+x_s)ds\right.\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,z_s+x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB(\eta))ds\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \quad \varepsilon\epsilon(0,\varepsilon_1],\\ h_k(\varepsilon,\Gamma(\varepsilon)), \quad \varepsilon\epsilon(\varepsilon_k,s_k], \quad k=1,2,3,\dots,m,\\ (\psi(\varepsilon)-s_k)^{\nu-1}M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right)h_k(s_k,\gamma(s_k))+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,z_s+x_s)ds\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,z_s+x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB(\eta))ds\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \quad \varepsilon\epsilon(s_k,\varepsilon_{k+1}].\end{cases}$$

Let $\Phi:\mathbb{Y}\to\mathbb{Y}$ be an operator given by

$$\Phi z(\varepsilon)=\begin{cases}0, \quad \varepsilon\in[0-r,0],\\ \Gamma(0), \quad \varepsilon=0,\\ \psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma})\left[\Gamma_0-h(\varepsilon,0)\right]+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,z_s+x_s)ds\right.\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,z_s+x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)dB(\eta))ds\\ +\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \quad \varepsilon\epsilon(0,\varepsilon_1],\\ h_k(\varepsilon,\Gamma(\varepsilon)), \quad \varepsilon\epsilon(\varepsilon_k,s_k], \quad k=1,2,3,\dots,m,\\ (\psi(\varepsilon)-s_k)^{\nu-1}M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right)h_k(s_k,\gamma(s_k))\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,z_s+x_s)ds\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,z_s+x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB(\eta))ds\\ +\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,z_\eta+x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \quad \varepsilon\epsilon(s_k,\varepsilon_{k+1}],\\ 0, \quad \varepsilon\in[b,b+h].\end{cases}$$

To show that the operator $\mathscr{H}$ has a fixed point, for this it is sufficient to show that the operator Φ has a fixed point and this fixed point will correspond to a solution of problem (1.1).

Theorem 4.1. *Assume that the Hypothesis* $(H1)$ *holds, then the solution (1.1) has a unique solution of the problem (1.1), provided*

$$L_0=\max\{L_1,L_k,L_k^*\}<1, \quad \mathrm{k}=1,2,\dots,m, \tag{4.1}$$

where

$$L_1=3M_2\frac{\psi(\varepsilon)^{2\gamma}}{\gamma^2}\left[(M_h+M_{f_1}+\psi(\varepsilon_1)M_{g_1}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_1})+(M_{f_2}+\psi(\varepsilon_1)M_{g_2}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_2})\right.$$
$$\left.\times\frac{\|A\|+M_h+M_{f_1}+\psi(\varepsilon)M_{g_1}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_1}}{1-(M_{f_2}+\psi(\varepsilon)M_{g_2}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_2})}\right],$$

$$L_k = M_{hk},$$
$$L_k^* = 4\Bigg[b^{\nu-1}M_1M_{h_k} + \frac{M_2(b)^{2\gamma}}{\gamma^2}\Bigg\{(M_h + M_{f_1} + bM_{g_1} + 2Hb^{2H}M_{\lambda_1}) + (M_{f_2} + bM_{g_2} + 2Hb^{2H}M_{\lambda_2}) \times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + 2Hb^{2H}M_{\lambda_1}}{1-(M_{f_2} + bM_{g_2} + 2Hb^{2H}M_{\lambda_2})}\Bigg\}\Bigg].$$

Proof. Consider the operator $\Phi : \mathbb{Y} \to \mathbb{Y}$ defined by

$$\Phi z(\varepsilon) = \begin{cases} 0, \quad \varepsilon \in [0-r, 0], \\ \Gamma(0), \quad \varepsilon = 0, \\ \psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\,(A(\psi(\varepsilon)^{\gamma})\,[\Gamma_0 - h(\varepsilon,0)] + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s, z_s + x_s)ds \\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s, z_s + x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s) \\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta, z_\eta + x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)dB(\eta))ds \\ + \int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta, z_\eta + x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \quad \varepsilon\epsilon(0, \varepsilon_1], \\ h_k(\varepsilon, \Gamma(\varepsilon)), \quad \varepsilon\epsilon(\varepsilon_k, s_k], \quad k = 1,2,3,\ldots,m, \\ (\psi(\varepsilon)-s_k)^{\nu-1}M_{\gamma,\nu}\,(A(s_k)^{\gamma})\,h_k(s_k, \gamma(s_k)) + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s, z_s + x_s)ds \\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s, z_s + x_s)ds, D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s) \\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta, z_\eta + x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB(\eta))ds \\ + \int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta, z_\eta + x_\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(\eta)dB^H)ds, \ \varepsilon\epsilon(s_k, \varepsilon_{k+1}], \\ 0, \quad \varepsilon \in [b, b+h]. \end{cases} \tag{4.2}$$

As Δ, h, g, λ, h_k are all continuous, we have to prove that Φ is a contraction.

Case 1. For $z, y \in \mathbb{Y}$ and for $\varepsilon \in [0, \varepsilon_1]$, we have

$$\begin{aligned} E\|(\Phi z)(\varepsilon) - (\Phi y)(\varepsilon)\|^2 \leq & 3\Bigg\{\Bigg(\frac{M_2\psi(\varepsilon)^{2\gamma}}{\gamma^2}M_h\|z(\epsilon) - y(\varepsilon)\|^2 + \frac{M_2\psi(\varepsilon)^{2\gamma}}{\gamma^2}M_{f_1}\|z(\epsilon) - y(\varepsilon)\|^2 \\ & + \frac{M_2\psi(\varepsilon)^{2\gamma+1}}{\gamma^2}M_{g_1}\|z(\epsilon) - y(\varepsilon)\|^2 + \frac{M_2\psi(\varepsilon)^{2\gamma+1}}{\gamma^2}2H\psi(\varepsilon)^{2H-1}M_{\lambda_1}\|z(\epsilon) - y(\varepsilon)\|^2\Bigg) \\ & + \Bigg(\frac{M_2\psi(\varepsilon)^{2\gamma}}{\gamma^2}M_{f_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s) - D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2 \\ & + \frac{M_2\psi(\varepsilon)^{2\gamma+1}}{\gamma^2}M_{g_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s) - D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2 \\ & + \frac{M_2\psi(\varepsilon)^{2\gamma+1}}{\gamma^2}2H\psi(\varepsilon)^{2H-1}M_{\lambda_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s) - D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2\Bigg)\Bigg\}. \end{aligned}$$

This implies

$$\|(\Phi z)(\varepsilon) - (\Phi y)(\varepsilon)\|^2_{\mathbb{Y}} \leq \frac{3M_2\psi(\varepsilon)^{2\gamma}}{\gamma^2}\Bigg[\Bigg(M_h + M_{f_1} + \psi(\varepsilon_1)M_{g_1} + 2Hb^{2H}M_{\lambda_1}\Bigg) \times \|z(\epsilon) - y(\varepsilon)\|^2_{\mathbb{Y}}$$

$$+\frac{3M_2\psi(\varepsilon)^{2\gamma}}{\gamma^2}\left(M_{f_2}+\psi(\varepsilon_1)M_{g_2}+2Hb^{2H}M_{\lambda_2}\right)\times\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s)-D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2\Bigg],$$

$$\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2_{\mathbb{Y}}\leq 3M_2\frac{\psi(\varepsilon)^{2\gamma}}{\gamma^2}\Bigg[(M_h+M_{f_1}+\psi(\varepsilon_1)M_{g_1}+H\psi(\varepsilon_1)^{2H}M_{\lambda_1})+(M_{f_2}+\psi(\varepsilon_1)M_{g_2}$$
$$+H\psi(\varepsilon_1)^{2H}M_{\lambda_2})\times\frac{\|A\|+M_h+M_{f_1}+\psi(\varepsilon)M_{g_1}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_1}}{1-(M_{f_2}+\psi(\varepsilon)M_{g_2}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_2})}\Bigg]\|z(\varepsilon)-y(\varepsilon)\|^2.$$

