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Research article

Reachable set estimation for neutral semi-Markovian jump systems with time-varying delay

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Abstract: This work addresses the issue of finding ellipsoidal bounds of reachable sets for neutral semi-Markovian jump systems with time-varying delay and bounded peak disturbances, for which the related result has been rarely proposed for neutral semi-Markovian jump systems. Based on the modified improved Lyapunov-Krasovskii functional, a boundary of the reachable set for neutral semi-Markovian jump systems was obtained with the aid of utilizing a novel integral inequality and combining with the time-delay segmentation technique. The numerical examples are supplied to verify the effectiveness of the obtained results.

Keywords: bound of reachable set; neutral semi-Markovian jump systems; Lyapunov-Krasovskii; Linear matrix inequalities
Mathematics Subject Classification: 93E03

1. Introduction

The reachable set of a dynamic system is first mentioned in [1], which is defined as the set of all state trajectories that may be achieved from the origin. The reachable set is a particularly valuable direction for further research in the field of control theory, which is closely related to stability, and it is crucial to many practical systems, such as ensuring circuit safety, safety verification, and avoiding aircraft collision [2–4]. In addition, many control methods has been proposed to improve the performance of the systems [5–7]. Because the exact shape of the reachable set for the actual system is difficult to obtain, it also causes scholars to study the estimation of the reachable set. So far, the commonly used methods for estimating reachable sets include the ellipsoid method and the polyhedron method. However, in the application of practical systems, time-delay often leads to the deterioration of system performance and even instability, but this is an unavoidable phenomenon [8]. Therefore, theoretical research on time-delay systems has attracted the attention of multitudinous researchers [9–15], and many achievements have been made in switched systems [16–20].

As a class of special hybrid system, Markovian jump systems can describe this kind of situation with sudden change (i.e., state sudden change and signal to lag) well. It has been widely used in actual manufacturing processes, network transmission systems, power circuit systems, economic systems, and so on. The sojourn time of Markovian jump systems obeys the exponential distribution, and the transition rate is constant and memoryless, in other words, the transition probability is a random process independent of the past. As a matter of fact, the transition rate of many practical systems is not constant, and the application field of Markovian jump systems is limited to some extent. Thus, a semi-Markovian jump system with a time-varying transition rate is proposed, which can better describe the general system. The semi-Markovian jump system obeys non-exponential distributions, such as the Weber distribution [21] and the Gaussian distribution [22], which relaxes the limitation of probability distribution function and reduces the conservatism of the system. Therefore, it has a wider application value. To date, a lot of research work has been done on the stability of semi-Markovian jump systems [23–30], but the reachable set of such systems is still in the stage of continuous development [31–34]. The issues of reachable set estimation and reachable set control for semi-Markovian jump systems under bounded peak disturbance have been addressed in [32]. The problem of reachable set estimation for a class of singular semi-Markovian jump systems with time-varying delay under zero initial condition was considered in [33].

On the other hand, it is worth noting that a particularly distinctive feature of many dynamic processes of physics, chemistry, biology, and engineering is that they are not only affected by past and present states, but are also fully affected by the derivative of the delay. Therefore, in order to describe this feature, the neutral time delay can be introduced into dynamic systems, called neutral time-delay systems. Since neutral systems have time delay in both the state and the derivative of the state, most systems with time delay can be regarded as a special case of neutral systems, which is a kind of more general system with time-delay. In the last twenty years, time-delay systems have been deeply studied by many scholars [35–37]. The ellipsoidal bound of reachable sets for linear neutral systems with bounded peak disturbances has been investigated in [36]. The exponential stability in the mean square of neutral stochastic delayed systems with switching and distributed-delay dependent impulses was studied in [37].

Furthermore, a less conservative result can be obtained by using a matrix inequality to enlarge the derivative of the Lyapunov functional to different degrees. Jensen's inequality [38], the Wirtinger integral inequality [39], the reciprocally convex combination inequality [40], and some improved integral inequalities have been generally used to reduce derivatives of the Lyapunov functional [41–45]. In [44], the author has investigated the boundary of the reachable set for a class linear systems with mixed delays and state constraints by the Wirtinger-based integral inequality and extended reciprocally convex combination approach. In [45], a novel quadratic generalized multiple-integral inequality based on free matrices was proposed to make the stability criterion of the system less conservative.

At present, few scholars have applied advanced methods to the neutral semi-Markovian jump system. Moreover, it is well known that the triple integral form of the Lyapunov functional can effectively reduce the conservatism of the criterion. Based on [46], a new integral inequality is derived by using the integral inequality and time-delay segmentation technique. Inspired by existing results and combined with the semi-Markovian jumping system, this paper will study the reachable set boundary of the neutral semi-Markovian jump system by utilizing a novel integral equality.

Notations: The superscript 'A^T' and 'A^{-1}' represent the transpose and inverse matrix representing matrix A; $\mathbb{R}^n$ stands for the n-dimensional Euclidean space; $\mathbb{R}^{p\times q}$ is the set all $p \times q$ real matrices; the symbol $P > 0$ ($P \geq 0$) means that P is a positive definite (semi-positive definite) matrix, and similarly, $P < 0$ ($P \leq 0$) denotes that P is a negative (semi-negative definite) definite matrix; diag$\{\cdot\}$ is a diagonal matrix; an asterisk ($*$) in the symmetric block represents a symmetric term; $E\{\cdot\}$ denotes the expectation operator and $\mathcal{L}$ means the weak infinitesimal generator; $P_r(\cdot|\cdot)$ is the conditional probability.

2. Problem statement

Consider the following neutral semi-Markovian jump system with disturbances:

$$\begin{cases} \dot{x}(t) - C_{(t,r_t)}\dot{x}(t-\tau(t)) = A_{(t,r_t)}x(t) + B_{(t,r_t)}x(t-h(t)) + D_{(t,r_t)}\omega(t), \\ x(t_0+\theta) = 0, \quad \forall\theta \in [-\rho^*, 0] \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\tau(t)$ is the time-varying neutral delay, the time-varying delay $h(t)$ is a time-varying function, and $w(t) \in \mathbb{R}^m$ is the system disturbance satisfying

$$\tau(t) \leq \tau_M,\ \dot{\tau}(t) \leq \tau_D < 1,\ 0 < h(t) \leq h,\ h_d \leq \dot{h}(t) \leq h_D < 1, w^T(t)w(t) \leq w_m^2, \tag{2.2}$$

$\rho^* = max\{\tau_M, h\}$, $\{r_t, t \geq 0\}$ is a semi-Markovian process taking values on the probability space in a finite state $\wp = \{1, 2, 3, ..., N\}$ with the following transition probability:

$$Pr(r_{t+\Delta} = j | r_t = i) = \begin{cases} \lambda_{ij}(\delta)\Delta + o(\Delta), & i \neq j, \\ 1 + \lambda_{ii}(\delta)\Delta + o(\Delta), & i = j, \end{cases} \tag{2.3}$$

where $\delta > 0$ is the sojourn time between two jumps, $\Delta > 0$, and $\lim\limits_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$. $\lambda_{ij}(\delta)$ is the transition rate from i to j at time t for $i \neq j$. In addition, $\lambda_{ii}(\delta) = -\sum\limits_{j\in\wp\backslash\{i\}} \lambda_{ij}(\delta)$ for $j = i$. $A_{(t,r_t)}$, $B_{(t,r_t)}$, $C_{(t,r_t)}$ and $D_{(t,r_t)}$ are known constant matrices of the semi-Markovian process.

Remark 1. As described in [24], in the application of the actual system, $\lambda_{ij}(\delta)$ is assumed as $\underline{\lambda}_{ij} \leqslant \lambda_{ij}(\delta) \leqslant \bar{\lambda}_{ij}$, where $\underline{\lambda}_{ij}$ and $\bar{\lambda}_{ij}$ are real constants. Then, we have $\lambda_{ij}(\delta) = \lambda_{ij} + \Delta\lambda_{ij}$, where $\lambda_{ij} = \frac{1}{2}(\bar{\lambda}_{ij} + \underline{\lambda}_{ij})$ and $\left|\Delta\lambda_{ij}\right| \leqslant l_{ij}$ with $l_{ij} = \frac{1}{2}(\bar{\lambda}_{ij} - \underline{\lambda}_{ij})$.

Remark 2. The neutral systems are suitable for describing the turbojet control systems [47], ship dynamic positioning systems [48], etc. Neutral systems are a class of more general time-delay systems, where the change rate of the actual system's state is not only related to the current and past time states, but also to the rate of change of past states. When matrix C becomes a zero matrix, system (2.1) can be rewritten as a general time-delay system, thus almost all time-delay systems can be described as neutral systems.

