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Research article

Interface vanishing of $d - \delta$ systems

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Abstract: We introduce $d - \delta$ systems on differential forms in Euclean spaces and show the interface vanishing of the solution. This result generalizes previous theorems on stationary and non-stationary Maxwell's equation. Other applications are also given.

Keywords: $d - \delta$ systems; differential forms; interface vanishing; Maxwell equation
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1. Introduction

Interface vanishing has been observed in both the stationary and non-stationary Maxwell's equation [1–4], which says that some components of the solution do not detect the interface. More precisely, if this equation holds piecewise in a region with interface $\mathcal{M}$, then these components extend across $\mathcal{M}$.

This equation is concerned with the electric field E, the magnetic field B, the current density J, and the electric charge density ρ, depending on the space-time variables $(x, t) \in \mathbf{R}^3 \times \mathbf{R}$. Using the gradient operator

$$\nabla = \nabla_x \equiv \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \quad x = (x_1, x_2, x_3),$$

and the outer and the inner products in $\mathbf{R}^3$ denoted by $\times$ and $\cdot$, respectively, it is given by

$$\nabla \times H - \frac{\partial D}{\partial t} = J, \quad \nabla \cdot D = \rho, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \nabla \cdot B = 0$$

with

$$D = \varepsilon E, \quad B = \mu H, \quad J = \sigma E,$$

where ε, μ, and σ denote the permittivity, the permeability, and the conductivity, respectively. Assuming that the first two physical constants ε and μ are independent of the medium, we set them to be 1, giving

$$\begin{array}{ll} \nabla \times B - \frac{\partial E}{\partial t} = J, \quad \nabla \cdot E = \rho & \\ \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \nabla \cdot B = 0 & \text{in } \Omega, \end{array} \tag{1.1}$$

with the normalized light speed $c = \mu\varepsilon = 1$, where $\Omega \subset \mathbf{R}^4$ is a domain in space-time.

We take the case that this Ω is composed of two media, with the interface $\mathcal{M}$ forming a part of a smooth hyper-surface, and that the variables J and ρ are only piecewisely regular. In [4], it is shown that, if J and ρ are provided with piecewise regularity, singularities of some components of B and E propagate across the interface without suffering any effects from it.

Remark 1.1. *The discontinuity of J arises in accordance with the secondary current of magnetoencephalography. The standard theory [5, 6] assumes the quasi-static state*

$$\nabla \times B = \mu_0 J, \;\; \nabla \cdot B = 0 \quad \text{in } \Omega,$$

where Ω is composed of two domains Ω_+ and Ω_- indicating the outside and inside of the head, respectively. Here, μ_0 denotes the permeability assumed to be a common constant in both $\Omega_\pm$. There also arises

$$J = J_p - \sigma(x)\nabla V \tag{1.2}$$

for the primary current J_p evoked by the neuron activity in Ω_- and the secondary current $-\sigma(x)\nabla V$ due to the electric field $-\nabla V$ associated with the voltage $V = V(x)$. The electric conductivity $\sigma(x)$ is thus assumed to be piecewise constant. In one-layer model, for example, it takes the form

$$\sigma(x) = \begin{cases} \sigma_I, & x \in \Omega_- \\ 0, & x \in \Omega_+, \end{cases}$$

where $\sigma_I > 0$ is a constant. By this discontinuity of $\sigma(x)$, the total current density J in (1.2) has a discontinuty on the interface $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$. In spite of this discontinuity, we noted in [1] that the normal component of the magnetic field B is continuously distributed across Γ.

For later use, we formulate the geometric profile of such Ω in a general setting as follows:

Definition 1.1. *The Lipschitz bounded domain $\Omega \subset \mathbf{R}^n$ is said to have an interface, denoted by $\mathcal{M}$, if this $\mathcal{M}$ is non-compact hyper-surface in $\mathbf{R}^n$ without boundary such that $\Gamma \equiv \mathcal{M} \cap \Omega \neq \emptyset$ is connected.*

This domain Ω is divided into two domains by $\mathcal{M}$, denoted by $\Omega_\pm$, and then Γ is distinguished as a subset of $\partial\Omega_\pm$ as $\Gamma_\pm$:

$$\Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm \setminus \partial\Omega (= \Gamma).$$

Henceforth, ν denotes the outer unit normal vector on Γ_-. It is extended on Ω as a Lipschitz continuous vector field if $\mathcal{M}$ is $C^{1,1}$.

Coming back to (1.1), we assume the piecewise regularity of $J \in L^2(\Omega)$ and $\rho \in L^2(\Omega)$ in (1.1), precisely,

$$\nabla \times J \in L^2(\Omega_\pm)^3, \quad \frac{\partial J}{\partial t} + \nabla\rho \in L^2(\Omega_\pm)^3, \tag{1.3}$$

where the differentiations are taken in the sense of distributions in $\Omega_\pm$.