Thus we take

$$L_1=3M_2\frac{\psi(\varepsilon)^{2\gamma}}{\gamma^2}\Bigg[(M_h+M_{f_1}+\psi(\varepsilon_1)M_{g_1}+H\psi(\varepsilon_1)^{2H}M_{\lambda_1})+(M_{f_2}$$
$$+\psi(\varepsilon_1)M_{g_2}+H\psi(\varepsilon_1)^{2H}M_{\lambda_2})$$
$$\times\frac{\|A\|+M_h+M_{f_1}+\psi(\varepsilon)M_{g_1}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_1}}{1-(M_{f_2}+\psi(\varepsilon)M_{g_2}+2H\psi(\varepsilon_1)^{2H}M_{\lambda_2})}\Bigg].$$

So, we get

$$\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2_{\mathbb{Y}}\leq L_1\|z(\varepsilon)-y(\varepsilon)\|^2_{\mathbb{Y}}.$$

Case 2. For $\varepsilon\in(\varepsilon_k,s_k]$, $k=1,2,\dots,m$, we have

$$E\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2_{\mathbb{Y}}\leq E\|h_k(\varepsilon,z(\varepsilon))-h_k(\varepsilon,y(\varepsilon))\|^2,$$
$$\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2_{\mathbb{Y}}\leq M_{hk}\|z(\varepsilon)-y(\varepsilon)\|^2_{\mathbb{Y}}.$$

Take $L_k:=M_{hk}$ and therefore,

$$\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2_{\mathbb{Y}}\leq L_k\|z(\varepsilon)-y(\varepsilon)\|^2_{\mathbb{Y}}.$$

Case 3. For $z,y\in\mathbb{Y}$ and for $\varepsilon\in(s_k,\varepsilon_{k+1}]$, we have

$$E\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2\leq 4\Bigg\{(\psi(\varepsilon)-s_k)^{\nu-1}M_1M_{hk}\|z(\varepsilon)-y(\varepsilon)\|^2+\Bigg(\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}M_h\|z(\epsilon)-y(\varepsilon)\|^2$$
$$+\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}M_{f_1}\|z(\epsilon)-y(\varepsilon)\|^2+\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}M_{g_1}\|z(\epsilon)-y(\varepsilon)\|^2$$
$$+\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}2H\psi(\varepsilon)^{2H-1}M_{\lambda_1}\|z(\epsilon)-y(\varepsilon)\|^2\Bigg)$$
$$+\Bigg(\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}M_{f_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s)-D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2$$
$$+\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}M_{g_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s)-D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2$$
$$+\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}2H\psi(\varepsilon)^{2H-1}M_{\lambda_2}\|D^{\gamma,\beta}_{\psi(\varepsilon)}z(s)-D^{\gamma,\beta}_{\psi(\varepsilon)}y(s)\|^2\Bigg)\Bigg\}.$$

$$\|(\Phi z)(\varepsilon)-(\Phi y)(\varepsilon)\|^2\leq 4\Bigg\{(\psi(\varepsilon)_{k+1}-s_k)^{\nu-1}M_1M_{hk}+\Bigg(\frac{M_2(\psi(\varepsilon)-s_k)^{2\gamma}}{\gamma^2}$$

$$\times [M_h + M_{f_1} + (\psi(\varepsilon_{k+1}) - s_k)M_{g_1} + 2H\psi(\varepsilon)^{2H} M_{\lambda_1}] \times \|z(\varepsilon) - y(\varepsilon)\|^2\Big)$$
$$+\left(\frac{M_2(\psi(\varepsilon) - s_k)^{2\gamma}}{\gamma^2} \times [M_{f_2} + (\psi(\varepsilon_{k+1}) - s_k)M_{g_2} + 2H\psi(\varepsilon)^{2H} M_{\lambda_2}]\right.$$
$$\left.\times \|D^{\gamma,\beta}_{\psi(\varepsilon)} z(s) - D^{\gamma,\beta}_{\psi(\varepsilon)} y(s)\|^2\right)\Bigg\}.$$

Thus,

$$\begin{aligned}\|(\Phi z)(\varepsilon) - (\Phi y)(\varepsilon)\|^2 \leq & 4\Bigg\{ b^{\nu-1} M_1 M_{h_k} + \frac{M_2 (b)^{2\gamma}}{\gamma^2}\{M_h + M_{f_1} + bM_{g_1} + 2Hb^{2H} M_{\lambda_1} + (M_{f_2} \\ & + bM_{g_2} + 2Hb^{2H} M_{\lambda_2})\} \\ & \times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + 2Hb^{2H} M_{\lambda_1}}{1 - (M_{f_2} + bM_{g_2} + 2Hb^{2H} M_{\lambda_2})}\Bigg\} \|z(\varepsilon) - y(\varepsilon)\|^2 \\ & := L^*_k \|z(\varepsilon) - y(\varepsilon)\|^2_{\mathbb{Y}}.\end{aligned}$$

From the above three cases, we obtain that $\|(\Phi z)(\varepsilon) - (\Phi y)(\varepsilon)\|^2 \leq L_0 \|z(\varepsilon) - y(\varepsilon)\|^2_{\mathbb{Y}}$ as per (4.1), Φ is contraction, and, thus, (1.1) has a unique fixed point z, which is a solution to problem (1.1).

Remark 4.1. *Banach contraction principle provides not only the existence results, but also uniqueness is assured. Also, the nonlinear functions could satisfy only Lipschitz conditions to prove existence and uniqueness results, even though conditions are stronger.*

Remark 4.2. *It is to be noted that the assumption $L_0 < 1$ in Theorem 4.1 shows a restrictive smallness on Lipschitz constants for the nonlinear functions h, g, Δ, and λ when compared with the periods of time, while the impulses are active or vice-versa. In order to relax such a kind of smallness, the generalized Schaefer's FPT is introduced.*

Theorem 4.2. *Assume that $(H2) - (H5)$ hold, then the nonlinear operator $\Phi : \mathbb{Y} \to \mathbb{Y}$ has a fixed point, which is a solution of problem (1.1)*

Proof. Since Φ is well defined, we will present the proof by the following four steps.

First, we prove that Φ is completely continuous (CC), that is, $\mathcal{T}$ is continuous, maps bounded sets into bounded sets, and maps bounded sets into quasi-equicontinuous sets.

Step 1. To prove Φ is continuous.

Since h, g, Δ, λ, and $h_{\mathcal{K}}$ are all continuous, then it is clear that the operator Φ is continuous on $\mathcal{J}$.

Step 2. To prove Φ maps bounded sets in $\mathbb{Y}$.

In fact, it is sufficient to show that for any $r > 0$, there exists an $\eta > 0$ such that for each $z \in \mathcal{B}_r = \{z \in \mathbb{Y} : \|z\|^2_{\mathbb{Y}} \leq z\}$.

