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Research article

A complete classification of weakly Dedekind groups

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Abstract: A finite group is called a weakly Dedekind group if all its noncyclic subgroups are normal. In this paper, we determine the complete classification of weakly Dedekind groups.

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1. Introduction

The groups involved in this paper are always finite. We denote by $\Omega_1(G)$ the subgroup of G generated by its elements of order p for a fixed prime p, $\pi(G)$ denotes the set of all prime divisors of $|G|$, C_n denotes a cyclic group of order n, and

$$\mho(G) = \langle a^p \mid a \in G \rangle.$$

All unexplained notations and terminologies are standard as in [1].

The normality of subgroups plays an important role in group theory. An important topic in group theory is to investigate the groups in which certain subgroups are assumed to be normal. There are many remarkable examples about this topic, and the so-called Dedekind group is one typical result. A group G is called a Dedekind group if every subgroup of G is normal in G. The structure of Dedekind groups has been completely determined by Dedekind and Baer (see [1, Theorem 5.3.7]). A group G is Dedekind if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group, and an abelian group with all its elements of odd order. Subsequently, many authors dealt with the generalization of Dedekind groups. Here, we mention some of them. Pic [2] considered groups in which every subgroup S is quasinormal, that is, S satisfies $SH = HS$ for all subgroups H of G. Buckley et al. [3] dealt with groups in which every subgroup has at most two conjugacy classes. Bozikov and Janko [4] gave a complete classification of p-groups, all of whose noncyclic subgroups are normal. The research [5] classified p-groups whose non-normal abelian subgroups are cyclic. Brandl [6] and Han et al. [7] classified groups in which all non-normal subgroups are conjugate. The further results can be found in [8–13].

One of the aims of this paper is to classify p-groups whose noncyclic subgroups are normal completely. Our method of proof is elementary, which is different from a result in [4, Theorem 1.1]. The other aim is to give a complete classification of non-nilpotent groups whose noncyclic subgroups are normal.

Definition 1.1. *A group is called a weakly Dedekind group if all noncyclic subgroups are normal.*

It is clear that the class of weakly Dedekind groups is closed under taking subgroups and quotient groups.

Remark 1.2. *A weakly Dedekind group G has a normal Sylow subgroup.*

Proof. Let p be the smallest prime dividing $|G|$. If a Sylow p-subgroup P is normal in G, then we are done. If P is cyclic, then P has a normal p-complement N in G. By induction, N has a normal Sylow subgroup Q, so Q is normal in G. □

Since the structure of Dedekind groups is well known, we discuss weakly Dedekind groups, which are not Dedekind.

Remark 1.3. *Let π be a set of primes. A direct product of a π-group and a π'-group is weakly Dedekind, but not Dedekind if and only if one factor is weakly Dedekind but not Dedekind, and the other is cyclic.*

Proof. Let $G = M \times N$ be a weakly Dedekind but not Dedekind group with M a π-group and N a π'-group. Then G has a non-normal subgroup $H \times K$, where $H \leq M$ and $K \leq N$. Hence, $H \ntrianglelefteq M$ or $K \ntrianglelefteq N$. Without loss of generality, let $H \ntrianglelefteq M$, then $H \times N \ntrianglelefteq G$ and $H \times N$ is cyclic by hypothesis. Thus, N is cyclic, so G has the required form.

The converse is clear. □

Definition 1.4. *A weakly Dedekind group that is not Dedekind and cannot be expressed as a proper direct product of a π-group and a π'-group is called a weakly primitive Dedekind group.*

Finally, we give the complete classification of weakly primitive Dedekind groups. The specific results are as shown below.

Theorem A. *Let G be a p-group. Then G is weakly primitive Dedekind if and only if G is isomorphic to one of the following groups:*

(1) *the quaternion group Q_{16} of order 16;*

(2) $G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, [a, b] = a^2, [a, c] = [b, c] = 1, \langle c\rangle \cap \langle a, b\rangle \leq \langle a^2\rangle, |c| > 2\rangle$;

(3) $G = \langle a, b, c, d \mid a^4 = d^2 = 1, c^2 = b^2 = a^2, [a, b] = [c, d] = a^2, [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle$;

(4) $G = \langle a, b \mid a^{p^m} = 1, b^{p^n} = 1, [b, a] = a^{p^{m-1}}\rangle$, *where $m, n \geq 2$;*

(5) $G = \langle a, b \mid a^p = b^{p^n} = 1, [a, b] = c, [a, c] = [b, c] = c^p = 1\rangle$, *where* $c = b^{p^{n-1}}$ *if* $n \neq 1$*;*

(6) $G = \langle a, b \mid a^9 = 1, b^3 = a^3, [a, b] = c, c^3 = 1, [a, c] = 1, [b, c] = a^6\rangle$*;*

(7) $G = \langle a, b, c \mid a^p = b^{p^n} = c^p = 1, [a, b] = 1\rangle$, *where* $\langle[a, c]\rangle = \langle b^{p^{n-1}}\rangle, [b, c] \in \langle b^{p^{n-1}}\rangle$ *and* $n \geq 2$*;*

(8) $G = \langle a, b \mid a^8 = 1, a^4 = b^4, [a, b] = a^2$ *or* $a^6\rangle$.

Theorem B. *Let G be a group with* $|\pi(G)| > 1$*. Then G is weakly primitive Dedekind if and only if G is isomorphic to one of the following groups:*

(1) $G = \langle a, b \mid a^p = b^r = 1, [a, b] = a^{t+1}\rangle$, *where*

$$t^r \equiv 1(\mathrm{mod}\ p), \ \ r = \prod_{i=1}^{k} q_i^{n_i}, \ \ t^{\frac{r}{q_i^{n_i}}} \not\equiv 1(\mathrm{mod}\ p)$$

for any $1 \leq i \leq n$*;*

(2) $G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, [a, b] = a^2, c^{3^n} = 1, a^c = b, b^c = ab$ *or* $ba\rangle$*;*

(3) $G = \langle a, b, c \mid a^{p^m} = b^p = c^r = 1, a^b = a^c = a, b^c = b^t\rangle$, *where*

$$t^r \equiv 1(\mathrm{mod}\ p), \ \ r = \prod_{i=1}^{k} q_i^{n_i}, \ \ t^{\frac{r}{q_i^{n_i}}} \not\equiv 1(\mathrm{mod}\ p)$$

for any $1 \leq i \leq n$*;*

(4) $G = \langle a, b, c \mid a^p = b^p = c^r = 1, [a, b] = 1, a^c = b, b^c = a^m b^n\rangle$, *where p is a prime. Write*

$$M = \begin{bmatrix} 0 & 1 \\ m & n \end{bmatrix}$$

and

$$r = \prod_{i=1}^{k} q_i^{n_i},$$

then

$$M^r \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{mod}\ p),$$

where M^k *has no eigenvalues (over the field of p elements* F_p*) or* M^k *(taken modulo p) is the identity matrix for any* $k \mid r$*, and* $M^{\frac{r}{q_i^{t_i}}}$ *has no eigenvalues for any* $1 \leq i \leq n$*.*

2. Some preliminary results

We collect some lemmas, which will be frequently used in the sequel.