Theorem 1.1. *[4] Let $\Omega \subset \mathbf{R}^4$ be a bounded Lipschitz domain with $C^{2,1}$ interface $\mathcal{M}$. Let ν be the outer unit normal vector on Γ_-, extended smoothly on Ω. Let $B, E \in H^1(\Omega)^3$ be the solution to (1.1), for $J \in L^2(\Omega)^3$ and $\rho \in L^2(\Omega)$ satisfying (1.3). Then, it holds that*

$$\Box(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3, \quad \Box(\tilde{\nu} \cdot B) \in L^2(\Omega), \tag{1.4}$$

where

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \quad \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}, \quad \Box = -\frac{\partial^2}{\partial t^2} + \Delta_x.$$

Remark 1.2. *Since $B, E \in H^1(\Omega)^3$ is assumed, the system of Eq (1.1) is valid across the interface. The conclusion (1.4), however, is derived without assuming $B, E \in H^2(\Omega)^3$, or, $J, \rho \in H^1(\Omega)^3$. In fact, since system (1.1) implies*

$$-\Delta B - \frac{\partial}{\partial t}\nabla \times E = \nabla \times J, \quad \nabla\rho - \Delta E + \frac{\partial}{\partial t}\nabla \times B = 0,$$

it arises that

$$\Box B = -\nabla \times J, \quad \Box E = \nabla\rho + \frac{\partial J}{\partial t}$$

in the sense of distributions in Ω. The assumption (1.3) thus implies

$$\Box B,\ \Box E \in L^2(\Omega_\pm)^3,$$

which, however, does not mean $\Box B$, $\Box E \in L^2(\Omega)^3$. The conclusion (1.4), therefore, assures that H^2-singularities of $\nu^0 B + \tilde{\nu} \times E$ and $\tilde{\nu} \cdot B$ propagate through the interface under light speed, without suffering any effects from it.

Remark 1.3. *Theorem 1.1 includes results on the stationary state, $E_t = B_t = 0$ [1–3]. These results are reduced to a problem on a 1-form, where its decomposition to tangential and normal components works effectively. Although this argument is valid for arbitrary space dimension n, difficulty arises in (1.1) which is reduced to an equation of 2-forms, not 1-forms, on Minkowski spaces in (3, 1)-dimension.*

Our purpose is to generalize Theorem 1.1 on (1.1) of 2-forms to arbitrary p-forms, formulating (1.1) as a $d - \delta$ system. Our results are stated in the context of Euclidean spaces, but are valid to Minkowski spaces in any dimension. Hence, they include all the above results on stationary and non-stationary Maxwell equations associated with 1- and 2-forms, respectively.

Remark 1.4. *Several models in mathematical physics, besides the Maxwell system (1.1), are formulated as the $d - \delta$ system, as described in [7]. Among them are the Stokes system and hyperbolic equations. See §3.*

Before stating our results, we recall the following facts.

Remark 1.5. *If $D \subset \mathbf{R}^n$ is a Lipschitz bounded domain, then $C^\infty(\overline{D})$ is dense in $H^1(D)$, where $\varphi \in C^\infty(\overline{D})$ if and only if there is $\tilde{\varphi}$, smooth in a neighbourhood of $\overline{D}$, such that*

$$\tilde{\varphi}|_{\overline{D}} = \varphi \text{ on } \overline{D}.$$

Then the trace operator $\gamma : H^1(D) \to H^{1/2}(\partial D)$ is well-defined, and it holds that

$$H^1(D)/H_0^1(D) \cong H^{1/2}(\partial D) \tag{1.5}$$

under this operator, where $H_0^1(D)$ denotes the closure of $C_0^\infty(D)$ in $H^1(D)$ [8–10].

Remark 1.6. *The p-form on the bounded domain $D \subset \mathbf{R}^n$ takes the form*

$$\omega = \sum_{i_1<\cdots<i_p} \omega^{i_1\cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for $1 \le p \le n$. We say that $\omega \in H^1(D)$ if and only if $\omega^{i_1\cdots i_p} \in H^1(D)$ for any $i_1 < \cdots < i_p$, and, similarly, $\omega \in L^2(D)$ if and only if $\omega^{i_1\cdots i_p} \in L^2(D)$ for any $i_1 < \cdots < i_p$.

Remark 1.7. *The set of p-forms on the bounded domain $D \subset \mathbf{R}^n$ is denoted by $\Lambda^p(D)$. There, the outer derivative*

$$d : \Lambda^p(D) \to \Lambda^{p+1}(D), \quad 0 \le p \le n-1$$

and the wedge product

$$\wedge : \Lambda^p(D) \times \Lambda^q(D) \to \Lambda^{p+q}(D), \quad p, q \ge 0,\ p+q \le n$$

are defined. The Hodge operator

$$* : \Lambda^p(D) \to \Lambda^{n-p}(D), \quad 0 \le p \le n$$

is also defined by

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = sgn\, \sigma \cdot dx_{i_{p+1}} \wedge dx_{j_n},$$

where $\sigma : (1, \cdots, n) \mapsto (i_1, \cdots, i_n)$. The co-derivative is then defined by

$$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D),$$

and it holds that [7, 11]

$$d\delta + \delta d = -\Delta : \Lambda^p(D) \to \Lambda^p(D), \quad 0 \le p \le n.$$

In the following theorems, the derivatives d, δ, and Δ are taken in the sense of distributions. Hence, the H^2-interface on $\mathcal{M}$ vanishes for special components of ω if it solves the $d-\delta$ system formulated below. Here, the outer unit normal vector $\nu = (\nu^i)$ on Γ_-, extended on Ω, is identified with the 1-form

$$\nu = \nu^1 dx_1 + \cdots + \nu^n dx_n.$$

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega \in H^1(\Omega)$ and $\theta \in L^2(\Omega)$ be p- and $(p+1)$-forms on Ω, respectively. Assume*

$$d\omega = \theta,\ \delta\omega = 0 \text{ in } \Omega, \quad \delta\theta \in L^2(\Omega_\pm). \tag{1.6}$$

Then it holds that

$$\Delta(\nu \wedge *\omega) \in L^2(\Omega). \tag{1.7}$$

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega \in H^1(\Omega)$ and $\theta \in L^2(\Omega)$ be p- and $(p-1)$-forms on Ω, respectively. Assume*

$$d\omega = 0,\ \delta\omega = \theta \text{ in } \Omega, \quad d\theta \in L^2(\Omega_\pm). \tag{1.8}$$