Case 1. For $\varepsilon \in [0, \varepsilon_1], z \in \mathbb{Y}$

$$\begin{aligned}\|(\Phi z)(\varepsilon)\|^2_{\mathcal{Y}} \leq & 4\Bigg\{ \psi(\varepsilon_1)^{\nu-1} M_1 \|\Gamma_0 - h(\varepsilon, 0)\|^2_{\mathcal{Y}} + M_2 \frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2} \|h(\varepsilon, z_s + x_s), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s)\| \\ & + M_2 \frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2} \|\Delta(\varepsilon, z_s + x_s, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s))\| + M_2 \frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2} \|g(\varepsilon, z_\eta + x_\eta, D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s))\|\end{aligned}$$

$$+2H\psi(\varepsilon)^{2H-1}M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}\|\lambda(\varepsilon,z_\eta+x_\eta,D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))\|\bigg\}.$$

Using $(H2)-(H4)$ and Lemmas 2.1 and 2.2, one can enumerate

$$\begin{aligned}\|(\Phi z)(\varepsilon)\|^2_{\mathcal{Y}} \leq & 4\bigg\{M_1\|\Gamma_0-h(\varepsilon,0)\|^2_{\mathcal{Y}}+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(l_h(\varepsilon)+m_h(\varepsilon)\|z_s+x_s\|^2_{\mathbb{Y}})+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(l_f(\varepsilon)\\
&+m_f(\varepsilon)\|z_s+x_s\|^2_{\mathbb{Y}}+n_f(\varepsilon)\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2_{\mathbb{Y}})\\
&+M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}(l_g(\varepsilon)+m_g(\varepsilon)\|z_\eta-x_\eta\|^2_{\mathbb{Y}}+n_g\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2)\\
&+2H\psi(\varepsilon)^{2H-1}M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}(l_\lambda(\varepsilon)+m_\lambda(\varepsilon)\|z_\eta+x_\eta\|^2_{\mathbb{Y}}+\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2)\bigg\}\\
\leq & 4\bigg\{M_1\|\Gamma_0-h(\varepsilon,0)\|^2_{\mathcal{Y}}+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(l^*_h)+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(l^*_f)+M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}(l^*_g)\\
&+M_2\frac{2H\psi(\varepsilon_1)^{2H}}{\gamma^2}(l^*_\lambda)\bigg\}\\
&+4\bigg\{M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(m^*_h)+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(m^*_f)+M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}(m^*_g)+M_2\frac{2H\psi(\varepsilon_1)^{2H}}{\gamma^2}(m^*_\lambda)\bigg\}\|z\|\\
&+4\bigg\{M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}n^*_f+M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}n^*_g+M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}n*_\lambda\bigg\}\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2\\
\leq & 4\bigg\{M_1\|\Gamma_0-h(\varepsilon,0)\|^2+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(l^*_h+l^*_f+\psi(\varepsilon)l^*_g+2H\psi(\varepsilon)^{2H}l^*_\lambda)\bigg\}\\
&+4\bigg\{M_1\|\Gamma_0-h(\varepsilon,0)\|^2+M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}(m^*_h+m^*_f+\psi(\varepsilon)m^*_g+2H\psi(\varepsilon)^{2H}m^*_\lambda)\\
&+(\frac{l^*_h+l^*_f+\psi(\varepsilon)l^*_g+\psi(\varepsilon)l^*_\lambda+(m^*_h+m^*_f+\psi(\varepsilon)m^*_g+\psi(\varepsilon)m^*_\lambda)}{1-(n^*_f+\psi(\varepsilon)n^*_g+\psi(\varepsilon)m^*_\lambda})\bigg\}z\\
&:=\eta_0.\end{aligned}$$

Case 2. For $\varepsilon\in(\varepsilon_k,s_k]$, k=1,2,…,m,

$$\begin{aligned}\|(\Phi z)(\varepsilon)\|^2 \leq & E\|h_k(\varepsilon,z_s+x_s)\|^2\leq M_{h_k}(1+\|z\|^2)\\
\leq & \psi(\varepsilon)^{1-v}M_{h_k}(1+\|z\|^2)\\
\leq & \max\psi(\varepsilon)^{1-v}M_{h_k}(1+z)\\
:= & \eta_k,\quad k=1,2,3\ldots,m.\end{aligned}$$

Case 3. For $\varepsilon\in(s_k,\varepsilon_{k+1}]$, $k=1,2,\ldots,m$, $\Gamma\in\mathbb{Y}$, we have

$$\begin{aligned}E\|(\Phi z)(\varepsilon)\|^2 \leq 4\bigg\{ & E\|(\psi(\varepsilon)_{k+1}-s_k)^{v-1}M_{\gamma,v}(A(s_k)^\gamma)h_k(s_k,\Gamma(s_k))\|^2\\
&+E\|\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^\gamma)h(s,z_s+x_s)ds\|^2\end{aligned}$$

$$+ E\|\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,z_s+x_s,D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\|^2$$
$$+ E\|\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,z_\eta+x_\eta,D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds\|^2$$
$$+ E\|\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda((\eta,z_\eta+x_\eta,D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\|^2\bigg\}.$$

Thus

$$\begin{aligned}
E\|(\Phi\Gamma)(\varepsilon)\|^2 \le& 4\bigg\{(\psi(\varepsilon)^{\nu-1}M_1M_{hk}(1+\|\Gamma(s_k)\|^2_{\mathbb{Y}})) + M_2\frac{(\psi(\varepsilon)_{k+1}-s_k)^{2\gamma}}{\gamma^2}[l_h^*\\
&+ l_f^* + (\psi(\varepsilon)_{k+1}-s_k)l_g^* + 2H(\psi(\varepsilon)_{k+1}-s_k)^{2H}l_\lambda^*]\bigg\}\\
&+ 4\bigg\{(\psi(\varepsilon)^{\nu-1}M_1M_{hk}(1+\|\Gamma(s_k)\|^2_{\mathbb{Y}})) + M_2\frac{(\psi(\varepsilon)_{k+1}-s_k)^{2\gamma}}{\gamma^2}[m_h^*\\
&+ m_f^* + (\psi(\varepsilon)_{k+1}-s_k)m_g^* + 2H(\psi(\varepsilon)_{k+1}-s_k)^{2H}l_\lambda^*]\bigg\}\\
&+ 4\bigg\{M_2\frac{\psi(\varepsilon_1)^{2\gamma}}{\gamma^2}n_f^* + M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}n_g^* + M_2\frac{\psi(\varepsilon_1)^{2\gamma+1}}{\gamma^2}n*_\lambda\bigg\}\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2\\
\le& 4\bigg\{b^{\nu-1}M_1M_{hk} + M_2\frac{b^{2\gamma}}{\gamma^2}[l_h^* + l_f^* + bl_g^* + 2Hb^{2H}l_\lambda^*]\bigg\}\\
&+ 4\bigg\{b^{\nu-1}M_1M_{hk} + M_2\frac{b^{2\gamma}}{\gamma^2}[m_h^* + m_f^* + bm_g^* + 2Hb^{2H}m_\lambda^*]\\
&+ [\frac{l_h^*+l_f^*+bl_g^*+bl_\lambda^*+(m_h^*+m_f^*+bm_g^*+bm_\lambda^*)}{1-4(n_f^*)+bn_g^*+bm_\lambda^*}]\bigg\}z\\
:=& \eta_b.
\end{aligned}$$