Lemma 2.1. *Let G be a group with* $|\pi(G)| > 1$*. If G is weakly primitive Dedekind, then* $G = P \rtimes C$*, where C is cyclic and* $q \mid |C/C_C(P)|$ *for any prime divisor q of* $|C|$*.*

Proof. By Remark 1.2, there exists a normal Sylow subgroup P of G. Hence, P has a complement C since G is solvable. By hypothesis, C is not normal in G, so it is cyclic.

Let Q be a Sylow q-subgroup of C for a prime q. If $C = Q$, the proof is complete. If $C \neq Q$, then there exists a Hall subgroup D of C such that $C = D \times Q$. It is clear that Q acts on P nontrivially. Otherwise, $G = PD \times Q$, a contradiction. Hence, $q \mid |C/C_C(P)|$. □

Lemma 2.2. *Let G be a 2-group and $|\Omega_1(Z(G))| \neq 2$. If G is weakly primitive Dedekind, then $G/\Omega_1(G)$ has no subgroup isomorphic to the quaternion group Q_8 of order 8.*

Proof. Assume that there is a quaternion group $K\Omega_1(G)/\Omega_1(G)$ of order 8 contained in $G/\Omega_1(G)$. By hypothesis, $\Omega_1(G)$ is an elementary abelian subgroup of G, which is contained in $Z(G)$. Hence,

$$K \cap \Omega_1(G) = \Omega_1(K).$$

Writing $\Omega = \Omega_1(K)$, then

$$K\Omega_1(G)/\Omega_1(G) \cong K/K \cap \Omega_1(G) = K/\Omega$$

is a quaternion group of order 8. Choosing a minimal generating system $a\Omega, b\Omega$ of K/Ω, then

$$a^4\Omega = \Omega, \;\; b^2\Omega = a^2\Omega \;\text{ and }\; [a,b]\Omega = a^2\Omega.$$

Noting that Ω is an elementary abelian subgroup, which is contained in $Z(K)$, then a, b are all of order 8, $a^2, b^2, [a,b]$ are contained in $Z(K)$, and $[a,b]^2 = a^4$. Thus,

$$1 = [a, b^2] = [a,b]^2 = a^4,$$

a contradiction. □

Lemma 2.3. *Let G be a p-group. Suppose that G does not contain the quaternion group Q_8 and $Z(G)$ is noncyclic. If G is weakly primitive Dedekind, then $\Omega_1(G)$ is an elementary abelian subgroup of type (p, p) and*

$$G' \leq \Omega_1(G) \leq \Phi(G) \leq Z(G).$$

Proof. Since

$$|\Omega_1(Z(G))| \neq p,$$

and all noncyclic subgroups of G are normal, we have that $\Omega_1(G)$ is an elementary abelian subgroup of G which is contained in $Z(G)$. Hence,

$$\langle a\rangle\Omega_1(G) \trianglelefteq G$$

for any element a of G, so $G/\Omega_1(G)$ is a Dedekind group. By Lemma 2.2, $G/\Omega_1(G)$ is abelian. Therefore, $[a,b] \in \Omega_1(G)$, and the order of $[a,b]$ is p for any two elements a, b of G, which implies $G' \leq Z(G)$. Since the order of $[a,b]$ is p, we have

$$[a^p, b] = [a,b]^p = 1.$$

Thus, we get $a^p \in Z(G)$. Furthermore,

$$G' \leq \Omega_1(G) \leq Z(G) \;\text{ and }\; \Phi(G) \leq Z(G)$$

by the argument above.

If $\Omega_1(G)$ is not an elementary abelian subgroup of type (p, p), then there are three different subgroups, A_1, A_2, A_3, of order p in $\Omega_1(G)$ such that

$$A_1A_2A_3 = A_1 \times A_2 \times A_3.$$

Thus, for any cyclic subgroup A of G, we have

$$A = AA_1 \cap AA_2 \cap AA_3 \trianglelefteq G,$$

a contradiction, which implies that $\Omega_1(G)$ is of type (p, p).

If $\Omega_1(G) \not\leq \Phi(G)$, then $G = H \times B$, where B is of order p. By hypothesis again, H is cyclic, which implies that G is abelian, a contradiction. □

Lemma 2.4. *Let $G = P \rtimes C$ with $(|P|, |C|) = 1$. If G is weakly primitive Dedekind, then*

(1) $\Phi(P)$ *is cyclic and C acts on* $\Phi(P)$ *trivially;*

(2) *The number of elements of any minimal generating system of P is less than or equal to* 2.

Proof. (1) Suppose that $\Phi(P)$ is noncyclic or C acts on $\Phi(P)$ as nontrivial. We have $\Phi(P)C \trianglelefteq G$ and hence, $\Phi(P)$ must act on $\{C^g \mid g \in G\}$ transitively. By Frattini's argument,

$$P = \Phi(P)N_P(C) = N_P(C),$$

which implies $C \trianglelefteq G$, so $G = P \times C$, a contradiction.