For the sake of brevity, $x(t)$ is used to represent the solution of the system under initial conditions $x(t_0 + \theta) = 0$, $\theta \in [-\rho^*, 0]$, and its weak infinitesimal generator, acting on the function $V(x_t, t, i)$, is defined in [49].

$$\mathcal{L}V(x_t, t, i) = \lim_{\Delta\to 0} \frac{1}{\Delta}[E(V(x_{t+\Delta}, t+\Delta, r_{t+\Delta})|(x_t, r_t = i)) - V(x_t, t, i)]. \tag{2.4}$$

This paper aims to find a reachable set for neutral semi-Markovian jump system (2.1) based on the Lyapunov-Krasovskii functional approach. We denote the set of reachable states with $w(t)$ that satisfies

Eq (2.2) by

$$\Re_x \triangleq \{x(t) \in \mathbb{R}^n | x(t), w(t) \text{ satisfy Eqs (2.1) and (2.2)}\}. \tag{2.5}$$

We will bound $\Re_x$ by an ellipsoid of the form

$$\Im(P, 1) \triangleq \{x(t) \in \mathbb{R}^n : x^T(t)Px(t) \leq 1, P > 0\}. \tag{2.6}$$

For simplicity, there are the following representations:

$$A_i = A_{(t,r_t)}, B_i = B_{(t,r_t)}, C_i = C_{(t,r_t)}, D_i = D_{(t,r_t)}, P_i = P_{(t,r_t)}, \forall i \in \wp.$$

In this paper, the following lemmas are needed.

Lemma 1. [50] For any positive-definite matrix $\Phi \in \mathbb{R}^{n\times n}$, if there is a scalar $\gamma > 0$ and a vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$(\int_0^\gamma \omega(s)ds)^T \Phi (\int_0^\gamma \omega(s)ds) \leq \gamma \int_0^\gamma \omega^T(s)\Phi\omega(s)ds. \tag{2.7}$$

Lemma 2. Consider a scalar $h > 0$ and any continuously differentiable function $x(t) \in \mathbb{R}^n$. For any positive definite matrix Q, the following inequality holds:

$$-\int_{t-h}^t \dot{x}^T(s)Q\dot{x}(s)ds \leq \frac{1}{h}\eta^T(t)\Xi\eta(t), \tag{2.8}$$

where

$$\Xi = \begin{bmatrix} -18Q & 6Q & 0 & 0 & -96Q & 0 & 480Q \\ * & -36Q & 6Q & -96Q & 144Q & 480Q & -480Q \\ * & * & -18Q & 144Q & 0 & -480Q & 0 \\ * & * & * & -1536Q & 0 & 5760Q & 0 \\ * & * & * & * & -1536Q & 0 & 5760Q \\ * & * & * & * & * & -2304Q & 0 \\ * & * & * & * & * & * & -2304Q \end{bmatrix},$$

$\eta(t) = [x^T(t) \quad x^T(t-\frac{h}{2}) \quad x^T(t-h) \quad \frac{1}{h}(\int_{t-h}^{t-\frac{h}{2}} x(s)ds)^T \quad \frac{1}{h}(\int_{t-\frac{h}{2}}^t x(s)ds)^T \quad \frac{1}{h^2}(\int_{t-h}^{t-\frac{h}{2}} \int_u^{t-\frac{h}{2}} x(s)dsdu)^T$
$\frac{1}{h^2}(\int_{t-\frac{h}{2}}^t \int_u^t x(s)dsdu)^T]^T.$

Proof of Lemma 2. For any continuously differentiable function $x(t) \in \mathbb{R}^n$ and positive definite matrix Q, the following equality holds:

$$\int_{t-h}^t \dot{x}^T(s)Q\dot{x}(s)ds = \int_{t-\frac{h}{2}}^t \dot{x}^T(s)Q\dot{x}(s)ds + \int_{t-h}^{t-\frac{h}{2}} \dot{x}^T(s)Q\dot{x}(s)ds. \tag{2.9}$$

Based on Lemma 2 in [46], the following inequalities hold:

$$\begin{aligned} \int_{t-\frac{h}{2}}^t \dot{x}^T(s)Q\dot{x}(s)ds &\geq \tfrac{2}{h}[x(t) - x(t-\tfrac{h}{2})]^T Q[x(t) - x(t-\tfrac{h}{2})] + \tfrac{6}{h}[x(t) + x(t-\tfrac{h}{2}) \\ &- \tfrac{4}{h}\int_{t-\frac{h}{2}}^t x(s)ds]^T Q[x(t) + x(t-\tfrac{h}{2}) - \tfrac{4}{h}\int_{t-\frac{h}{2}}^t x(s)ds] + \tfrac{10}{h}[x(t) - x(t-\tfrac{h}{2}) + \tfrac{12}{h}\int_{t-\frac{h}{2}}^t x(s)ds \\ &- \tfrac{48}{h^2}\int_{t-\frac{h}{2}}^t \int_u^t x(s)ds]^T Q[x(t) - x(t-\tfrac{h}{2}) + \tfrac{12}{h}\int_{t-\frac{h}{2}}^t x(s)ds - \tfrac{48}{h^2}\int_{t-\frac{h}{2}}^t \int_u^t x(s)ds], \end{aligned} \tag{2.10}$$

$$\begin{aligned}\int_{t-h}^{t-\frac{h}{2}} \dot{x}^T(s)Q\dot{x}(s)ds \geq{}& \tfrac{2}{h}[x(t-\tfrac{h}{2})-x(t-h)]^TQ[x(t-\tfrac{h}{2})-x(t-h)]+\tfrac{6}{h}[x(t-\tfrac{h}{2})+x(t-h)\\&-\tfrac{4}{h}\int_{t-h}^{t-\frac{h}{2}}x(s)ds]^TQ[x(t-\tfrac{h}{2})+x(t-h)-\tfrac{4}{h}\int_{t-h}^{t-\frac{h}{2}}x(s)ds]+\tfrac{10}{h}[x(t-\tfrac{h}{2})-x(t-h)\\&+\tfrac{12}{h}\int_{t-h}^{t-\frac{h}{2}}x(s)ds-\tfrac{48}{h^2}\int_{t-h}^{t-\frac{h}{2}}\int_{u}^{t-\frac{h}{2}}x(s)ds]^TQ[x(t-\tfrac{h}{2})-x(t-h)+\tfrac{12}{h}\int_{t-h}^{t-\frac{h}{2}}x(s)ds\\&-\tfrac{48}{h^2}\int_{t-h}^{t-\frac{h}{2}}\int_{u}^{t-\frac{h}{2}}x(s)ds].\end{aligned} \tag{2.11}$$

Replacing Eq (2.10) and (2.11) into Eq (2.9) yields Eq (2.8). This completes the proof.

Lemma 3. [34] For system (2.1) with constraints (2.2), if there is a Lyapunov functional $V(x_t, r_t)$ with $V(x_0, r_0) = 0$ and a positive scalar α, such that

$$\mathcal{L}V(x_t, r_t) + \alpha V(x_t, r_t) - \frac{\alpha}{w_m^2} w(t)^T w(t) \leq 0, \tag{2.12}$$

then $V(x_t, r_t) \leq 1$ for any $t \geq 0$.

Lemma 4. [32] Given any constant ϵ and square matrix $P \in \mathbb{R}^{n\times n}$, the inequality

$$\epsilon\left(P + P^T\right) \leqslant \epsilon^2 T + PT^{-1}P^T, \tag{2.13}$$

holds for any symmetric matrix $T > 0$.

Lemma 5. [51] (Schur Complement) Given constant symmetric matrices Σ_1, Σ_2, and Σ_3, where $\Sigma_1 = \Sigma_1^T$ and $\Sigma_2 = \Sigma_2^T > 0$, then $\Sigma_1 + \Sigma_3^T\Sigma_2^{-1}\Sigma_3 < 0$ holds if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3^T \\ \Sigma_3 & \Sigma_1 \end{bmatrix} < 0. \tag{2.14}$$

3. Main results

Our aim is to find an ellipsoid set as small as possible to bound the reachable set defined in Eq (2.6). Based on the Lyapunov method and linear matrix inequality techniques, the following theorems are derived.

Theorem 1. Consider the time-delayed system (2.1) with constraints (2.2). If there exist real matrices P_{2i} and P_{3i}, symmetric matrices $P_{1i} > 0$ for each mode $i \in \wp$, $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \geq 0$, $R_4 \geq 0$, $S_1 \geq 0$, $S_2 \geq 0$, $S_3 \geq 0$, $M_1 \geq 0$, $M_2 \geq 0$, and $Q \geq 0$, and a scalar $\alpha > 0$ satisfying the following matrix inequalities:

$$\Phi^i = \begin{bmatrix} \Phi^i_{1,1} & \Phi^i_{1,2} & \Phi^i_{1,3} & \Phi^i_{1,4} & \Phi^i_{1,5} & 0 & \Phi^i_{1,7} & 0 & \Phi^i_{1,9} & \Phi^i_{1,10} \\ * & \Phi^i_{2,2} & 0 & \Phi^i_{2,4} & \Phi^i_{2,5} & 0 & 0 & 0 & 0 & \Phi^i_{2,10} \\ * & * & \Phi^i_{3,3} & \Phi^i_{3,4} & 0 & \Phi^i_{3,6} & \Phi^i_{3,7} & \Phi^i_{3,8} & \Phi^i_{3,9} & 0 \\ * & * & * & \Phi^i_{4,4} & 0 & \Phi^i_{4,6} & 0 & \Phi^i_{4,8} & 0 & 0 \\ * & * & * & * & \Phi^i_{5,5} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi^i_{6,6} & 0 & \Phi^i_{6,8} & 0 & 0 \\ * & * & * & * & * & * & \Phi^i_{7,7} & 0 & \Phi^i_{7,9} & 0 \\ * & * & * & * & * & * & * & \Phi^i_{8,8} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Phi^i_{9,9} & 0 \\ * & * & * & * & * & * & * & * & * & \Phi^i_{10,10} \end{bmatrix} \leq 0, \tag{3.1}$$

where

$\Phi^i_{1,1} = \alpha P_{1i} + P^T_{2i}A_i + A^T_i P_{2i} + R_2 + R_3 + h(S_1 + \frac{S_2}{2}) + \frac{h^2}{8}M_2 - \frac{18Q}{h} + \sum_{j\in\wp} \lambda_{ij}(\delta)P_{1j}$,

