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Research article

Dual Brunn-Minkowski inequality for C-star bodies

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Abstract: In this paper, we introduced the concept of C-star bodies in a fixed pointed closed convex cone C and studied the dual mixed volume for C-star bodies. For C-star bodies, we established the corresponding dual Brunn-Minkowski inequality, dual Minkowski inequality, and dual Aleksandrov-Fenchel inequality. Moveover, we found that the dual Brunn-Minkowski inequality for C-star bodies can strengthen the Brunn-Minkowski inequality for C-coconvex sets.

Keywords: C-star bodies; Lutwak sum; dual mixed volume; dual Brunn-Minkowski inequality
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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the standard inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. We write $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$. Denote by $\mathscr{K}^n$ the class of all compact convex sets with nonempty interior in $\mathbb{R}^n$. In this paper, the volume (i.e., the n-dimensional Lebesgue measure) of $K \in \mathscr{K}^n$ is written by $V(K)$.

For $K, L \in \mathscr{K}^n$ and $\lambda, \mu > 0$, the Minkowski combination $\lambda K + \mu L$ is defined by

$$\lambda K + \mu L = \{\lambda x + \mu y \,|\, x \in K,\ y \in L\}.$$

The famous Brunn-Minkowski inequality in the classical Brunn-Minkowski theory states that for $K, L \in \mathscr{K}^n$ and $\lambda \in (0, 1)$, there is (see, e.g., [24])

$$V((1-\lambda)K + \lambda L)^{\frac{1}{n}} \geqslant (1-\lambda)V(K)^{\frac{1}{n}} + \lambda V(L)^{\frac{1}{n}}, \tag{1.1}$$

with equality if, and only if, K and L are homothetic. It is worth noting that the Brunn-Minkowski inequality (1.1) is equivalent to the following form:

$$V((1-\lambda)K + \lambda L) \geqslant V(K)^{(1-\lambda)} V(L)^{\lambda},$$

with equality if, and only if, K and L are parallel. This shows that the volume functional on the commutative semi-group $\mathscr{K}^n$ is log-concave. More generally, the corresponding analytical version of the Brunn-Minkowski inequality, the Prékopa-Leindler inequality, ensures that the inequality (1.1) holds for two Borel measurable sets K and L if $K + L$ is still a Borel measurable set. See [6, 24] for more discussion on the Brunn-Minkowski inequality and its rich applications.

Lutwak [10] first presented the dual mixed volume for star bodies. Therefore, the dual Brunn-Minkowski theory has achieved significant developments on many problems; see[3, 4, 7, 12, 13, 31, 32, 33]. Moreover, there are a number of new geometric measures (e.g., the q-th dual curvature measure) induced by the dual mixed volume in the dual Brunn-Minkowski theory; see [8, 20]. This duality between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory plays an important role in convex geometric analysis. For example, there is the following dual Brunn-Minkowski inequality (see [10, 12])

$$V((1-\lambda)M\tilde{+}\lambda N)^{\frac{1}{n}} \leqslant (1-\lambda)V(M)^{\frac{1}{n}} + \lambda V(N)^{\frac{1}{n}}, \tag{1.2}$$

for star bodies M, N and $\lambda \in (0, 1)$, where the Lutwak combination $(1-\lambda)M\tilde{+}\lambda N$ is defined by

$$(1-\lambda)M\tilde{+}\lambda N = \bigcup_{u\in\mathbb{S}^{n-1}} \{M \cap \{tu \,|\, t \geqslant 0\} + N \cap \{tu \,|\, t \geqslant 0\}\}.$$

Compared with the inequalities (1.1) and (1.2), we can obtain

$$V((1-\lambda)K + \lambda L)^{\frac{1}{n}} \geqslant (1-\lambda)V(K)^{\frac{1}{n}} + \lambda V(L)^{\frac{1}{n}} \geqslant V((1-\lambda)K\tilde{+}\lambda L)^{\frac{1}{n}}, \tag{1.3}$$

for $K, L \in \mathscr{K}^n$ with the origin $o \in K, L$.