Then it holds that

$$\Delta(\nu \wedge \omega) \in L^2(\Omega). \tag{1.9}$$

Remark 1.8. *From the assumption, ν is extended as a $C^{1,1}$ vector field on Ω.*

Remark 1.9. *In both cases of the above theorems, it holds always that $\Delta\omega \in L^2(\Omega_\pm)$, which, however, does not mean $\Delta\omega \in L^2(\Omega)$.*

Remark 1.10. *Theorems 1.2 and 1.3 are equivalent. If (1.8) holds in the setting of Theorem 1.3, for example, then,*

$$\alpha = *\omega$$

is an $(n-p)$-form, and there arises that

$$\begin{aligned}\delta\alpha &= (-1)^{n-p} *^{-1} d * \alpha = (-1)^{n-p} *^{-1} d * *\omega \\ &= (-1)^{n-p} *^{-1} d[(-1)^{p(n-p)}\omega] = (-1)^{(p+1)(n-p)} *^{-1} d\omega = 0.\end{aligned}$$

We obtain, also,

$$\theta = \delta\omega = (-1)^p *^{-1} d * \omega = (-1)^p *^{-1} d\alpha$$

and hence

$$d\alpha = (-1)^p * \theta,$$

which implies

$$\begin{aligned}\delta d\alpha &= (-1)^{n-p+1} *^{-1} d * d\alpha = (-1)^{n+1} *^{-1} d * *\theta \\ &= (-1)^{n+1} *^{-1} d[(-1)^{(n+p-1)(p-1)}\theta] \\ &= (-1)^{n+1} \cdot (-1)^{(n-p+1)(p-1)} *^{-1} d\theta \in L^2(\Omega_\pm).\end{aligned}$$

If we apply Theorem 1.2 to $\omega = \alpha$, we get

$$\Delta(\nu \wedge *\alpha) \in L^2(\Omega),$$

and hence (1.9), the conclusion of Theorem 1.3.

This paper is composed of four sections. Taking preliminaries in Section 2, Theorem 1.2 is proven in §3. Section 3 is devoted to applications. There, we confirm that Theorems 1.2 and 1.3 imply all the results on interface vanishing obtained so far [1–4]. We will extend these results to differential forms on manifolds as in [12], to deal with systems of variable coefficients, arising often in the theory of electromagnetism [13] in the future. We will also develop the L^p theory and its applications to nonlinear problems, such as the Navier–Stokes equation, to refine [2] derived from the L^2 theory. The authors thank the referees for pointing out these challenges.

2. Preliminaries

Here, we show the Gauss and the Stokes formulae in the context of H^1-theories. Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Recall that, if $\theta \in H^1(D)$ is a p-form, its trace $\theta|_{\partial D}$ belongs to $H^{1/2}(\partial D)$.

The Euclidean inner product of 1-forms

$$\alpha = \sum_\ell \alpha^\ell dx_\ell, \quad \beta = \sum_\ell \beta^\ell dx_\ell$$

is given by

$$(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell.$$

If $\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p$ and $\mu = \beta_1 \wedge \cdots \wedge \beta_p$ are p-forms made by 1-forms α_i and β_i for $1 \le i \le p$, we put

$$(\lambda, \mu) = \det \left((\alpha_i, \beta_j)\right)_{i,j}. \tag{2.1}$$

Then, it holds that [7, 11]

$$\omega \wedge \tau = (*\omega, \tau) dx_1 \wedge \cdots dx_n, \quad \omega \in \Lambda^p(D),\ \tau \in \Lambda^{n-p}(D).$$

Given $\omega \in \Lambda^p(D)$ and $\theta \in \Lambda^{p-1}(D)$, we have

$$\begin{aligned} &(d\theta, \omega)\, dx_1 \wedge \cdots \wedge dx_n = d\theta \wedge *\omega \\ &= d(\theta \wedge *\omega) + (-1)^p \theta \wedge d * \omega \end{aligned}$$

and hence

$$\begin{aligned} &\int_D (d\theta, \omega)\, dx_1 \wedge \cdots \wedge dx_n = \int_{\partial D} \theta \wedge *\omega \\ &+ \int_D (\theta, \delta\omega)\, dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{2.2}$$

if $\theta, \omega \in H^1(D)$.

The volume and area elements on D and ∂D are given by

$$dx = dx_1 \wedge \cdots \wedge dx_n$$

and

$$ds = \sum_i \nu^i * dx_i,$$

respectively, where $\nu = (\nu^i)$ denotes the outer unit normal vector on ∂D. We thus obtain the vector area element

$$\nu ds = (*dx_1, \cdots, *dx_n)^T.$$

Henceforth, we write

$$\int_D \cdots\ dx_1 \wedge \cdots \wedge dx_n = \int_D \cdots$$

and

$$\int_{\partial D} \cdots\ ds = \int_{\partial D} \cdots,$$

in short.

The following lemmas are nothing but Propositions 1 and 2 of [4]. Here, we provide the proof for completeness.

