AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 8134–8147.
DOI: 10.3934/math.2024395
Received: 03 December 2023
Revised: 11 January 2024
Accepted: 06 February 2024
Published: 26 February 2024

Research article

Coefficient bounds for certain families of bi-Bazilevič and bi-Ozaki-close-to-convex functions

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Abstract: The aim of this work is to introduce two families, $\mathcal{B}_\Sigma(\wp;\vartheta)$ and $\mathcal{O}_\Sigma(\varkappa;\vartheta)$, of holomorphic and bi-univalent functions involving the Bazilevič functions and the Ozaki-close-to-convex functions, by using generalized telephone numbers. We determine upper bounds on the Fekete-Szegö type inequalities and the initial Taylor-Maclaurin coefficients for functions in these families. We also highlight certain edge cases and implications for our findings.

Keywords: bi-univalent function; holormorphic function; Bazilevič function; Ozaki-close-to-convex functions; upper bounds; telephone numbers; Fekete-Szegö problem
Mathematics Subject Classification: 30C45, 30C50

1. Introduction and preliminaries

Indicate by $\mathcal{A}$ the family of holomorphic functions in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k\geq 2} a_k z^k. \tag{1.1}$$

The $\mathcal{A}$ subfamily of functions that are univalent in $\mathbf{U}$ is denoted by $\mathcal{S}$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{K}$ the families of functions that starlike, convex, and close-to-convex in $\mathbf{U}$, respectively, and given as follows:

A function $f \in \mathcal{S}$ is said to be starlike if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbf{U},$$

a function $f \in \mathcal{S}$ is said to be convex if

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \qquad z \in \mathbf{U},$$

and a function $f \in \mathcal{S}$ is said to be a close-to-convex functions if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0; g \in \mathcal{S}^*, \qquad z \in \mathbf{U}.$$

Note that, $C \subset \mathcal{S}^* \subset \mathcal{K}$.

The families of functions that are starlike of order $\aleph(0 \leqq \aleph < 1)$ and convex of order $\aleph(0 \leqq \aleph < 1)$ are denoted by $\mathcal{S}^*(\aleph)$ and $C(\aleph)$, respectively, and are given as follows:

$$\mathcal{S}^*(\aleph) = \left\{f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \aleph, \qquad z \in \mathbf{U}\right\}$$

and

$$C(\aleph) = \left\{f \in \mathcal{S} : \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \aleph, \qquad z \in \mathbf{U}\right\}.$$

A well known subclass of S, named as class of Bazilevič function [36], is defined as:

$$\mathcal{B}(\wp) = \left\{f \in \mathcal{A} : \Re\left(\frac{z^{1-\wp} f'(z)}{(f(z))^{1-\wp}}\right) > 0; \qquad z \in \mathbf{U};\ \wp \geqq 0\right\}.$$

For $0 \leq \alpha \leq \pi$, Kaplan [25] defined a subfamily of A called close-to-convex functions, given as:

$$\mathcal{K}(\alpha) = \left\{f \in \mathcal{A} : \Re\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0; g \in \mathcal{S}^*\ z \in \mathbf{U}\right\}.$$

In 1935, Ozaki [34] considered functions in $\mathcal{A}$ satisfying the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \qquad z \in \mathbf{U},$$

whose members are known to be close-to-convex and therefore univalent.

Lately, Kargar and Ebadian [26] considered the generalization of Ozaki's condition as the following:

Definition 1.1. *Let $f \in \mathcal{A}$ be locally univalent for $z \in \mathbf{U}$ and let $-\frac{1}{2} \leqq \varkappa \leqq 1$. A function f is called an Ozaki-close-to-convex function in $\mathbf{U}$ if*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \varkappa, \qquad z \in \mathbf{U}.$$

We denote by $F(\varkappa)$ the family of all Ozaki-close-to-convex functions. It is clear that, for $-\frac{1}{2} \leqq \varkappa \leqq \frac{1}{2}$, we have $F(\varkappa) \subset K \subset \mathcal{S}^*$.

In $\mathbf{U}$, a function $f \in \mathcal{A}$ is considered bi-univalent if f and f^{-1} are both univalent functions. Denote by Σ all bi-univalent functions in $\mathbf{U}$. We revisit a few functions in the family Σ from Srivastava et al.'s work [41]. We notice that the family Σ consists of

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so is not empty, however the Koebe function is not a member of Σ. Also, the functions $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are not bi-univalent. A very large number of works related to the class Σ have been presented in the recent papers (one may see [1,3,6,8,10–12,14,16,18,19,22–24,27–29,31–33,38–40,42–46,49–51]).

The Littlewood-Paley conjecture (see [21]) that the coefficients of odd univalent functions are bounded by unity has been refuted by Fekete and Szegö, a fact widely recognized in the field of Geometric Function Theory (GFT). Consequently, given $f \in \mathcal{S}$, we get the Fekete-Szegö problem $\left|a_3 - \hbar a_2^2\right|$. Many authors studied and deduced the Fekete-Szegö inequality for different families of functions (see [2–5,9,13,17,30,35,38,40,41,47,48,53]) in GFT.