$\Phi^i_{1,2} = P_{1i} - P^T_{2i} + A^T_i P_{3i}$, $\Phi^i_{1,3} = \frac{6Q}{h}$, $\Phi^i_{1,4} = P^T_{2i}B_i$, $\Phi^i_{1,5} = P^T_{2i}C_i$, $\Phi^i_{1,7} = -\frac{96Q}{h}$, $\Phi^i_{1,9} = \frac{480Q}{h}$,

$\Phi^i_{1,10} = P^T_{2i}D_i$, $\Phi^i_{2,2} = hQ + R_4 - P^T_{3i} - P_{3i}$, $\Phi^i_{2,4} = P^T_{3i}B_i$, $\Phi^i_{2,5} = P^T_{3i}C_i$, $\Phi^i_{2,10} = P^T_{3i}D_i$,

$\Phi^i_{3,3} = (1 - \frac{h_d}{2})e^{-\frac{\alpha h}{2}}R_1 - e^{-\frac{\alpha h}{2}}(1 - \frac{h_D}{2})R_2 + \frac{h}{2}(1 - \frac{h_D}{2})e^{-\frac{\alpha h}{2}}S_3 + h^2 e^{-\frac{\alpha h_d}{2}}M_1 - \frac{36Q}{h}$, $\Phi^i_{3,4} = \frac{6Q}{h}$,

$\Phi^i_{3,6} = -\frac{96Q}{h}$, $\Phi^i_{3,7} = \frac{144Q}{h}$, $\Phi^i_{3,8} = \frac{480Q}{h}$, $\Phi^i_{3,9} = -\frac{480Q}{h}$, $\Phi^i_{4,4} = -(1 - h_D)e^{-\frac{-\alpha h}{2}}R_1 - (1 - h_D)e^{-\alpha h}R_3 - \frac{18Q}{h}$,

$\Phi^i_{4,6} = \frac{144Q}{h}$, $\Phi^i_{4,8} = -\frac{480Q}{h}$, $\Phi^i_{5,5} = -(1 - \tau_D)e^{-\alpha\tau_M}R_4$, $\Phi^i_{6,6} = -2h(1 - h_D)e^{-\alpha h}[S_1 + S_3] - \frac{1536Q}{h}$,

$\Phi^i_{6,8} = \frac{5760Q}{h}$, $\Phi^i_{7,7} = -2he^{-\frac{\alpha h}{2}}[S_1 + S_2] - \frac{1536Q}{h}$, $\Phi^i_{7,9} = \frac{5760Q}{h}$, $\Phi^i_{8,8} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_1 - \frac{2304Q}{h}$,

$\Phi^i_{9,9} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_2 - \frac{2304Q}{h}$, $\Phi^i_{10,10} = -\frac{\alpha}{w_m^2}I$.

Then, the reachable sets of system (2.1) having constraints (2.2) is bounded by an ellipsoidal bound $\bigcap_{i\in\wp} \Im(P_{1i}, 1)$ defined in Eq (2.6).

Proof of Theorem 1. Choose a new class of functional candidate for system (2.1) as follows:

$$V(x_t, t, r_t) = \sum_{k=1}^{5} V_k(x_t, t, r_t), \tag{3.2}$$

where

$$V_1(x_t, t, r_t) = x^T(t)P_{1r_t}x(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{1r_t} & 0 \\ P_{2r_t} & P_{3r_t} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

$$V_2(x_t, t, r_t) = \int_{t-h(t)}^{t-\frac{h(t)}{2}} e^{\alpha(s-t)}x^T(s)R_1x(s)ds + \int_{t-\frac{h(t)}{2}}^{t} e^{\alpha(s-t)}x^T(s)R_2x(s)ds + \int_{t-h(t)}^{t} e^{\alpha(s-t)}x^T(s)R_3x(s)ds + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)}\dot{x}^T(s)R_4\dot{x}(s)ds,$$

$$V_3(x_t, t, r_t) = \int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}x^T(s)S_1x(s)dsd\theta + \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}x^T(s)S_2x(s)dsd\theta + \int_{-h(t)}^{-\frac{h(t)}{2}}\int_{t+\theta}^{t-\frac{h(t)}{2}} e^{\alpha(s-t)}x^T(s)S_3x(s)dsd\theta,$$

$$V_4(x_t, t, r_t) = \int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{\theta}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}} e^{\alpha(s-t)}x^T(s)M_1x(s)dsdud\theta + \int_{t-\frac{h(t)}{2}}^{t}\int_{\theta}^{t}\int_{u}^{t} e^{\alpha(s-t)}x^T(s)M_2x(s)dsdud\theta,$$

$$V_5(x_t, t, r_t) = \int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{x}^T(s)Q\dot{x}(s)dsd\theta.$$

Furthermore, $P_{1r_t} > 0$, P_{2r_t}, P_{3r_t}, $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \geq 0$, $S_1 \geq 0$, $S_2 \geq 0$, $S_3 \geq 0$, $M_1 \geq 0$, $M_2 \geq 0$, and $Q \geq 0$, and a scalar $\alpha > 0$ are solutions of Eq (3.1).

First, we show that $V(x_t, t, r_t)$ in Eq (3.2) is a fine Lyapunov-Krasovskii functional candidate. From Eq (3.2), we have $\sum_{k=2}^{5} V_k(x_t, t, r_t) \geq 0$. Therefore, we get

$$\begin{cases} V(x_t, t, r_t) = \sum_{k=1}^{5} V_k(x_t, t, r_t) \geq V_1(x_t, t, r_t) = x^T(t)P_{1i}x(t), \\ V(x_t, t, r_t) = 0, \quad when \quad x(\theta) = 0, \quad \theta \in [t - \rho^*, t]. \end{cases} \tag{3.3}$$

Next, the derivative of $V(x_t, t, r_t)$ along the trajectory of system (2.1) is given by

$$\mathcal{L}V(x_t, t, r_t) = \sum_{k=1}^{5} \mathcal{L}V_k(x_t, t, r_t). \tag{3.4}$$

From Eq (2.4), we have

$$\begin{aligned}\mathcal{L}V_1(x_t,t,i) &= \lim_{\Delta\to 0}\frac{1}{\Delta}\left[\sum_{j\in\wp}Pr\{j\mid i\}x^T(t+\Delta)P_jx(t+\Delta)-x^T(t)P_ix(t)\right]\\ &= \lim_{\Delta\to 0}\frac{1}{\Delta}[\sum_{j\in\wp\setminus\{i\}}\tfrac{Pr\{j,i\}}{Pr\{i\}}x^T(t+\Delta)P_jx(t+\Delta)+\tfrac{Pr\{i,i\}}{Pr\{i\}}x^T(t+\Delta)\\ &\quad\cdot P_ix(t+\Delta)-x^T(t)P_ix(t)]\\ &= \lim_{\Delta\to 0}\frac{1}{\Delta}[\sum_{j\in\wp\setminus\{i\}}\tfrac{q_{ij}(F_i(\delta+\Delta)-F_i(\delta))}{1-F_i(\delta)}x^T(t+\Delta)P_jx(t+\Delta)+\tfrac{1-F_i(\delta+\Delta)}{1-F_i(\delta)}\\ &\quad\cdot x^T(t+\Delta)P_ix(t+\Delta)-x^T(t)P_ix(t)]\end{aligned}\tag{3.5}$$

where $F_i(t)$ is the cumulative distribution function of the sojourn time δ in mode i, and q_{ij} is the probability density intensity of the system jump from mode i to mode j. When Δ is small, $x(t+\Delta) = x(t)+\dot{x}(t)\Delta+o(\Delta) = (A_i\Delta+I)\,x(t)+B_i\Delta x(t-h(t))+C_i\Delta\dot{x}(t-\tau(t))+D_i\Delta w(t)+o(\Delta)$. Then, Eq (3.5) becomes

$$\begin{aligned}\mathcal{L}V_1(x_t,t,i) &= \lim_{\Delta\to 0}\frac{1}{\Delta}[\sum_{j\in\wp\setminus\{i\}}\tfrac{q_{ij}(F_i(\delta+\Delta)-F_i(\delta))}{1-F_i(\delta)}\xi_1^T(t)G_i^TP_jG_i\xi_1(t)+\tfrac{1-F_i(\delta+\Delta)}{1-F_i(\delta)}\\ &\quad\cdot\xi_1^T(t)G_i^TP_iG_i\xi_1(t)-x^T(t)P_ix(t)],\end{aligned}\tag{3.6}$$

where $G_i = [A_i\Delta \quad B_i\Delta \quad C_i\Delta \quad D_i\Delta]$, $\xi_1(t) = [x^T(t) \quad x^T(t-h(t)) \quad \dot{x}^T(t-\tau(t)) \quad \omega^T(t)]$.

