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Research article

Random dynamics for a stochastic nonlocal reaction-diffusion equation with an energy functional

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Abstract: In this paper, the asymptotic behavior of solutions to a fractional stochastic nonlocal reaction-diffusion equation with polynomial drift terms of arbitrary order in an unbounded domain was analysed. First, the stochastic equation was transformed into a random one by using a stationary change of variable. Then, we proved the existence and uniqueness of solutions for the random problem based on pathwise uniform estimates as well as the energy method. Finally, the existence of a unique pullback attractor for the random dynamical system generated by the transformed equation is shown.

Keywords: nonlocal reaction-diffusion equations; fractional Laplacian; unbounded domain; random attractors
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1. Introduction

In real world applications, there might exist several nonlocal effects that influence the evolution of a system. For instance, we usually do not have enough information about the systems under study and its features at every point. In reality, the measurements are not made pointwise, but through some local average. Actually, during recent decades, many mathematicians have been studying nonlocal problems motivated by its various applications in physics, biology, and population dynamics [1–13].

Given $\gamma \in (0, 1)$ and an initial time $\tau \in \mathbb{R}$, in this work the following problem is considered:

$$\begin{cases} \partial_t u + a(l(u))(-\Delta)^\gamma u + \lambda u = f(t, x, u) + h(t, x) + \alpha u \circ \dfrac{dW}{dt}, & x \in \mathbb{R}^n,\ t \geq \tau, \\ u(x, \tau) = u_\tau(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where λ and α are positive constants, $a(l(u))$ is a more general nonlocal operator (cf. see [14] for more details), $l \in \mathcal{L}(\mathbb{R}; L^2(\mathbb{R}^n))$, $h \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$, f is a continuous function satisfying standard dissipative and growth conditions, W is a two-sided real-valued Wiener process in a probability space, and the symbol $\circ$ indicates the stochastic equation in the sense of Stratonovich integration.

Moreover, let us consider the following conditions imposed on the function $a \in C(\mathbb{R}; \mathbb{R}^+)$. Suppose there exist some constants $0 < m \le M$, such that

$$m \le a(s) \le M, \qquad s \in \mathbb{R}^+. \tag{1.2}$$

Assume that the function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth nonlinearity which satisfies, for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t, x, u)u \le -\beta |u|^p + \psi_1(t, x), \tag{1.3}$$

$$|f(t, x, u)| \le \psi_2(t, x)|u|^{p-1} + \psi_3(t, x), \tag{1.4}$$

$$\frac{\partial f}{\partial u}(t, x, u) \le \psi_4(t, x), \tag{1.5}$$

where $\beta > 0$ and $p \ge 2$ are constants and

$$\psi_1 \in L^1_{loc}(\mathbb{R}; L^1(\mathbb{R}^n)),\ \psi_2, \psi_4 \in L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^n)),\ \psi_3 \in L^q_{loc}(\mathbb{R}; L^q(\mathbb{R}^n)),$$

with $\frac{1}{p} + \frac{1}{q} = 1$. The identification $l(u)$ is in fact (l, u), however we keep the usual notation as in the existing previous literature $l(u)$ instead of (l, u) for the operator l acting on u.

We will establish the existence of a continuous cocycle for the non-autonomous fractional stochastic differential equation with $\gamma \in (0, 1)$,

$$\partial_t u + a(l(u))(-\Delta)^\gamma u + \lambda u = f(t, x, u) + h(t) + \alpha u \circ \frac{dW}{dt}, \qquad x \in \mathbb{R}^n,\ t \ge \tau, \tag{1.6}$$

with initial condition

$$u(\tau, x) = u_\tau(x), \qquad x \in \mathbb{R}^n. \tag{1.7}$$

For our purpose, we need to convert the stochastic equation into a deterministic one parameterized by $\omega \in \Omega$. To that end, we introduce a new variable $v = v(t, \tau, \omega, v_\tau)$ by,

$$v(t, \tau, \omega, v_\tau) = e^{-\alpha z(\theta_t \omega)} u(t, \tau, \omega, u_\tau), \tag{1.8}$$

with

$$v_\tau = e^{-\alpha z(\theta_\tau)\omega} u_\tau, \tag{1.9}$$

where $\tau \in \mathbb{R}$ is a deterministic time, $t \ge \tau$, $\omega \in \Omega$, $u_\tau \in L^2(\mathbb{R}^n)$, and $u = u(t, \tau, \omega, u_\tau)$ is a solution of (1.1). Then, we find that, for $t > \tau$,

$$\begin{cases} \partial_t v + a(l(e^{\alpha z(\theta_t \omega)} v))(-\Delta)^\gamma v + \lambda v = \alpha z(\theta_t \omega) v \\ \qquad + e^{-\alpha z(\theta_t \omega)} f(t, x, e^{\alpha z(\theta_t \omega)} v) + e^{-\alpha z(\theta_t \omega)} h(t), \quad x \in \mathbb{R}^n,\ t \ge \tau, \\ v(x, \tau) = v_\tau(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{1.10}$$

In fact, our final goal is to prove the existence of random attractors of problem (1.1) via proving the one of (1.10). Indeed, to state the existence of random attractors in $H^\gamma(\mathbb{R}^n)$, we need to establish the pullback asymptotic compactness of solutions in $H^\gamma(\mathbb{R}^n)$. The main technique is approaching the whole domain $\mathbb{R}^n$ by a sequence of bounded domains O_k, $k \in \mathbb{N}$. For each k, it is well-known that $H^\gamma(O_k)$ is compactly embedded in $L^2(O_k)$. Then, letting $k \to \infty$, the tail estimates will help us overcome this compactness difficulty.

We emphasize that the main innovation of our analysis is that the model contains two kinds of nonlocal terms: one is the nonlocal diffusion coefficient $a(l(u))$, and the other is the nonlocal fractional Laplace operator. Fractional partial differential equations arise from a variety of applications in physics, finance, probability, and materials sciences. Hence, there are a great number of works about fractional models which are analyzed theoretically or by computer simulations, such as [15–18] and the references therein.

This paper is organized as follows: In the next section, we will recall definitions of non-autonomous random dynamical systems and the fractional Laplacian operator, and introduce notation which will be used frequently in this manuscript. In Section 3, we show the existence and uniqueness of solutions to problem (1.1) driven by multiplicative noise. Some uniform estimates are given in Section 4 mainly by energy estimates and a random transformation. In the last section, we establish the existence and uniqueness of random attractors to problem (1.1).

2. Preliminaries

2.1. Non-autonomous random dynamical systems

First, we briefly review some notation and results for non-autonomous random dynamical systems for the sake of readers' convenience. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and (X, d) is a separable metric space. We use $d(A, B)$ to denote the Hausdorff semi-distance for nonempty subsets A and B of X.

Definition 2.1. [19, Definition 2.1] Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and t, $s \in \mathbb{R}^+$, the following conditions are satisfied:

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping;
(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;
(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
(iv) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Definition 2.2. [19, Definition 2.2] Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then, Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and any sequence $t_n \to +\infty$, $x_n \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X.$$

Definition 2.3. [19, Definition 2.3] Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then, $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor of Φ if the following conditions are satisfied:

(i) $\mathcal{A}$ is measurable and $\mathcal{A}(\tau, \omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$;

(ii) $\mathcal{A}$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t \omega), \quad t \geq 0;$$

(iii) $\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, given $B \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to \infty} d(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

The following results can be found in [20, 21] (see also [22–24]) for related results.