(2) Assume that the number of elements of any minimal generating system of Q is more than 2. We choose a proper generating system $\{b_1, \cdots, b_k, a_1, \cdots, a_l\}$ of P such that $b_i \in N_P(C)$, but $a_j \notin N_P(C)$ and the value of k are to be the maximum. By hypothesis, $l \geq 1$. Let

$$K = \langle b_1, \cdots, b_k, a_2, \cdots, a_l \rangle.$$

Clearly, K is noncyclic. Moreover, $KC \trianglelefteq G$ and K acts on $\{C^g \mid g \in G\}$ transitively. Therefore, $P = KN_P(C)$, and there exists $b \in N_P(C)$ such that

$$P = \langle b_1, \cdots, b_k, b, a_2, \cdots, a_l \rangle,$$

which contradicts the maximality of k. □

3. Some relevant results and proofs

In this section, we give some relevant results of proofs of Theorems A and B.

Theorem 3.1. *Suppose that G is a* 2*-group containing a quaternion group of order* 16*. Then G is weakly primitive Dedekind if and only if G is isomorphic to Q_{16}.*

Proof. Let

$$H = \langle a, b \mid a^8 = 1, b^2 = a^4, a^b = a^{-1} \rangle$$

be a quaternion group of order 16 contained in G. Obviously $\langle b \rangle \ntrianglelefteq G$. Since all noncyclic subgroups of G are normal, we have $H \trianglelefteq G$.

Now, we claim $|\Omega_1(G)| = 2$. Suppose $|\Omega_1(G)| \neq 2$ and let $c \notin H$ be an element of order 2, then $\langle b^2, c \rangle \trianglelefteq G$, so $c^b = c$ or b^2c, which implies

$$\langle b \rangle = H \cap \langle b, c \rangle \trianglelefteq G,$$

a contradiction. Therefore, $|\Omega_1(G)| = 2$ and G is a quaternion group.

If $|G| \geq 32$, then we can assume

$$G = \langle c, b \mid c^{16} = 1, b^2 = c^8, c^b = c^{-1} \rangle.$$

So

$$b^c = bc^2 \notin \langle b, c^4 \rangle \text{ and } \langle b, c^4 \rangle \ntrianglelefteq G,$$

a contradiction. Thus, G is isomorphic to Q_{16}.

The converse is clear. □

Theorem 3.2. *Let G be a 2-group containing a quaternion group K of order 8 but not 16, and G/K be cyclic. Then G is weakly primitive Dedekind if and only if*

$$G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, [a, b] = a^2, [a, c] = [b, c] = 1 \rangle,$$

where

$$\langle c \rangle \cap \langle a, b \rangle \leq \langle a^2 \rangle \textit{ and } |c| > 2.$$

Proof. Let

$$G = \langle a, b, c \rangle \text{ and } K = \langle a, b \mid a^4 = 1, b^2 = a^2, [a, b] = a^2 \rangle,$$

where

$$c^{2^m} = 1 \text{ and } H = \langle c \rangle \cap K.$$

We first claim $|H| \neq 4$. If $|H| = 4$, we write $H = \langle a \rangle$ and let $u \in \langle c \rangle$ be an element of order 8, then $u^2 = a$. Since K is noncyclic, we have $K \trianglelefteq G$ and hence, $[b, u] \in K$. Let $[b, u] = w$. If $|w| = 2$, then $w = b^2$, $b^u = b^3$, and $b^{u^2} = b$, contrary to $u^2 = a$. If $|w| = 4$ and $\langle w \rangle \neq \langle u^2 \rangle$, then $\langle u^2, w \rangle = K$ and hence, $\langle c, w \rangle = G$, contrary to $G' \leq \Phi(G)$. Assume $|w| = 4$ and $\langle w \rangle = \langle u^2 \rangle$. If $w = u^2$, then $\langle u, b \rangle$ is a quaternion group of order 16, a contradiction. If $w = u^6$, then $\langle ub, b^2 \rangle \ntrianglelefteq G$, contrary to the hypothesis. Therefore, we get $|H| \leq 2$.

We next claim that every subgroup of K is normal in G. If not, the generator c induces an automorphism of order 2 on K, then we can choose proper generators a, b of K such that $a^c = b, b^c = a$. Thus, $\langle c^2, a \rangle \ntrianglelefteq G$ and $\langle c^2, a \rangle$ is cyclic. Since $|H| \leq 2$, we have $c^2 = 1$ or $c^2 = a^2$. If $c^2 = 1$, then

$$G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, [a, b] = a^2, c^2 = 1, a^c = b, b^c = a \rangle.$$

It is easy to see that $\langle a^2, c \rangle \ntrianglelefteq G$, a contradiction. If $c^2 = a^2$, then $c^4 = 1$. Now, let

$$T = \langle cb, c \mid (cb)^8 = 1, (cb)^4 = c^2, (cb)^c = (cb)^{-1} \rangle.$$

T is a quaternion group of order 16, a contradiction. If the generator c induces an automorphism of order 4 on K, then we get $\langle c^2, a \rangle \ntrianglelefteq G$, which is noncyclic, a contradiction also.

Finally, since every subgroup of K is normal in G, we can choose a proper generator c such that c acts on K trivially. Hence,

$$G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, [a, b] = a^2, [a, c] = [b, c] = 1 \rangle,$$

where

$$\langle c\rangle \cap \langle a,b\rangle \le \langle a^2\rangle.$$

By the structure of Dedekind groups, we have $|c| > 2$.

The converse is clear. □

Remark 3.3. *By the proof of Theorem* 3.2*, let* K *be a quaternion group of order* 8 *and* $G = \langle K, a\rangle$ *be a* 2*-group. If* G *is weakly primitive Dedekind, then we can choose some proper generator* a *such that* $G = \langle K, a\rangle$*,* a *acts on* K *trivially, and* $|\langle a\rangle \cap K| \le 2$*.*

Theorem 3.4. *Let* G *be a* 2*-group containing a quaternion group of order* 8 *but not* 16*, and* G/K *be noncyclic. Then* G *is weakly primitive Dedekind if and only if*

$$G = \langle a,b,c,d \mid a^4 = d^2 = 1, c^2 = a^2 = b^2, [a,b] = [c,d] = a^2, [a,c] = [a,d] = [b,c] = [b,d] = 1\rangle.$$

Proof. Let $K = \langle a,b\rangle$ be a quaternion group of order 8 and $H = \langle c,d,\cdots,x\rangle$ such that $\{cK, dK, \cdots, xK\}$ is a minimal generating system of G/K. By Remark 3.3 and the structure of Dedekind groups, we can choose a proper H such that $|H \cap K| \le 2$ and H acts trivially on K.