Lemma 2.1. *If φ, B, and J are 0-form, 1-form, and 2-form belonging to $H^1(D)$, respectively, it holds that*

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B, \nu)\varphi \tag{2.3}$$

and

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J). \tag{2.4}$$

Proof. Having

$$\varphi|_{\partial D}\,, \; B|_{\partial D}\,, \; J|_{\partial D} \in H^{1/2}(\partial D),$$

we apply (2.2) to $\omega = B$ and $\theta = \varphi$. Since

$$\theta \wedge *\omega = \varphi \cdot *B = \varphi \cdot (B, \nu)\, ds \quad \text{on } \partial D,$$

equality (2.3) holds. For (2.4), we put $\omega = J$ and $\theta = B$ in (2.2). It arises that

$$\theta \wedge *\omega = B \wedge *J = (\nu \wedge B, J)\, ds \quad \text{on } \partial D,$$

and hence the conclusion. □

Henceforth, X' denotes the dual space of the Banach space X over **R**, and $\langle\ ,\ \rangle$ denotes the paring between X and X'. We put, in particular,

$$H^{-1/2}(\partial D) = H^{1/2}(\partial D)'.$$

Lemma 2.2. *Let p be 0-form in $H^1(D)$.*

(1) If $\Delta p \in H^1(D)'$, then

$$(dp, \nu)|_{\partial D} \in H^{-1/2}(\partial D) \tag{2.5}$$

is well-defined, and it holds that

$$\langle \varphi, (dp, \nu) \rangle = \int_D (d\varphi, dp) + \langle \varphi, \Delta p \rangle, \quad \forall \varphi \in H^1(D). \tag{2.6}$$

(2) The 2-form

$$\nu \wedge dp|_{\partial D} \in H^{-1/2}(\partial D)$$

is well-defined, and is continuous in $p \in H^1(D)$. It holds that

$$\langle J, \nu \wedge dp \rangle = -\int_D (\delta J, dp) \tag{2.7}$$

for any 2-form $J \in H^1(D)$.

Proof. In the first case we have $p \in H^1(D)$ and $\Delta p \in H^1(D)'$, and hence the mapping

$$\varphi \in H^1(D) \mapsto \int_D (d\varphi, dp) + \langle \varphi, \Delta p \rangle$$

is bounded linear. We show that this mapping is reduced to

$$\varphi \in H^{1/2}(\partial D) \mapsto \int_D (\varphi, dp) + \langle \varphi, \Delta p \rangle, \tag{2.8}$$

to define $(dp, \nu)|_{\partial D}$ in (2.5) by (2.8).

In fact, since (1.5) holds, the well-posedness of (2.8) follows from

$$\int_D (d\varphi, dp) + \langle \varphi, \Delta p \rangle = 0, \quad \forall \varphi \in H_0^1(D).$$

This equality is reduced to

$$\int_D (d\varphi, dp) + \langle \varphi, \Delta p \rangle = 0, \quad \forall \varphi \in C_0^\infty(D),$$

or

$$\int_D (d\varphi, dp) + (\Delta \varphi, p) = 0, \quad \forall \varphi \in C_0^\infty(D), \tag{2.9}$$

which is valid to $p \in H^1(D)$ by (2.3).

If $p \in H^2(D)$, the above $(dp, \nu)|_{\partial D}$ coincides with

$$(dp|_{\partial D}, \nu) \in H^{1/2}(D), \tag{2.10}$$

by (2.3) for $B = dp$ and

$$\delta d = -\Delta \quad \text{on } \Lambda^0(D),$$

and therefore, this

$$(dp, \nu)|_{\partial D}$$

in (2.5) for $p \in H^1(D)$ with $\Delta p \in H^1(D)'$ is consistent with (2.10) for $p \in H^2(D)$.

The proof of the second case is similar. First, given $p \in H^1(D)$, we regard the right-hand side of (2.6) as a bounded linear mapping of 2-forms belonging to $H^1(D)$:

$$J \in H^1(D) \ \mapsto \ -\int_D (dp, \delta J).$$

We note that this mapping is continuous in $p \in H^1(D)$ in the operator norm. Second, this mapping is regarded as an element in $H^{-1/2}(\partial D)$ by (1.5), because the right-hand side is 0 for $J \in H_0^1(D)$:

$$J \in H^{1/2}(D) \ \mapsto \ -\int_D (dp, \delta J),$$

which ensures the well-posedness of $\nu \wedge dp \in H^{-1/2}(\partial D)$ by

$$\langle J, \nu \wedge dp \rangle = -\int_D (dp, \delta J).$$

Finally, we observe that equality (2.7) for $p \in H^2(D)$ holds with

$$(\nu \wedge dp|_{\partial D}, \nu) \in H^{1/2}(\partial D)$$

by equality (2.4) applied to $B = dp$ by $d^2 = 0$, and hence the above

$$\nu \wedge dp \in H^{-1/2}(\partial D)$$

is identified with

$$\nu \wedge dp|_{\partial D} \in H^{1/2}(\partial D)$$

if $p \in H^2(\Omega)$. □

Remark 2.1. *Writing*

$$\langle dp, \nu\rangle|_{\partial D} = \frac{\partial p}{\partial \nu} \in H^{-1/2}(\partial D)$$

in (2.5), we obtain Green's formula

$$\begin{aligned}&\langle g, \Delta h\rangle_{H^1(D),H^1(D)'} - \langle h, \Delta g\rangle_{H^1(D),H^1(D)'}\\ &= \left\langle g, \tfrac{\partial h}{\partial \nu}\right\rangle_{H^{1/2}(\partial D),H^{-1/2}(\partial D)} - \left\langle h, \tfrac{\partial g}{\partial \nu}\right\rangle_{H^{1/2}(\partial D),H^{-1/2}(\partial D)}\end{aligned} \tag{2.11}$$

valid to $g, h \in H^1(D)$ *with* $\Delta g, \Delta h \in H^1(D)'$.