Proposition 2.1. *[19, Proposition 2.4] Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and Φ be a continuous cocyle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$, then the $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau, \omega) = \bigcap_{r \geq 0} \overline{\bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega))}.$$

2.2. Fractional Laplacian

Now, we recall some notation related to the fractional derivatives and fractional Sobolev spaces. Given $0 < \gamma < 1$, the fractional Laplace operator $(-\Delta)^\gamma$ is defined by,

$$(-\Delta)^\gamma u(x) = -\frac{1}{2} C(n, \gamma) \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2\gamma}} dy, \quad x \in \mathbb{R}^n,$$

provided the integral exists, where $C(n, \gamma)$ is a positive constant depending on n and γ as given by,

$$C(n, \gamma) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2\gamma}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n. \tag{2.1}$$

It follows from [25] that

$$(-\Delta)^\gamma u = \mathcal{F}^{-1}(|\xi|^{2\gamma}(\mathcal{F} u)), \quad \xi \in \mathbb{R}^n,$$

where $\mathcal{F}$ is the Fourier transform. Let $H^\gamma(\mathbb{R}^n)$ be the fractional Sobolev space defined by,

$$H^\gamma(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\gamma}} dx dy < \infty \right\},$$

with norm

$$\|u\|_{H^\gamma(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\gamma}} dx dy \right)^{\frac{1}{2}}.$$

Throughout this paper, we denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}^n)$ for some $p \geq 2$. Especially, we denote the norm and the inner product of $L^2(\mathbb{R}^n)$ by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. For convenience, the Gagliardo semi-norm of $H^\gamma(\mathbb{R}^n)$ is denoted by $\|\cdot\|_{\dot{H}^\gamma(\mathbb{R}^n)}$, i.e.,

$$\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy, \quad u \in H^\gamma(\mathbb{R}^n).$$

We also use the notation

$$(u, v)_{\dot{H}^\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2\gamma}} dxdy, \quad u, v \in H^\gamma(\mathbb{R}^n).$$

Then, for all $u \in H^\gamma(\mathbb{R}^n)$, we have $\|u\|^2_{H^\gamma(\mathbb{R}^n)} = \|u\|^2 + \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}$. Note that $H^\gamma(\mathbb{R}^n)$ is a Hilbert space with inner product given by

$$(u, v)_{H^\gamma(\mathbb{R}^n)} = (u, v) + (u, v)_{\dot{H}^\gamma(\mathbb{R}^n)}, \quad u, v \in H^\gamma(\mathbb{R}^n).$$

By [25], we have

$$\|(-\Delta)^{\frac{\gamma}{2}} u\|^2 = \frac{C(n, \gamma)}{2} \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}, \quad \forall u \in H^\gamma(\mathbb{R}^n),$$

hence,

$$\|u\|^2_{H^\gamma(\mathbb{R}^n)} = \|u\|^2 + \frac{2}{C(n, \gamma)} \|(-\Delta)^{\frac{\gamma}{2}} u\|^2, \quad \forall u \in H^\gamma(\mathbb{R}^n).$$

This implies that $\left(\|u\|^2 + \|(-\Delta)^{\frac{1}{2}} u\|^2\right)^{\frac{1}{2}}$ is an equivalent norm of $H^\gamma(\mathbb{R}^n)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the standard probability space with $\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ the Wiener measure on $(\Omega, \mathcal{F})$. Denote by $\theta_t : \Omega \to \Omega$ the transformation

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad \omega \in \Omega.$$

Then, $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. Consider the following one-dimensional stochastic equations:

$$dy + ydt = dW.$$

It follows from [26] that this equation has a unique stationary solution $y(t) = z(\theta_t\omega)$, where $z : \Omega \to \mathbb{R}$ is a random variable given by $z(\omega) = -\int_{-\infty}^0 e^\tau \omega(\tau)d\tau$ for $\omega \in \Omega$. Moreover, there exists a θ_t-invariant set of full measure Ω_0 such that $z(\theta_t\omega)$ is pathwise continuous for every $\omega \in \Omega_0$, and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0 \qquad \text{and} \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_\tau\omega)d\tau = 0. \tag{2.2}$$

For convenience, in the following, we will not distinguish Ω_0 and Ω and use the same notation Ω for both Ω_0 and Ω.

3. Main result

To define a continuous cocycle for the fractional stochastic reaction-diffusion equation (1.1), we first need to prove the existence and uniqueness of solutions to problem (1.10). By a solution of (1.10), we mean that v satisfies the equation in the following sense:

Definition 3.1. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_\tau \in L^2(\mathbb{R}^n)$, a continuous function $v(\cdot, \tau, \omega, v_\tau) : [\tau, \infty) \to L^2(\mathbb{R}^n)$ is called a solution of problem (1.10), if $v(\tau, \tau, \omega, v_\tau) = v_\tau$, and

$$v \in L^2_{loc}((\tau, \infty); H^\gamma(\mathbb{R}^n)) \cap L^p_{loc}((\tau, \infty); L^p(\mathbb{R}^n)),$$

$$\frac{dv}{dt} \in L^2_{loc}((\tau,\infty); H^{-\gamma}(\mathbb{R}^n)) \cap L^q_{loc}((\tau,\infty); L^q(\mathbb{R}^n)),$$

and v satisfies, for every $\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$\begin{aligned} &\frac{d}{dt}(v,\xi) + \frac{C(n,\gamma)}{2} a(l(e^{\alpha z(\theta_t\omega)}v)) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x)-v(y))(\xi(x)-\xi(y))}{|x-y|^{n+2\gamma}} dxdy + \lambda(x,\xi) \\ &= \alpha z(\theta_t\omega)(v,\xi) + e^{-\alpha z(\theta_t\omega)} \int_{\mathbb{R}^n} f(t,x,e^{\alpha z(\theta_t\omega)}v)\xi(x)dx + e^{-\alpha z(\theta_t\omega)} \int_{\mathbb{R}^n} h(t)\xi(x)dx, \end{aligned} \tag{3.1}$$

in the sense of distributions on (τ,∞).

To prove the existence of solutions of (1.10) in the sense of Definition 3.1, we will approximate the entire space $\mathbb{R}^n$ by a sequence of bounded domains $O_k = \{x \in \mathbb{R}^n : |x| < k\}$, and then take the limit as $k \to \infty$. Let $\rho : [0,\infty) \to \mathbb{R}$ be a smooth function such that $0 \le \rho(s) \le 1$ for all $0 \le s < \infty$, and

$$\rho(s) = 1, \text{ for } 0 \le s \le \frac{1}{2}, \qquad \text{and} \qquad \rho(s) = 0, \text{ for } s \ge 1.$$

Let us consider the non-autonomous fractional stochastic differential equation on O_k,

$$\begin{aligned} &\frac{dv_k}{dt} + a(l(e^{\alpha z(\theta_t\omega)}v_k))(-\Delta)^\gamma v_k + \lambda v_k = \alpha z(\theta_t\omega)v_k \\ &\quad + e^{-\alpha z(\theta_t\omega)} f(t,x,e^{\alpha z(\theta_t\omega)}v_k) + e^{-\alpha z(\theta_t\omega)} h(t), \qquad x \in O_k,\ t \ge \tau, \end{aligned} \tag{3.2}$$

with boundary condition

$$v_k(t,x) = 0, \qquad x \in \mathbb{R}^n \backslash O, \quad t \ge \tau, \tag{3.3}$$

and initial condition

$$v_k(\tau,x) = \rho\left(\frac{|x|}{k}\right) v_\tau(x), \qquad x \in O_k, \tag{3.4}$$

where $v_\tau \in L^2(\mathbb{R}^n)$. Note that, in the boundary condition (3.3), we require $v_k = 0$ on the complement of O_k (i.e., on $\mathbb{R}^n \backslash O_k$), not just on the boundary of O_k. This boundary condition is consistent with the definition of the nonlocal fractional Laplace operator $(-\Delta)^\gamma$. To present the existence of solutions of problem (3.2), for every $k \in \mathbb{N}$ we set $H_k = \{v \in L^2(\mathbb{R}^n) : v = 0 \text{ a.e. for } |x| \ge k\}$ and $V_k = \{v \in H^\gamma(\mathbb{R}^n) : v = 0 \text{ a.e. for } |x| \ge k\}$. The dual space of V_k is denoted by V_k^*.