If $H \cap K = 1$, then

$$\langle u\rangle = \langle a^2, u\rangle \cap H \trianglelefteq G$$

for any element u in H, which implies that H is a Dedekind group. If the exponent of H is 2, then G itself is a Dedekind group, a contradiction. Hence, the exponent of H is greater than 2. Assume that v is an element of order 4 and $w \ne v^2$ is an element of order 2 in H, then $\langle va, w\rangle \ntrianglelefteq G$, a contradiction. Hence, there is a unique element of order 2 in H, which implies that $H = \langle c,d\rangle$ is a quaternion group of order 8 by hypothesis. Thus, $\langle ca, db\rangle \ntrianglelefteq G$, a contradiction, so $|H \cap K| = 2$. In this case, there exists an element c of order 4 in H such that $c^2 = a^2$. Hence, $\langle ca\rangle \ntrianglelefteq G$.

Now we claim that H is not a Dedekind group. If H is Dedekind but not a quaternion group, then there are two elements v, w in H of order 2 that are in the center of G. Thus, we have

$$\langle ca\rangle = \langle ca, v\rangle \cap \langle ca, w\rangle \trianglelefteq G,$$

a contradiction. If H is a quaternion group, then

$$H = \langle c,d\rangle \text{ and } K \cap H = \langle a^2\rangle.$$

Hence, $\langle ca, db\rangle \ntrianglelefteq G$, a contradiction also. Thus, our claim holds.

By the same argument as above, H cannot contain any quaternion group, then there exists an element d of order 2 in H such that $d \ne a^2$. If $[c,d] = 1$, then

$$\langle ca\rangle = \langle ca, a^2\rangle \cap \langle ca, d\rangle \trianglelefteq G,$$

a contradiction. By hypothesis, $\langle a^2, d\rangle \trianglelefteq G$ and hence, $[c,d] = a^2$.

Writing $H_1 = \langle c,d\rangle$, we prove $H = H_1$. If not, we choose $H_2 \le H$ such that $|H_2/H_1| = 2$, and let e be an element of smallest possible order satisfying $e \in H_2 \backslash H_1$. Our proof will be divided into three cases:

(1) $|e| = 2$.

Note $\langle a^2, ca\rangle \trianglelefteq G$. We have

$$ca^3 = (ca)^e = c^e a,$$

so $[c, e] = a^2$. Similarly, $[d, e] = a^2$, and it makes $[cd, e] = 1$. On the other hand, $\langle e\rangle \trianglelefteq G$ holds by $[a^2, e] = 1$, then $e \in Z(G)$, a contradiction.

(2) $|e| = 4$.

Assume $a^2 = e^2$. If $[c, e] = 1$, then $|ce| = 2$. Therefore, $H_2 = \langle H_1, ce\rangle$, which contradicts the choice of e. Hence, $[c, e] \neq 1$. Since $\langle a^2, ca\rangle \trianglelefteq G$, we have

$$ca^3 = (ca)^e = c^e a.$$

Hence, $[c, e] = a^2$, which implies that $\langle c, e\rangle$ is a quaternion group, a contradiction. Assume $e^2 \neq a^2$. We can choose a proper generator d of H_1 such that $d = e^2$. Hence, $[d, c] = a^2$, thus $[e, c] \neq 1$. On the other hand, since $\langle a^2, ca\rangle \trianglelefteq G$, we get

$$(ca)^e = ca^3 = c^e a.$$

Thus, $c^e = c^3$, which implies $c^d = c$, a contradiction.

(3) $|e| = 8$.

Clearly, $e^2 = c$. Since $\langle d\rangle \ntrianglelefteq G$ and $\langle a^2, d\rangle \trianglelefteq G$, we get $d^e = da^2$, which implies $d^c = d$, a contradiction. Thus,

$$H = H_1 = \langle c, d\rangle.$$

In a word,

$$G = \langle a, b, c, d \mid a^4 = d^2 = 1, c^2 = b^2 = a^2, [a, b] = [c, d] = a^2, [a, c] = [a, d] = [b, c] = [b, d] = 1\rangle.$$

Conversely, since

$$G' = \langle [u, v]^g \mid u, v \in \{a, b, c, d\}, g \in G\rangle = \langle a^2\rangle,$$

we have that all elements of G can be written as $a^i b^j c^k d^l$ by computations, and where $0 \leq i \leq 3, 0 \leq j, k, l \leq 1$. Furthermore, if one of i, j, k is an odd number, then

$$(a^i b^j c^k d^l)^2 = a^2$$

holds certainly. Now we consider any binary generated subgroup G_1 of G. Clearly, if there exists an odd number of i, j, k such that $a^i b^j c^k d^l \in G_1$, then $G_1 \trianglelefteq G$. For any $a^i b^j c^k d^l \in G_1$, if all of i, j, and k are even numbers, then $G_1 = \langle a^2, d\rangle$, so $G_1 \trianglelefteq G$. That is, G is weakly primitive Dedekind. □

Theorem 3.5. *Let G be a p-group such that G does not contain a quaternion group of order* 8 *and $Z(G)$ is noncyclic. Then G is weakly primitive Dedekind if and only if*

$$G = \langle a, b \mid a^{p^m} = 1, b^{p^n} = 1, [a, b] = a^{p^{m-1}}\rangle,$$

where $m, n \geq 2$.

Proof. Our proof will be divided into three steps:

(1) G is a 2-generated group.

If not, choose three generators u, v, w in a minimal generating system of G. By Lemma 2.3, $u^p \neq 1$, $v^p \neq 1$, u^p and v^p are contained in $Z(G)$, which implies that

$$H = \langle u^p, v^p, w\rangle$$

is abelian. Again by Lemma 2.3, $\Omega_1(G)$ is of type (p, p), so H is cyclic or a direct product of two cyclic subgroups. If H is cyclic, then $u^p \in \langle w\rangle$. Hence, there exists a maximal cyclic subgroup $\langle w\rangle$ in $\langle u, w\rangle$. By hypothesis, G does not contain any quaternion group and there exists an element $u_1 \in \langle u, w\rangle$ of order p such that

$$\langle u, w\rangle = \langle u_1, w\rangle.$$

By Lemma 2.3, we know $\Omega_1(G) \leq \Phi(G)$. Hence, $u \in \langle w\rangle$, which contradicts the choice of u. If H is a direct product of two cyclic subgroups, then H is 2-generated. Assume $H = \langle u^p, v^p\rangle$, then $w \in \langle u, v\rangle$, which contradicts the choice of w. By the same argument as above, we get $H \neq \langle u^p, w\rangle$ and $H \neq \langle v^p, w\rangle$ also, a contradiction.