Furthermore, utilizing the same technique as in [28], it is obtained that

$$\begin{cases}\lim\limits_{\Delta\to 0}\frac{F_i(\delta+\Delta)-F_i(\delta)}{1-F_i(\delta)} = 0,\\ \lim\limits_{\Delta\to 0}\frac{1-F_i(\delta+\Delta)}{1-F_i(\delta)} = 1,\\ \lim\limits_{\Delta\to 0}\frac{F_i(\delta+\Delta)-F_i(\delta)}{(1-F_i(\delta))\Delta} = \lambda_i(\delta).\end{cases}\tag{3.7}$$

Here, $\lambda_i(\delta)$ is the transition rate of the system jumping from mode i, and we define $\lambda_{ij}(\delta) = \lambda_i(\delta)q_{ij}$ for $j\neq i$ and $\lambda_{ii}(\delta) = -\sum\limits_{j=1,j\neq i}\lambda_{ij}(\delta)$. Next, $\mathcal{L}V_1(x_t,t,i)$ can be rewritten as

$$\begin{aligned}\mathcal{L}V_1(x_t,t,i) &= 2\begin{bmatrix} x^T(t) & \dot{x}^T(t)\end{bmatrix}\begin{bmatrix} P_{1i} & P_{2i}^T\\ 0 & P_{3i}^T\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ 0\end{bmatrix}+x^T(t)[\sum_{j\in\wp}\lambda_{ij}(\delta)P_{1j}]x(t)\\ &= 2\begin{bmatrix} x^T(t) & \dot{x}^T(t)\end{bmatrix}\begin{bmatrix} P_{1i} & P_{2i}^T\\ 0 & P_{3i}^T\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \begin{pmatrix}-\dot{x}(t)+A_ix(t)+B_ix(t-h(t))\\ +C_i\dot{x}(t-\tau(t))+D_iw(t)\end{pmatrix}\end{bmatrix}+x^T(t)[\sum_{j\in\wp}\lambda_{ij}(\delta)P_{1j}]x(t)\end{aligned}$$

$$\begin{aligned}&= x^T(t)[P_{2i}^TA_i+A_i^TP_{2i}]x(t)+2x^T(t)[P_{1i}-P_{2i}^T+A_i^TP_{3i}]\dot{x}(t)+2x^T(t)P_{2i}^TB_ix(t-h(t))\\ &\quad+2x^T(t)P_{2i}^TC\dot{x}(t-\tau(t))+2x^T(t)P_{2i}^TD_iw(t)-\dot{x}^T(t)[P_{3i}^T+P_{3i}]\dot{x}(t)+2\dot{x}^T(t)P_{3i}^TB_ix(t-h(t))\\ &\quad+2\dot{x}^T(t)P_{3i}^TC_i\dot{x}(t-\tau(t))+2\dot{x}^T(t)P_{3i}^TD_iw(t)+x^T(t)[\sum_{j\in\wp}\lambda_{ij}(\delta)P_{1j}]x(t),\end{aligned}\tag{3.8}$$

$$\begin{aligned}\mathcal{L}V_2(x_t,t,i) &\leq x^T(t)[R_2+R_3]x(t)+(1-\tfrac{h_d}{2})e^{-\frac{\alpha h}{2}}x^T(t-\tfrac{h(t)}{2})R_1x(t-\tfrac{h(t)}{2})+\dot{x}^T(t)R_4\dot{x}^T(t)\\ &\quad-(1-h_D)e^{-\frac{\alpha h}{2}}x^T(t-h(t))R_1x(t-h(t))-(1-\tfrac{h_D}{2})e^{-\frac{\alpha h}{2}}x^T(t-\tfrac{h(t)}{2})R_2x(t-\tfrac{h(t)}{2})\\ &\quad-(1-h_D)e^{-\alpha h}x^T(t-h(t))R_3x(t-h(t))-(1-\tau_D)e^{-\alpha\tau_M}\dot{x}^T(t-\tau(t))R_4\dot{x}(t-\tau(t))-\alpha V_2,\end{aligned}\tag{3.9}$$

$$\mathcal{L}V_3(x_t,t,i) \le hx^T(t)[S_1+\tfrac{S_2}{2}]x(t)+\tfrac{h(t)}{2}(1-\tfrac{\dot h(t)}{2})e^{-\frac{\alpha h(t)}{2}}x^T(t-\tfrac{h(t)}{2})S_3x(t-\tfrac{h(t)}{2})$$
$$-\int_{t-\frac{h(t)}{2}}^{t}e^{\alpha(s-t)}x^T(s)[S_1+S_2]x(s)ds-\int_{t-h(t)}^{t-\frac{h(t)}{2}}e^{\alpha(s-t)}x^T(s)[S_1+(1-\dot h(t))S_3]x(s)ds-\alpha V_3$$
$$\le hx^T(t)[S_1+\tfrac{S_2}{2}]x(t)+\tfrac{h}{2}(1-\tfrac{h_D}{2})e^{-\frac{\alpha h}{2}}x^T(t-\tfrac{h(t)}{2})S_3x(t-\tfrac{h(t)}{2})-2he^{-\frac{\alpha h}{2}}(\tfrac{1}{h}\int_{t-\frac{h(t)}{2}}^{t}x(s)ds)^T$$
$$\cdot[S_1+S_2](\tfrac{1}{h}\int_{t-\frac{h(t)}{2}}^{t}x(s)ds)-2h(1-h_D)e^{-\frac{\alpha h}{2}}(\tfrac{1}{h}\int_{t-h(t)}^{t-\frac{h(t)}{2}}x(s)ds)^T[S_1+S_3](\tfrac{1}{h}\int_{t-h(t)}^{t-\frac{h(t)}{2}}x(s)ds)-\alpha V_3, \tag{3.10}$$

$$\mathcal{L}V_4(x_t,t,i) = -(1-\dot h(t))\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}}e^{\alpha(s-t)}x^T(s)M_1x(s)dsdu+(1-\tfrac{\dot h(t)}{2})$$
$$\cdot\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{\theta}^{t-\frac{h(t)}{2}}e^{-\frac{\alpha h(t)}{2}}x^T(t-\tfrac{h(t)}{2})M_1x(t-\tfrac{h(t)}{2})dsdu-(1-\tfrac{\dot h(t)}{2})\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}e^{\alpha(s-t)}x^T(s)M_2x(s)dsdu$$
$$+\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}x^T(t)M_2x(t)ds-\alpha V_4 \tag{3.11}$$
$$\le \tfrac{h^2}{8}e^{-\frac{\alpha h_d}{2}}x^T(t-\tfrac{h(t)}{2})M_1x(t-\tfrac{h(t)}{2})+\tfrac{h^2}{8}x^T(t)M_2x(t)-\tfrac{8(1-\frac{h_D}{2})e^{-\frac{\alpha h}{2}}}{h^2}(\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}}x(s)dsdu)^T$$
$$\cdot M_1(\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}}x(s)dsdu)-\tfrac{8(1-\frac{h_D}{2})}{h^2}(\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}x(s)dsdu)^TM_2(\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}x(s)dsdu)-\alpha V_4,$$

$$\mathcal{L}V_5(x_t,t,i) \le h\dot x^T(t)Q\dot x(t)-\int_{t-h(t)}^{t}\dot x^T(s)Q\dot x(s)ds-\alpha V_5. \tag{3.12}$$

From Lemma 2, we have

$$-\int_{t-h(t)}^{t}\dot x^T(s)Q\dot x(s)ds \le \tfrac{1}{h}\xi^T(t)\Xi\xi(t), \tag{3.13}$$

where Ξ is the same as defined in Lemma 2, and

$$\xi(t) = [x^T(t)\quad x^T(t-\tfrac{h(t)}{2})\quad x^T(t-h(t))\quad \tfrac{1}{h}(\int_{t-h(t)}^{t-\frac{h(t)}{2}}x(s)ds)^T\quad \tfrac{1}{h}(\int_{t-\frac{h(t)}{2}}^{t}x(s)ds)^T\quad \tfrac{1}{h^2}(\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}}x(s)dsdu)^T$$
$$\tfrac{1}{h^2}(\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}x(s)dsdu)^T]^T.$$

Finally, by combining Eqs (3.5)–(3.13), we further have

$$\mathcal{L}V(x_t,t,i)+\alpha V(x_t,t,i)-\tfrac{\alpha}{w_m^2}w(t)^Tw(t) \le X^T(t)\Phi^iX(t), \tag{3.14}$$

where Φ^i is the same as that defined in Theorem 1 for any $i \in \wp$, and

$$X(t) = [x^T(t)\quad \dot x^T(t)\quad x^T(t-\tfrac{h(t)}{2})\quad x^T(t-h(t))\quad \dot x^T(t-\tau(t))\quad \tfrac{1}{h}(\int_{t-h(t)}^{t-\frac{h(t)}{2}}x(s)ds)^T\quad \tfrac{1}{h}(\int_{t-\frac{h(t)}{2}}^{t}x(s)ds)^T$$
$$\tfrac{1}{h^2}(\int_{t-h(t)}^{t-\frac{h(t)}{2}}\int_{u}^{t-\frac{h(t)}{2}}x(s)dsdu)^T\quad \tfrac{1}{h^2}(\int_{t-\frac{h(t)}{2}}^{t}\int_{u}^{t}x(s)dsdu)^T\quad w^T(t)]^T.$$

Thus, from the matrix inequalities (3.1), we get

$$\mathcal{L}V(x_t,t,i)+\alpha V(x_t,t,i)-\tfrac{\alpha}{w_m^2}w(t)^Tw(t) \le 0,\ \forall i \in \wp. \tag{3.15}$$

which means, by Lemma 3, that $V(x_t,t,i) = V_1(x_t,t,i)+V_2(x_t,t,i)+V_3(x_t,t,i)+V_4(x_t,t,i)+V_5(x_t,t,i) \le 1$, and this results in $V_1(x_t,t,i) = x^T(t)P_{1i}x(t) \le 1$ for any $i \in \wp$, since $V_2(x_t,t,i)+V_3(x_t,t,i)+V_4(x_t,t,i)+V_5(x_t,t,i) \ge 0$. This completes the proof.