2. Generalized telephone numbers(GTN)

The recurrence relation quantifies traditional telephone numbers:

$$\mathcal{X}(\ell) = \mathcal{X}(\ell-1) + (\ell-1)\mathcal{X}(\ell-2), \qquad \ell \geqq 2.$$

Initial conditions:

$$\mathcal{X}(0) = \mathcal{X}(1) = 1.$$

For integers $\ell \geqq 0$ and $\varsigma \geqq 1$, Wloch and Wolowiec-Musial [52] defined generalized telephone numbers $\mathcal{X}(\varsigma, \ell)$ by the recurrence relation:

$$\mathcal{X}(\varsigma, \ell) = \varsigma\mathcal{X}(\varsigma, \ell-1) + (\ell-1)\mathcal{X}(\varsigma, \ell-2),$$

with initial conditions

$$\mathcal{X}(\varsigma, 0) = 1 \quad \text{and} \quad \mathcal{X}(\varsigma, 1) = \varsigma.$$

Bednarz and Wolowiec-Musial [7] recently examined the accessible generalization of phone numbers using the formula

$$\mathcal{X}_\varsigma(\ell) = \mathcal{X}_\varsigma(\ell-1) + \varsigma(\ell-1)\mathcal{X}_\varsigma(\ell-2),$$

where $\ell \geqq 2$ and $\varsigma \geqq 1$ with initial conditions

$$\mathcal{X}_\varsigma(0) = \mathcal{X}_\varsigma(1) = 1.$$

According to Deniz [17], who conducted this investigation very recently, $\mathcal{X}_\varsigma(\ell)$ has an exponential generating function

$$e^{\left(r+\varsigma\frac{r^2}{2}\right)} = \sum_{\ell=0}^{\infty} \mathcal{X}_\varsigma(\ell)\frac{r^\ell}{\ell!}.$$

As $\varsigma = 1$, classical phone numbers $\mathcal{X}_{\varsigma}(\ell) \equiv \mathcal{X}(\ell)$ are evident.

Now, we study the function

$$\vartheta(z) = e^{\left(z+\varsigma\frac{z^2}{2}\right)} = 1 + z + (1+\varsigma)\frac{z^2}{2} + (1+3\varsigma)\frac{z^3}{6} + \cdots \tag{2.1}$$

with its domain of definition as the open unit disk $\mathbf{U}$. Note that the holomorphic function $\vartheta(z)$ in $\mathbf{U}$, with a positive real part such that $\vartheta(0) = 1, \quad \vartheta'(0) > 0$, and ϑ maps $\mathbf{U}$ onto an area that is symmetric with respect to the real axis and starlike with respect to 1. Using the generalized telephone numbers, we now give the following subfamilies of holomorphic bi-Ozaki-close-to-convex and bi-Bazilevič functions.

Definition 2.1. If a function $f \in \Sigma$ satisfies the following subordinations, it belongs to the family $\mathcal{B}_\Sigma(\wp;\vartheta)$:

$$\frac{z^{1-\wp} f'(z)}{(f(z))^{1-\wp}} \prec e^{\left(z+\varsigma\frac{z^2}{2}\right)} =: \vartheta(z)$$

and

$$\frac{w^{1-\wp} g'(w)}{(g(w))^{1-\wp}} \prec e^{\left(w+\varsigma\frac{w^2}{2}\right)} =: \vartheta(w),$$

where $\wp$ is non-negative integer, $\varsigma \geqq 1$, and $g(w) = f^{-1}(w)$.

Remark 2.1. *If we take $\wp = 0$ in Definition 2.1, the family $\mathcal{B}_\Sigma(\wp;\vartheta)$ reduces to the family $\mathcal{S}^*_\Sigma(\vartheta)$, which was studied recently by Cotîrlă and Wanas (see [15]).*

Definition 2.2. The family $O_\Sigma(\varkappa;\vartheta)$ contains all the functions $f \in \Sigma$ if the next subordinations satisfy:

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^{\left(z+\varsigma\frac{z^2}{2}\right)} =: \vartheta(z)$$

and

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec e^{\left(w+\varsigma\frac{w^2}{2}\right)} =: \vartheta(w),$$

where $\frac{1}{2} \leqq \varkappa \leqq 1$, $\varsigma \geqq 1$, and $g(w) = f^{-1}(w)$.

Remark 2.2. *If we take $\varkappa = \frac{1}{2}$ in Definition 2.2, the family $O_\Sigma(\varkappa;\vartheta)$ reduces to the family $C_\Sigma(\vartheta)$, which was introduced recently by Cotîrlă and Wanas (see [15]).*

In the following sections we determine the upper bounds on the Fekete-Szegö type inequalities and the initial Taylor-Maclaurin coefficients for functions in these families in Definitions 2.1 and 2.2.