(2) G is a meta-cyclic group.

Suppose that G is not meta-cyclic. Let $\{a, b\}$ be a generating system satisfying

$$a^{p^m} = 1, \;\; b^{p^n} = a^{p^t}, \;\; [a, b] = c$$

and let b be a generator of smallest possible order. Obviously, $c \notin \langle a\rangle$ or $\langle b\rangle$. If $p = 2, n = t = 1$, then

$$(ab^{-1})^2 = a^2b^{-2}[b^{-1}, a] = [ab^{-1}, a].$$

Hence,

$$G = \langle a, ab^{-1}\rangle \text{ and } \langle ab^{-1}\rangle \trianglelefteq G,$$

that is, G is meta-cyclic, a contradiction. If $p = 2$, $n = 1$, $t > 1$, then by Lemma 2.3, we have $a^{2^{t-1}} \in Z(G)$, $(b^{-1}a^{2^{t-1}})^2 = 1$, so

$$G = \langle a, b^{-1}a^{2^{t-1}}\rangle,$$

which contradicts Lemma 2.3. Assume $p \neq 2$ or $n \neq 1$. If $a^{p^t} \neq 1$, then $|b| > p^n$. By the choice of b, $n \leq t$. Since

$$\left(ba^{-p^{t-n}}\right)^{p^n} = b^{p^n}a^{-p^t}\left[a^{-p^{t-n}},\ b\right]^{\binom{p^n}{2}} = \left[a^{-p^{t-n}},\ b\right]^{\binom{p^n}{2}},$$

we can get

$$(ba^{-p^{t-n}})^{p^n} = 1$$

provided $p \neq 2$ or $n \neq 1$. Now,

$$G = \langle a, ba^{-p^{t-n}}\rangle,$$

which contradicts the choice of b. Hence, $a^{p^t} = 1$, that is, $\langle a\rangle \cap \langle b\rangle = 1$. By Lemma 2.3, we assume that

$$c = a^{ip^{m-1}}b^{jp^{n-1}}, p \nmid i, p \nmid j.$$

Let r be a natural number satisfying $jr \equiv 1 (\mathrm{mod}\ p)$. Then,

$$[a^r, a^{irp^{m-n}} b] = a^{irp^{m-1}} b^{jrp^{n-1}} = (a^{irp^{m-n}} b)^{p^{n-1}}.$$

Hence,

$$G = \langle a^r, a^{irp^{m-n}} b\rangle \text{ and } \langle a^{irp^{m-n}} b\rangle \trianglelefteq G,$$

a contradiction.

(3) Completing the proof.

By (1) and (2), we can choose two generators a, b of G such that $a^{p^m} = 1, b^{p^n} = a^{p^t}$, and $[a, b] \in \langle a\rangle$, where the order of b comes to the smallest. Now, we show $a^{p^t} = 1$. If $a^{p^t} \neq 1$, then $[a, b] \in \langle b\rangle$. By the choice of b, we get $n \leq t$. If $p = 2$, $n = t = 1$, and $m = 2$, then G is a quaternion group of order 8, a contradiction. If $p = 2$, $n = t = 1$, and $m > 2$, then

$$(a^{-1}b)^2 = a^{-2}b^2[b, a^{-1}] = a^{2^{m-1}}.$$

Writing $d = a^{-1}b$, then

$$G = \langle a, d\rangle = \langle a, a^{-2^{m-2}} d\rangle.$$

Hence,

$$(a^{-2^{m-2}} d)^2 = a^{-2^{m-1}} d^2 = 1,$$

contrary to the choice of b. If $p = 2$, $n = 1$, and $t > 1$, then by Lemma 2.3, we have

$$a^{2^{t-1}} \in Z(G), \quad (b^{-1}a^{2^{t-1}})^2 = 1 \text{ and } G = \langle a, b^{-1}a^{2^{t-1}}\rangle,$$

contrary to Lemma 2.3. So, we assume $p \neq 2$ or $n \neq 1$. Since

$$\left(ba^{-p^{t-n}}\right)^{p^n} = b^{p^n} a^{-p^t} \left[a^{-p^{t-n}},\ b\right]^{\binom{p^n}{2}} = \left[a^{-p^{t-n}},\ b\right]^{\binom{p^n}{2}},$$

we get

$$(ba^{-p^{t-n}})^{p^n} = 1$$

provided $p \neq 2$ or $n \neq 1$. Now,

$$G = \langle a, ba^{-p^{t-n}}\rangle,$$

which contradicts the choice of b. Thus, $a^{p^t} = 1$, and therefore,

$$G = \langle a, b \mid a^{p^m} = 1, b^{p^n} = 1, [b, a] = a^{p^{m-1}}\rangle,$$

where $m, n \geq 2$.

Conversely, it is clear that

$$G' = \langle a^{p^{m-1}}\rangle.$$

For any proper subgroup of G, which is binary generated, since G is also a binary generated group, we have

$$|H\Phi(G)/\Phi(G)| \leq p.$$

If

$$H\Phi(G)/\Phi(G) = 1,$$

then

$$H \leq \Phi(G) = Z(G),$$

so $H \trianglelefteq G$. If

$$|H\Phi(G)/\Phi(G)| = p,$$

then we can choose the appropriate generating system u, v of H such that $u \notin \Phi(G), v \in \Phi(G)$. Let

$$u = a^{i_1 p^{m_1}} b^{j_1 p^{n_1}}, v = a^{i_2 p^{m_2}} b^{j_2 p^{n_2}},$$

where at least one of i_1 and j_1 is more than 0, as well as i_2 and j_2, which prime to p, at least one of m_1, n_1 is 0, m_2 and n_2 are both more than 0. If j_1 or j_2 is 0, then $a^{p^{m-1}} \in H$, so $H \trianglelefteq G$. Now we can assume that j_1 and j_2 are both more than 0. Without loss of generality, we can choose the appropriate power of u, v (still written as u, v) such that $H = \langle u, v\rangle$, where

$$u = a^{i_1 p^{m_1}} b^{p^{n_1}}, \quad v = a^{i_2 p^{m_2}} b^{p^{n_2}}.$$

If $m_1 \leq m_2$, then

$$1 \neq u^{p^{m_2 - m_1}} v^{-1}.$$

Since H is noncyclic, then

$$u^{p^{m_2 - m_1}} v^{-1} \in \langle a\rangle$$

by computations. Thus,

$$a^{p^{m-1}} \in \langle u^{p^{m_2 - m_1}} v^{-1}\rangle.$$

Therefore, $H \trianglelefteq G$. Similarly, if $m_1 > m_2$, the result holds. So G is weakly primitive Dedekind. □

By Theorem 3.5, the following remark holds.