Remark 2.2. *If* $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ *is a bounded Lipschitz domain with* $C^{0,1}$ *interface* $\mathcal{M}$ *and* $\Gamma = \Gamma_\pm = \partial\Omega_\pm$, *any* 0*-form* $p \in H^1(\Omega_\pm)$ *admits* 2*-forms on* $\Gamma_\pm$ *as in*

$$\nu \wedge dp|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm) = H_0^{1/2}(\Gamma_\pm)'.$$

3. Proof of Theorem 1.2

To begin with, let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{0,1}$ interface. If $p \in H^1(\Omega)$ is 0-form, the 2-forms

$$\nu \wedge dp|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)$$

are well-defined by Remark 2.2. Identifying $H^{-1/2}(\Gamma_\pm)$ with $H^{-1/2}(\Gamma)$, we define 2-form on Γ by

$$[\nu \wedge dp]_-^+ = \nu \wedge dp|_{\Gamma_+} - \nu \wedge dp|_{\Gamma_-} \in H^{-1/2}(\Gamma).$$

Recall that ν is the outer unit normal vector on Γ_- extended smoothly on Ω. Then we use the following lemma proven by [1].

Lemma 3.1. *If* $p \in H^1(\Omega)$, *it holds that*

$$[\nu \wedge dp]_-^+ = 0 \quad \text{in } H^{-1/2}(\Gamma). \tag{3.1}$$

Proof. Given 2-form J on Ω of which coefficients are in $C_0^\infty(\Omega)$, we obtain

$$\pm \langle J, \nu \wedge dp\rangle_{H_0^{1/2}(\Gamma_\pm),H^{-1/2}(\Gamma_\pm)} = \int_{\Omega_\pm} (\delta J, dp)$$

by Lemma 2.2, which implies

$$\left[\langle J, \nu \wedge dp\rangle_{H_0^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\right]_-^+ = \int_\Omega (\delta J, dp). \tag{3.2}$$

The right-hand side of (3.2) is equal to 0 if $p \in H^2(\Omega)$ by (2.4) for $B = dp$ and $D = \Omega$, because the coefficients of J are in $C_0^\infty(\Omega)$:

$$\left[\langle J, \nu \wedge dp\rangle_{H_0^{1/2}(\Gamma), H^{-1/2}(\Gamma)}\right]_-^+ = 0. \tag{3.3}$$

Then, equality (3.1) follows, because $J \in C_0^\infty(\Omega)$ is arbitrary. This equality (3.1) is extended to $p \in H^1(\Omega)$ from the continuity of

$$p \in H^1(\Omega) \ \mapsto \nu \wedge dp \in H^{-1/2}(\Gamma_\pm)$$

because $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. □

Henceforth, we write

$$a_i = \frac{\partial a}{\partial x_i}$$

for $a = a(x_1, \cdots, x_n)$ in short.

Lemma 3.2. *If $p \in H^1(\Omega)$ is 0-form, it holds that*

$$\left[\nu^i p_j - \nu^j p_i\right]_-^+ = 0, \ 1 \le i, j \le n, \quad in\ H^{-1/2}(\Gamma),$$

where $p_i = \frac{\partial p}{\partial x_i}$.

Proof. The result is a direct consequence of Lemma 3.1 because

$$dp = \sum_i p_i dx_i$$

and

$$\nu \wedge dp = \sum_{i<j} (\nu^i p_j - \nu^j p_i) dx_i \wedge dx_j$$

hold. □

Now, we show the key lemma. Let $1 \le p \le n-1$ and

$$\omega = \sum_{i_1 < \cdots < i_p} \omega^{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{3.4}$$

be a p-form in $H^1(\Omega)$. Given $1 \le i_1, \cdots, i_p \le n$, we put

$$\tilde{\omega}^{i_1 \cdots i_p} = \text{sgn}\,\sigma \cdot \omega^{i'_1 \cdots i'_p} \tag{3.5}$$

for $i'_1 < \cdots < i'_p$, where $\sigma : (i_1, \cdots, i_p) \mapsto (i'_1, \cdots, i'_p)$. Then, it follows that

$$\delta\omega = -\sum_{i_2 < \cdots < i_p} \sum_\ell \tilde{\omega}^{\ell i_2 \cdots i_p} dx_{i_2} \wedge \cdots \wedge dx_{i_p}. \tag{3.6}$$

Here, we define the 1-form in $H^1(\Omega)$ by

$$\hat{\omega}^{i_2 \cdots i_p} = \sum_\ell \tilde{\omega}^{\ell i_2 \cdots i_p} dx_\ell. \tag{3.7}$$

Henceforth, we say $A \sim B$ if

$$[A - B]_-^+ = 0$$

for $A, B \in H^{-1/2}(\Gamma)$.

Lemma 3.3. *If $\Omega \subset \mathbf{R}^n$ is a bounded Lipshitz domain with $C^{1,1}$ interface, any p-form $\omega \in H^1(\Omega)$ admits*

$$\left[\delta\omega + \sum_{i_2<\cdots<i_p} (\nu, d)(\hat{\omega}^{i_2\cdots i_p}, \nu) dx_{i_2} \wedge \cdots dx_{i_p}\right]_-^+ = 0 \quad \text{in } H^{-1/2}(\Gamma), \tag{3.8}$$

where

$$(\nu, d) = \sum_\ell \nu^\ell \frac{\partial}{\partial x_\ell}.$$

Proof. Take ω as in (3.4), and fix $i_2 < \cdots < i_p$. We put

$$B = \sum_\ell B^\ell dx_\ell, \quad B^\ell = \tilde{\omega}^{\ell i_2 \cdots i_p},$$

recalling (3.5). Then, it holds that

$$\begin{aligned} &\textstyle\sum_\ell \tilde{\omega}^{\ell i_2\cdots i_p} - (\nu, d)(\hat{\omega}^{i_2\cdots i_p}, \nu) = \sum_\ell \left\{B^\ell_\ell - \nu^\ell (B, \nu)_\ell\right\} \\ &\sim \textstyle\sum_\ell B^\ell_\ell - \sum_{\ell,k} \nu^k \nu^\ell B^k_\ell = \sum_\ell B^\ell_\ell - \sum_{\ell,k} \nu^k \nu^\ell B^\ell_k \\ &= \textstyle\sum_\ell \left\{B^\ell_\ell - \nu^\ell (\nu, d) B^\ell\right\} \end{aligned}$$

because ν is extended as a $C^{0,1}$ vector field on Ω from the assumption.