Let $\eta=\{\eta_0,\eta_k,\eta_b\}$, $k=1,2,\ldots,m$, then $\{\psi^{1-\nu}(\Phi z)(\varepsilon): z\in\mathscr{B}\}$ is a bounded set in $\mathbb{Y}$, i.e., $\|\Phi z\|^2_{\mathbb{Y}}\le\eta$.
Step 3. Φ is a quasi-equicontinous set in $\mathbb{Y}$.
Case 1. Let $\tau_1,\tau_2\in[0,\varepsilon_1]$ with $0\le\tau_1\le\tau_2\le\varepsilon_1$. One could establish the following estimate:

$$\begin{aligned}
\|(\Phi z)(\tau_2)-(\Phi z)(\tau_1)\|^2 =& \sup\{E\|\tau_2^{1-\nu}(\Phi\Gamma)(\tau_2)-\tau_1^{1-\nu}(\Phi\Gamma)(\tau_1)\|^2\}\\
\le& 8\bigg\{E\|M_{\gamma,\nu}(A\tau_2^{\gamma})\|\Gamma_0-h(\varepsilon,0)\|-M_{\gamma,\nu}(A\tau_1^{\gamma})\|\Gamma_0-h(\varepsilon,0)\|\|\|^2\\
&+ \tau_2\int_0^{\tau_2}\|\tau_2^{1-\nu}(\tau_2-s)^{\gamma-1}M_{\gamma,\gamma}(A(\tau_2-s)^{\gamma})\\
&- \tau_1^{1-\nu}(\tau_1-s)^{\gamma-1}M_{\gamma,\gamma}(A(\tau_1-s)^{\gamma})\|^2(l_h^*+m_h^*E\|z_s+x_s\|^2+n_h^*\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2)ds\\
&+ (\tau_2-\tau_1)\int_0^{\tau_2}\tau_2^{2(1-\nu)}(\tau_2-s)^{2\gamma-2}\|M_{\gamma,\gamma}(A(\tau_2-s)^{\gamma})\|^2(l_h^*+m_h^*E\|z_s+x_s\|^2\\
&+ n_h^*\|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2)ds + \tau_2\int_0^{\tau_2}\|\tau_2^{1-\nu}(\tau_2-s)^{\gamma-1}M_{\gamma,\gamma}(A(\tau_2-s)^{\gamma})
\end{aligned}$$

$$
\begin{aligned}
&- \tau_1^{1-\nu}(\tau_1 - s)^{\gamma-1} M_{\gamma,\gamma}(A(\tau_1 - s)^{\gamma})\|^2 (l_f^* + m_f^* E\|z_s + x_s\|^2 + n_f^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds \\
&+ (\tau_2 - \tau_1) \int_0^{\tau_2} \tau_2^{2(1-\nu)}(\tau_2 - s)^{2\gamma-2} \|M_{\gamma,\gamma}(A(\tau_2 - s)^{\gamma})\|^2 (l_f^* + m_f^* E\|z_s + x_s\|^2 \\
&+ n_f^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds + \tau_2 \int_0^{\tau_2} \|\tau_2^{1-\nu}(\tau_2 - s)^{\gamma-1} M_{\gamma,\gamma}(A(\tau_2 - s)^{\gamma}) \\
&- \tau_1^{1-\nu}(\tau_1 - s)^{\gamma-1} M_{\gamma,\gamma}(A(\tau_1 - s)^{\gamma})\|^2 (l_g^* + m_g^* E\|z_\eta + x_\eta\|^2 + n_g^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds \\
&+ (\tau_2 - \tau_1) \int_0^{\tau_2} \tau_2^{2(1-\nu)}(\tau_2 - s)^{2\gamma-2} \|M_{\gamma,\gamma}(A(\tau_2 - s)^{\gamma})\|^2 (l_g^* + m_g^* E\|z_s + x_s\|^2 \\
&+ n_g^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds + 2H\tau_2^{2H-1} \int_0^{\tau_2} \|\tau_2^{1-\nu}(\tau_2 - s)^{\gamma-1} M_{\gamma,\gamma}(A(\tau_2 - s)^{\gamma}) \\
&- \tau_1^{1-\nu}(\tau_1 - s)^{\gamma-1} M_{\gamma,\gamma}(A(\tau_1 - s)^{\gamma})\|^2 (l_\lambda^* + m_\lambda^* E\|z_s + x_s\|^2 + n_\lambda^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds \\
&+ 2H(\tau_2 - \tau_1)^{2H-1} \int_0^{\tau_2} \tau_2^{2(1-\nu)}(\tau_2 - s)^{2\gamma-2} \|M_{\gamma,\gamma}(A(\tau_2 - s)^{\gamma})\|^2 \\
&\times (l_\lambda^* + m_\lambda^* E\|z_s + x_s\|^2 + n_\lambda^* \|D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)\|^2) ds \Big\}.
\end{aligned}
$$

We conclude that as $\tau_2 - \tau_1 \to 0$ with ε sufficiently small, the right hand side of the above inequality tends to zero independently of $z \in \mathscr{B}$. Furthermore, the similar results are true for $\varepsilon_k < \tau_1 < \tau_2 \leq s_k$ and $s_k < \tau_1 < \tau_2 \leq \varepsilon_{k+1}$ for $k = 1, 2, \ldots, m$. It proves the equicontinuous of Φ on $\mathbb{Y}$. Thus, for $\tau_1, \tau_2 \in [0, b] \bigcap (\varepsilon_k, \varepsilon_{k+1}]$, $k = 1, 2, \ldots, m$, whenever $\mathscr{B}$ is a bounded set of $\mathbb{Y}$ as in Step 2, let $z \in \mathscr{B}_r$, then

$$\|(\Phi z)(\tau_1) - (\Phi y)(\tau_2)\|^2_{\mathbb{Y}} \leq M(r)(\tau_2 - \tau_1).$$

Thus, Φ is quasi-equicontinuous, then $\Phi(z)$ is relatively compact by Lemma 3.1, which implies that $\Phi(z)$ is CC.

Step 4. Φ has a Prior bound.

It remains to estimate that the set $\prod(\Phi) = \{z \in \mathbb{Y} : z = \Theta\Phi z, 0 < \Theta < 1\}$ is bounded.

Let $z \in \Pi(\Phi)$, then $z = \Theta\Phi(z)$ for some $\Theta \in (0, 1)$, by following the proof of Step 2 that $\|z\|_{\mathbb{Y}} \leq \eta$. This proves that the set $\Pi(\Phi)$ is bounded. Hence, by Theorem 3.1, Φ has a fixed point, which is the required solution on $\mathscr{J}$.