Remark 3.6. *Let G be a p-group such that G does not contain a quaternion group of order* 8 *and $Z(G)$ is noncyclic. If G is weakly primitive Dedekind, then*

$$\Omega_1(G) = \langle \Omega_1(\langle a\rangle) \mid a \in G, |a| > p\rangle.$$

Theorem 3.7. *Let G be a p-group such that G does not contain a quaternion group of order* 8 *and $Z(G)$ is cyclic. Then G is weakly primitive Dedekind if and only if G is isomorphic to one of the following groups:*

(1) $G = \langle a, b \mid a^p = b^{p^n} = 1, [a, b] = c, [a, c] = [b, c] = c^p = 1\rangle$, *where* $c = b^{p^{n-1}}$ *if* $n \neq 1$;

(2) $G = \langle a, b \mid a^9 = 1, b^3 = a^3, [a, b] = c, c^3 = 1, [a, c] = 1, [b, c] = a^6\rangle$;

(3) $G = \langle a, b, c \mid a^p = b^{p^n} = c^p = 1, [a, b] = 1\rangle$, *where* $\langle [a, c]\rangle = \langle b^{p^{n-1}}\rangle, [b, c] \in \langle b^{p^{n-1}}\rangle, n \geq 2$;

(4) $G = \langle a, b \mid a^8 = 1, a^4 = b^4, [a, b] = a^2$ *or* $a^6\rangle$.

Proof. Let C be a subgroup of order p in $Z(G)$. By hypothesis, there exists another subgroup A of order p, which is different from C. Since $Z(G)$ is cyclic, we have $A \ntrianglelefteq G$. Hence, we obtain that $AC \trianglelefteq G$ by hypothesis, which implies

$$|G/N_G(A)| = p.$$

Thus,

$$\Phi(G) \leq N_G(A) = C_G(A).$$

We claim $\langle a^p \rangle \trianglelefteq G$ for any $a \in G$. If $C \nleq \langle a \rangle$, then $\langle a \rangle C \trianglelefteq G$ and

$$\langle a^p \rangle = \mho(\langle a \rangle C) \trianglelefteq G.$$

If $C \leq \langle a \rangle$, then

$$A\langle a \rangle = AC\langle a \rangle \trianglelefteq G,$$

so

$$\langle a^p \rangle = \mho(A\langle a \rangle) \trianglelefteq G.$$

Let H be any cyclic subgroup of G whose order is greater than p. Then C is the unique subgroup of order p in H since $\langle a^p \rangle \trianglelefteq G$ for any $a \in G$. Hence,

$$\langle \Omega_1(\langle a \rangle) \mid a \in G, |a| > p \rangle = C.$$

We claim that $C_G(A)$ is abelian. If not, then there exists at least one non-normal subgroup in $C_G(A)$ by hypothesis. Since A and C are both contained in $Z(C_G(A))$, we get that

$$\Omega_1(C_G(A)) = \langle \Omega_1(\langle a \rangle) \mid a \in C_G(A), |a| > p \rangle \leq \langle \Omega_1(\langle a \rangle) \mid a \in G, |a| > p \rangle = C$$

by Remark 3.6, which contradicts $A \leq \Omega_1(C_G(A))$.

Since $C_G(A)$ is abelian, then $C_G(A)$ is a direct product of some cyclic subgroups. In fact, the number of direct product factors of $C_G(A)$ is at most two by hypothesis. On the other hand, A cannot be contained in any cyclic subgroup. Thus, there is an element u such that

$$C_G(A) = A\langle u \rangle = A \times \langle u \rangle \text{ and } \Omega_1(\langle u \rangle) = C.$$

For any $H \leq G$, we have $HAC \trianglelefteq G$, so

$$HAC/(AC) \trianglelefteq G/(AC),$$

which implies that $G/(AC)$ is a Dedekind group. Our proof will be divided into two cases as shown below.

Case 1. $G/(AC)$ is abelian.

Obviously, for any $g, h \in G$, we have $[g, h] \in AC$ and $[g, h]^p = 1$ by the structure of Dedekind groups.

We claim $g^p \in Z(G)$ for any $g \in G$. If $g \notin C_G(A)$, then

$$G = \langle g, C_G(A) \rangle.$$

By what has been shown in the first paragraph, we know

$$|G/C_G(A)| = p.$$

Thus, $g^p \in C_G(A)$ and $g^p \in Z(G)$ by the fact that $C_G(A)$ is abelian. If $g \in C_G(A)$, then $g \in C_G(AC)$. So $g \in C_G([g,h])$ and

$$[g^p, h] = [g,h]^p = 1$$

for any $h \in G$. Furthermore, $g^p \in Z(G)$ and $\mho(G) \leq Z(G)$.

Let $A = \langle a \rangle$. Since $C_G(A) = \langle a, u \rangle$ is a maximal subgroup of G, there exists a generator b of G such that $G = \langle a, u, v \rangle$. Hence, there is a subset of $\{a, u, v\}$ which can form a minimal generating system of G. Obviously, anyone of $\{a\}, \{u\}, \{v\}, \{a, u\}$ is not a minimal generating system of G.