Here we fix ℓ, set $p = B^\ell$, and notice

$$\begin{aligned} &B^\ell_\ell - \nu^\ell(\nu, d)B^\ell = p_\ell - \nu^\ell(\nu, d)p = \textstyle\sum_k \left\{(\nu_k)^2 p_\ell - \nu^\ell \nu^k p_k\right\} \\ &= \textstyle\sum_k \nu^k(\nu^k p_\ell - \nu^\ell p_k). \end{aligned}$$

Then, it follows that

$$[B^\ell_\ell - \nu^\ell(\nu, d)B^\ell]^+_- = 0 \quad \text{in } H^{-1/2}(\Gamma)$$

from Lemma 3.2. Thus, we obtain

$$\left[\sum_\ell \tilde{\omega}^{\ell i_2\cdots i_p}_\ell - (\nu, d)(\hat{\omega}^{i_2\cdots i_p}, \nu)\right]^+_- = 0 \quad \text{in } H^{-1/2}(\Gamma),$$

and then (3.6) implies (3.8). □

We are ready to give the following proof:

Proof of Theorem 1.2. Using $\delta\omega = 0$ in (1.6), we obtain

$$\left[(\nu, d)(\hat{\omega}^{i_2\cdots i_p}, \nu)\right]^+_- = 0 \quad \text{in } H^{-1/2}(\Gamma) \tag{3.9}$$

for any $i_2 < \cdots < i_p$ by (3.8). It holds also that

$$-\Delta\omega = (d\delta + \delta d)\omega = \delta\theta \in L^2(\Omega_\pm).$$

Then, we get

$$\Delta\omega^{i_1\cdots i_p} \in L^2(\Omega_\pm)$$

for any $i_1 < \cdots < i_p$ by (3.4), and hence

$$h^{i_2\cdots i_p} \equiv -\Delta(\hat{\omega}^{i_2\cdots i_p}, \nu) \in L^2(\Omega_\pm)$$

for any $1 \leq i_2, \cdots, i_p \leq n$ by (3.7), because ν is extended as a $C^{1,1}$ vector field on Ω from the assumption.

Then equality (3.9) implies

$$\begin{aligned} -\int_\Omega (\hat{\omega}^{i_2\cdots i_p}, \nu)\Delta\varphi &= \left(\int_{\Omega_+} + \int_{\Omega_-}\right) h^{i_2\cdots i_p}\varphi \\ &= \int_\Omega h^{i_2\cdots i_p}\varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$ by Green's formula, (2.11). Thus, we obtain

$$\Delta(\hat{\omega}^{i_2\cdots i_p}, \nu) \in L^2(\Omega)$$

and hence

$$\Delta\beta \in L^2(\Omega)$$

for the $(p-1)$-form β defined by

$$\beta = \sum_{i_2<\cdots<i_p} (\hat{\omega}^{i_2\cdots i_p}, \nu) dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

The conclusion (1.7) is thus reduced to the following lemma: □

Lemma 3.4. *It holds that*

$$\nu \wedge *\omega = *\beta \tag{3.10}$$

as $(n-p+1)$-forms.

Proof. It suffices to show (3.10) for

$$\omega = dx_1 \wedge \cdots \wedge dx_p. \tag{3.11}$$

In this case it holds that

$$\tilde{\omega}^{i_1\cdots i_p} = \begin{cases} \operatorname{sgn}\sigma, & \{i_1, \cdots, i_p\} = \{1, \cdots, p\} \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma : (i_1, \cdots, i_p) \mapsto (1, \cdots, p)$. Then, (3.7) implies

$$\begin{aligned} \hat{\omega}^{i_2\cdots i_p} &= \sum_\ell \tilde{\omega}^{\ell i_2\cdots i_p}\, dx_\ell \\ &= \begin{cases} \sum_\ell \operatorname{sgn}\sigma_\ell\, dx_\ell, & 1 \leq i_2, \cdots, i_p \leq p \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\sigma_\ell : (\ell, i_2, \cdots, i_p) \mapsto (1, \cdots, p)$. We have, for example,

$$(\hat{\omega}^{2,\cdots,p}, \nu) = \nu^1,$$

and hence

$$\begin{aligned} \beta &= \sum_{i_2<\cdots<i_p} (\hat{\omega}^{i_2\cdots i_p}, \nu) dx_{i_2} \wedge \cdots \wedge dx_{i_p} \\ &= \nu^1 dx_2 \wedge \cdots \wedge dx_p + \nu^2 dx_3 \wedge \cdots \wedge dx_p \wedge dx_1 \\ &\quad + \cdots + \nu^p dx_1 \wedge \cdots \wedge dx_{p-1}. \end{aligned}$$

It thus holds that

$$\begin{aligned}*\beta = &\nu^1 dx_1 \wedge dx_{p+1} \wedge \cdots dx_n + \nu^2 dx_2 \wedge dx_{p+1} \wedge \cdots \wedge dx_n \\ &+\nu^p dx_p \wedge dx_{p+1} \wedge \cdots \wedge dx_n.\end{aligned}$$