3. Initial Taylor coefficient estimates

We recall the following lemma where we obtain the upper bounds on the Fekete-Szegö type inequalities and the initial Taylor-Maclaurin coefficients for functions in $f \in \mathcal{B}_\Sigma(\wp;\vartheta)$, where $\wp$ is non-negative integer.

Lemma 3.1. *([20], p.41)* Let $h \in \mathcal{P}$ be given by the following series:

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad , \quad where \quad z \in \mathbf{U}$$

then

$$|c_n| \leqq 2 \quad , for\ all \quad n \in \mathbb{N}.$$

Theorem 3.1. If f given by (1.1) is in the family $\mathcal{B}_\Sigma(\wp;\vartheta)$, where $\wp$ is non-negative integer, then

$$|a_2| \leqq \min\left\{\frac{1}{\wp+1}, \sqrt{\frac{2}{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{\wp+\varsigma+2}{(\wp+2)(\wp+1)}, \frac{1}{(\wp+1)^2}+\frac{1}{\wp+2}\right\}.$$

Proof. Assume that $f \in \mathcal{B}_\Sigma(\wp;\vartheta)$ and that $f^{-1} = g$. We consider the holomorphic functions Θ, Υ : $\mathbf{U} \longrightarrow \mathbf{U}$, where $\Theta(0) = \Upsilon(0) = 0$, satisfying the following criteria:

$$\frac{z^{1-\wp} f'(z)}{(f(z))^{1-\wp}} = \vartheta(\Theta(z)), \quad z \in \mathbf{U} \tag{3.1}$$

and

$$\frac{w^{1-\wp} g'(w)}{(g(w))^{1-\wp}} = \vartheta(\Upsilon(w)), \quad w \in \mathbf{U}. \tag{3.2}$$

Define the functions $\mathbf{m}$ and $\mathbf{n}$ by

$$\mathbf{m}(z) = \frac{1+\Theta(z)}{1-\Theta(z)} = 1 + m_1 z + m_2 z^2 + \cdots$$

and

$$\mathbf{n}(z) = \frac{1+\Upsilon(z)}{1-\Upsilon(z)} = 1 + n_1 z + n_2 z^2 + \cdots .$$

It follows that $\mathbf{m}$, $\mathbf{n}$ are analytic functions in $\mathbf{U}$, where $\mathbf{m}(0) = 1 = \mathbf{n}(0)$. Then, we get $\Theta, \Upsilon : \mathbf{U} \longrightarrow \mathbf{U}$, where $\mathbf{m}$ and $\mathbf{n}$ are functions with a positive real part in $\mathbf{U}$.
But, we have

$$\Theta(z) = -\frac{1-\mathbf{m}(z)}{\mathbf{m}(z)+1} = \frac{1}{2}\left[m_1 z + \left(m_2 - \frac{m_1^2}{2}\right)z^2\right] + \cdots \quad , z \in \mathbf{U} \tag{3.3}$$

and

$$\Upsilon(z) = -\frac{1-\mathbf{n}(z)}{\mathbf{n}(z)+1} = \frac{1}{2}\left[n_1 z + \left(n_2 - \frac{n_1^2}{2}\right)z^2\right] + \cdots \quad , z \in \mathbf{U}. \tag{3.4}$$

By substituting (3.3) and (3.4) into (3.1) and (3.2) and applying (2.1), we get

$$\frac{z^{1-\wp} f'(z)}{(f(z))^{1-\wp}} = \vartheta(\Theta(z)) = e^{\left(\frac{\mathbf{m}(z)-1}{\mathbf{m}(z)+1} + \varsigma\frac{\left(\frac{\mathbf{m}(z)-1}{\mathbf{m}(z)+1}\right)^2}{2}\right)} = 1 + \frac{1}{2}m_1 z + \left[\frac{m_2}{2} + \frac{(\varsigma-1)m_1^2}{8}\right]z^2 + \cdots \tag{3.5}$$

and

$$\frac{w^{1-\wp} g'(w)}{(g(w))^{1-\wp}} = \vartheta(\Upsilon(w)) = e^{\left(\frac{\mathbf{n}(w)-1}{1+\mathbf{n}(w)} + \varsigma\frac{\left(\frac{\mathbf{n}(w)-1}{\mathbf{n}(w)+1}\right)^2}{2}\right)} = 1 + \frac{1}{2}n_1 w + \left[\frac{n_2}{2} + \frac{(\varsigma-1)n_1^2}{8}\right]w^2 + \cdots \tag{3.6}$$

Equating the coefficients in (3.5) and (3.6) yields

$$(\wp+1)a_2 = \frac{1}{2}m_1, \tag{3.7}$$

$$(\wp+2)a_3+\frac{1}{2}(\wp+2)(\wp-1)a_2^2=\frac{m_2}{2}+\frac{(\varsigma-1)m_1^2}{8},\tag{3.8}$$

$$-(\wp+1)a_2=\frac{1}{2}n_1\tag{3.9}$$

and

$$(\wp+2)(2a_2^2-a_3)+\frac{1}{2}(\wp+2)(\wp-1)a_2^2=\frac{n_2}{2}+\frac{(\varsigma-1)n_1^2}{8}.\tag{3.10}$$