Remark 3. Since $\lambda_{ij}(\delta)$ in Theorem 1 is time-varying and contains an infinite number of inequalities, it is impossible to solve by using the Linear Matrix Inequalities (LMIs). At this point, we will obtain the boundary of the reachable set according to the upper and lower bounded method in [32].

Corollary 1. Consider the time-delayed system (2.1) with constraints (2.2), and real matrices P_{2i} and P_{3i}, symmetric matrices $P_{1i} > 0$ for each mode $i \in \wp$, $T_{ij} > 0$, $R_1 \ge 0$, $R_2 \ge 0$, $R_3 \ge 0$, $R_4 \ge 0$, $S_1 \ge 0$, $S_2 \ge 0$, $S_3 \ge 0$, $M_1 \ge 0$, $M_2 \ge 0$, and $Q \ge 0$, and a scalar $\alpha > 0$ satisfying the following matrix inequalities:

$$\Phi^i = \begin{bmatrix} \widehat{\Phi}^i_{1,1} & \Phi^i_{1,2} & \Phi^i_{1,3} & \Phi^i_{1,4} & \Phi^i_{1,5} & 0 & \Phi^i_{1,7} & 0 & \Phi^i_{1,9} & \Phi^i_{1,10} & \Phi^i_{1,11} \\ * & \Phi^i_{2,2} & 0 & \Phi^i_{2,4} & \Phi^i_{2,5} & 0 & 0 & 0 & 0 & \Phi^i_{2,10} & 0 \\ * & * & \Phi^i_{3,3} & \Phi^i_{3,4} & 0 & \Phi^i_{3,6} & \Phi^i_{3,7} & \Phi^i_{3,8} & \Phi^i_{3,9} & 0 & 0 \\ * & * & * & \Phi^i_{4,4} & 0 & \Phi^i_{4,6} & 0 & \Phi^i_{4,8} & 0 & 0 & 0 \\ * & * & * & * & \Phi^i_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi^i_{6,6} & 0 & \Phi^i_{6,8} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Phi^i_{7,7} & 0 & \Phi^i_{7,9} & 0 & 0 \\ * & * & * & * & * & * & * & \Phi^i_{8,8} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Phi^i_{9,9} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Phi^i_{10,10} & 0 \\ * & * & * & * & * & * & * & * & * & * & \Phi^i_{11,11} \end{bmatrix} \le 0, \tag{3.16}$$

where

$\Phi^i_{1,1} = \alpha P_{1i} + P^T_{2i}A_i + A^T_i P_{2i} + R_2 + R_3 + h(S_1 + \frac{S_2}{2}) + \frac{h^2}{8}M_2 - \frac{18Q}{h} + + \sum_{j\in\wp} \lambda_{ij}P_{1j} + \sum_{j\in\wp\setminus\{i\}} \frac{l^2_{ij}}{4}T_{ij}$,

$\Phi^i_{1,2} = P_{1i} - P^T_{2i} + A^T_i P_{3i}$, $\Phi^i_{1,3} = \frac{6Q}{h}$, $\Phi^i_{1,4} = P^T_{2i}B_i$, $\Phi^i_{1,5} = P^T_{2i}C_i$, $\Phi^i_{1,7} = -\frac{96Q}{h}$, $\Phi^i_{1,9} = \frac{480Q}{h}$,

$\Phi^i_{1,10} = P^T_{2i}D_i$, $\Phi^i_{2,2} = hQ + R_4 - P^T_{3i} - P_{3i}$, $\Phi^i_{2,4} = P^T_{3i}B_i$, $\Phi^i_{2,5} = P^T_{3i}C_i$, $\Phi^i_{2,10} = P^T_{3i}D_i$,

$\Phi^i_{3,3} = (1 - \frac{h_d}{2})e^{-\frac{\alpha h}{2}}R_1 - e^{-\frac{\alpha h}{2}}(1 - \frac{h_D}{2})R_2 + \frac{h}{2}(1 - \frac{h_D}{2})e^{-\frac{\alpha h}{2}}S_3 + h^2 e^{-\frac{\alpha h_d}{2}}M_1 - \frac{36Q}{h}$, $\Phi^i_{3,4} = \frac{6Q}{h}$,

$\Phi^i_{3,6} = -\frac{96Q}{h}$, $\Phi^i_{3,7} = \frac{144Q}{h}$, $\Phi^i_{3,8} = \frac{480Q}{h}$, $\Phi^i_{3,9} = -\frac{480Q}{h}$, $\Phi^i_{4,4} = -(1-h_D)e^{-\frac{-\alpha h}{2}}R_1 - (1-h_D)e^{-\alpha h}R_3 - \frac{18Q}{h}$,

$\Phi^i_{4,6} = \frac{144Q}{h}$, $\Phi^i_{4,8} = -\frac{480Q}{h}$, $\Phi^i_{5,5} = -(1-\tau_D)e^{-\alpha\tau_M}R_4$, $\Phi^i_{6,6} = -2h(1-h_D)e^{-\alpha h}[S_1 + S_3] - \frac{1536Q}{h}$,

$\Phi^i_{6,8} = \frac{5760Q}{h}$, $\Phi^i_{7,7} = -2he^{-\frac{\alpha h}{2}}[S_1 + S_2] - \frac{1536Q}{h}$, $\Phi^i_{7,9} = \frac{5760Q}{h}$, $\Phi^i_{8,8} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_1 - \frac{2304Q}{h}$,

$\Phi^i_{9,9} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_2 - \frac{2304Q}{h}$, $\Phi^i_{10,10} = -\frac{\alpha}{w^2_m}I$,

$\Phi^i_{1,11} = [P_{11} - P_{1i}, \cdots, P_{1i-1} - P_{1i}, P_{1i+1} - P_{1i}, \cdots, P_{1N} - P_{1i}]$,

$\Phi^i_{11,11} = -diag\{T_{i1}, \cdots, T_{i(i-1)}, T_{i(i+1)}, \cdots, T_{iN}\}$.

Other unknown parameters are the same as those defined in Theorem 1. Then, the reachable sets of system (2.1) having constraints (2.2) are bounded by an ellipsoidal bound $\bigcap_{i\in\wp} \Im(P_{1i}, 1)$ defined in Eq (2.6).

Proof of Corollary 1. According to Remark 1, the item $\sum_{j\in\wp} \lambda_{ij}(\delta)P_{1j}$ will be handled separately, and we can get that

$$\begin{aligned} \sum_{j\in\wp} \lambda_{ij}(\delta)P_{1j} &= \sum_{j\in\wp} \left(\lambda_{ij} + \Delta\lambda_{ij}\right) P_{1j} \\ &= \sum_{j\in\wp} \lambda_{ij}P_{1j} + \sum_{j\in\wp\setminus\{i\}} \Delta\lambda_{ij}\left(P_{1j} - P_{1i}\right) \\ &= \sum_{j\in\wp} \lambda_{ij}P_{1j} + \sum_{j\in\wp\setminus\{i\}} \left[\tfrac{1}{2}\Delta\lambda_{ij}\left(P_{1j} - P_{1i}\right) + \tfrac{1}{2}\Delta\lambda_{ij}\left(P_{1j} - P_{1i}\right)\right]. \end{aligned} \tag{3.17}$$

Meanwhile, by Lemma 4, there exist symmetric positive definite matrix T_{ij} for any $| \Delta\lambda_{ij} |\le l_{ij}$, and we have

$$\sum_{i\in\wp} \lambda_{ij}(\delta)P_{1j} \leqslant \sum_{j\in\wp} \lambda_{ij}P_{1j} + \sum_{j\in\wp\setminus\{i\}} \left[\frac{l^2_{ij}}{4}T_{ij} + \left(P_{1j} - P_{1i}\right)T^{-1}_{ij}\left(P_{1j} - P_{1i}\right)\right]. \tag{3.18}$$

Thus, by the Schur complement, inequality (3.1) can be written as inequality (3.16). The proof is complete.

Remark 4. Inspired by reference [52], the mathematical expectation method is used to solve the transfer rate $\lambda_{ij}(\delta)$, Corollary 2 is derived from this approach, and the simulation result is worse than that of Corollary 1.