Recently, Schneider [25, 26] established the Brunn-Minkowski theory for the unbounded closed convex sets in a fixed pointed closed convex cone C. Let $\mathbb{A}$ be an unbounded closed convex set in C and $A = C \setminus \mathbb{A}$ be called a C-coconvex set if A has finite volume. For $0 < \lambda < 1$, the co-sum $(1-\lambda)A_1 \oplus \lambda A_2$ of two C-coconvex sets A_1, A_2 is defined by

$$C \setminus ((1-\lambda)A_1 \oplus \lambda A_2) = (1-\lambda)(C \setminus A_1) + \lambda(C \setminus A_2).$$

There is the following Brunn-Minkowski inequality for C-coconvex sets (see [25]):

$$V((1-\lambda)A_1 \oplus \lambda A_2)^{\frac{1}{n}} \leqslant (1-\lambda)V(A_1)^{\frac{1}{n}} + \lambda V(A_2)^{\frac{1}{n}}, \tag{1.4}$$

with equality if, and only if, $A_1 = \alpha A_2$ for some $\alpha > 0$. Later, the L_p Brunn-Minkowski theory for C-coconvex sets was established in [30] and the dual Minkowski problem for C-coconvex sets was solved in [21], which further refined the Brunn-Minkowski theory for C-coconvex sets. Moreover, the authors in [1] pointed out that there is a duality to the polarity in the classical Brunn-Minkowski theory and they called it the copolarity. By the wonderful works in [21, 23, 27, 28, 29], there is a close connection between the copolarity and the theory of C-coconvex sets.

Inspired by the works of Schneider and Lutwak, we extend the concept of C-coconvex sets to C-star bodies. Denote by $\mathscr{S}^n_C$ the class of all C-star bodies. Importantly, we can prove that the C-radial sum $M\tilde{+}_c N$ of C-star bodies M, N is also a C-star body. That is, $\mathscr{S}^n_C$ is a commutative semi-group with respect to the Lutwak addition. Moreover, if we consider the scalar multiplication, then $\mathscr{S}^n_C$ is a closed convex cone in the space of the continuous functions on the relative interior of $C \cap \mathbb{S}^{n-1}$. Similar to the duality between convex bodies and star bodies in the classical setting, we establish the dual Brunn-Minkowski inequality for C-star bodies, which can strengthen the Brunn-Minkowski inequality (1.4).

Theorem 1.1 (Dual Brunn-Minkowski inequality for C-star bodies)**.** *Let $M, N \in \mathscr{S}_C^n$ and $0 < \lambda < 1$, then*

$$V((1-\lambda)M\tilde{+}_c\lambda N)^{\frac{1}{n}} \leqslant (1-\lambda)V(M)^{\frac{1}{n}} + \lambda V(N)^{\frac{1}{n}}, \tag{1.5}$$

with equality if, and only if, $M = \lambda N$ for some $\lambda > 0$. Moveover, for C-coconvex sets A_1, A_2, there is

$$V((1-\lambda)A_1 \oplus \lambda A_2)^{\frac{1}{n}} \leqslant V((1-\lambda)A_1\tilde{+}_c\lambda A_2)^{\frac{1}{n}} \leqslant (1-\lambda)V(A_1)^{\frac{1}{n}} + \lambda V(A_2)^{\frac{1}{n}}. \tag{1.6}$$

Remark 1.1. *Our inequality* (1.5) *has the same form as the inequality* (1.2)*, but they have different meanings. That is, the inequality* (1.5) *holds for the non-compact star-shaped sets.*

Remark 1.2. *In the classical setting, the inequality* (1.2) *can't strengthen the inequality* (1.1)*, so we can only obtain the continued inequality* (1.3)*. However, for C-coconvex sets, our inequality* (1.5) *can strengthen the inequality* (1.4)*. That is, we can obtain the inequality* (1.6)*.*

This paper is organized as follows. In Section 2, we provide some necessary backgrounds for the C-coconvex sets. In Section 3, we introduce the C-star bodies, C-radial sum, and dual mixed volume for C-star bodies. Section 4 contains the dual Brunn-Minkowski inequality, dual Minkowski inequality, and dual Aleksandrov-Fenchel inequality for C-star bodies. In Section 5, we provide a supplementary note on the equality case of the Brunn-Minkowski inequality for C-coconvex sets in [25]. As applications, we show the Brunn-Minkowski type inequalities for volume difference and volume sum in Section 6.