From (3.7) and (3.9), we have

$$m_1=-n_1\tag{3.11}$$

and

$$2\,(\wp+1)^2\,a_2^2=\frac{1}{4}(m_1^2+n_1^2).\tag{3.12}$$

If we add (3.8) to (3.10), we obtain

$$(\wp+2)\,(\wp+1)\,a_2^2=\frac{1}{2}(m_2+n_2)+\frac{1}{8}(\varsigma-1)(m_1^2+n_1^2).\tag{3.13}$$

Substituting from (3.12) the value of $m_1^2+n_1^2$ in the relation (3.13), we get

$$a_2^2=\frac{m_2+n_2}{2\left[(\wp+2)\,(\wp+1)+(1-\varsigma)\,(\wp+1)^2\right]}.\tag{3.14}$$

Applying Lemma 3.1 for the coefficients m_1, m_2, n_1, and n_2 in (3.12) and (3.14), we get

$$|a_2|\leqq\frac{1}{\wp+1},\qquad |a_2|\leqq\sqrt{\frac{2}{\left|(\wp+2)\,(\wp+1)+(1-\varsigma)\,(\wp+1)^2\right|}}.$$

Applying (3.11) and subtracting (3.10) from (3.8) yields $m_1^2=n_1^2$, which is the bound on $|a_3|$,

$$2(\wp+2)(a_3-a_2^2)=\frac{1}{2}(m_2-n_2).\tag{3.15}$$

Substituting a_2^2 from (3.12) into (3.15) yields the following result:

$$a_3=\frac{m_1^2+n_1^2}{8\,(\wp+1)^2}+\frac{m_2-n_2}{4(\wp+2)}.$$

So, we have

$$|a_3|\leqq\frac{1}{(\wp+1)^2}+\frac{1}{\wp+2}.$$

Also, substituting the value of a_2^2 from (3.13) into (3.15), we get

$$a_3=\frac{m_2-n_2}{4(\wp+2)}+\frac{m_2+n_2}{2\,(\wp+2)\,(\wp+1)}+\frac{(\varsigma-1)(m_1^2+n_1^2)}{8\,(\wp+2)\,(\wp+1)}$$

and we have

$$|a_3|\leqq\frac{\wp+\varsigma+2}{(\wp+2)\,(\wp+1)}.$$

□

Theorem 3.2. Let$f \in \mathcal{O}_\Sigma(\varkappa;\vartheta)$ $(\frac{1}{2} \leqq \varkappa \leqq 1)$ and f be given by (1.1). Then,

$$|a_2| \leqq \min\left\{\frac{(2\varkappa+1)^2}{16}, \frac{2\varkappa+1}{2\sqrt{|2(\varkappa-\varsigma)+3|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{(2\varkappa+1)(3\varsigma+5)}{24}, \frac{2\varkappa+1}{12} + \frac{(2\varkappa+1)^2}{16}\right\}.$$

Proof. Assume that $f \in \mathcal{O}_\Sigma(\varkappa;\vartheta)$ and $g = f^{-1}$. Specifically, there exist holomorphic functions $\Theta, \Upsilon : \mathbf{U} \longrightarrow \mathbf{U}$, hence

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \vartheta(\Theta(z)), \quad \textit{where} \quad z \in \mathbf{U} \tag{3.16}$$

and

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) = \vartheta(\Upsilon(w)), \quad \textit{where} \quad w \in \mathbf{U}, \tag{3.17}$$

where $\Theta(z)$ and $\Upsilon(z)$ have the forms (3.3) and (3.4). From (3.16), (3.17), and (2.1), we deduce that

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \vartheta\left(\Theta(z)\right) = e^{\left(\frac{\mathbf{m}(z)-1}{\mathbf{m}(z)+1} + \varsigma\frac{\left(\frac{\mathbf{m}(z)-1}{\mathbf{m}(z)+1}\right)^2}{2}\right)}$$
$$= 1 + \frac{1}{2}m_1 z + \left[\frac{m_2}{2} + \frac{(\varsigma-1)m_1^2}{8}\right]z^2 + \cdots \tag{3.18}$$

and

$$\frac{2\varkappa-1}{2\varkappa+1} + \frac{2}{2\varkappa+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) = \vartheta\left(\Upsilon(w)\right) = e^{\left(\frac{\mathbf{n}(w)-1}{\mathbf{n}(w)+1} + \varsigma\frac{\left(\frac{\mathbf{n}(w)-1}{\mathbf{n}(w)+1}\right)^2}{2}\right)}$$
$$= 1 + \frac{1}{2}n_1 w + \left[\frac{n_2}{2} + \frac{(\varsigma-1)n_1^2}{8}\right]w^2 + \cdots \tag{3.19}$$