Let $b : H^\gamma(\mathbb{R}^n) \times H^\gamma(\mathbb{R}^n) \to \mathbb{R}$ be a bilinear form given by, for $v_1, v_2 \in H^\gamma(\mathbb{R}^n)$,

$$b(v_1,v_2) = \lambda(v_1,v_2) + \frac{m}{2} C(n,\gamma) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v_1(x)-v_1(y))(v_2(x)-v_2(y))}{|x-y|^{n+2\gamma}} dxdy.$$

By using the bilinear form b, we define $A : H^\gamma(\mathbb{R}^n) \to H^{-\gamma}(\mathbb{R}^n)$ by

$$(A(v_1), v_2)_{(H^{-\gamma},H^\gamma)} = b(v_1,v_2), \quad \forall v_1, v_2 \in H^\gamma(\mathbb{R}^n),$$

where $(\cdot,\cdot)_{(H^{-\gamma},H^\gamma)}$ is the duality pairing of $H^{-\gamma}(\mathbb{R}^n)$ and $H^\gamma(\mathbb{R}^n)$. Since H_k and V_k are subspaces of $L^2(\mathbb{R}^n)$ and $H^\gamma(\mathbb{R}^n)$, respectively, we find that $b : V_k \times V_k \to \mathbb{R}$ and $A : V_k \to V_k^*$ are well defined. Indeed, we have

$$(A(v_1), v_2)_{(V_k^*,V_k)} = b(v_1,v_2), \qquad \forall v_1, v_2 \in V_k,$$

where $(\cdot,\cdot)_{(V_k^*,V_k)}$ is the duality pairing of V_k^* and V_k.

By means of conditions (1.2)–(1.5), it follows from [27] that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_\tau \in L^2(\mathbb{R}^n)$, problem (3.2)-(3.4) has a unique solution v_k in the sense that $v_k(\cdot,\tau,\omega,v_k(\tau)) : [\tau,\infty) \to H_k$ is continuous, $v_k(\tau,\tau,\omega,v_k(\tau))(x) = \rho\left(\frac{|x|}{k}\right) v_k(x)$, and

$$v_k \in L^2_{loc}((\tau,\infty);V_k) \cap L^p_{loc}((\tau,\infty);L^p(\mathbb{R}^n)), \quad \frac{dv_k}{dt} \in L^2_{loc}((\tau,\infty);V_k^*) + L^q_{loc}((\tau,\infty);L^q(\mathbb{R}^n)), \tag{3.5}$$

and v_k, satisfies, for every $\xi \in V_k \cap L^p(\mathbb{R}^n)$,

$$\begin{aligned} &\frac{d}{dt}(v_k,\xi) + \frac{C(n,\gamma)}{2} a(l(e^{\alpha z(\theta_t\omega)}v_k)) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v_k(x)-v_k(y))(\xi(x)-\xi(y))}{|x-y|^{n+2\gamma}} dxdy + \lambda(v_k,\xi) \\ &= \alpha z(\theta_t\omega)(v_k,\xi) + e^{-\alpha z(\theta_t\omega)} \int_{O_k} f(t,x,e^{\alpha z(\theta_t\omega)}v_k)\xi(x)dx + e^{-\alpha z(\theta_t\omega)} \int_{O_k} h(t)\xi(x)dx, \end{aligned} \tag{3.6}$$

in the sense of distributions on (τ,∞). Next, we derive uniform estimates of the solution v_k with respect to $k \in \mathbb{N}$ and prove the existence of solutions of (1.10) by taking the limit of v_k when $k \to \infty$.

Theorem 3.2. *Let* (1.2)–(1.5) *hold. Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_\tau \in L^2(\mathbb{R}^n)$, problem* (1.10) *has a unique solution $v(t,\tau,\omega,v_\tau)$ in the sense of Definition 3.1. This solution is $(\mathcal{F},\mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in ω and continuous in initial data v_τ in $L^2(\mathbb{R}^n)$. Moreover, the solution v satisfies the energy equation,*

$$\begin{aligned} \frac{d}{dt}\|v\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v))C(n,\gamma)\|v\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + 2\lambda\|v\|^2 &= 2\alpha z(\theta_t\omega)\|v\|^2 \\ &+ 2e^{-\alpha z(\theta_t\omega)} \int_{\mathbb{R}^n} f(t,x,e^{\alpha z(\theta_t\omega)}v)vdx + 2e^{-\alpha z(\theta_t\omega)} \int_{\mathbb{R}^n} h(t)vdx, \end{aligned} \tag{3.7}$$

for almost all $t > \tau$.

Proof. The proof is similar to the case of bounded domains as in [27] by modifying appropriately the conditions of the nonlinear term f. Of course, for problem (1.10) defined on the unbounded domain $\mathbb{R}^n$, we must show that all estimates on the solutions of (3.2)–(3.4) are uniform with respect to $k \in \mathbb{N}$.

Step 1. *Uniform estimates of solutions of* (3.2)–(3.4). By (3.2), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{O_k} |v_k(x)|^2 dx + a(l(e^{\alpha z(\theta_t\omega)}v_k)) \int_{O_k} v_k(x)(-\Delta)^\gamma v_k(x)dx + \lambda \int_{O_k} |v_k(x)|^2 dx \\ &= \alpha z(\theta_t\omega) \int_{O_k} |v_k(x)|^2 dx + e^{-\alpha z(\theta_t\omega)} \int_{O_k} f(t,x,e^{\alpha z(\theta_t\omega)}v_k)v_k dx + e^{-\alpha z(\theta_t\omega)} \int_{O_k} h(t)v_k(x)dx. \end{aligned}$$

By the boundary condition (3.3), all of the above integrals over the bounded domain O_k can be replaced by that over the entire space $\mathbb{R}^n$, and hence we have

$$\begin{aligned} &\frac{d}{dt}\|v_k\|^2 + a(l(e^{\alpha z(\theta_t\omega)}v_k))C(n,\gamma)\|v_k\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + 2\lambda\|v_k\|^2 \\ &= 2\alpha z(\theta_t\omega)\|v_k\|^2 + 2e^{-\alpha z(\theta_t\omega)}(f(t,x,e^{\alpha z(\theta_t\omega)}v_k),v_k) + 2e^{-\alpha z(\theta_t\omega)}(h(t),v_k). \end{aligned} \tag{3.8}$$

By (1.4), the nonlinear term in (3.8) satisfies,

$$\begin{aligned} e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} f(t,x,e^{\alpha z(\theta_t\omega)}v_k)v_k dx &\le e^{-2\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}(\psi_1(t,x)-\beta|e^{\alpha z(\theta_t\omega)}v_k|^p)dx \\ &\le e^{-2\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\psi_1(t,x)dx-\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v_k|^p dx. \end{aligned} \tag{3.9}$$

By Young's inequality, we derive

$$e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n} h(t)v_k dx \le \frac{1}{2}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2+\frac{1}{2}\|v_k\|^2. \tag{3.10}$$

It follows from (1.2) and (3.8)–(3.10) that

$$\begin{aligned} &\frac{d}{dt}\|v_k\|^2+C(n,\gamma)m\|v_k\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+2\lambda\|v_k\|^2+2\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v_k|^p dx \\ &\le (2\alpha z(\theta_t\omega)+1)\|v_k\|^2+2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}\psi_1(t,x)dx+e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2. \end{aligned} \tag{3.11}$$

By the above inequality, we see that for every fixed $\omega\in\Omega$ and $T>0$, $\{v_k\}_{k=1}^\infty$ is bounded in

$$L^\infty(\tau,\tau+T;L^2(\mathbb{R}^n))\cap L^2(\tau,\tau+T;H^\gamma(\mathbb{R}^n))\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n)), \tag{3.12}$$

and

$$\{A(v_k)\}_{k=1}^\infty \text{ is bounded in } L^2(\tau,\tau+T;H^{-\gamma}(\mathbb{R}^n)). \tag{3.13}$$

By (1.4) and (3.12), one can verify that

$$\{f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k)\}_{k=1}^\infty \text{ is bounded in } L^q(\tau,\tau+T;L^q(\mathbb{R}^n)). \tag{3.14}$$

As a consequence of (3.2) and (3.12)-(3.13), we find that, for each fixed $K\in\mathbb{N}$,

$$\left\{\frac{dv_k}{dt}\right\}_{k=1}^\infty \text{ is bounded in } L^q(\tau,\tau+T;(V_k\cap L^p(\mathbb{R}^n))^*). \tag{3.15}$$

Note that $1<q\le 2$ since $p\ge 2$ and p and q are conjugate exponents.