Corollary 2. Consider the time-delayed system (2.1) with constraints (2.2), and real matrices P_{2i} and P_{3i}, symmetric matrices $P_{1i} > 0$ for each mode $i \in \wp$, $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \geq 0$, $R_4 \geq 0$, $S_1 \geq 0$, $S_2 \geq 0$, $S_3 \geq 0$, $M_1 \geq 0$, $M_2 \geq 0$, and $Q \geq 0$, and a scalar $\alpha > 0$ satisfying the following matrix inequalities:

$$\Phi^i = \begin{bmatrix} \widetilde{\Phi}^i_{1,1} & \Phi^i_{1,2} & \Phi^i_{1,3} & \Phi^i_{1,4} & \Phi^i_{1,5} & 0 & \Phi^i_{1,7} & 0 & \Phi^i_{1,9} & \Phi^i_{1,10} \\ * & \Phi^i_{2,2} & 0 & \Phi^i_{2,4} & \Phi^i_{2,5} & 0 & 0 & 0 & 0 & \Phi^i_{2,10} \\ * & * & \Phi^i_{3,3} & \Phi^i_{3,4} & 0 & \Phi^i_{3,6} & \Phi^i_{3,7} & \Phi^i_{3,8} & \Phi^i_{3,9} & 0 \\ * & * & * & \Phi^i_{4,4} & 0 & \Phi^i_{4,6} & 0 & \Phi^i_{4,8} & 0 & 0 \\ * & * & * & * & \Phi^i_{5,5} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi^i_{6,6} & 0 & \Phi^i_{6,8} & 0 & 0 \\ * & * & * & * & * & * & \Phi^i_{7,7} & 0 & \Phi^i_{7,9} & 0 \\ * & * & * & * & * & * & * & \Phi^i_{8,8} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Phi^i_{9,9} & 0 \\ * & * & * & * & * & * & * & * & * & \Phi^i_{10,10} \end{bmatrix} \leq 0, \tag{3.19}$$

where

$\widetilde{\Phi}^i_{1,1} = \alpha P_{1i} + P^T_{2i}A_i + A^T_i P_{2i} + R_2 + R_3 + h(S_1 + \frac{S_2}{2}) + \frac{h^2}{8}M_2 - \frac{18Q}{h} + \sum_{j\in\wp} \widetilde{\lambda}_{ij}P_{1j}$, $\widetilde{\lambda}_{ij} = E\left[\lambda_{ij}(\delta)\right]$,
$\Phi^i_{1,2} = P_{1i} - P^T_{2i} + A^T_i P_{3i}$, $\Phi^i_{1,3} = \frac{6Q}{h}$, $\Phi^i_{1,4} = P^T_{2i}B_i$, $\Phi^i_{1,5} = P^T_{2i}C_i$, $\Phi^i_{1,7} = -\frac{96Q}{h}$, $\Phi^i_{1,9} = \frac{480Q}{h}$,
$\Phi^i_{1,10} = P^T_{2i}D_i$, $\Phi^i_{2,2} = hQ + R_4 - P^T_{3i} - P_{3i}$, $\Phi^i_{2,4} = P^T_{3i}B_i$, $\Phi^i_{2,5} = P^T_{3i}C_i$, $\Phi^i_{2,10} = P^T_{3i}D_i$,
$\Phi^i_{3,3} = (1 - \frac{h_d}{2})e^{-\frac{\alpha h}{2}}R_1 - e^{-\frac{\alpha h}{2}}(1 - \frac{h_D}{2})R_2 + \frac{h}{2}(1 - \frac{h_D}{2})e^{-\frac{\alpha h}{2}}S_3 + h^2e^{-\frac{\alpha h_d}{2}}M_1 - \frac{36Q}{h}$, $\Phi^i_{3,4} = \frac{6Q}{h}$,
$\Phi^i_{3,6} = -\frac{96Q}{h}$, $\Phi^i_{3,7} = \frac{144Q}{h}$, $\Phi^i_{3,8} = \frac{480Q}{h}$, $\Phi^i_{3,9} = -\frac{480Q}{h}$, $\Phi^i_{4,4} = -(1-h_D)e^{-\frac{\alpha h}{2}}R_1 - (1-h_D)e^{-\alpha h}R_3 - \frac{18Q}{h}$,
$\Phi^i_{4,6} = \frac{144Q}{h}$, $\Phi^i_{4,8} = -\frac{480Q}{h}$, $\Phi^i_{5,5} = -(1 - \tau_D)e^{-\alpha\tau_M}R_4$, $\Phi^i_{6,6} = -2h(1 - h_D)e^{-\alpha h}[S_1 + S_3] - \frac{1536Q}{h}$,
$\Phi^i_{6,8} = \frac{5760Q}{h}$, $\Phi^i_{7,7} = -2he^{-\frac{\alpha h}{2}}[S_1 + S_2] - \frac{1536Q}{h}$, $\Phi^i_{7,9} = \frac{5760Q}{h}$, $\Phi^i_{8,8} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_1 - \frac{2304Q}{h}$,
$\Phi^i_{9,9} = -\frac{8(1-\frac{h_D}{2})e^{-\alpha h}}{h^2}M_2 - \frac{2304Q}{h}$, $\Phi^i_{10,10} = -\frac{\alpha}{w^2_m}I$.

The other parameters are the same as those defined in Theorem 1. Then, the reachable sets of system (2.1) having constraints (2.2) are bounded by an ellipsoidal bound $\bigcap_{i\in\wp} \Im(P_{1i}, 1)$ defined in Eq (2.6).

Proof of Corollary 2. $\lambda_{ij}(\delta)$ is handled by the same method as in [52]. The $\widetilde{\lambda}_{ij}$ can be obtained through the probability density function $f_i(\delta) = \frac{b}{a^b}\delta^{b-1}e^{-(\delta/a)^b}$ with respect to the sojourn time ($\delta > 0$). It is worth noting that a represents the scale parameter and b represents the shape parameter. Then, the expectation of λ_{ij} is $E\left[\lambda_{ij}(\delta)\right] = \int_0^\infty \lambda_{ij}(\delta)f_i(\delta)d\delta$. After $\widetilde{\lambda}_{ij}$ is trivial to obtain, Corollary 2 can be proved based on Theorem 1.

Next, we consider the neutral semi-Markovian jump system with uncertainties as follows:

$$\begin{cases} \dot{x}(t) - (C_i + \Delta C_i(t))\dot{x}(t - \tau(t)) = (A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t)) \\ \qquad\qquad \cdot x(t - h(t)) + (D_i + \Delta D_i(t))w(t), \\ x(t_0 + \theta) \equiv 0, \quad \forall\theta \in [-\rho^*, 0], \end{cases} \tag{3.20}$$

where the uncertainties of the form A_i, B_i, C_i, and D_i are the known mode-dependent matrices with appropriate dimensions, and the uncertainties $\Delta A_i(t)$, $\Delta B_i(t)$, $\Delta C_i(t)$, and $\Delta D_i(t)$ are expressed as

$$\begin{bmatrix} \Delta A_i(t) & \Delta B_i(t) & \Delta C_i(t) & \Delta D_i(t) \end{bmatrix} = L_i K_i(t) \begin{bmatrix} E_{1i} & E_{2i} & E_{3i} & E_{4i} \end{bmatrix},$$

where $K_i(t) \in \mathbb{R}^{p\times q}$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$K_i^T(t)K_i(t) \le I,$$

and L_i, E_{1i}, E_{2i}, E_{3i}, and E_{4i} are known real constant matrices which characterize how the uncertainty enters the nominal matrices A_i, B_i, C_i, and D_i. Before proceeding further, system (3.20) can be written as:

$$\begin{cases} \dot{x}(t) - C_i\dot{x}(t-\tau(t)) = A_ix(t) + B_ix(t-h(t)) + D_iw(t) + L_iu_i, \\ z_i(t) = E_{1i}x(t) + E_{2i}x(t-h(t) + E_{3i}\dot{x}(t-\tau(t)) + E_{4i}w(t), \end{cases} \tag{3.21}$$

with the constraint $u_i = K_i(t)z_i(t)$. We further have

$$\begin{aligned} u^Tu \le\ & [E_{1i}x(t) + E_{2i}x(t-h(t) + E_{3i}\dot{x}(t-\tau(t)) + E_{4i}w(t)]^T \\ & \cdot[E_{1i}x(t) + E_{2i}x(t-h(t) + E_{3i}\dot{x}(t-\tau(t)) + E_{4i}w(t)]. \end{aligned} \tag{3.22}$$

Based on Theorem 1, we can obtain the reachable sets of uncertain neutral systems (3.21). The following Theorem 2 is a result for the no-ellipsoidal bound of a reachable set for an uncertain time-delayed system (3.21) having the constraints (2.2).

Theorem 2. Consider the uncertain time-delayed system (3.21) with constraints (2.2), and real matrices P_{2i} and P_{3i}, symmetric matrices $P_{1i} > 0$ for each mode $i \in \wp$, $R_1 \ge 0$, $R_2 \ge 0$, $R_3 \ge 0$, $R_4 \ge 0$, $S_1 \ge 0$, $S_2 \ge 0$, $S_3 \ge 0$, $M_1 \ge 0$, $M_2 \ge 0$, and $Q \ge 0$, and scalars $\alpha > 0$, $\varepsilon_i > 0$ satisfying the following matrix inequalities:

$$\Psi^i = \begin{bmatrix} \Phi^i & \Psi^i_{1,2} & \Psi^i_{1,3} \\ * & -\varepsilon_iI & 0 \\ * & * & -\varepsilon_iI \end{bmatrix} \le 0, \tag{3.23}$$

where

$$\Psi^i_{1,2} = \begin{bmatrix} L_i^TP_{2i} & L_i^TP_{3i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\Psi^i_{1,3} = \begin{bmatrix} \varepsilon_iE_{1i} & 0 & 0 & \varepsilon_iE_{2i} & \varepsilon_iE_{3i} & 0 & 0 & 0 & 0 & 0 & \varepsilon_iE_{4i} \end{bmatrix}^T.$$

The other parameters are the same as those defined in Theorem 1. Then, the reachable sets of system (3.21) having constraints (2.2) are bounded by an ellipsoidal bound $\bigcap_{i\in\wp} \Im(P_{1i}, 1)$ defined in Eq (2.6).