Equating the coefficients in (3.18) and (3.19), yields

$$\frac{4}{2\varkappa+1}a_2 = \frac{1}{2}m_1, \tag{3.20}$$

$$\frac{12}{2\varkappa+1}a_3 - \frac{8}{2\varkappa+1}a_2^2 = \frac{m_2}{2} + \frac{(\varsigma-1)m_1^2}{8}, \tag{3.21}$$

$$-\frac{4}{2\varkappa+1}a_2 = \frac{1}{2}n_1 \tag{3.22}$$

and

$$\frac{12}{2\varkappa+1}(2a_2^2 - a_3) - \frac{8}{2\varkappa+1}a_2^2 = \frac{n_2}{2} + \frac{(\varsigma-1)n_1^2}{8}. \tag{3.23}$$

From (3.20) and (3.22), we have

$$m_1 = -n_1 \tag{3.24}$$

and

$$\frac{32}{(2\varkappa+1)^2}a_2^2 = \frac{1}{4}(m_1^2+n_1^2). \tag{3.25}$$

If we add (3.21) to (3.23), we obtain

$$\frac{8}{2\varkappa+1}a_2^2 = \frac{1}{2}(m_2+n_2)+\frac{1}{8}(\varsigma-1)(m_1^2+n_1^2). \tag{3.26}$$

Substituting from (3.25) the value of $m_1^2+n_1^2$ in the relation (3.26), we deduce that

$$a_2^2 = \frac{(2\varkappa+1)^2(m_2+n_2)}{16(2(\varkappa-\varsigma)+3)}. \tag{3.27}$$

Applying Lemma 3.1 for the coefficients m_1, m_2, n_1, and n_2 in (3.25) and (3.27), we get

$$|a_2| \leqq \frac{(2\varkappa+1)^2}{16}, \qquad |a_2| \leqq \frac{2\varkappa+1}{2\sqrt{|2(\varkappa-\varsigma)+3|}}.$$

Subtracting (3.23) from relation (3.21) and applying (3.24), we get $|a_3|$.
This yields $m_1^2 = n_1^2$, hence

$$\frac{24}{2\varkappa+1}(a_3-a_2^2) = \frac{1}{2}(m_2-n_2), \tag{3.28}$$

then by substituting from (3.25) the value of a_2^2 into (3.28), we get

$$a_3 = \frac{(2\varkappa+1)(m_2-n_2)}{48}+\frac{(2\varkappa+1)^2(m_1^2+n_1^2)}{128}.$$

So, we have

$$|a_3| \leqq \frac{2\varkappa+1}{12}+\frac{(2\varkappa+1)^2}{16}.$$

Also, substituting the value of a_2^2 from (3.26) into (3.28), we get

$$a_3 = \frac{(2\varkappa+1)(m_2-n_2)}{48}+\frac{(2\varkappa+1)(m_2+n_2)}{16}+\frac{(2\varkappa+1)(\varsigma-1)(m_1^2+n_1^2)}{64}$$

and we have

$$|a_3| \leqq \frac{(2\varkappa+1)(3\varsigma+5)}{24}.$$

□

Utilizing a_2^2 and a_3 values, and spurred by Zaprawa's recent work [53], we prove the Fekete-Szegö problem for $f \in \mathcal{B}_\Sigma(\wp;\vartheta)$ and $f \in \mathcal{O}_\Sigma(\varkappa;\vartheta)$ in the following theorems.

Theorem 3.3. For a non-negative integer $\wp$ and $\hbar \in \mathbb{R}$, let $f \in \mathcal{B}_\Sigma(\wp;\vartheta)$ be of the form (1.1). Then,

$$\left|a_3-\hbar a_2^2\right| \leqq \begin{cases} \frac{1}{\wp+2}; & |\hbar-1| \leqq \frac{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}{2(\wp+2)}, \\ \frac{2|\hbar-1|}{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}; & |\hbar-1| \geqq \frac{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}{2(\wp+2)}. \end{cases}$$

Proof. It follows from (3.14) and (3.15) that

$$\begin{aligned} a_3 - \hbar a_2^2 &= \frac{m_2 - n_2}{4(\wp + 2)} + (1-\hbar)\, a_2^2 \\ &= \frac{m_2 - n_2}{4(\wp + 2)} + \frac{(m_2 + n_2)(1-\hbar)}{2\left[(\wp+2)(\wp+1) + (1-\varsigma)(\wp+1)^2\right]} \\ &= \frac{1}{2}\left[\left(\psi(\hbar,\varsigma) + \frac{1}{2(\wp+2)}\right) m_2 + \left(\psi(\hbar,\varsigma) - \frac{1}{2(\wp+2)}\right) n_2\right], \end{aligned}$$

where

$$\psi(\hbar,\varsigma) = \frac{1-\hbar}{(\wp+2)(\wp+1) + (1-\varsigma)(\wp+1)^2}.$$