5. UHR stability results

Here, we derive the UHR stability for (1.1). Let $\omega > 0$, $\phi \geq 0$, and $\zeta \in PC(\mathscr{J}, R^n)$ be nondecreasing. Consider the following inequalities.

$$
\begin{aligned}
&E\|D^{\gamma,\beta}_{\psi(\varepsilon)}[\tilde{\Gamma}(\varepsilon) - h(\varepsilon, \tilde{\Gamma}(\varepsilon))] - A\tilde{\Gamma}(\varepsilon) - \Delta(\varepsilon, \tilde{\Gamma}(\varepsilon), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) - \int_0^{\varepsilon} g(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) dB(s) \\
&- \int_0^{\varepsilon} \lambda(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) dB^H(s)\| \leq \omega\zeta(\varepsilon), \quad \varepsilon \in (s_k, \varepsilon_{k+1}], \quad k = 0, 1, 2, \ldots, m, \\
&E\|\Gamma(\varepsilon) - h_k(\varepsilon, \Gamma(\varepsilon))\|^2 \leq \omega\phi, \quad \varepsilon \in (\varepsilon_k, s_k], \quad k = 1, 2, \ldots, m, \\
&E\|I^{1-\nu}_{0^+}\Gamma(\varepsilon) - \Gamma_0\|^2 \leq \omega\phi, \\
&\Gamma(\varepsilon) = \phi(\varepsilon), \varepsilon \in [0 - r, 0],
\end{aligned}
\tag{5.1}
$$

$\Gamma(\varepsilon) = \varphi(\varepsilon), \varepsilon \in [b, b+h]$.

Define a vector space as χ:

$$\chi = PC(\mathscr{J}, R^n) \cap C((s_k, t_{k+1}], R^n).$$

Definition 5.1. *System (1.1) is UHR stable with respect to* (ζ, ϕ) *if there exists* $C_{(M,L,P,\zeta)} > 0$ *such that for each solution* $\tilde{\Gamma} \in \chi$ *of the inequality (5.1), there exists solution* $\Gamma \in PC(\mathscr{J}, R^n)$ *of (1.1) with*

$$E\|\tilde{\Gamma}(\varepsilon) - \Gamma(\varepsilon)\|^2 \leq C_{(M,L,p,\zeta)}\omega(\zeta(\varepsilon) + \phi), \varepsilon \in \mathscr{J}.$$

Remark 5.1. *A function* $\tilde{\Gamma} \in \chi$ *is a solution of (5.1)* $\Leftrightarrow$ *there is* $\mathscr{Q} \in \bigcap_{i=0}^{\mathscr{K}} ((s_k, \varepsilon_{k+1}], R^n]$ *and* $q \in \bigcap_{i=0}^{\mathscr{K}}((\varepsilon_k, s_k], R^n)$ *such that:*

(1) $E\|\mathscr{Q}(\varepsilon)\|^2 \leq \omega\zeta(\varepsilon), \varepsilon \in \bigcap_{i=0}^{\mathscr{K}}(s_k, \varepsilon_{k+1}]; E\|q(\varepsilon)\|^2 \leq \omega\phi, \varepsilon \in \bigcap_{i=0}^{\mathscr{K}}(t_k, s_k]$;

(2) $\mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}[\tilde{\Gamma}(\varepsilon) - h(\varepsilon, \tilde{\Gamma}(\varepsilon))] = A\tilde{\Gamma}(\varepsilon) + \Delta(\varepsilon, \tilde{\Gamma}(\varepsilon), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s))) + \int_0^\varepsilon g(s, \tilde{\Gamma}(s), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s)))dB(s)$
$+ \int_0^\varepsilon \lambda(s, \tilde{\Gamma}(s), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s)))dB^H(s) + \mathscr{Q}(\varepsilon), \quad \varepsilon \in (s_k, \varepsilon_{k+1}], \quad k = 0, 1, 2, \ldots, m,$

(3) $\tilde{\Gamma}(\varepsilon) = h_k(\varepsilon, \tilde{\Gamma}(\varepsilon)) + q(\varepsilon), \varepsilon \in (\varepsilon_k, s_k], k = 0, 1, 2, \ldots, m.$

By Remark 5.1, we have

$$\begin{aligned}
D^{\gamma,\beta}_{\psi(\varepsilon)}[\tilde{\tilde{\Gamma}}(\varepsilon) - h(\varepsilon, \tilde{\Gamma}(\varepsilon))] =& A\tilde{\Gamma}(\varepsilon) + \Delta(\varepsilon, \tilde{\Gamma}(\varepsilon), D^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s))) + \int_0^\varepsilon g(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s)))dB(s) \\
&+ \int_0^\varepsilon \lambda(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\tilde{\Gamma}(s)))dB^H(s) + \mathscr{Q}(\varepsilon), \\
&\varepsilon \in (s_k, \varepsilon_{k+1}], k = 0, 1, 2, \ldots, m, \\
\tilde{\Gamma}(\varepsilon) =& h_k(\varepsilon, \tilde{\Gamma}(\varepsilon)) + q(\varepsilon), \quad \varepsilon \in (\varepsilon_k, s_k], k = 0, 1, 2, \ldots, m, \\
&I^{1-\nu}_{0^+}\tilde{\Gamma}(\varepsilon)/_{\varepsilon=0} = \Gamma_0.
\end{aligned}$$

Lemma 5.1. *Let* $\beta \in [0, 1]$, $\gamma \in (0, 1)$. *If a function* $\tilde{\Gamma} \in \chi$ *is a solution of (5.1) then we have:*

$$\begin{aligned}
(i) \quad & E\|\tilde{\Gamma}(\varepsilon) - \psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\,(A(\psi(\varepsilon)^\gamma)\,[\Gamma_0 - h(\varepsilon, 0)] \\
& - \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma)h(s, \tilde{\Gamma}(s), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds \\
& - \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma)\Delta(s, \tilde{\Gamma}(s), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds \\
& - \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma)(\int_0^s g(\eta, \tilde{\Gamma}(\eta), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds \\
& - \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma)(\int_0^s \lambda(\eta, \tilde{\Gamma}(\eta), \mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\|^2 \\
& \leq \frac{\psi^{2\gamma}(\varepsilon)}{\gamma^2}M_2\omega\int_0^\varepsilon \zeta(s)ds, \quad \varepsilon \in [0, \varepsilon_1].
\end{aligned}$$

$$(ii) \quad E\|(\psi(\varepsilon) - s_k)^{\nu-1}M_{\gamma,\nu}\,(A(s_k)^\gamma)\,h_k(s_k, \tilde{\Gamma}(s_k))$$

$$
\begin{aligned}
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,\tilde{\Gamma}(s),\mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,\tilde{\Gamma}(s),\mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,\tilde{\Gamma}(\eta),\mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,\tilde{\Gamma}(\eta),\mathrm{D}^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\|^2\\
&\leq\frac{b^2\gamma}{\gamma^2}M_2\omega\int_{s_k}^{\varepsilon}\zeta(s)ds,\ \varepsilon\in(s_k,\varepsilon_{k+1}],\quad k=1,2,\ldots,m.
\end{aligned}
$$

By Remark 5.1, we have:

Case 1. For $\varepsilon\in[0,\varepsilon_1]$, we have

$$
\begin{aligned}
D^{\gamma,\beta}_{\psi(\varepsilon)}[\tilde{\Gamma}(\varepsilon)-h(\varepsilon,\tilde{\Gamma}(\varepsilon))]=&A\tilde{\Gamma}(\varepsilon)+\Delta(\varepsilon,\tilde{\Gamma}(\varepsilon),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))+\int_0^{\varepsilon}g(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(s)\\
&+\int_0^{\varepsilon}\lambda(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H(s)+\mathscr{Q}(\varepsilon).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\tilde{\Gamma}(\varepsilon)=&\psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)[\Gamma_0-h(\varepsilon,0)]\\
&+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds\\
&+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s \lambda(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\\
&+\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\mathscr{Q}(s)ds.
\end{aligned}
$$