By (3.11), on the other hand, we obtain

$$*\omega = dx_{p+1} \wedge \cdots \wedge dx_n$$

and hence

$$\begin{aligned}&\nu \wedge *\omega = (\nu^1 dx_1 + \cdots + \nu^n dx_n) \wedge dx_{p+1} \wedge \cdots \wedge dx_n \\ &= \nu^1 dx_1 \wedge dx_{p+1} \wedge \cdots \wedge dx_n + \nu^2 dx_2 \wedge dx_{p+1} \wedge \cdots \wedge dx_n \\ &+ \cdots + \nu^p dx_p \wedge dx_{p+1} \wedge \cdots \wedge dx_n = *\beta,\end{aligned}$$

which completes the proof. □

4. Applications

4.1. Theorem 1.2 for 1*-forms*

Theorem 1.2 for 1-forms recovers a result obtained by [3].

Theorem 4.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega \in H^1(\Omega)$ be* 1*-form. Assume (1.6). Then, it holds that*

$$\Delta(\nu \cdot \omega) \in L^2(\Omega), \tag{4.1}$$

where $\cdot$ denotes the $\mathbf{R}^n$-inner product.

Proof. Writing

$$\omega = \sum_i \omega^i dx_i, \tag{4.2}$$

we obtain

$$*\omega = \sum_i (-1)^{i+1} \omega^i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$$

and hence

$$\begin{aligned}&\nu \wedge *\omega = \textstyle\sum_{i,j} \nu^j (-1)^{i+1} \omega^i dx_j \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_n \\ &= \textstyle\sum_j \nu^j w^j dx_1 \wedge \cdots \wedge dx_n = (\nu, \omega) dx_1 \wedge \cdots \wedge dx_n.\end{aligned}$$

Then, the result follows immedately from Theorem 1.2. □

For ω in (4.2), we obtain

$$\begin{aligned}&d * \omega = \textstyle\sum_i (-1)^{i+1} \omega^i_i dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \\ &= \textstyle\sum_i \omega^i_i dx_1 \wedge \cdots \wedge dx_n\end{aligned} \tag{4.3}$$

and hence $\delta\omega = 0$ if and only if div $\omega = \sum_i \omega^i_i = 0$. Then the 2-form θ is defined by

$$d\omega = \sum_{i<j} (-\omega^i_j + \omega^j_i) dx_i \wedge dx_j = \theta. \tag{4.4}$$

An elementary calculation now ensures

$$d * \theta = \sum_{i,j} (-1)^{j+1} \chi_{ij} (-\omega^i_{ij} + \omega^j_{ii}) dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots dx_n$$

for

$$\chi_{ij} = \begin{cases} 1, & i < j \\ 0, & i = j \\ -1. & i > j \end{cases}$$

Hence, $\delta\theta = *^{-1} d * \theta \in L^2(\Omega_\pm)$ if and only if

$$\sum_i \chi_{ij} (-\omega^i_{ij} + \omega^j_{ii}) \in L^2(\Omega_\pm), \quad 1 \leq j \leq n. \tag{4.5}$$

By these observations, Theorem 4.1 on a 1-form induces the following theorem on a vector field:

Theorem 4.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega = (\omega^i) \in H^1(\Omega)$ be a vector field satisfying*

$$div\ \omega = 0 \ in\ \Omega, \quad \Delta\omega \in L^2(\Omega_\pm).$$

Then it holds that (4.1).

Proof. By Theorem 4.1, if $\omega = (\omega^i) \in H^1(\Omega)$ satisfies div $\omega = 0$ in Ω and (4.5), then (4.1) holds. Here, Eq (4.5) with $j = 1$ means

$$\sum_{i=2}^n (-\omega^i_{i1} + \omega^1_{ii}) = \omega^2_{11} + \sum_{i=2}^n \omega^1_{ii} = \Delta\omega^1 \in L^2(\Omega_\pm)$$

by div$\omega = 0$. The other case of j is similar, and we obtain the result. □

Theorem 4.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface. Assume that $v \in H^1(\Omega; \mathbf{R}^n)$ and $p \in H^1(\Omega_\pm)$ satisfy the stationary Stokes system*

$$\Delta v = \nabla p \ in\ \Omega_\pm, \quad div\ v = 0 \ in\ \Omega.$$

Then, it holds that

$$\Delta(v, v) \in L^2(\Omega).$$

Proof. This theorem is an immediate consequence of Theorem 4.2 applied to $\omega = v$. □

Theorem 4.4. *Let $\Omega \subset \mathbf{R}^n \times \mathbf{R}$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and assume that $u \in H^2(\Omega)$ satisfy*

$$\Box u = 0 \ in\ \Omega, \quad u \in H^3(\Omega_\pm),$$

where $\Box = \Delta_x - \partial_t^2$. Then, it holds that

$$\Delta(\vec{v}, v) \in L^2(\Omega)$$

for

$$\vec{v} = \begin{pmatrix} \nabla_x u \\ -u_t. \end{pmatrix}.$$

Proof. This theorem follows from Theorem 4.1 for

$$\omega = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx_i - u_t dt.$$

$\square$

Remark 4.1. *If $\Omega = D \times (-T, T)$ with bounded Lipschitz domain $D \subset \mathbf{R}^n$ and $\mathcal{M} = \{t = 0\}$, it holds that $\Omega_- = D \times (-T, 0)$ and $\Omega_+ = D \times (0, T)$. In this case, we obtain*

$$\Delta u_t \in L^2(\Omega),$$

provided that

$$u \in H^2(\Omega), \quad \Box u = 0 \text{ in } \Omega, \quad u \in H^3(\Omega_\pm).$$

4.2. Theorem 1.3 for 1-forms

Theorem 1.3 for 1-forms also recovers a result obtained by [3].