According to Lemma 3.1, we find that

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{1}{\wp+2}, & 0 \leqq |\psi(\hbar,\varsigma)| \leqq \frac{1}{2(\wp+2)}, \\ \\ 2\,|\psi(\hbar,\varsigma)|, & |\psi(\hbar,\varsigma)| \geqq \frac{1}{2(\wp+2)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{1}{\wp+2}; & |\hbar - 1| \leqq \frac{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}{2(\wp+2)}, \\ \\ \frac{2|\hbar-1|}{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}; & |\hbar - 1| \geqq \frac{\left|(\wp+2)(\wp+1)+(1-\varsigma)(\wp+1)^2\right|}{2(\wp+2)}. \end{cases}$$

□

Putting $\hbar = 1$ in Theorem 3.3, we get the next result:

Corollary 3.1. If $f \in \mathcal{B}_\Sigma(\wp;\vartheta)$ is of the form (1.1), then we have that

$$\left|a_3 - a_2^2\right| \leqq \frac{1}{\wp+2}.$$

Theorem 3.4. For $\frac{1}{2} \leqq \varkappa \leqq 1$ and $\hbar \in \mathbb{R}$, let $f \in O_\Sigma(\varkappa;\vartheta)$ be of the form (1.1). Then,

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{1}{6}; & |\hbar - 1| \leqq \frac{|2-\varsigma|}{3}, \\ \\ \frac{|\hbar-1|}{2|2-\varsigma|}; & |\hbar - 1| \geqq \frac{|2-\varsigma|}{3}. \end{cases}$$

Proof. It follows from (3.27) and (3.28) that

$$\begin{aligned} a_3 - \hbar a_2^2 &= \frac{(2\varkappa+1)(m_2 - n_2)}{48} + (1-\hbar)\, a_2^2 \\ &= \frac{(2\varkappa+1)(m_2 - n_2)}{48} + \frac{(2\varkappa+1)^2 (m_2+n_2)(1-\hbar)}{16(2(\varkappa-\varsigma)+3)} \end{aligned}$$

$$= \frac{1}{16}\left[\left(\phi(\hbar,\varsigma) + \frac{2\varkappa+1}{3}\right)m_2 + \left(\phi(\hbar,\varsigma) - \frac{2\varkappa+1}{3}\right)n_2\right],$$

where

$$\phi(\hbar,\varsigma) = \frac{(2\varkappa+1)^2(1-\hbar)}{2(\varkappa-\varsigma)+3}.$$

According to Lemma 3.1, we find that

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{2\varkappa+1}{12}, & 0 \leqq |\phi(\hbar,\varsigma)| \leqq \frac{2\varkappa+1}{3}, \\ \\ \frac{1}{4}\left|\phi(\hbar,\varsigma)\right|, & |\phi(\hbar,\varsigma)| \geqq \frac{2\varkappa+1}{3}. \end{cases}$$

After some computations, we obtain

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{2\varkappa+1}{12}; & |\hbar-1| \leqq \frac{(2\varkappa+1)|2(\varkappa-\varsigma)+3|}{3(2\varkappa+1)^2}, \\ \\ \frac{(2\varkappa+1)^2|\hbar-1|}{4|2(\varkappa-\varsigma)+3|}; & |\hbar-1| \geqq \frac{(2\varkappa+1)|2(\varkappa-\varsigma)+3|}{3(2\varkappa+1)^2}. \end{cases}$$

□

Fixing $\hbar = 1$ in Theorem 3.4, we get the following result:

Corollary 3.2. If $f \in \mathcal{O}_\Sigma(\varkappa;\vartheta)$ is of the form (1.1), then

$$\left|a_3 - a_2^2\right| \leqq \frac{2\varkappa+1}{12}.$$

Remark 3.1. *$\wp = 0$ and $\varkappa = \frac{1}{2}$ give the results of Cotîrlă and Wanas (see [15]).*