2. Preliminaries

We write B^n for the unit ball in $\mathbb{R}^n$. The topological boundary and the topological interior of a subset $E \subset \mathbb{R}^n$ are denoted by ∂E and $\mathrm{int}(E)$, respectively. Denote by H_u^- a negative half-space with the outer normal vector $u \in \mathbb{S}^{n-1}$. A compact convex set K in $\mathbb{R}^n$ with nonempty interior is called a convex body and its support function is defined by $h_K(u) = \max\{\langle x, u\rangle \,|\, x \in K\}$ for each $u \in \mathbb{S}^{n-1}$. The Minkowski sum $K + L$ of convex bodies K and L is also given by

$$h_{K+L}(u) = h_K(u) + h_L(u),$$

for any $u \in \mathbb{S}^{n-1}$. A set M in $\mathbb{R}^n$ is called a star-shaped set with respect to the origin if the intersection of M with any line passing the origin is a line segment. The radial function of a star-shaped set M is defined by $\rho_M(u) = \sup\{t \geqslant 0 \,|\, tu \in M\}$ for any $u \in \mathbb{S}^{n-1}$. A star-shaped set M is called a star body if the radial function ρ_M is a positive continuous function on $\mathbb{S}^{n-1}$. The Lutwak sum $M\tilde{+}N$ of star bodies M and N is also given by

$$\rho_{M\tilde{+}N}(u) = \rho_M(u) + \rho_N(u),$$

for any $u \in \mathbb{S}^{n-1}$. For two sets A and B in $\mathbb{R}^n$, A and B are homothetic if there exist $\lambda > 0$ and $x \in \mathbb{R}^n$ such that $A = \lambda B + x$. Similarly, A and B are dilated if there exists $\lambda > 0$ such that $A = \lambda B$.

For a closed convex set $E \subset \mathbb{R}^n$, we call E a closed convex cone if $\lambda x \in E$ for any $\lambda > 0$ and $x \in E$. A closed convex cone $C \subset \mathbb{R}^n$ is called pointed if $C \cap (-C) = \{o\}$, which is equivalent to saying that C is line-free. From now on, we denote by C a pointed closed convex cone. The polar cone of C is defined by

$$C^\circ = \{y \in \mathbb{R}^n \,|\, \langle y, x\rangle \leqslant 0 \text{ for any } x \in C\}.$$

Denote by $\Omega_C = \mathrm{int}(C) \cap \mathbb{S}^{n-1}$ and $\Omega_{C^\circ} = \mathrm{int}(C^\circ) \cap \mathbb{S}^{n-1}$. An unbounded closed convex set $o \notin \mathbb{A} \subset C$ is called a C-close set if $C \setminus \mathbb{A}$ has finite volume, and $A = C \setminus \mathbb{A}$ is called a C-coconvex set. Specifically, a C-close set $\mathbb{A}$ is called a C-full set if $C \setminus \mathbb{A}$ is bounded. An unbounded closed convex set $o \notin \mathbb{A} \subset C$ is called a C-asymptotic set if $\mathbb{A}$ is asymptotic to the boundary of C at infinity, i.e.,

$$\lim_{x \in \mathbb{A}, \|x\| \to +\infty} d(x, \partial C) = 0,$$

where $d(x, \partial C)$ is the distance from x to ∂C.