Step 2. *Existence of solutions of problem* (3.2)–(3.4). By a diagonal process, from (3.12)–(3.14), we find that there exists $\tilde{v}\in L^2(\mathbb{R}^n)$, such that

$$v\in L^\infty(\tau,\tau+T;L^2(\mathbb{R}^n))\cap L^2(\tau,\tau+T;H^\gamma(\mathbb{R}^n))\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n)),$$

and $\chi\in L^q(\tau,\tau+T;L^q(\mathbb{R}^n))$ such that, up to a subsequence,

$$v_k\to v \text{ weak-star in } L^\infty(\tau,\tau+T;L^2(\mathbb{R}^n)), \tag{3.16}$$

$$v_k\to v \text{ weakly in } L^2(\tau,\tau+T;H^\gamma(\mathbb{R}^n))\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n)), \tag{3.17}$$

$$f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k)\to\chi \text{ weakly in } L^q(\tau,\tau+T;L^q(\mathbb{R}^n)), \tag{3.18}$$

$$\frac{dv_k}{dt} \to \frac{dv}{dt} \text{ weakly in } L^q(\tau, \tau+T; (V_k \cap L^p(\mathbb{R}^n))^*), \tag{3.19}$$

and

$$v_k(\tau, \tau+T, \omega) \to \tilde{v} \text{ weakly in } L^2(\mathbb{R}^n). \tag{3.20}$$

Note that the embedding $H^\gamma(O_K) \hookrightarrow L^2(O_K)$ is compact, and also note that $L^2(O_K) \hookrightarrow (V_K \cap L^p(\mathbb{R}^n))^*$ is continuous. Then by (3.12), (3.15), and the compactness result in [28], after an appropriate diagonal process, we find that, up to a subsequence,

$$v_k \to v \quad \text{strongly in } L^2(\tau, \tau+T; L^2(O_K)), \quad \forall K \in \mathbb{N}. \tag{3.21}$$

By (3.21) and a diagonal process again, there exists a further subsequence (which is still denoted by $\{v_k\}_{k=1}^\infty$) such that

$$v_k \to v \quad \text{for almost every } (t,x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.22}$$

On the one hand, since a is a continuous function and $l \in L^2(\mathbb{R}^n)$, it follows from the above inequality that

$$a(l(e^{\alpha z(\theta_t\omega)}v_k)) \to a(l(e^{\alpha z(\theta_t\omega)}v)), \quad \text{for almost every } (t,x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.23}$$

On the other hand, as f is continuous, by (3.22), we obtain

$$f(t,x,e^{\alpha z(\theta_t\omega)}v_k) \to f(t,x,e^{\alpha z(\theta_t\omega)}v), \quad \text{for almost every } (t,x) \in (\tau, \tau+T) \times \mathbb{R}^n. \tag{3.24}$$

By (3.14) and (3.24), we infer from Mazur's lemma that,

$$f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k) \to f(t,\cdot,e^{\alpha z(\theta_t\omega)}v), \quad \text{weakly in } \ L^q(\tau, \tau+T; L^q(\mathbb{R}^n)). \tag{3.25}$$

It follows from (3.18) and (3.25) that

$$\chi = f(t,\cdot,e^{\alpha z(\theta_t\omega)}v). \tag{3.26}$$

Now, given $\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, denote by

$$\xi_K(x) = \rho\left(\frac{|x|}{K}\right)\xi(x), \quad \forall x \in \mathbb{R}^n.$$

By simple computations, one can verify that, for each $K \in \mathbb{N}$, $\xi_K \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and

$$\xi_K \to \xi \ \text{ in } \ H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \tag{3.27}$$

For every $k > K$ and $\phi \in C_0^\infty(\tau, \tau+T)$, by (3.2)–(3.4), we deduce

$$\begin{aligned}
&-\int_\tau^{\tau+T} (v_k, \xi_K)\phi' dt + C(n,\gamma)\int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)}v_k))(v_k,\xi_K)_{\dot{H}^\gamma(\mathbb{R}^n)}\phi dt + \lambda\int_\tau^{\tau+T} (v_k,\xi_K)\phi dt \\
&= \alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v_k,\xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v_k),\xi_K)_{(L^q,L^p)}\phi dt \\
&\quad + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t),\xi_K)\phi dt.
\end{aligned} \tag{3.28}$$

Taking the limit of (3.28) as $k \to \infty$, by (3.16)-(3.18) and (3.26), we obtain

$$\begin{aligned}
&-\int_\tau^{\tau+T}(v,\xi_K)\phi' dt + C(n,\gamma)\int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)}v))(v,\xi_K)_{\dot H^\gamma(\mathbb{R}^n)}\phi dt + \lambda\int_\tau^{\tau+T}(v,\xi_K)\phi dt\\
&=\alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,\xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi_K)_{(L^q,L^p)}\phi dt \qquad (3.29)\\
&\quad+\int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t),\xi_K)\phi dt.
\end{aligned}$$

Taking the limit of (3.29) as $K \to \infty$, by (3.27), we have

$$\begin{aligned}
&-\int_\tau^{\tau+T}(v,\xi)\phi' dt + C(n,\gamma)\int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)}v))(v,\xi)_{\dot H^\gamma(\mathbb{R}^n)}\phi dt + \lambda\int_\tau^{\tau+T}(v,\xi)\phi dt\\
&=\alpha\int_\tau^{\tau+T} z(\theta_t\omega)(v,\xi)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi)_{(L^q,L^p)}\phi dt \qquad (3.30)\\
&\quad+\int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)}(h(t),\xi)\phi dt.
\end{aligned}$$

Hence, we obtain that for all $\xi \in H^\gamma(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$,

$$\begin{aligned}
&\frac{d}{dt}(v,\xi) + \frac{C(n,\gamma)}{2}a(l(e^{\alpha z(\theta_t\omega)}v))(v,\xi)_{\dot H^\gamma(\mathbb{R}^n)} + \lambda(v,\xi)\\
&=\alpha z(\theta_t\omega)(v,\xi) + e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),\xi)_{(L^q,L^p)} + e^{-\alpha z(\theta_t\omega)}(h(t),\xi), \qquad (3.31)
\end{aligned}$$

in the sense of distribution on $(\tau,\tau+T)$.

To prove the continuity of $v : [\tau,\infty) \to L^2(\mathbb{R}^n)$, we notice that $v \in L^2(\tau,\tau+T;H^\gamma(\mathbb{R}^n))\cap L^p(\tau,\tau+T;L^p(\mathbb{R}^n))$ and $\frac{dv}{dt} \in L^2(\tau,\tau+T;H^{-\gamma}(\mathbb{R}^n))+L^q(\tau,\tau+T;L^q(\mathbb{R}^n))$ by (3.17) and (3.19), respectively. Then, by the argument of [28], we infer that $v \in C([\tau,\tau+T];L^2(\mathbb{R}^n))$ and

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = \left(\frac{dv}{dt},v\right)_{(H^{-\gamma}+L^q,H^\gamma+L^p)}, \quad \text{for almost every } t \in (\tau,\tau+T). \qquad (3.32)$$

It follows from (3.31)–(3.32), by taking $\xi = v$, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{C(n,\gamma)}{2}a(l(e^{\alpha z(\theta_t\omega)}))\|v\|^2_{\dot H^\gamma(\mathbb{R}^n)} + \lambda\|v\|^2\\
&=\alpha z(\theta_t\omega)\|v\|^2 + e^{-\alpha z(\theta_t\omega)}(f(t,\cdot,e^{\alpha z(\theta_t\omega)}v),v)_{(L^q,L^p)} + e^{-\alpha z(\theta_t\omega)}(h,v), \qquad (3.33)
\end{aligned}$$

which yields the desired energy equality (3.7).