4. Conclusions

Motivated by many recent advances on the Fekete-Szegö functional and Taylor-Maclaurin coefficient estimations, we defined new families of holormorphic bi-univalent functions $\mathcal{B}_\Sigma(\wp;\vartheta)$ and $\mathcal{O}_\Sigma(\varkappa;\vartheta)$ associated with generalized telephone numbers are presented and thoroughly examined in this article. For functions in these families, we determined Taylor-Maclaurin coefficient inequalities and examined the well-known Fekete-Szegö issue. Furthermore, the generic coefficients $|a_n|$, $n \geqq 4$, for the functions of these new classes remain unbounded. We also opted to utilize a significant finding from a recently released evaluate-cum-explanatory paper by Srivastava ([37], p. 340) to extend our study based on the q-difference operator. This observation pointed out that using some seemingly parametric and argumentative versions of the extra parameter p is redundant; the effects for the new or previously mentioned $q-$analogs could be easily (and possibly trivially) translated into corresponding effects for the so-called $(p;q)-$analogues (with $0 < |q| \leq 1$). Further, one can obtain the second Hankel determinant inequalities for function classes studied in this article (see [42–47] and references cited therein).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a large group research project under grant number RGP2/432/44.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. I. Al-Shbeil, A. K. Wanas, A. Saliu, A. Cătaş, Applications of beta negative binomial distribution and Laguerre polynomials on Ozaki bi-close-to-convex functions, *Axioms*, **11** (2022), 451.
2. Ş. Altınkaya, S. Yalçin, Fekete-Szegö inequalities for certain classes of biunivalent functions, *Int. Scholarly Res. Not.*, **2014** (2014), 327962. https://doi.org/10.1155/2014/327962
3. A. G. Al-Amoush, Coefficient estimates for a new subclasses of λ-pseudo biunivalent functions with respect to symmetrical points associated with the Horadam Polynomials, *Turk. J. Math.*, **43** (2019), 2865–2875.
4. A. Amourah, Fekete-Szegö inequalities for analytic and bi-univalent functions subordinate to (p, q)-Lucas Polynomials, *Complex Var*, 2020, arXiv:2004.00409.
5. A. Amourah, B. A. Frasin, T. Abdeljaward, Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials, *J. Funct. Space.*, **2021** (2021), 5574673. http://doi.org/10.1155/2021/5574673
6. A. Amourah, B. A. Frasin, G. Murugusundaramoorthy, T. Al-Hawary, Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with $(p; q)$-Lucas polynomials, *AIMS Mathematics*, **6** (2021), 4296–4305. http://doi.org/10.3934/math.2021254
7. U. Bednarz, M. Wolowiec-Musial, On a new generalization of telephone numbers, *Turk. J. Math.*, **43** (2019), 1595–1603.
8. D. Breaz, G. Murugusundaramoorthy, K. Vijaya, L. I. Cotîrlă, Certain class of bi-univalent functions defined by Salagean q-difference operator related with involution numbers, *Symmetry*, **15** (2023), 1302.
9. D. Breaz, A. K. Wanas, F. M. Sakar, S. M. Aydoğan, On a Fekete–Szegö problem associated with generalized telephone numbers, *Mathematics*, **11** (2023), 3304.
10. S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions, *Filomat*, **30** (2016), 1567–1575.

11. M. Çağlar, E. Deniz, H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, *Turk. J. Math.*, **41** (2017), 694–706.
12. M. Çağlar, G. Palpandy, E. Deniz, Unpredictability of initial coefficient bounds for m-fold symmetric bi-univalent starlike and convex functions defined by subordinations, *Afr. Mat.*, **29** (2018), 793–802.
13. A. Cataş, A note on subclasses of univalent functions defined by a generalized Sălăgean operator, *Acta Univ. Apulensis*, **12** (2006), 73–78.
14. L. I. Cotîrlă, New classes of analytic and bi-univalent functions, *AIMS Mathematics*, **6** (2021), 10642–10651. http://doi.org/10.3934/math.2021618
15. L.I. Cotîrlă, A. K. Wanas, Coefficient-related studies and Fekete–Szegö inequalities for new classes of bi-Starlike and bi-convex functions, *Symmetry*, **14** (2022), 2263.
16. E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Classical Anal.*, **2** (2013), 49–60.
17. E. Deniz, Sharp coefficient bounds for starlike functions associated with generalized telephone numbers, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 1525–1542.
18. E. Deniz, M. Kamali, S. Korkmaz, A certain subclass of bi-univalent functions associated with Bell numbers and q-Srivastava Attiya operator, *AIMS Mathematics*, **5** (2020), 7259–7271. https://doi.org/10.3934/math.2020464
19. E. Deniz, M. Çağlar, H. Orhan, Second Hankel determinant for bi-starlike and bi-convex functions of order β, *Appl. Math. Comput.*, **271** (2015), 301–307.
20. P. L. Duren, *Univalent Functions, Grundlehren der Mathematischen Wissenschaften*, New York, Berlin, Heidelberg, Tokyo: Springer, 1983.
21. M. Fekete, G. Szegö, Eine bemerkung uber ungerade schlichte funktionen, *J. London Math. Soc.*, **2** (1933), 85–89.
22. B. A. Frasin, S. R. Swamy, J. Nirmala, Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to k-Fibonacci numbers involving modified Sigmoid activation function, *Afr. Mat.*, **32** (2020), 631–643.
23. J. O. Hamzat, M. O. Oluwayemi, A. A. Lupaş, A. K. Wanas, Bi-univalent problems involving generalized multiplier transform with respect to symmetric and conjugate points, *Fractal Fract.*, **6** (2022), 483.
24. H. Ö. Güney, G. Murugusundaramoorthy, J. Sokół, Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers, *Acta Univ. Sapient. Math.*, **10** (2018), 70–84.
25. W. Kaplan, Close-to-convex schlicht functions, *Michi. Math J.*, **1** (1952), 169–185.
26. R. Kargar, A. Ebadian, Ozaki's conditions for general integral operator, *Sahand Commun. Math. Anal.*, **5** (2017), 61–67.
27. S. Kazimoğlu, E. Deniz, Fekete-Szegö problem for generalized bi-subordinate functions of complex order, *Hacet. J. Math. Stat.*, **49** (2020), 1695–1705.
28. B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Mathematics*, **6** (2021), 1024–1039. http://doi.org/10.3934/math.2021061