A closed convex set $o \notin E \subset \mathbb{R}^n$ is called a pseudo-cone if $\lambda x \in E$ for any $\lambda \geqslant 1$ and $x \in E$. The recession cone of a nonempty closed convex set E is defined by

$$\mathrm{rec}(E) = \{x \in \mathbb{R}^n | E + x \subset E\}.$$

If E is unbounded, then $\mathrm{rec}(E)$ is a closed convex cone. Moreover, a closed convex set $o \notin E \subset \mathbb{R}^n$ is a pseudo-cone if, and only if, $E \subset \mathrm{rec}(E)$. An unbounded closed convex set $o \notin E \subset C$ is called a C-compatible set if E is a pseudo-cone and $\mathrm{rec}(E) = C$. For C-compatible sets, there is the following proposition:

Proposition 2.1. *(see [21]) Let $o \notin E \subset C$ be a nonempty set, then E is a C-compatible set if, and only if,*

$$\begin{aligned} E &= \bigcap \{\tilde{E} | \tilde{E} \text{ is an unbounded closed convex set in } C \text{ containing } x+C \text{ for all } x \in E\} \\ &= \bigcap \{\tilde{E} | \tilde{E} \text{ is a } C\text{-close set such that } E \subset \tilde{E}\} \\ &= \bigcap \{\tilde{E} | \tilde{E} \text{ is a } C\text{-full set such that } E \subset \tilde{E}\} \\ &= C \cap \bigcap_{u \in \Omega_{C^\circ}} \{H_u^- | E \subset H_u^-\}. \end{aligned}$$

For $0 \neq q \in \mathbb{R}$, a C-compatible set E is called a (C, q)-close set if

$$\tilde{V}_q(E) = \frac{1}{n} \int_{\Omega_C} \rho_E^q(u) du < +\infty,$$

where $\rho_E(u) = \inf\{\lambda \geqslant 0 | \lambda u \in E\}$ is the radial function of E. See [21] for more details of (C, q)-close sets.

3. *C*-star bodies, Lutwak sum, and dual mixed volume

Similar to the classical case, we can also consider the corresponding star-shaped sets in a fixed pointed closed convex cone C as follows.

Definition 3.1. *A nonempty set $M \subset C$ is called a C-star-shaped set if for any $u \in \Omega_C$, the intersection $\{tu \,|\, t \geqslant 0\} \cap M$ is a line segment passing through the origin.*

For a C-star-shaped set M, the radial function of M is defined by

$$\rho_M(u) = \sup\{t \geqslant 0 \,|\, tu \in M\}, \text{ for any } u \in \Omega_C.$$

If ρ_M is a continuous function on Ω_C, then the volume of the C-star-shaped set M is

$$V(M) = \frac{1}{n}\int_{\Omega_C} \rho_M^n(u)du,$$

by using the polar coordinate transformation.

Definition 3.2. *A C-star-shaped set M is called a C-star body if ρ_M is a positive continuous function on Ω_C and $V(M) < +\infty$.*

We denote by $\mathscr{S}_C^n$ the class of all C-star bodies.

Remark 3.1. *Compared with the classical star bodies in $\mathbb{R}^n$, C-star bodies may be unbounded sets.*

Remark 3.2. *Since a C-coconvex set A is generally unbounded, we can only prove that the radial function of A is Lipschitz continuous on any compact set $\omega \subset \Omega_C$. Hence, C-coconvex sets are all C-star bodies.*

Similar to Lutwak's radial sum, we define the C-radial sum $M_1\tilde{+}_c M_2$ of two C-star bodies M_1 and M_2 as follows:

$$\rho_{M_1\tilde{+}_c M_2}(u) = \rho_{M_1}(u) + \rho_{M_2}(u), \text{ for any } u \in \Omega_C.$$

Next, we prove that the C-radial sum $M_1\tilde{+}_c M_2$ is still a C-star body. That is, the set of all C-star bodies are closed under the C-radial addition.