Suppose that $\{a, v\}$ is a minimal generating system of G. Let $b^{p^n} = 1$ and $[a, v] = c$. Since $c \in AC$ and $[a, v^p] = 1$, then

$$[a, c] = [v, c] = c^p = 1.$$

If $n \neq 1$, then $c \in \langle v^p \rangle$ as $c \in Z(G)$, so $v^p \in Z(G)$ and $Z(G)$ is cyclic. Moreover, we can choose some proper a such that $c = v^{p^{n-1}}$. Hence,

$$G = \langle a, v \mid a^p = v^{p^n} = 1, [a, v] = c, [a, c] = [v, c] = c^p = 1 \rangle,$$

where $c = v^{p^{n-1}}$ if $n \neq 1$. By choosing some proper symbols, G is a group of type (1).

Suppose that $\{u, v\}$ is a minimal generating system of G. Since $\Omega_1(\langle u \rangle) = C$, we have $u^p \neq 1$. If $a \notin \Phi(G)$, then there exists a generator w such that $\langle a, w \rangle = G$. By the same argument as above, we also get that G is a group of type (1). Assume $a \in \Phi(G)$. Since $\mho(G) \leq Z(G)$, we get $a \in G'$. Let $a = [u, v]$. Then

$$\langle [a, v] \rangle = C \leq \langle u \rangle.$$

On the other hand, since $v^p \in Z(G)$, we have $\langle v^p \rangle \lneqq \langle u \rangle$.

We claim $\langle u^p \rangle = \langle v^p \rangle$. If not, then $\langle v^p \rangle \lneqq \langle u^p \rangle$. Since $u^p \in Z(G)$, then $\langle u^p \rangle$ is a maximal subgroup of $\langle u^p, v \rangle$. Hence, there exists an element $w \in \langle u^p, v \rangle$ of order p such that $v \in \langle u^p, w \rangle$, so $G = \langle u, w \rangle$. Now, $\langle [u, w] \rangle = C$ since $\langle w, C \rangle \trianglelefteq G$, which implies $G' = C$, contrary to $a \in G'$. Thus, our claim holds and

$$G = \langle u, v \mid u^{p^n} = 1, \langle u^p \rangle = \langle v^p \rangle, a = [u, v], c_1 = [a, v], a^p = 1, c_1 \in \Omega_1(\langle u \rangle), [a, u] = 1 \rangle.$$

Obviously, $|G| = p^{n+2}$. If $p = 2$, then

$$(vu)^2 = v^2 u^2 a$$

and $a \in \mho(G)$, which contradicts $\mho(G) \leq Z(G)$. If $p \geq 5$, then

$$(vu)^p = v^p u^p a^{p(p-1)/2} c_1^{(p-2)(p-1)p/6} = v^p u^p.$$

By choosing proper elements u, v such that $v^p = u^{-p}$, we get $(vu)^p = 1$. Since $\langle vu, C \rangle \trianglelefteq G$ and $C \leq \langle u \rangle$, we have

$$G = \langle v, u \rangle = \langle vu, u \rangle \text{ and } |\langle vu, u \rangle| = p^{n+1},$$

contrary to $|G| = p^{n+2}$. Hence, $p = 3$. By choosing proper elements u, v such that $u^3 = v^3$, we have $c_1 = u^6$ by

$$(vu)^3 = v^3 u^3 a^3 c_1 = u^6 c_1,$$

so

$$G = \langle u, v \mid u^9 = 1, v^3 = u^3, a = [u, v], a^3 = 1, [a, u] = 1, [a, v] = u^6\rangle.$$

By choosing some proper symbols, G is a group of type (2).

Suppose that $\{a, u, v\}$ is a minimal generating system of G. Let $|u| = p^n$. Since $\langle a, u\rangle$ is a maximal subgroup of G, we have $|G| = p^{n+2}$. On the other hand,

$$\langle u\rangle \lneqq \langle u, v\rangle \lneqq G,$$

so $\langle u\rangle$ is a maximal subgroup of $\langle u, v\rangle$. Hence, there exists an element w of order p satisfying $\langle u, v\rangle = \langle u, w\rangle$, so $G = \langle a, u, w\rangle$. Obviously, $A\langle w\rangle$ cannot be a subgroup of G. If not, then

$$A = AC \cap A\langle w\rangle \trianglelefteq G,$$

a contradiction. On the other hand, $AC\langle w\rangle$ is a subgroup of G, so $\langle [a, w]\rangle = C$. Since $[u, w] \in \langle w\rangle C$ and $[u^p, w] = 1$, then $[u, w] \in C$. Thus,

$$G = \langle a, u, w \mid a^p = u^{p^n} = w^p = 1, [a, u] = 1\rangle,$$

where

$$\langle [a, w]\rangle = \langle u^{p^{n-1}}\rangle, [u, w] \in \langle u^{p^{n-1}}\rangle, \quad n \geq 2.$$

By choosing some proper symbols, G is a group of type (3).

Case 2. $G/(AC)$ is non-abelian.

In this case, we have that $G/(AC)$ is a direct product of a quaternion group and an elementary abalian 2-group. Write $K = AC$, $C = \langle c_1\rangle$, and $A = \langle a\rangle$. Let

$$T/K = \langle uK, vK \mid u^4K = K, v^2K = u^2K, [u, v]K = u^2K\rangle$$

be a quaternion subgroup of G/K. Then $u^2v^{-2} \in K$ and $[u, v] \in u^2K$. Since

$$\langle \Omega_1(\langle g\rangle) \mid g \in G, |g| > p\rangle = C,$$

we have $u^4 = v^4 = c_1$ and $u^8 = 1$. By the symmetry of u and v, we choose some proper generators (we still call them u and v) of T such that

$$T = \langle u, v \mid u^8 = 1, u^4 = v^4 = c_1, u^2v^2 = a, [u, v] = u^2 \text{ or } u^6\rangle.$$

Obviously, $u \in C_G(a)$ and $v \notin C_G(a)$.

Now, we show $G = T$. If $G \neq T$, then there exists an element $w \in G\backslash T$. By hypothesis, the exponent of G is less than 8. On the other hand,

$$C_G(A) = A \times \langle u\rangle$$

and $u \in C_G(a)$, hence, we get $C_G(a) = \langle a, u\rangle$. If $w \in C_G(a)$, then $w \in \langle a, u\rangle$, contrary to $w \notin T$. If $w \notin C_G(a)$, then $wv \in C_G(a)$ and $wv \in \langle a, u\rangle$, a contradiction also. Thus, we get $G = T$. By choosing some proper symbols, G is a group of type (4).