29. A. A. Lupas, S. M. El-Deeb, Subclasses of bi-univalent functions connected with integral operator based upon Lucas polynomial, *Symmetry*, **14** (2022), 622.
30. S. S. Miller, P. T. Mocanu, *Differential Subordinations: Theory and Applications*, New York, Basel: Marcel Dekker Incorporated, 2000.
31. G. Murugusundaramoorthy, K. Vijaya, Certain subclasses of analytic functions associated with generalized telephone numbers, *Symmetry*, **14** (2022), 1053.
32. G. Murugusundaramoorthy, K. Vijaya, Bi-starlike function of complex order involving mathieu-type series associated with telephone numbers, *Symmetry*, **15** (2023), 638.
33. G. Murugusundaramoorthy, J. Sokół, On $\wp-$ Pseudo Bi-Starlike Functions related to some conic domains, *Bull. Transilvania Univ. Braşov Ser. III: Math. Inf. Phys.*, **12** (2019), 381–392 https://doi.org/10.31926/but.mif.2019.12.61.2.15
34. S. Ozaki, On the theory of multivalent functions, *Sci. Rep. Tokyo Bunrika Daigaku Sec. A*, **2** (1935), 167–188.
35. R. K. Raina, J. Sokol, Fekete-Szegö problem for some starlike functions related to shell-like curves, *Math. Slovaca*, **66** (2016), 135–140.
36. R. Singh, On Bazilevič functions, *Proc. Amer. Math. Soc.*, **38** (1973), 261–271.
37. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in Geometric Function theory of Complex Analysis, *Iran J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344.
38. H. M. Srivastava, Ş. Altınkaya, S. Yalçin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, *Iran. J. Sci. Technol. Trans. A*: *Sci.*, **43** (2019), 1873–1879.
39. H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, *Bull. Iran. Math. Soc.*, **44** (2018), 149–157.
40. H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, *Afr. Mat.*, **28** (2017), 693–706.
41. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), 1188–1192.
42. H. M. Srivastava, A. Motamednezhad, E. A. Adegani, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, *Mathematics*, **8** (2020), 172.
43. H. M. Srivastava, A. K. Wanas, Applications of the Horadam polynomials involving $\wp$-pseudo-starlike bi-univalent functions associated with a certain convolution operator, *Filomat*, **35** (2021), 4645–4655.
44. H. M. Srivastava, A. K. Wanas, H. Ö. Güney, New families of bi-univalent functions associated with the Bazilevič functions and the λ-Pseudo-starlike functions, *Iran. J. Sci. Technol. Trans. A*: *Sci.*, **45** (2021), 1799–1804.
45. H. M. Srivastava, A. K. Wanas, R. Srivastava, Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials, *Symmetry*, **13** (2021), 1230.

46. H. M. Srivastava, A. K. Wanas, G. Murugusundaramoorthy, A certain family of bi-univalent functions associated with the Pascal distribution series based upon the Horadam polynomials, *Surv. Math. Appl.*, **16** (2021), 193–205.

47. H. M. Srivastava, G. Murugusundaramoorty, T. Bulboacă, The second Hankel determinant for subclasses of Bi-univalent functions associated with a nephroid domain, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, **116** (2022), 145. http://doi.org/10.1007/s13398-022-01286-6

48. H. M. Srivastava, T. G. Shaba, G. Murugusundaramoorthy, A. K. Wanas, G. I. Oros, The Fekete-Szegö functional and the Hankel determinant for a certain class of analytic functions involving the Hohlov operator, *AIMS Mathematics*, **8** (2022), 340–360. https://doi.org/10.3934/math.2023016

49. S. R. Swamy, A. K. Wanas, A comprehensive family of bi-univalent functions defined by (m, n)-Lucas polynomials, *Bol. Soc. Mat. Mex.*, **28** (2022), 34.

50. A. K. Wanas, F. M. Saka, A. A. Lupaş, Applications Laguerre polynomials for families of bi-univalent functions defined with (p, q)-Wanas operator, *Axioms*, **12** (2023), 430.

51. A. K. Wanas, G. S. Sălăgean, P.-S. A. Orsolya, Coefficient bounds and Fekete-Szegö inequality for a certain family of holomorphic and bi-univalent functions defined by (M,N)-Lucas polynomials, *Filomat*, **37** (2023), 1037–1044.

52. A. Wloch, M. Wolowiec-Musial, On generalized telephone number, their interpretations and matrix generators, *Util. Math.*, **10** (2017), 531–539.

53. P. Zaprawa, On the Fekete-Szegö problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **21** (2014), 169–178. http://doi.org/10.36045/bbms/1394544302