Proof of Theorem 2. Applying a similar method to that in the proof of Theorem 1, we can obtain

$$\mathcal{L}V(x_t,t,i) + \alpha V(x_t,t,i) - \tfrac{\alpha}{w_m^2}w(t)^Tw(t) \le X^T(t)\Phi^iX(t) + 2x^T(t)P_{2i}^TL_iu_i + 2\dot{x}^T(t)P_{3i}^TL_iu_i, \tag{3.24}$$

where Φ^i is the same as defined in Theorem 1 for any $i \in \wp$.

From inequalities (3.22), one can see that the following equation holds for any nonnegative scalar ε_i:

$$\begin{aligned} \mathcal{L}V(x_t,t,i) + \alpha V(x_t,t,i) - \tfrac{\alpha}{w_m^2}w(t)^Tw(t) \le\ & \begin{bmatrix} X^T(t) & u_i^T \end{bmatrix} \begin{bmatrix} \Phi^i & \Psi^i_{1,2} \\ * & -\varepsilon_iI \end{bmatrix} \\ & \cdot \begin{bmatrix} X(t) \\ u_i \end{bmatrix} + \varepsilon_i\{[E_{1i}x(t) + E_{2i}x(t-h(t) + E_{3i}\dot{x}(t-\tau(t)) \\ & + E_{4i}w(t)]^T[E_{1i}x(t) + E_{2i}x(t-h(t)) + E_{3i}\dot{x}(t-\tau(t)) + E_{4i}w(t)], \end{aligned} \tag{3.25}$$

where Φ^i and $\Psi^i_{1,2}$ are the same as defined in Theorem 2. By using Lemma 5, the matrix inequalities (3.23) imply

$$\mathcal{L}V(x_t,t,i)+\alpha V(x_t,t,i)-\frac{\alpha}{w_m^2}w(t)^Tw(t)\le 0,\ \forall i\in\wp \tag{3.26}$$

which means, by Lemma 3, that $V(x_t,t,i)=V_1(x_t,t,i)+V_2(x_t,t,i)+V_3(x_t,t,i)+V_4(x_t,t,i)+V_5(x_t,t,i)\le 1$, and this results in $V_1(x_t,t,i)=x^T(t)P_{1i}x(t)\le 1$ for any $i\in\wp$. This completes the proof.

Remark 5. When the reachable set is estimated by an ellipsoidal technique, the smaller the ellipsoidal boundary set is, the closer it is to the actual reachable set boundary. As in reference [53], that is, maximizing ρ subject to $\rho I\le P_{1i}$, is equivalent to the following optimization problem:

$$\begin{aligned}&\text{minimize}\quad \bar{\rho}\quad (\bar{\rho}=\tfrac{1}{\rho})\\ &s.t.\begin{cases}(a) & \begin{bmatrix}\bar{\rho}I & I\\ I & P_{1i}\end{bmatrix}\ge 0,\\ (b) & \text{Eqs (3.1), or (3.16), or (3.19), or (3.23).}\end{cases}\end{aligned} \tag{3.27}$$

Remark 6. The matrix inequalities in Theorems 1 and 2 contain only one non-convex scalar $\alpha>0$, and these become LMIs by fixing the scalar α. The feasibility check of a matrix inequality having only one non-convex scalar parameter is numerically tractable, and a local optimum value of α can be found by fminsearch.m.

4. Numerical examples

In this section, the validity of the main results derived above is illustrated by the following three examples.

Example 1. Consider system (2.1) with time-varying delays as follows:

$$\begin{cases}\dot{x}(t)-C_i\dot{x}(t-\tau(t))=A_ix(t)+B_ix(t-h(t))+D_iw(t),\\ x(t_0+\theta)\equiv 0,\quad \forall\theta\in[-\rho^*,0],\end{cases} \tag{4.1}$$

where $w^T(t)w(t)\le 1$. The parameters of system (4.1) are introduced as follows: $A_1=\begin{bmatrix}-2 & -1\\ 0 & -2\end{bmatrix}$, $A_2=\begin{bmatrix}-3 & 0\\ 0 & -2\end{bmatrix}$, $A_3=\begin{bmatrix}-1 & 0\\ -1 & -2\end{bmatrix}$, $B_1=\begin{bmatrix}-1.2 & 0\\ -1 & -1\end{bmatrix}$, $B_2=\begin{bmatrix}-2 & 0\\ -1.5 & -0.5\end{bmatrix}$, $B_3=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$, $C_1=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}$, $C_2=\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}$, $C_3=\begin{bmatrix}0.3 & 0\\ 0 & 0.3\end{bmatrix}$, $D_1=\begin{bmatrix}-0.13\\ 0.15\end{bmatrix}$, $D_2=\begin{bmatrix}-0.12\\ 0.35\end{bmatrix}$, $D_3=\begin{bmatrix}-0.2\\ 0.3\end{bmatrix}$, $h=\tau_M=0.2$, $h_d=0.1$, $h_D=\tau_D=0.75$, $\tau(t)=h(t)=0.1+0.1sin(t)$, $w(t)=sin(t)$.

According to the same method in [32], parameters for the three modes are chosen as $i=1$, $a=2$, $b=1.8$, $\lambda_{11}(\delta)=-1.04\delta^{0.8}$, $\lambda_{12}(\delta)=0.52\delta^{0.8}$, $\lambda_{13}(\delta)=0.52\delta^{0.8}$; $i=2$, $a=3$, $b=1.8$, $\lambda_{22}(\delta)=-0.5\delta^{0.8}$, $\lambda_{21}(\delta)=0.25\delta^{0.8}$, $\lambda_{23}(\delta)=0.25\delta^{0.8}$; $i=3$, $a=4.5$, $b=1.8$, $\lambda_{33}(\delta)=-0.24\delta^{0.8}$, $\lambda_{31}(\delta)=0.12\delta^{0.8}$, $\lambda_{32}(\delta)=0.12\delta^{0.8}$. Then, λ_{ij} and l_{ij} can be obtained as in Remark 1, and the bounds of $\lambda_{ij}(\delta)$ are denoted by the following two matrices:

$$\underline{\lambda}_{ij}=\begin{bmatrix}-1.4 & 0.7 & 0.7\\ 0.4 & -0.8 & 0.4\\ 0.1 & 0.1 & -0.2\end{bmatrix},\quad \bar{\lambda}_{ij}=\begin{bmatrix}-2.6 & 1.3 & 1.3\\ 1 & -2 & 1\\ 0.7 & 0.7 & -1.4\end{bmatrix}.$$

Moreover, the $\widetilde{\lambda}_{ij}$ are obtained by the same method as in [52], the details of which are as follows:

$$\widetilde{\lambda}_{ij} = \begin{bmatrix} -1.6038 & 0.8019 & 0.8019 \\ 0.5333 & -1.0666 & 0.5333 \\ 0.3540 & 0.3540 & -0.7080 \end{bmatrix}.$$

By solving the optimization problem (3.27), the maximum value of ρ in different methods and the corresponding feasible matrices are obtained in Table 1. Using the LMIs toolbox to solve the theoretical results of Corollaries 1 and 2, the computational time is 3.9930 seconds and 3.8212 seconds.

Table 1. The results of Corollaries 1 and 2.

α	ρ	P_{11}	P_{12}	P_{13}
3.1	5.4430	$\begin{bmatrix} 52.1001 & 21.6797 \\ 21.6797 & 41.4409 \end{bmatrix}$	$\begin{bmatrix} 35.8567 & 4.9769 \\ 4.9769 & 18.5069 \end{bmatrix}$	$\begin{bmatrix} 14.1468 & 7.6204 \\ 7.6204 & 17.0268 \end{bmatrix}$
1.4	5.3122	$\begin{bmatrix} 33.6548 & -5.5096 \\ -5.5096 & 17.0706 \end{bmatrix}$	$\begin{bmatrix} 30.5665 & -4.2398 \\ -4.2398 & 21.0895 \end{bmatrix}$	$\begin{bmatrix} 30.1417 & -5.2802 \\ -5.2802 & 21.3094 \end{bmatrix}$

Figures 1 and 2 show a possible mode evolution and the reachable state from the origin of the neutral semi-Markovian jump system respectively. Figure 3 manifests that Corollary 1 is less conservative than Corollary 2. Simulation results are shown in Table 1, and it is not difficult to see from Figure 1 that the reachable set is in the intersection of ellipsoidal bounds, and thus both methods are valid.

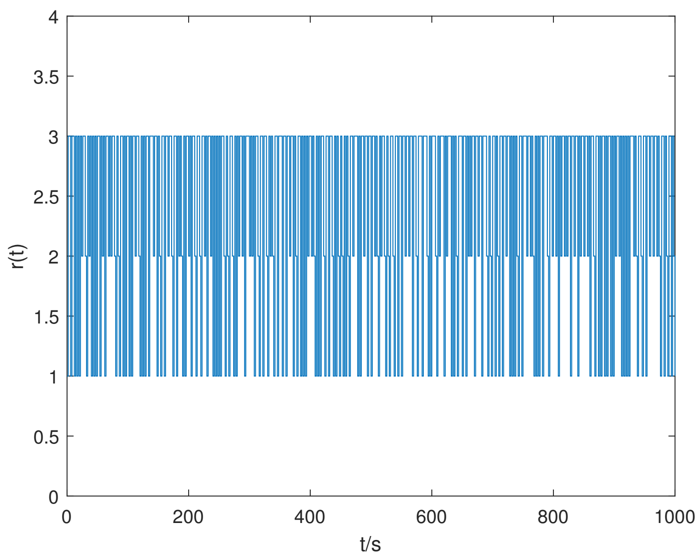

Figure 1. Random jumping mode r(t) of the neutral semi-Markovian jump system (4.1).