Conversely, it is easy to check that anyone of types (1)–(4) is weakly primitive Dedekind. □

4. Proofs of Theorems A and B

In this section, we complete the proofs of Theorems A and B.

Proof of Theorem A. It follows from Theorems 3.1–3.7 clearly. □

proof of Theorem B. By Lemma 2.1, Lemma 2.4, and the structure of Dedekind groups, we have $G = P \rtimes C$ with P, C of coprime order, P is of type (1) or (5) in Theorem A, or a cyclic p-group, a quaternion group Q_8, or $C_{p^m} \times C_p$, C is a cyclic subgroup of G, and $q \mid |C/C_C(P)|$ for any prime divisor q of $|C|$. In the first place, P can't be the group of type (1) in Theorem A since

$$C/C_C(P) \lesssim \ Aut(P)$$

and $\mathrm{Aut}(Q_{16})$ is a 2-group.

If P is a group of type (5) in Theorem A, let R

$$P = \langle u, v \mid u^p = v^{p^n} = 1, [u, v] = w, [u, w] = [v, w] = w^p = 1\rangle,$$

where $w = v^{p^{n-1}}$ if $n \neq 1$, then $\langle u, w\rangle \trianglelefteq G$. If C acts on $\langle u, w\rangle$ trivially, then $C \trianglelefteq G$, a contradiction. Hence, C acts on $\langle u, w\rangle$ nontrivially. By Lemma 2.4, C acts trivially on $\langle w\rangle$. Hence, there exists a C-invariant subgroup in $\langle u, w\rangle$ (without loss of generality, write this subgroup $\langle u\rangle$) such that C acts nontrivially on $\langle u\rangle$. Hence, $\langle u\rangle C \trianglelefteq G$ and

$$\langle u\rangle = \langle u, w\rangle \cap \langle u\rangle C \trianglelefteq G,$$

a contradiction.

If P is cyclic and $|P| = p^m$, then $p \geq 3$ since the automorphism group of a 2-group is a 2-group and

$$C/C_C(P) \lesssim \ Aut(P).$$

We claim $m = 1$. If not, then $\Omega_1(P) \leq \Phi(P)$. By Lemma 2.4, C acts trivially on $\Phi(P)$. Of course, C acts trivially on $\Omega_1(P)$. Thus, C must act trivially on P, a contradiction. Therefore, P is of order p.

Let

$$G = \langle a, b \mid a^p = 1, b^r = 1, a^b = b^t\rangle$$

and

$$r = \prod_{i=1}^{k} q_i^{n_i}.$$

For any $1 \leq i \leq n$ and any prime divisor q of $|C|$, we have

$$t^{\frac{r}{q_i^{n_i}}} \not\equiv 1 (\mathrm{mod}\ p)$$

since $q \mid |C/C_C(P)|$, so G is a group of type (1).

If

$$P = \langle a, b \mid a^4 = 1, b^2 = a^2, [a, b] = a^2\rangle$$

and $C = \langle c\rangle$, then $a^c = b, b^c = ab$ or ba by the structure of $\mathrm{Aut}(P)$ and $2 \nmid r$. Furthermore, $r = 3^n$ since $q \mid |C/C_C(P)|$ for any prime divisor q of $|C|$, so G is a group of type (2).

If P is a group of type $C_{p^m} \times C_p$ and C acts decomposably on P, then by Lemma 2.4, C acts on $\Phi(P)$ trivially and all noncyclic subgroups are normal in G. Hence, we choose proper generators a, b, c such that

$$G = \langle a, b, c \mid a^{p^m} = b^p = c^r = 1, a^b = a^c = a, b^c = b^t \rangle,$$

where $t \not\equiv 1(\mathrm{mod}\ p), t^r \equiv 1(\mathrm{mod}\ p)$. Clearly, $\langle b, c \rangle$ is of type (1). Thus,

$$G = \langle a, b, c \mid a^{p^m} = b^p = c^r = 1, a^b = a^c = a, b^c = b^t \rangle,$$

where

$$t^r \equiv 1(\mathrm{mod}\ p), \quad r = \prod_{i=1}^{k} q_i^{n_i}, t^{\frac{r}{q_i^{n_i}}} \not\equiv 1(\mathrm{mod}\ p)$$

for any $1 \leq i \leq n$, so G is a group of type (3).

If P is of type $C_{p^m} \times C_p$ and C acting on P is indecomposable, then by Lemma 2.4, C acts on $\Phi(P)$ trivially and all noncyclic subgroups are normal in G. Hence, $m = 1$, and we can choose three proper generators a, b, c such that $a^p = b^p = c^r = 1$, where $a^b = a, a^c = b$, and $b^c = a^m b^n$. Now, take P as a vector space in F_p and c as a linear transformation of P, then the matrix induced by c is

$$M = \begin{bmatrix} 0 & 1 \\ m & n \end{bmatrix}.$$

Since $c^r = 1$, we have

$$M^r \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{mod}\ p).$$

On the other hand, for any subgroup K of C, we easily know that K must act on P trivially if K acts on P reducibly. Thus, the matrix M^k corresponding to the generator of K either has no eigenvalues or is an identity matrix in F_p. Therefore,

$$G = \langle a, b, c \mid a^p = b^p = c^r = 1, [a, b] = 1, a^c = b, b^c = a^m b^n \rangle.$$

Let

$$M = \begin{bmatrix} 0 & 1 \\ m & n \end{bmatrix}.$$

Thus,

$$M^r \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{mod}\ p),$$

where M^k has no eigenvalues or is an identity matrix in F_p for any $k \mid r$. Let

$$r = \prod_{i=1}^{k} q_i^{n_i}.$$

$M^{\frac{r}{q_i^{t_i}}}$ has no eigenvalues since $q \mid |C/C_C(P)|$ for any $1 \leq i \leq n$ and any prime divisor q of $|C|$, so G is a group of type (4).

Conversely, it is easy to check that anyone of types (1)–(4) is weakly primitive Dedekind. □

5. Conclusions

We give the complete classification of p-groups whose noncyclic subgroups are normal. Our method of proof is elementary, which is different from Bozikov and Janko's in [4], and give the complete classification of non-nilpotent groups whose noncyclic subgroups are normal.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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