Theorem 4.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega \in H^1(\Omega)$ be a 1-form. Assume (1.8). Then, it holds that*

$$\Delta \omega_\tau \in L^2(\Omega), \tag{4.6}$$

where $\omega_\tau = \omega - (\omega, \nu)\nu$.

Proof. Since Theorem 1.3 implies

$$\Delta(\nu \wedge \omega) \in L^2(\Omega),$$

this theorem is reduced to the following lemma applied to $\theta = \omega_\tau$: $\square$

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and ν be a $C^{1,1}$ vector field on Ω. Assume that $\theta \in H^1(\Omega)$ is 1-form satisfying*

$$(\nu, \theta) = 0 \tag{4.7}$$

and

$$\Delta(\nu \wedge \theta) \in L^2(\Omega). \tag{4.8}$$

Then, it holds that $\Delta\theta \in L^2(\Omega)$.

Proof. Writing

$$\theta = \sum_i \theta^i dx_i,$$

we obtain

$$\nu \wedge \theta = \sum_{i<j} (\nu^i \theta^j - \nu^j \theta^i)\, dx_i \wedge dx_j.$$

Then, assumption (4.8) with $\nu \in C^{1,1}$ ensures

$$\nu^i \Delta\theta^j - \nu^j \Delta\theta^i \in L^2(\Omega), \quad 1 \leq i, j \leq n,$$

which means

$$\vec{f} \equiv \nu(\Delta\theta)^T - (\Delta\theta)^T \nu \in L^2(\Omega).$$

Equality (4.7), on the other hand, implies

$$g \equiv (\nu, \Delta\theta) \in L^2(\Omega)$$

by $\nu \in C^{1,1}$ and $\theta \in H^1(\Omega)$. Thus, we obtain

$$F_\nu(\Delta\theta) = (\vec{f}, g), \tag{4.9}$$

where

$$F_\nu : b \in \mathbf{R}^n \mapsto (\nu b^T - b\nu^T, b^T\nu) \in \mathbf{A}_n \times \mathbf{R}$$

and A_n denotes the set of (n, n) skew-symmetric real matrices.

To examine the invertibility of this F_ν, we assume the case

$$\nu_0 = (1, 0, \cdots, 0)^T.$$

Then, it holds that

$$F_\nu(b) = (B, b_1), \quad B = \begin{pmatrix} 0 & -b_2 & \cdots & -b_n \\ b_2 & 0 & \cdots & -b_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & b_{n-1} & \cdots & \cdot \end{pmatrix}$$

and hence F_{ν_0} is an isomorphism. Since ν is realized by the rotation of ν_0 with $\in C^{1,1}$ regularity, we obtain

$$\Delta\theta = \vec{c} \cdot \vec{f} + dg$$

by (4.9), where $\vec{c} = \vec{c}(x) \in \mathbf{R}^n$ and $d = d(x) \in \mathbf{R}$ are $C^{1,1}$ in $x \in \Omega$. Then, we obtain $\Delta\theta \in L^2(\Omega)$. □

Theorem 4.5 takes the following form concerning the vector field:

Theorem 4.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $\omega = (\omega^i) \in H^1(\Omega)$ be a vector field satisfying*

$$\omega^i_j = \omega^j_i \text{ in } \Omega, \quad 1 \le i, j \le n$$

and

$$(div\ \omega)_j \in L^2(\Omega_\pm), \quad 1 \le j \le n.$$

Then, it follows that (4.6).

Proof. We identify ω as the 1-form defined by (4.2), to get (4.3) and (4.4). Then, the result follows from Theorem 4.5. □

Theorem 4.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $C^{2,1}$ interface, and $p \in H^2(\Omega)$ be a 0-form. Assume*

$$\frac{\partial}{\partial x_i}(\Delta p) \in L^2(\Omega_\pm), \quad 1 \le i \le n.$$

Then it holds that

$$\Delta\frac{\partial p}{\partial\tau} \in L^2(\Omega),$$

where

$$\frac{\partial p}{\partial\tau} = \nabla p - (\nu\cdot\nabla p)\nu.$$

Proof. This theorem is a direct consequence of Theorem 4.6, applied to $\omega = \nabla p$. □

Remark 4.2. *If we modify Theorem 1.3 to a* $(n, 1)$*-Minkowski space, the above* Δ *in Theorem 4.7 is replaced by*

$$\Box = \Delta_x - \partial_t^2.$$

The gradient and inner product are changed accordingly as

$$\nabla = \begin{pmatrix} \nabla_x \\ -\partial_t \end{pmatrix}$$

and

$$(a, b) = \sum_{i=1}^{n} a_i b_i - a_0 b_0$$

for $a = (a_1, \cdots, a_n, a_0)^T$ *and* $b = (b_1, \cdots, b_n, b_0)$*, respectively.*

4.3. 2-forms on $(3, 1)$-Minkowski spaces

Theorems 1.2 and 1.3 are modified as theorems in Minkowski spaces. The non-stationary Maxwell equation (1.1) is then reduced to (1.6) or (1.8) for a 2-form ω on a $(3, 1)$-Minkowski space, which ensures Theorem 1.1. See [4] for details.

5. Conclusions

We introduced $d - \delta$ systems of differential forms in Euclidean spaces, to describe several models in mathematical physics. We obtained interface vanishing of the solution, which means that if outer forces are piecewise regular, then several combinations of the components of the solution accordingly gain the regularity across the interface. This result was applied to non-stationary Maxwell, Stokes, and hyperbolic systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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