From above, we obtain

$$
\begin{aligned}
&E\|\tilde{\Gamma}(\varepsilon)-\psi(\varepsilon)^{\nu-1}M_{\gamma,\nu}\left(A(\psi(\varepsilon)^{\gamma}\right)[\Gamma_0-h(\varepsilon,0)]\\
&-\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s\lambda(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\|^2\\
\le& E\|\int_0^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\mathscr{Q}(s)ds\|^2\\
\le&\frac{\psi^{2\gamma}(\varepsilon)}{\gamma^2}M_2\omega\int_0^{\varepsilon}\zeta(s)ds.
\end{aligned}
$$

Case 2. For $\varepsilon\in(\varepsilon_k,s_k]$, we have

$$
E\|\tilde{\Gamma}(\varepsilon)-h_k(\varepsilon,\tilde{\Gamma}(\varepsilon))\|^2\le\omega\phi.
$$

Case 3. For $\varepsilon\in(s_k,\varepsilon_{k+1}]$, we get

$$
\begin{aligned}
D^{\gamma,\beta}_{\psi(\varepsilon)}[\tilde{\Gamma}(\varepsilon)-h(\varepsilon,\tilde{\Gamma}(\varepsilon))]=&A\tilde{\Gamma}(\varepsilon)+\Delta(\varepsilon,\tilde{\Gamma}(\varepsilon),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))+\int_0^{\varepsilon}g(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(s)\\
&+\int_0^{\varepsilon}\lambda(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H(s)+\mathscr{Q}(\varepsilon),\\
\text{then,}\quad\tilde{\Gamma}(\varepsilon)=&(\psi(\varepsilon)-s_k)^{\nu-1}M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right)h_k(s_k,\gamma(s_k))\\
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds\\
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s\lambda(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds,\\
&+\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\mathscr{Q}(s)ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&E\|\tilde{\Gamma}(\varepsilon)-(\psi(\varepsilon)-s_k)^{\nu-1}M_{\gamma,\nu}\left(A(s_k)^{\gamma}\right)h_k(s_k,\tilde{\Gamma}(s_k))\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})h(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\Delta(s,\tilde{\Gamma}(s),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s g(\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB(\eta))ds\\
&-\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})(\int_0^s\lambda((\eta,\tilde{\Gamma}(\eta),D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s))dB^H)ds\|^2\\
\le& E\|\int_{s_k}^{\psi(\varepsilon)}(\psi(\varepsilon)-s)^{\gamma-1}M_{\gamma,\gamma}(A(\psi(\varepsilon)-s)^{\gamma})\mathscr{Q}(s)ds\|^2
\end{aligned}
$$

$$\leq \frac{(\psi(\varepsilon) - s_k)^{2\gamma}}{\gamma^2} M_2 \omega \int_0^{\varepsilon} \zeta(s) ds$$
$$\leq \frac{b^{2\gamma}}{\gamma^2} M_2 \varepsilon \int_0^{\varepsilon} \zeta(s) ds.$$

In order to prove stability, we assume the following assumptions:
(H7) There exist positive constants $M_k (k = 1, 2, \dots, m)$ such that

$$E\|h_k(\varepsilon, \Gamma(\varepsilon)) - h_k(\varepsilon, z(\varepsilon))\|^2 \leq \sum_{k=1}^{m} M_k E\|\Gamma(\varepsilon) - z(\varepsilon)\|_{\mathbb{Y}}^2 \forall \varepsilon \in (\varepsilon_k, s_k].$$

(H8) Let $\zeta \in C(\mathscr{J}, R^n)$ be a nondecreasing function if there exists $c_\zeta > 0$ such that $\int_0^{\varepsilon} \zeta(s)ds < c_\zeta \zeta(\varepsilon)$, $\forall\, \varepsilon \in \mathscr{J}$.

Lemma 5.2. *Let $P_0 = P \cup 0$ where $p = 1, 2, \dots, m$, and the following inequality holds*

$$\Gamma(\varepsilon) \leq a(\varepsilon) + \int_0^{\varepsilon} b(s)\Gamma(s)ds + \Sigma_{0<\varepsilon_k<\varepsilon} \alpha_k \Gamma(\varepsilon_{\bar{k}}), \quad \varepsilon \geq 0,$$

where $\Gamma, a, b \in PC(R^+, R^+)$, a is nondecreasing and $b(\varepsilon) > 0$, $\alpha_k > 0$, $k \in P$. For $\varepsilon \in R^+$,

$$\Gamma(\varepsilon) \leq a(\varepsilon)(1 + \alpha)^K e\left(\int_0^{\varepsilon} b(s)ds\right), \varepsilon \in (\varepsilon_K, \varepsilon_{K+1}], \quad K \in P_0,$$

where $\alpha = \sup_{K \in P}\{\alpha_K\}$ and $\varepsilon_0 = 0$.

Theorem 5.1. *If the assumptions $(H1), (H7)$, and $(H8)$ are satisfied, then (1.1) is UHR stable with respect to (ζ, ϕ).*

Proof. Let $\tilde{\Gamma} \in \chi$ be a solution of inequality (5.1) and Γ be the unique solution of (1.1).
Case 1. For $\varepsilon \in [0, \varepsilon_1]$, we have:

$$\begin{aligned}
&E\|\tilde{\Gamma}(\varepsilon) - \psi(\varepsilon)^{\nu-1} M_{\gamma,\nu} \left(A(\psi(\varepsilon)^{\gamma}) \left[\Gamma_0 - h(\varepsilon, 0)\right]\right. \\
&- \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) h(s, \tilde{\Gamma}(s)) ds \\
&- \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \Delta(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) ds \\
&- \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^{s} g(\eta, \tilde{\Gamma}(\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) dB(\eta)) ds \\
&- \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) (\int_0^{s} \lambda(\eta, \tilde{\Gamma}(\eta), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)) dB^H) ds\|^2 \\
=&E\| \int_0^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^{\gamma}) \mathscr{Q}(s) ds\|^2 \\
\leq& \frac{\psi^{2\gamma}(\varepsilon)}{\gamma^2} M_2 \varepsilon \int_0^{\varepsilon} \psi(s) ds \leq \frac{\psi^{2\gamma}(\varepsilon)}{\gamma^2} M_2 \omega c_\zeta \zeta(\varepsilon) \leq \frac{b^{2\gamma}}{\gamma^2} M_2 \omega c_\zeta \zeta(\varepsilon) \leq c_p c_\zeta \zeta(\varepsilon),
\end{aligned}$$

where $c_p = \frac{b^{2\gamma}}{\gamma^2} M_2$.
Case 2. For $\varepsilon \in (\varepsilon_k, s_k]$, we have

$$E\|\tilde{\Gamma}(\varepsilon) - h_k(\varepsilon, \tilde{\Gamma}(\varepsilon))\|^2 \leq \omega\phi.$$