In what follows, we show $v(\tau) = v_\tau$ and $v(\tau+T) = \tilde v$. To this end, we take $\phi \in C^1([\tau,\tau+T])$ and

$\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Similar to (3.28), by (3.2)–(3.4) we deduce, for every $k > K$,

$$\begin{aligned}
&(v_k(\tau+T), \xi_K)\phi(\tau+T) - (v_k(\tau), \xi_K)\phi(\tau) = \int_\tau^{\tau+T} (v_k, \xi_K)\phi' dt \\
&- \frac{C(n,\gamma)}{2} \int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)} v_k))(v_k, \xi_K)\phi dt - \lambda \int_\tau^{\tau+T} (v_k, \xi_K)\phi dt \\
&+ \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v_k, \xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f(t, \cdot, e^{\alpha z(\theta_t\omega)} v_k), \xi_K)_{(L^q, L^p)}\phi dt \\
&+ \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (h, \xi_K)\phi dt.
\end{aligned} \tag{3.34}$$

Proceeding as before, by (3.4), (3.16)–(3.18), (3.20), and (3.26) we obtain from the above equality that, as $k \to \infty$,

$$\begin{aligned}
&(\tilde{v}, \xi_K)\phi(\tau+T) - (v_\tau, \xi_K)\phi(\tau) = \int_\tau^{\tau+T} (v, \xi_K)\phi' dt \\
&- \frac{C(n,\gamma)}{2} \int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)} v))(v, \xi_K)\phi dt - \lambda \int_\tau^{\tau+T} (v, \xi_K)\phi dt \\
&+ \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, \xi_K)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \xi_K)_{(L^q, L^p)}\phi dt \\
&+ \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (h, \xi_K)\phi dt.
\end{aligned} \tag{3.35}$$

As $K \to \infty$ in the above equality, by (3.27) we find, for all $\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, that

$$\begin{aligned}
&(\tilde{v}, \xi)\phi(\tau+T) - (v_\tau, \xi)\phi(\tau) = \int_\tau^{\tau+T} (v, \xi)\phi' dt \\
&- \frac{C(n,\gamma)}{2} \int_\tau^{\tau+T} a(l(e^{\alpha z(\theta_t\omega)} v))(v, \xi)\phi dt - \lambda \int_\tau^{\tau+T} (v, \xi)\phi dt \\
&+ \alpha \int_\tau^{\tau+T} z(\theta_t\omega)(v, \xi)\phi dt + \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (f(t, \cdot, e^{\alpha z(\theta_t\omega)} v), \xi)_{(L^q, L^p)}\phi dt \\
&+ \int_\tau^{\tau+T} e^{-\alpha z(\theta_t\omega)} (h, \xi)\phi dt.
\end{aligned} \tag{3.36}$$

On the other hand, by (3.31) we find that the right hand side of equality (3.36) is given by

$$(v(\tau+T), \xi)\phi(\tau+T) - (v(\tau), \xi)\phi(\tau),$$

and therefore, we obtain

$$(v(\tau+T), \xi)\phi(\tau+T) - (v(\tau), \xi)\phi(\tau) = (\tilde{v}, \xi)\phi(\tau+T) - (v_\tau, \xi)\phi(\tau). \tag{3.37}$$

By choosing $\phi \in C^1([\tau, \tau + T])$ with $\phi(\tau) = 1$ and $\phi(\tau + T) = 0$, we obtain from (3.37), that for all $\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$(v(\tau), \xi) = (v_\tau, \xi). \tag{3.38}$$

Similarly, by choosing $\phi \in C^1([\tau, \tau + T])$ with $\phi(\tau) = 0$ and $\phi(\tau + T) = 1$, we infer from (3.37) that, for all $\xi \in H^\gamma(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$(v(\tau + T), \xi) = (\tilde{v}, \xi). \tag{3.39}$$

By (3.38)–(3.39), we have

$$v(\tau) = v_\tau \quad \text{and} \quad v(\tau + T) = \tilde{v} \quad \text{in} \quad L^2(\mathbb{R}^n), \tag{3.40}$$

which along with (3.20) implies that

$$v_k(\tau + T, \tau, \omega) \to v(\tau + T) \quad \text{weakly in} \quad L^2(\mathbb{R}^n). \tag{3.41}$$

Similar to (3.41), one can verify that, for every $t \geq \tau$, as $k \to \infty$

$$v_k(t, \tau, \omega) \to v(t) \quad \text{weakly in} \quad L^2(\mathbb{R}^n). \tag{3.42}$$

Note that (3.31) and (3.42) indicate that v is a solution of problem (3.2)–(3.4) in the sense of Definition 3.1, and (3.33) shows that v satisfies the energy equation (3.7).

Step 3. *Uniqueness and measurability of solutions*. Suppose v_1 and v_2 are solutions of (3.2)–(3.4). Then, for $\tilde{v} = v_1 - v_2$, we have

$$\begin{aligned} &\frac{d\tilde{v}}{dt} + \left(a(l(e^{\alpha z(\theta_t\omega)}v_1))(-\Delta)^\gamma v_1 - a(l(e^{\alpha z(\theta_t\omega)}v_2))(-\Delta)^\gamma v_2\right) + \lambda\tilde{v} \\ &= \alpha z(\theta_t\omega)\tilde{v} + e^{-\alpha z(\theta_t\omega)}\left(f(t, \cdot, e^{\alpha z(\theta_t\omega)}v_1) - f(t, \cdot, e^{\alpha z(\theta_t\omega)}v_2)\right), \end{aligned} \tag{3.43}$$

from which by (1.5) and the Lipschitz assumption imposed on function a, we find that for each $T > 0$, there exists $c_1 > 0$ such that, for all $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|\tilde{v}\|^2 \leq c_1\|\tilde{v}\|^2.$$

Then, the uniqueness and continuity of solutions for initial data in $L^2(\mathbb{R}^n)$ follow immediately.

Since the solution of problem (3.2)–(3.4) is unique, by (3.42), we see the whole sequence (not just a subsequence) $v_k(t, \tau, \omega, v_k(\tau)) \to v(t, \tau, \omega, v_\tau)$ weakly in $L^2(\mathbb{R}^n)$ for any $t \geq \tau$ and $\omega \in \Omega$. Because $v_k(t, \tau, \omega, v_k(\tau))$ is measurable in $\omega \in \Omega$ as provided in [26], we infer that the weak limit $v(t, \tau, \omega, v_\tau)$ is also measurable in ω, which completes the proof. □

Based on Theorem 3.2, we can define a continuous cocycle for problem (1.10). Note that if v is a solution of (3.2)-(3.4), then by (1.8) we see that u is a solution of (3.2)-(3.4), where u is given by

$$u(t, \tau, \omega, u_\tau) = e^{\alpha z(\theta_t\omega)}v(t, \tau, \omega, v_\tau),$$

with $u_\tau = e^{\alpha z(\theta_\tau\omega)}v_\tau$. Define a mapping $\Phi : \mathbb{R} \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $u_\tau \in L^2(\mathbb{R}^n)$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = e^{\alpha z(\theta_t\omega)}v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau), \tag{3.44}$$

where $v_\tau = e^{-\alpha z(\omega)}u_\tau$. It follows from Theorem 3.2 that Φ is a continuous cocycle in $L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. The main purpose of this paper is to prove the existence of attractors of Φ in $L^2(\mathbb{R}^n)$. To that end, we recall that a family of bounded nonempty subsets of $L^2(\mathbb{R}^n)$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, is tempered if for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{ct}\|D(\tau + t, \theta_t\omega)\| = 0,$$

where the notation $\|D\|$ for a subset D of $L^2(\mathbb{R}^n)$ is understood as $\|D\| = \sup_{u\in D} \|u\|$. The collection of all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^n)$ is denoted by $\mathcal{D}$, that is,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \ D \text{ is tempered in } L^2(\mathbb{R}^n)\}. \tag{3.45}$$

In this case, a $\mathcal{D}$-pullback attractor is also called a tempered attractor since $\mathcal{D}$ given by (3.45) contains all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^n)$.

From now on, we assume that, for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{0} e^{m\lambda s}\left(\|h(s + \tau, \cdot)\|^2 + \|\psi_1(s + \tau, \cdot)\|_{L^1(\mathbb{R}^n)}\right) ds < \infty. \tag{3.46}$$

When deriving the existence of tempered pullback absorbing sets, we will further assume that h and ψ_1 are tempered in the sense that, for every $c > 0$,

$$\lim_{r\to-\infty} e^{cr} \int_{-\infty}^{0} e^{\lambda s}\left(\|h(s + r, \cdot)\|^2 + \|\psi_1(s + r, \cdot)\|_{L^1(\mathbb{R}^n)}\right) ds = 0. \tag{3.47}$$

It is clear that (3.46)–(3.47) do not imply that h is bounded in $L^2(\mathbb{R}^n)$ when $t \to \infty$.

4. Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of the nonlocal fractional stochastic differential equations in $H^\gamma(\mathbb{R}^n)$ as well as the uniform estimates on the tails of solutions for large space and time variables. The estimates in $L^2(\mathbb{R}^n)$ are given below.

Lemma 4.1. *Under conditions* (1.3)-(1.5) *and* (3.46)*, for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that, for all $t \geq T$, the solution v of system* (1.10) *satisfies*

$$\begin{aligned} &\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \frac{1}{2}m \int_{-t}^{\sigma-\tau} \zeta(s)\|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2_{H^\gamma(\mathbb{R}^n)} ds \\ &+ 2\beta \int_{-t}^{\sigma-\tau} \zeta(s)e^{(p-2)\alpha z(\theta_s\omega)}\|v(s + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|_p^p ds \\ &\leq M_1 + M_1 \int_{-\infty}^{\sigma-\tau} \zeta(s)\left(h(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1(\mathbb{R}^n)}\right) ds, \end{aligned} \tag{4.1}$$

where $\zeta(s) = e^{\frac{5}{4}\lambda - 2\alpha\int_0^s z(\theta_r\omega)dr}$, $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, and M_1 is a positive constant independent of τ, ω and D.

Proof. The proof is similar to the case of bounded domains as in [27]. For the reader's convenience, we outline the main ideas here. First, by (1.2)-(1.3) and (3.7), we have

$$\begin{aligned}&\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2+mC(n,\gamma)\|v(t,\tau,\omega,v_\tau)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+2\lambda\|v(t,\tau,\omega,v_\tau)\|^2\\&+2\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v(t,\tau,\omega,v_\tau)|^p dx\le 2\alpha z(\theta_t\omega)\|v(t,\tau,\omega,v_\tau)\|^2\\&+2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}h(t)v(t,\tau,\omega,v_\tau)dx+2e^{-2\alpha z(\theta_t\omega)}\|\psi_1\|_{L^1(\mathbb{R}^n)}.\end{aligned}\tag{4.2}$$

Note that Young's inequality implies that

$$2e^{-\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}h(t)v(t,\tau,\omega,v_\tau)dx\le\frac{1}{4}\lambda\|v(t,\tau,\omega,v_\tau)\|^2+\frac{4}{\lambda}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2,\tag{4.3}$$

which, along with (4.2), yields

$$\begin{aligned}&\frac{d}{dt}\|v(t,\tau,\omega,v_\tau)\|^2+mC(n,\gamma)\|v(t,\tau,\omega,v_\tau)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+\frac{1}{2}\lambda\|v(t,\tau,\omega,v_\tau)\|^2\\&+(\frac{5}{4}\lambda-2\alpha z(\theta_t\omega))\|v(t,\tau,\omega,v_\tau)\|^2+2\beta e^{(p-2)\alpha z(\theta_t\omega)}\int_{\mathbb{R}^n}|v(t,\tau,\omega,v_\tau)|^p dx\\&\le\frac{4}{\lambda}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2+2e^{-2\alpha z(\theta_t\omega)}\|\psi_1\|_{L^1(\mathbb{R}^n)}.\end{aligned}\tag{4.4}$$

Solving (4.4) for $\|v\|^2$ on the interval $(\tau-t,\sigma)$ by introducing the integrating factor $e^{\frac{5}{4}\lambda-2\alpha\int_0^t z(\theta_r\omega)dr}$, and the replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+\frac{\lambda}{2}\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2ds\\&+2\beta\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{(p-2)\alpha z(\theta_{s-\tau}\omega)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_p^p ds\\&+mC(n,\gamma)\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}ds\\&\le e^{\frac{5}{4}\lambda(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t}z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t}\|^2+\frac{4}{\lambda}\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|h(s)\|^2ds\\&+2\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}\|\psi_1(s)\|_{L^1(\mathbb{R}^n)}ds.\end{aligned}\tag{4.5}$$

Since $e^{\alpha z(\theta_{-t}\omega)}v_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ with $D\in\mathcal{D}$, by (3.2) one can verify that

$$\lim_{t\to\infty}e^{\frac{5}{4}\lambda(\tau-t-\sigma)-2\alpha\int_\sigma^{\tau-t}z(\theta_{r-\tau}\omega)dr}\|v_{\tau-t}\|^2=0.\tag{4.6}$$

On the other hand, by (3.46) and (3.2) we find that, for all $\sigma\ge\tau-t$,

$$\begin{aligned}&\int_{\tau-t}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|h(s)\|^2ds\\&\le\int_{-\infty}^{\sigma}e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr}e^{-2\alpha z(\theta_{s-\tau}\omega)}\|h(s)\|^2ds<\infty,\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}&\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)} \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds \\ \le &\int_{\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)} \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds < \infty.\end{aligned} \tag{4.8}$$

It follows from (4.7)–(4.8) that there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that, for all $t \ge T$,

$$\begin{aligned}&\|v(\sigma, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \frac{\lambda}{2}\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} \|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\ &+ 2\beta \int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{(p-2)\alpha z(\theta_{s-\tau}\omega)} \|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|_p^p ds \\ &+ mC(n,\gamma)\int_{\tau-t}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} \|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t}\|_{\dot{H}^\gamma(\mathbb{R}^n)}^2 ds \\ &\le 1 + \frac{4}{\lambda}\int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} e^{-2\alpha z(\theta_{s-\tau}\omega)} \|h(s)\|^2 ds \\ &+ 2\int_{-\infty}^{\sigma} e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha\int_\sigma^s z(\theta_{r-\tau}\omega)dr} \|\psi_1(s)\|_{L^1(\mathbb{R}^n)} ds.\end{aligned}$$

After performing a change of variable, the desired estimates follow from the above inequality immediately; for more details, see [27, Lemma 4.1]. □

After a consequence of Lemma 4.1, we see that problem (1.10) has a tempered pullback absorbing set in $L^2(\mathbb{R}^n)$.

Corollary 4.2. *Under conditions* (1.3)–(1.5) *and* (3.47)*, for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D) > 0$ *such that the solution* v *of* (1.10) *with* $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ *satisfies, for all* $t \ge T$,

$$v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t}) \in B(\tau, \omega), \tag{4.9}$$

where $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ *is given by,*

$$B(\tau, \omega) = \{v \in L^2(\mathbb{R}^n) : \|v\|^2 \le R(\tau, \omega)\},$$

with $R = R(\tau, \omega)$ *being a positive number given by,*

$$R = M_1 + M_1 \int_{-\infty}^{0} e^{\frac{5}{4}\lambda s-2\alpha\int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)} \left(\|h(s+\tau)\|^2 + \|\psi_1(s+\tau)\|^2\right) ds. \tag{4.10}$$

Moreover, $R = \{R(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ *is tempered in the sense that, for any* $c > 0$,

$$\lim_{t\to\infty} e^{-ct} R(\tau - t, \theta_{-t}\omega) = 0. \tag{4.11}$$

Proof. (4.9) follows from Lemma 4.1 if we take $\sigma = \tau$, and the convergence of (4.11) can be proved in the same way as in the case of bounded domains, which can be found in [14, 27, 29]. The details are omitted here. □

Next, we derive uniform estimates of solutions in $H^{\gamma}(\mathbb{R}^n)$ for which we further assume that the function ψ_4 in (1.5) belongs to $L^{\infty}(\mathbb{R}; L^{\infty}(\mathbb{R}^n))$, and the nonlinearity f satisfies, for all t, $u \in \mathbb{R}$, x and $y \in \mathbb{R}^n$,

$$|f(t, x, u) - f(t, y, u)| \leq |\psi_5(x) - \psi_5(y)|, \tag{4.12}$$

where $\psi_5 \in H^{\gamma}(\mathbb{R}^n)$.