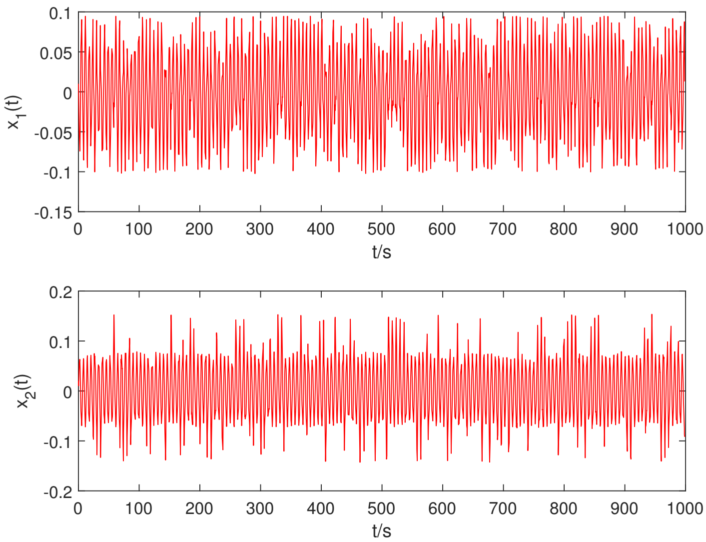

Figure 2. The time responses of state variable x(t) of the neutral semi-Markovian jump system (4.1).

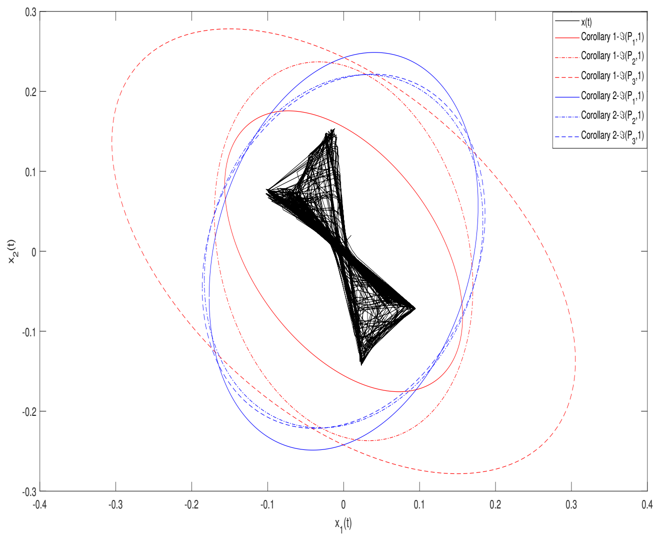

Figure 3. The comparative bounding ellipsoids and the state trajectories by Corollaries 1 and 2 for system (4.1).

Example 2. Consider the following semi-Markovian jump system studied in [32]:

$$\begin{cases} \dot{x}(t) = A_i x(t) + D_i w(t), \\ x(t_0 + \theta) \equiv 0, \quad \forall \theta \in [-\rho^*, 0], \end{cases} \tag{4.2}$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ -10.88 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix}$, $D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\omega^T(t)\omega(t) \leq \omega_m^2 = 1$.

By using Corollary 2 and solving the optimization problem (3.27), we can obtain $\rho = 1.9$ when $\alpha = 1.1$. The corresponding results are obtained in Table 2. Figure 4 indicates the ellipsoidal boundaries of system (4.2), and it is evident that the results of Corollary 2 get significant improvement over those in [32].

Table 2. Comparison results of Corollary 2 and [32].

α	ρ	P_1	P_2	Method
1.1	1.9	$\begin{bmatrix} 20.7347 & 1.0226 \\ 1.0226 & 1.9975 \end{bmatrix}$	$\begin{bmatrix} 20.7345 & 1.0226 \\ 1.0226 & 1.9975 \end{bmatrix}$	Corollary 2
0.9147	1.7101	$\begin{bmatrix} 17.2762 & 0.8429 \\ 0.8429 & 1.7557 \end{bmatrix}$	$\begin{bmatrix} 13.5858 & 1.0078 \\ 1.0078 & 1.7956 \end{bmatrix}$	[32]

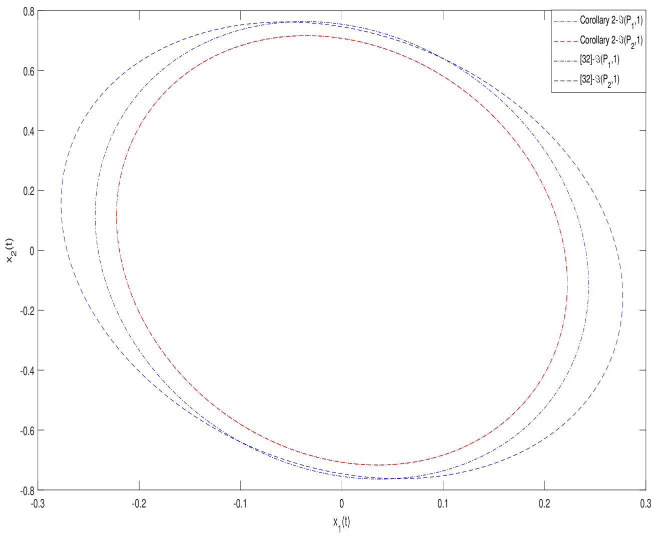

Figure 4. Comparision of the ellipsoidal bounds of Corollary 2 and [32].

Example 3. Consider the following uncertain neutral semi-Markovian jump systems (see Figure 5):

$$\begin{cases} \dot{x}(t) - (C_i + L_i K(t) E_{1i})\dot{x}(t - \tau(t)) = (A_i + L_i K(t) E_{2i})x(t) + (B_i + L_i K(t) E_{3i}) \\ \qquad \qquad \cdot x(t - h(t)) + (D_i + L_i K(t) E_{4i})w(t), \\ x(t_0 + \theta) \equiv 0, \quad \forall \theta \in [-\rho^*, 0], \end{cases} \tag{4.3}$$

where $L1 = L2 = L3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $E_{11} = E_{12} = E_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_{21} = E_{31} = E_{22} = E_{32} = E_{23} = E_{33} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{41} = E_{42} = E_{43} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $\varepsilon 1 = \varepsilon 2 = \varepsilon 3 = 1$, $K(t) = sin(t)$. The other parameters are introduced in Example 1.

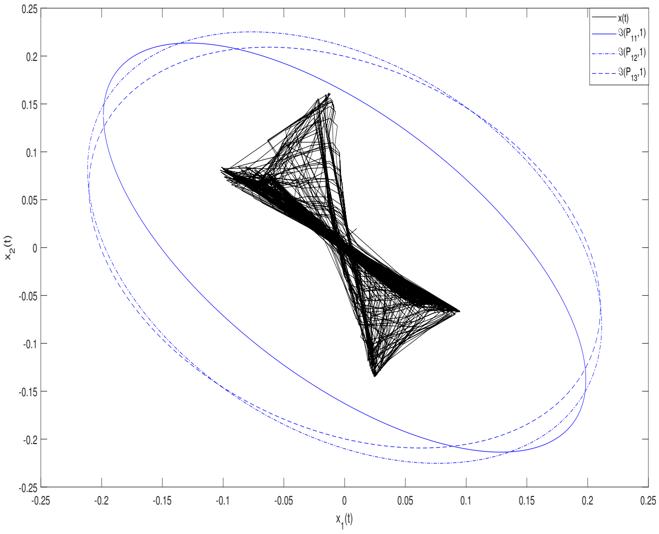

Figure 5. The boundaring ellipsoids of the uncertain neutral semi-Markovian jump systems (4.3).

The transition rate problem is solved using the expected technique in order to seek a less conservative boundary of the reachable set for uncertain systems, and the maximum value of ρ and the corresponding feasible matrices are obtained by finding the local optimal value of α. When $\alpha = 0.5$, $\rho = 13.2$, and the corresponding feasible matrices are $P_{11} = \begin{bmatrix} 43.6548 & 26.2615 \\ 26.2615 & 37.7218 \end{bmatrix}$, $P_{12} = \begin{bmatrix} 25.7136 & 8.7409 \\ 8.7409 & 22.6792 \end{bmatrix}$, and $P_{13} = \begin{bmatrix} 24.7843 & 7.4360 \\ 7.4360 & 25.0682 \end{bmatrix}$.

5. Conclusions

In this paper, the reachable set problem of neutral semi-Markovian jump systems with time-varying delays and uncertain neutral semi-Markovian jump systems is investigated. First, a novel and appropriate Lyapunov functional is constructed. Furthermore, its derivative is reduced by the improved integral inequality, and the reachable set boundary of the neutral semi-Markovian jump system under zero initial conditions is given by an ellipsoid in terms of LMIs. Finally, a numerical example is given to verify the effectiveness of the obtained results. Comparing the upper and lower bound method and the mathematical expectation method for dealing with the transition rate, we get the bound of the reachable set less conservatively.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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