Case 3. For $\varepsilon \in (s_k, \varepsilon_{k+1}]$, we have

$$\begin{aligned}
&E\|\tilde{\Gamma}(\varepsilon) - (\psi(\varepsilon) - s_k)^{\nu-1} M_{\gamma,\nu}\,(A(s_k)^\gamma)\, h_k(s_k, \Gamma(s_k)) \\
&- \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma) h(s, \tilde{\Gamma}(s)) ds \\
&- \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma) \Delta(s, \tilde{\Gamma}(s), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s)) ds \\
&- \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma) (\int_0^s g(\eta, \tilde{\Gamma}(\eta), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s)) dB(\eta)) ds \\
&- \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma) (\int_0^s \lambda((\eta, \tilde{\Gamma}(\eta), D^{\gamma,\beta}_{\psi(\varepsilon)} \Gamma(s)) dB^H) ds\|^2 \\
=&E\| \int_{s_k}^{\psi(\varepsilon)} (\psi(\varepsilon) - s)^{\gamma-1} M_{\gamma,\gamma}(A(\psi(\varepsilon) - s)^\gamma) \mathscr{Q}(s) ds\|^2 \\
\leq&\frac{(\psi(\varepsilon) - s_k)^{2\gamma}}{\gamma^2} M_2 \omega \int_0^\varepsilon \zeta(s) ds \leq \frac{b^{2\gamma}}{\gamma^2} M_2 \omega c_\zeta \zeta(\varepsilon) \leq c_p c_\zeta \zeta(\varepsilon).
\end{aligned}$$

Hence, for $\varepsilon \in [0, \varepsilon_1]$, we have:

$$\begin{aligned}
E\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 \leq&\frac{\psi(\varepsilon)^{2\gamma}}{\gamma^2} M_2 \Bigg[\omega c_\zeta \zeta(\varepsilon) + \Bigg((M_h + M_{f_1} + \psi(\varepsilon_1) M_{g_1} + H\psi(\varepsilon_1)^{2H} M_{\lambda_1}) \\
&+ (M_{f_2} + \psi(\varepsilon_1) M_{g_2} + 2H\psi(\varepsilon_1)^{2H} M_{\lambda_2}) \\
&\times \frac{\|A\| + M_h + M_{f_1} + \psi(\varepsilon) M_{g_1} + 2H\psi(\varepsilon_1)^{2H} M_{\lambda_1}}{1 - (M_{f_2} + \psi(\varepsilon) M_{g_2} + 2H\psi(\varepsilon_1)^{2H} M_{\lambda_2})}\Bigg) \|\Gamma(s) - \tilde{\Gamma}(s)\|^2\Bigg] \\
\leq&\frac{b^{2\gamma}}{\gamma^2} M_2 \Bigg[\omega c_\zeta \zeta(\varepsilon) + \Bigg((M_h + M_{f_1} + bM_{g_1} + Hb^{2H} M_{\lambda_1}) \\
&+ (M_{f_2} + bM_{g_2} + Hb^{2H} M_{\lambda_2}) \\
&\times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + Hb^{2H} M_{\lambda_1}}{1 - (M_{f_2} + bM_{g_2} + 2Hb^{2H} M_{\lambda_2})}\Bigg) \|\Gamma(s) - \tilde{\Gamma}(s)\|^2\Bigg].
\end{aligned} \tag{5.2}$$

For $\varepsilon \in (\varepsilon_k, s_k]$, we get

$$\begin{aligned}
E\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 =&E\|\Gamma(\varepsilon) - h_k(s_k, \tilde{\Gamma}(s_k))\|^2 \\
=&E\|\Gamma(\varepsilon) - h_k(s_k, \Gamma(s_k)) + h_k(s_k, \Gamma(s_k)) - h_k(s_k, \tilde{\Gamma}(s_k))\|^2 \\
\leq&2\{E\|\Gamma(\varepsilon) - h_k(s_k, \Gamma(s_k))\|^2 + E\|h_k(s_k, \Gamma(s_k)) - h_k(s_k, \tilde{\Gamma}(s_k))\|^2\} \\
\leq&2\{\omega\phi + \Sigma_{i=0}^k M_k \|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2\}.
\end{aligned} \tag{5.3}$$

For $\varepsilon \in (s_k, \varepsilon_{k+1}]$, k=1,2,...,m,

$$E\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 \leq 5\frac{b^{2\gamma}}{\gamma^2} M_2 \Bigg[\omega c_\zeta \zeta(\varepsilon) + \Bigg((M_h + M_{f_1} + bM_{g_1} + Hb^{2H} M_{\lambda_1})$$

$$
\begin{aligned}
&+ (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2}) \\
&\times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + Hb^{2H}M_{\lambda_1}}{1 - (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2})}\Bigg)\|\Gamma(s) - \tilde{\Gamma}(s)\|^2\Bigg]. \\
&+ 5b^{\nu-1}M_1\Sigma_{i=1}^{k}M_k\|\Gamma(s) - \Gamma\tilde{(s)}\|^2. \qquad (5.4)
\end{aligned}
$$

Combining (5.2)–(5.4), one can get an inequality of the form given in Lemma 5.2. For $\varepsilon \in \mathscr{J}$, since $\varepsilon \in (\varepsilon_k, \varepsilon_{k+1}]$, $K \in P_0$, we have

$$
\begin{aligned}
E\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 \leq &5\Bigg\{c_p\omega c_\zeta\zeta(\varepsilon) + b^{\nu-1}M_1\Sigma_{i=1}^{k}M_k\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 \\
&+ c_p\Bigg[(M_h + M_{f_1} + bM_{g_1} + Hb^{2H}M_{\lambda_1}) \\
&+ (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2}) \\
&\times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + Hb^{2H}M_{\lambda_1}}{1 - (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2})}\Bigg]\|\Gamma(s) - \tilde{\Gamma}(s)\|^2 \\
&+ 5b^{\nu-1}M_1\Sigma_{i=1}^{k}M_k\|\Gamma(s) - \Gamma\tilde{(s)}\|^2\Bigg\} + 2\omega\phi \\
\leq &5c_p\omega c_\zeta(\zeta(\varepsilon) + \phi)(1 + M)^k e^{\int_0^\varepsilon L ds},
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \sup\{b^{\nu-1}M_1M_k\}, \\
L &= c_p\Bigg[(M_h + M_{f_1} + bM_{g_1} + Hb^{2H}M_{\lambda_1}) + (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2}) \\
&\times \frac{\|A\| + M_h + M_{f_1} + bM_{g_1} + Hb^{2H}M_{\lambda_1}}{1 - (M_{f_2} + bM_{g_2} + Hb^{2H}M_{\lambda_2})}\Bigg].
\end{aligned}
$$

Thus,

$$
E\|\Gamma(\varepsilon) - \tilde{\Gamma}(\varepsilon)\|^2 \leq 5C_{(M,L,p,\zeta)}\omega(\zeta(\varepsilon) + \phi), \forall \varepsilon \in \mathscr{J},
$$

where $C_{(M,L,p,\zeta)}$ is a constant depending on M, L, p, ζ. Hence, (1.1) is UHR stable w.r.t (ζ, ϕ).