Lemma 4.3. *Under conditions* (1.3)–(1.5), (4.12)*, and* (3.46)*, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that, for any $t \geq T$, the solution v of problem* (1.1) *with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ satisfies*

$$\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2_{H^{\gamma}(\mathbb{R}^n)} \\
&\leq M_2 + M_2 \int_{-\infty}^{0} e^{\frac{5}{4}\lambda s - 2\alpha \int_0^s z(\theta_r \omega) dr} e^{-2\alpha z(\theta_s \omega)} \left(1 + \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1(\mathbb{R}^n)}\right) ds,
\end{aligned}$$

where M_2 is a positive constant independent of τ, ω, and D.

Proof. Multiplying (1.10) by $(-\Delta)^{\gamma} v$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^{\frac{\gamma}{2}} v\|^2 + 2a(l(e^{\alpha z(\theta_t \omega)} v))\|(-\Delta)^{\gamma} v\|^2 + 2(\lambda - \alpha z(\theta_t \omega))\|(-\Delta)^{\frac{\gamma}{2}} v\|^2 \\
&= 2e^{-\alpha z(\theta_t \omega)} \left(f(t, x, e^{\alpha z(\theta_t \omega)} v), (-\Delta)^{\gamma} v\right) + 2e^{-\alpha z(\theta_t \omega)} \left(h(t), (-\Delta)^{\gamma} v\right).
\end{aligned} \tag{4.13}$$

We now estimate the right-hand side of (4.13). For the first term, by (1.5) and (4.11) we have $C_{n\gamma} = C(n, \gamma)$, and

$$\begin{aligned}
&2e^{-\alpha z(\theta_t \omega)} \left(f(t, x, e^{\alpha z(\theta_t \omega)} v), (-\Delta)^{\gamma} v\right) = 2e^{-\alpha z(\theta_t \omega)} \left((-\Delta)^{\frac{\gamma}{2}} f(t, x, e^{\alpha z(\theta_t \omega)} v), (-\Delta)^{\frac{\gamma}{2}} v\right) \\
&= C_{n\gamma} e^{-\alpha z(\theta_t \omega)} \left(f(t, \cdot, e^{\alpha z(\theta_t \omega)} v), v\right)_{\dot{H}^{\gamma}(\mathbb{R}^n)} \\
&= C_{n,\gamma} e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(f(t, x, e^{\alpha z(\theta_t \omega)} v(x)) - f(t, y, e^{\alpha z(\theta_t \omega)} v(y))\right) B(x, y) dx dy \\
&= C_{n,\gamma} e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(f(t, x, e^{\alpha z(\theta_t \omega)} v(x)) - f(t, y, e^{\alpha z(\theta_t \omega)} v(x))\right) B(x, y) dx dy \\
&\quad + C_{n,\gamma} e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(f(t, y, e^{\alpha z(\theta_t \omega)} v(x)) - f(t, y, e^{\alpha z(\theta_t \omega)} v(y))\right) B(x, y) dx dy \\
&\leq C_{n,\gamma} e^{-\alpha z(\theta_t \omega)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\psi_5(x) - \psi_5(y)||v(x) - v(y)|}{|x - y|^{n+2\gamma}} dx dy \\
&\quad + C_{n\gamma} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\psi_4(t, y)(v(x) - v(y))^2}{|x - y|^{n+2\gamma}} dx dy \\
&\leq C_{n\gamma} e^{-\alpha z(\theta_t \omega)} \|\psi_5\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} \|v\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + C_{n,\gamma} \|\psi_4\|_{L^{\infty}(\mathbb{R}, L^{\infty}(\mathbb{R}^n))} \|v\|^2_{\dot{H}^{\gamma}(\mathbb{R}^n)} \\
&\leq \frac{1}{2m} C_{n\gamma} e^{-2\alpha z(\theta_t \omega)} \|\psi_5\|^2_{H^{\gamma}(\mathbb{R}^n)} + \left(\frac{m}{2} + \|\psi_4\|_{L^{\infty}(\mathbb{R}, L^{\infty}(\mathbb{R}^n))}\right) C_{n\gamma} \|v\|^2_{\dot{H}^{\gamma}(\mathbb{R}^n)} \\
&\leq \frac{1}{2m} C_{n\gamma} e^{-2\alpha z(\theta_t \omega)} \|\psi_5\|^2_{H^{\gamma}(\mathbb{R}^n)} + (m + 2\|\psi_4\|_{L^{\infty}(\mathbb{R}, L^{\infty}(\mathbb{R}^n))}) \|(-\Delta)^{\frac{\gamma}{2}} v\|^2,
\end{aligned} \tag{4.14}$$

where $B(x,y)=\frac{v(x)-v(y)}{|x-y|^{n+2\gamma}}$. For the last term on the right-hand side of (4.13), we have

$$2\left|e^{-\alpha z(\theta_t\omega)}(h(t),(-\Delta)^{\gamma}v)\right|\leq\frac{m}{2}\|(-\Delta)^{\gamma}v\|^2+\frac{2}{m}e^{-2\alpha z(\theta_t\omega)}\|h(t)\|^2. \tag{4.15}$$

It follows from (4.13)–(4.15) and (1.2) that,

$$\begin{aligned}&\frac{d}{dt}\|(-\Delta)^{\frac{\gamma}{2}}v\|^2+m\|(-\Delta)^{\gamma}v\|^2+2(\lambda-\alpha z(\theta_t\omega))\|(-\Delta)^{\frac{\gamma}{2}}v\|^2\\&\leq c_1\|(-\Delta)^{\frac{\gamma}{2}}v\|^2+\left(\frac{2}{m}\|h(t)\|^2+c_2\right)e^{-2\alpha z(\theta_t\omega)}.\end{aligned} \tag{4.16}$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$, let $s\in(\tau-1,\tau)$. Multiplying (4.16) by $e^{\int_0^t(\frac{5}{4}\lambda-2\alpha z(\theta_s\omega))ds}$ and integrating over (s,τ), we infer that

$$\begin{aligned}\|(-\Delta)^{\frac{\gamma}{2}}v(\tau,\tau-t,\omega,v_{\tau-t})\|^2&\leq e^{\int_\tau^s(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{\gamma}{2}}v(s,\tau-t,\omega,v_{\tau-t})\|^2\\&+c_1\int_s^\tau e^{\int_\tau^\zeta(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{\gamma}{2}}v(\zeta,\tau-t,\omega,v_{\tau-t})\|^2d\zeta\\&+\int_s^\tau e^{\int_\tau^\zeta(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\left(\frac{2}{m}\|h(\zeta)\|^2+c_2\right)e^{-2\alpha z(\theta_\zeta\omega)}d\zeta.\end{aligned} \tag{4.17}$$

Integrating again with respect to s on $(\tau-1,\tau)$, we obtain

$$\begin{aligned}&\|(-\Delta)^{\frac{\gamma}{2}}v(\tau,\tau-t,\omega,v_{\tau-t})\|^2\\&\leq\int_{\tau-1}^\tau e^{\int_\tau^s(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{\gamma}{2}}v(s,\tau-t,\omega,v_{\tau-t})\|^2ds\\&+c_1\int_{\tau-1}^\tau e^{\int_\tau^\zeta(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\|(-\Delta)^{\frac{\gamma}{2}}v(\zeta,\tau-t,\omega,v_{\tau-t})\|^2d\zeta\\&+\int_{\tau-1}^\tau e^{\int_\tau^\zeta(\frac{5}{4}\lambda-2\alpha z(\theta_\xi\omega))d\xi}\left(\frac{2}{m}\|h(\zeta)\|^2+c_2\right)e^{-2\alpha z(\theta_\zeta\omega)}d\zeta.\end{aligned} \tag{4.18}$$