Lemma 3.1. *If $M, N \in \mathscr{S}_C^n$, then $M\tilde{+}_c N \in \mathscr{S}_C^n$.*

Proof. Clearly, $M\tilde{+}_c N$ is C-star-shaped set and $\rho_{M_1\tilde{+}_c M_2}(u) = \rho_{M_1}(u) + \rho_{M_2}(u)$ is also a positive continuous function on Ω_C. Thus, we need only to show $V(M\tilde{+}_c N) < +\infty$. Since $M, N \in \mathscr{S}_C^n$, we have

$$\frac{1}{n}\int_{\Omega_C} \rho_M^n(u)du < +\infty \text{ and } \frac{1}{n}\int_{\Omega_C} \rho_N^n(u)du < +\infty.$$

Let $\{\omega_k\}_{k=1}^{+\infty}$ be a sequence of compact sets on Ω_C and $\omega_k \uparrow \Omega_C$, then

$$\begin{aligned} \frac{1}{n}\int_{\omega_k} \rho_{M\tilde{+}_c N}^n(u)du &= \frac{1}{n}\int_{\omega_k} \sum_{i=0}^{n} \binom{n}{i} \rho_M^{n-i}(u)\rho_N^i(u)du \\ &= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n}\int_{\omega_k} \rho_M^{n-i}(u)\rho_N^i(u)du. \end{aligned}$$

By the Hölder inequality, for each $0 \leqslant i \leqslant n$,

$$\frac{1}{n}\int_{\omega_k} \rho_M^{n-i}(u)\rho_N^i(u)du \leqslant \left(\frac{1}{n}\int_{\omega_k} \rho_M^n(u)du\right)^{\frac{n-i}{n}} \left(\frac{1}{n}\int_{\omega_k} \rho_N^n(u)du\right)^{\frac{i}{n}}.$$

Let $k \to +\infty$, We have

$$\frac{1}{n}\int_{\Omega_C} \rho_M^{n-i}(u)\rho_N^i(u)du \leqslant \left(\frac{1}{n}\int_{\Omega_C} \rho_M^n(u)du\right)^{\frac{n-i}{n}} \left(\frac{1}{n}\int_{\Omega_C} \rho_N^n(u)du\right)^{\frac{i}{n}} < +\infty.$$

Therefore,

$$\begin{aligned}V(M\tilde{+}_c N) &= \frac{1}{n}\int_{\Omega_C} \rho^n_{M\tilde{+}_c N}(u)du\\ &= \lim_{k\to+\infty}\frac{1}{n}\int_{\omega_k} \rho^n_{M\tilde{+}_c N}(u)du\\ &= \sum_{i=0}^{n}\binom{n}{i}\lim_{k\to+\infty}\frac{1}{n}\int_{\omega_k}\rho_M^{n-i}(u)\rho_N^i(u)du\\ &= \sum_{i=0}^{n}\binom{n}{i}\frac{1}{n}\int_{\Omega_C}\rho_M^{n-i}(u)\rho_N^i(u)du\\ &< +\infty.\end{aligned}$$

□

Remark 3.3. *This result shows that $\mathscr{S}_C^n$ is a commutative semi-group with respect to the Lutwak addition. For the scalar multiplication, we have $\rho_{\lambda M}(u) = \lambda\rho_M(u)$ for $M \in \mathscr{S}_C^n$, $\lambda > 0$, and $u \in \Omega_C$. Clearly, $\lambda M \in \mathscr{S}_C^n$ for $M \in \mathscr{S}_C^n$ and $\lambda > 0$. Therefore, $\mathscr{S}_C^n$ is a convex cone in the linear space consisting of the continuous functions on Ω_C.*

Similar to the proof of Lemma 3.1, we have the following expansion formulas:

Theorem 3.1. *Let $M_1, \cdots, M_m \in \mathscr{S}_C^n$ and $\lambda_1, \cdots, \lambda_m > 0$, then*

$$\begin{aligned}V(\lambda_1 M_1\tilde{+}_c\cdots\tilde{+}_c\lambda_m M_m) &= \sum_{i_1,\cdots,i_n=1}^{m}\lambda_{i_1}\cdots\lambda_{i_n}\frac{1}{n}\int_{\Omega_C}\rho_{M_{i_1}}(u)\cdots\rho_{M_{i_n}}(u)du\\ &= \sum\binom{n}{q_1,\cdots,q_m}\lambda_1^{q_1}\cdots\lambda_m^{q_m}\frac{1}{n}\int_{\Omega_C}\rho_{M_1}^{q_1}(u)\cdots\rho_{M_m}^{q_m}(u)du,\end{aligned}$$

where $q_1, \cdots, q_m \in \mathbb{N},\ 0 \leqslant q_1, \cdots, q_m \leqslant n,$ and $q_1 + \cdots + q_m = n$.