6. Example

Consider the following nonlinear ψ-HFSE with NI impulses driven by both noises. This type of fractional SDE can be applied in pharmacotherapy.

$$
\begin{aligned}
D_{\psi(\varepsilon)}^{\frac{1}{2},\frac{3}{4}}\Gamma_1(\varepsilon) = &\Gamma_2(\varepsilon) + \frac{1}{5}\left(\frac{\Gamma_1(\varepsilon)}{1 + \Gamma_1^2(\varepsilon) + \Gamma_2^2(\varepsilon)}\right) + \frac{1}{5}\left(\frac{\Gamma_1(\varepsilon)}{1 + \Gamma_1^2(\varepsilon) + \Gamma_2^2(\varepsilon)}, D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(s)\right) \\
&+ \frac{\Gamma_1(\varepsilon)e^{-\psi(\varepsilon)}}{3}\beta_1(\varepsilon), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(s) + \frac{\Gamma_1(\varepsilon)e^{-2\psi(\varepsilon)}}{5}\beta_1^H(\varepsilon), D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(s), \\
D_{\psi(\varepsilon)}^{\frac{1}{2},\frac{3}{4}}\Gamma_2(\varepsilon) = &\Gamma_1(\varepsilon) + \frac{1}{5}\left(\frac{\Gamma_2(\varepsilon)}{1 + \Gamma_1^2(\varepsilon) + \Gamma_2^2(\varepsilon)}\right) + \frac{1}{5}\left(\frac{\Gamma_2(\varepsilon)}{1 + \Gamma_1^2(\varepsilon) + \Gamma_2^2(\varepsilon)}, D_{\psi(\varepsilon)}^{\gamma,\beta}\Gamma(s)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma_2(\varepsilon)e^{-\psi(\varepsilon)}}{3}\beta_2(\varepsilon), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s)+\frac{\Gamma_2(\varepsilon)e^{-2\psi(\varepsilon)}}{3}\beta^H_2(\varepsilon), D^{\gamma,\beta}_{\psi(\varepsilon)}\Gamma(s),\\
&\varepsilon\in(s_k,\quad \varepsilon_{k+1}],\quad k=0,1,\ldots,m,\\
\Gamma(\varepsilon)=&\frac{1}{4}\Gamma(\varepsilon,\varepsilon_k),\quad \varepsilon(0.9,s_k),\quad k=1,2,\ldots,m,\\
\Gamma(0)=&\Gamma_0,\quad \psi(\varepsilon)=\sin(\varepsilon),
\end{aligned}
$$

where $\Gamma\in R^2$, $\Gamma\in\frac{1}{2}$, $\beta=\frac{3}{4}$, $0<\varepsilon_0<s_0<\varepsilon_1<\cdots<s_m=1$ are prefixed numbers, $\mathscr{J}=[0,1]$. Here

$$
A=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \Delta(\varepsilon,\Gamma(\varepsilon))=\begin{pmatrix}\frac{\Gamma_1(\varepsilon)}{5(1+\Gamma_1^2(\varepsilon)+\Gamma_2^2(\varepsilon))}&0\\0&\frac{\Gamma_2(\varepsilon)}{5(1+\Gamma_1^2(\varepsilon)+\Gamma_2^2(\varepsilon))}\end{pmatrix},
$$

$$
h(\varepsilon,\Gamma(\varepsilon))=\begin{pmatrix}\frac{\Gamma_1(\varepsilon)}{5(1+\Gamma_1^2(\varepsilon)+\Gamma_2^2(\varepsilon))}&0\\0&\frac{\Gamma_2(\varepsilon)}{5(1+\Gamma_1^2(\varepsilon)+\Gamma_2^2(\varepsilon))}\end{pmatrix},
$$

$$
g(\varepsilon,\Gamma(\varepsilon))=\begin{pmatrix}\frac{\Gamma_1(\varepsilon)e^{-\psi(\varepsilon)}}{3}&0\\0&\frac{\Gamma_2(\varepsilon)e^{-\psi(\varepsilon)}}{3}\end{pmatrix},\quad \lambda(\varepsilon,\Gamma(\varepsilon))=\begin{pmatrix}\frac{\Gamma_1(\varepsilon)e^{-2\psi(\varepsilon)}}{5}&0\\0&\frac{\varepsilon\Gamma_2(\varepsilon)e^{-\psi(\varepsilon)}}{5}\end{pmatrix}.
$$

Calculate the M-L function by using [12]. We have

$$
M_{\gamma,\nu}(A\psi(\varepsilon)^\gamma)=\begin{pmatrix}S_1&S_2\\-S_3&S_4\end{pmatrix},
$$

where

$$
\begin{aligned}
S_1=&S_4=\sum_{j=0}^{\infty}\frac{(-1)^j b^{2j\gamma}}{\Gamma(1+2j\gamma)}=-0.3377,\\
S_2=&-S_3=\sum_{j=0}^{\infty}\frac{(-1)^j b^{(2j+1)\gamma}}{\Gamma(1+(2j+1)\gamma)}=-1.666.
\end{aligned}
$$

Also calculate

$$
M_{\gamma,\nu}(A(b-s)^\gamma)=\begin{pmatrix}P_1&P_2\\-P_3&P_4\end{pmatrix},
$$

where

$$
\begin{aligned}
P_1=&P_4=\sum_{j=0}^{\infty}\frac{(-1)^j (b-s)^{2j\gamma}}{\Gamma(2j\gamma+\gamma)}=-0.7477,\\
P_2=&-P_3=\sum_{j=0}^{\infty}\frac{(-1)^j (b-s)^{(2j+1)\gamma}}{\Gamma[(2j+1)\gamma]}=-0.4203.
\end{aligned}
$$

Therefore, we need to check the hypotheses of nonlinear functions:

$$
E\|h(\varepsilon,u_1)-h(\varepsilon,u_2)\|^2\leq\frac{1}{30}\|u_1-u_2\|^2,
$$

$$E\|\Delta(\varepsilon,u_1,v_1)-\Delta(\varepsilon,u_2,v_2)\|^2 \leq \frac{1}{25}\|u_1-u_2\|^2+\frac{1}{35}\|v_1-v_2\|^2,$$
$$E\|g(\varepsilon,u_1,v_1)-g(\varepsilon,u_2,v_2)\|^2 \leq \frac{1}{9}e^{-2}\|u_1-u_2\|^2+\frac{1}{10}e^{-3}\|v_1-v_2\|^2,$$
$$E\|\lambda(\varepsilon,u_1,v_1-\lambda(\varepsilon,u_2,v_2)\|^2 \leq \frac{1}{25}e^{-4\psi(\varepsilon)}\|u_1-u_2\|^2+\frac{1}{20}e^{-5\psi(\varepsilon)}\|v_1-v_2\|^2,$$
$$E\|h_k(\varepsilon,u)-h_k(\varepsilon,v\|^2 \leq \frac{1}{16}\|u-v\|^2, k=1,2,\ldots,m.$$

We get $M_1 = 0.0862$, $M_2 = 0.3824$, $M_{hk} = 0.065$, and $\gamma = 0.5$. Hence we have $L = max\{L_0, L_k, L_k*\} = 0.52 < 1$. Also,

$$E\|\Gamma(\varepsilon)-\tilde{\Gamma}(\varepsilon)\|^2 \leq 5C_{(M,L,p,\zeta)}\varepsilon(\zeta(\varepsilon)+\phi), \forall\varepsilon\in\mathscr{J},$$
$$\leq 0.218.$$

In this example, all the conditions stated in Theorems 4.1 and 5.1 are satisfied, so the example has a unique solution and is also UHR stable.

7. Conclusions

With the help of Schaefer's FPT and the Banach contraction principles, we obtained the existence and uniqueness results for HSFEs with retarded and advanced arguments, selected the non-instantaneous impulses with both multiplicative and fractional noises, and obtained the UHR stability for HSFEs. UHR stability gives bounds between the exact and approximation solution, which is why this theory is very important in the numerical analysis as well as in approximation theory. We are hoping that our findings will have a great importance in the mentioned theories. Finally, a case study is provided to demonstrate the efficacy of the suggested outcomes. In the future, we can use the findings to investigate the controllability of HSFEs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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