Substituting $\theta_{-\tau}\omega$ for ω, we deduce from (4.18) that

$$
\begin{aligned}
&\|(-\Delta)^{\frac{\gamma}{2}} v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \\
\leq& \int_{\tau-1}^{\tau} e^{\int_\tau^s (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi-\tau}\omega))d\xi} \|(-\Delta)^{\frac{\gamma}{2}} v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\
&+ c_1 \int_{\tau-1}^{\tau} e^{\int_\tau^\zeta (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi-\tau}\omega))d\xi} \|(-\Delta)^{\frac{\gamma}{2}} v(\zeta, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 d\zeta \\
&+ \int_{\tau-1}^{\tau} e^{\int_\tau^\zeta (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi-\tau}\omega))d\xi} \left(\frac{2}{m}\|h(\zeta)\|^2 + c_2\right) e^{-2\alpha z(\theta_{\zeta-\tau}\omega)} d\zeta \\
\leq& \int_{-1}^{0} e^{\int_0^s (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi}\omega))d\xi} \|(-\Delta)^{\frac{\gamma}{2}} v(s+\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\
&+ c_1 \int_{\tau-1}^{\tau} e^{\int_0^\zeta (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi}\omega))d\xi} \|(-\Delta)^{\frac{\gamma}{2}} v(\zeta+\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 d\zeta \\
&+ \int_{\tau-1}^{\tau} e^{\int_0^\zeta (\frac{5}{4}\lambda - 2\alpha z(\theta_{\xi}\omega))d\xi} \left(\frac{2}{m}\|h(\zeta+\tau)\|^2 + c_2\right) e^{-2\alpha z(\theta_{\zeta}\omega)} d\zeta,
\end{aligned}
\tag{4.19}
$$

which, along with Lemma 4.1 for $\sigma = \tau$, implies the desired estimates. □

To prove the pullback asymptotic compactness of the cocycle associated with problem (1.10) on the unbounded domain $\mathbb{R}^n$, we need to derive the uniform estimate on the tail parts of the solutions for large space variables when the time is large enough.

Lemma 4.4. *Suppose the conditions of Lemma 4.1 hold. Then, for every $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \varepsilon, \alpha) > 0$, $K = K(\tau, \omega, \varepsilon) \leq 1$ such that, for all $t \geq T$ and $k \geq K$, the solution v of problem* (1.10) *with $e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ satisfies*

$$\int_{|x|\geq k} |v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})(x)|^2 dx \leq \varepsilon.$$

Proof. The proof of this lemma follows that of Lemma 4.4 in [19], so we omit the details here. □

5. Existence of random attractors

In this section, we prove the existence and uniqueness of tempered pullback attractors for the non-local fractional stochastic differential equation (1.1). To that end, we need to establish the existence of tempered random absorbing sets and the pullback asymptotic compactness of the cocycle Φ.

Lemma 5.1. *Under conditions* (1.2)-(1.5) *and* (3.46)*, the cocyle Φ has a closed measurable pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the set $K(\tau, \omega)$ is defined by*

$$K(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq e^{2\alpha z(\omega)} R(\tau, \omega)\},$$

where $R(\tau, \omega)$ is the same as in (4.10).

Proof. First, by (3.2) and (4.11), we see that $K \in \mathcal{D}$, that is, for every $c > 0$,

$$\lim_{t\to\infty} e^{-ct} \|K(\tau - t, \theta_{-t}\omega)\| = 0.$$

On the other hand, by (1.8), we have

$$u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) = e^{\alpha z(\omega)} v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \quad \text{with} \quad u_{\tau-t} = e^{\alpha z(\theta_{-t}\omega)} v_{\tau-t}. \tag{5.1}$$

Then, it follows from Corollary 4.2 that, for every $D \in \mathcal{D}$ and $u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$, there exists $T = T(\tau,\omega,D,\alpha) > 0$ such that for all $t \geq T$,

$$v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) \in B(\tau,\omega), \tag{5.2}$$

where $B(\tau,\omega)$ is the same set as in (4.9). By (5.1)-(5.2), we find that, for all $t \geq T$,

$$u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) \in K(\tau,\omega),$$

which along with (3.44) implies that, for all $t \geq T$,

$$\Phi(t,\tau-t,\theta_{-t}\omega,u_{\tau-t}) \in K(\tau,\omega).$$

This shows that K is a $\mathcal{D}$-pullback absorbing set of Φ. It is clear that $R(\tau,\omega)$ is measurable in $\omega \in \Omega$, which implies the measurability of $K(\tau,\omega)$ in $\omega \in \Omega$. □

The uniform estimates of the solutions of problem (1.1) in $H^\gamma(\mathbb{R}^n)$ is given below.

Lemma 5.2. *Assume the conditions of Lemma 4.3 hold. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ there exists $T = T(\tau,\omega,D,\alpha) > 0$ such that, for any $t \geq T$, the solution u of problem* (1.1) *with $u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ satisfies*

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^\gamma(\mathbb{R}^n)}\\ \leq\ & M_3 + M_3\int_{-\infty}^0 e^{\frac{5}{4}\lambda-2\alpha\int_0^s z(\theta_r\omega)dr} e^{-2\alpha z(\theta_s\omega)}\left(1+\|h(s+\tau)\|^2+\|\psi_1(s+\tau)\|_{L^1(\mathbb{R}^n)}\right)ds,\end{aligned}$$

where $M_3 = M_2 e^{2\alpha z(\omega)}$ and M_2 is the same positive constant as in Lemma 4.3.

Proof. This estimate follows from (5.1) and Lemma 4.3 immediately. □

Based on Lemma 4.4, one can derive the uniform estimates on the tails of solutions of problem (1.1) as stated below.

Lemma 5.3. *Suppose the conditions of Lemma 4.3 are true. Then, for every $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ there exists $T = T(\tau,\omega,D,\varepsilon,\alpha) > 0$, $K = K(\tau,\omega,\varepsilon) \geq 1$ such that, for all $t \geq T$ and $k \geq K$, the solution u of problem* (1.1) *with $u_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ satisfies*

$$\int_{|x|\geq k} |u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})(x)|^2 dx \leq \varepsilon.$$

Proof. This is an immediate consequence of Lemma 4.4 together with the arguments of the proof of Lemma 5.1. The details are omitted here. □

The next lemma is concerned with the $\mathcal{D}$-pullback asymptotic compactness of Φ.

Lemma 5.4. *Under conditions* (1.2)–(1.5), (4.12), *and* (3.47), *for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,u_{0,n})$ has a convergence subsequence in $L^2(\mathbb{R}^n)$ whenever $t_n \to \infty$ and $u_{0,n} \in D(\tau-t_n,\theta_{-t_n}\omega)$.*

Proof. By (3.44), we have

$$\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}),$$

which, along with Lemma 5.3, shows that for every $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, there exist $K = K(\tau, \omega, \varepsilon) \geq 1$ and $N_1 = N_1(\tau, \omega, D, \varepsilon) \geq 1$ such that, for all $n \geq N_1$,

$$\|\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})\|_{L^2(|x|\geq K)} \leq \frac{\varepsilon}{2}. \tag{5.3}$$

By Lemma 5.2, we find there exists $N_2 = N_2(\tau, \omega, D, \varepsilon) \geq N_1$ such that, for all $n \geq N_2$,

$$\|\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0n})\|_{H^\gamma(|x|<K)} \leq L(\tau, \omega),$$

where $L(\tau, \omega)$ is a positive constant. Since $\gamma \in (0, 1)$, the embedding $H^\gamma(|x| < K) \hookrightarrow L^2(|x| < K)$ is compact, which together with (5.3) implies that $\{\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n}\}_{n=1}^{\infty}$ has a finite covering in $L^2(\mathbb{R}^n)$ of balls of radii less than ε. As a consequence, we infer that the sequence $\{\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ is precompact in $L^2(\mathbb{R}^n)$. □

We now present our main result of this paper as follows.

Theorem 5.5. *Suppose* (1.2)–(1.5), (4.12)*, and* (3.47) *hold. Then, the cocycle* Φ *of problem* (1.1) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* $L^2(\mathbb{R}^n)$*.*

Proof. This is an immediate consequence of Lemmas 5.1 and 5.4 and Proposition 2.1. □

6. Conclusions and future work

We studied the existence and uniqueness of weak solutions, as well as the existence and uniqueness of random attractors, to a kind of nonlocal fractional stochastic reaction-diffusion equations in $\mathbb{R}^n$ by doing an appropriate change of variables. The limitation of this method is obvious: it is only helpful when the noise is additive or (linear) multiplicative. Therefore, in the next step, it is worth analyzing this model, but driven by a more general nonlinear noise. In this case, the method used in this paper fails. Instead, it is necessary to find another technique to establish the results, for example, the Wong-Zakai approximation can be a good option to handle this problem or the theory of mean weak random attractors.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Tomás Caraballo is an editorial board member for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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