Remark 3.4. *The above result can be calculated directly. Compared with the classical star bodies and the Lutwak addition, we can deal with the unbounded case in the pointed closed convex cone C.*

Corollary 3.1. *(Dual Steiner formula) Let $M \in \mathscr{S}_C^n$ and $\lambda > 0$, then*

$$V(M\tilde{+}_c\lambda(B^n\cap C)) = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{n}\int_{\Omega_C}\rho_M^{n-i}(u)du\right)\lambda^i. \tag{3.1}$$

Inspired by the above formulas, we can define some geometric functionals as follows.

Definition 3.3. *Let $M_1, \cdots, M_m \in \mathscr{S}_C^n$, $0 \leqslant q_1, \cdots, q_m \leqslant n$, and $q_1 + \cdots + q_m = n$, We define the $(q_1, \cdots, q_m)$-dual mixed volume of $M_1, \cdots, M_m$ by*

$$\tilde{V}^C_{q_1,\cdots,q_m}(M_1,\cdots,M_m) = \frac{1}{n}\int_{\Omega_C}\rho_{M_1}^{q_1}(u)\cdots\rho_{M_m}^{q_m}(u)du.$$

Specifically, for $0 \leqslant q \leqslant n$, the q-th dual mixed volume of $M, N \in \mathscr{S}_C^n$ is defined by

$$\tilde{V}^C_q(M,N) = \frac{1}{n}\int_{\Omega_C}\rho_M^{n-q}(u)\rho_N^q(u)du.$$

Definition 3.4. *Let $M \in \mathscr{S}^n_C$ and $0 \leqslant q \leqslant n$, We define the q-th dual quermassintegral of M by*

$$\tilde{W}^C_q(M) = \frac{1}{n}\int_{\Omega_C} \rho^{n-q}_M(u)du.$$

The q-th dual volume of M is defined by

$$\tilde{V}^C_q(M) = \frac{1}{n}\int_{\Omega_C} \rho^{q}_M(u)du.$$

Therefore, the q-norm of M for $q \geqslant 1$ in the space $L^q(\Omega_C)$ is $\|\cdot\|_{q,\Omega_C} = \tilde{V}^C_q(\cdot)^{\frac{1}{q}}$.

Remark 3.5. *The dual Steiner formula* (3.1) *can be written by*

$$V(M \tilde{+}_c \lambda(B^n \cap C)) = \sum_{i=0}^{n} \binom{n}{i} \tilde{W}^C_i(M)\lambda^i.$$

4. Dual Brunn-Minkowski inequality for *C*-star bodies

Similar to the classical case, the following dual Aleksandrov-Fenchel inequality can be obtained by the Hölder inequality.

Theorem 4.1. *(Dual Aleksandrov-Fenchel inequality for C-star bodies) Let $M_1, \cdots, M_n \in \mathscr{S}^n_C$, $0 \leqslant q_1, \cdots, q_m \leqslant n$, $1 \leqslant k \leqslant m$, $q_1 + \cdots + q_k > 0$, and $q_1 + \cdots + q_m = n$, then*

$$\tilde{V}^C_{q_1,\cdots,q_n}(M_1, \cdots, M_n)^{q_1+\cdots+q_k} \leqslant \prod_{i=1}^{k} \tilde{V}^C_{q_1,\cdots,q_n}(\overbrace{M_i, \cdots, M_i}^{k}, M_{k+1}, \cdots, M_n)^{q_i},$$

with equality if, and only if, $M_1, \cdots, M_k$ are dilates of each other.

Proof. Define the measure μ by

$$\mu(\omega) = \int_{\omega} \rho^{q_{k+1}}_{M_{k+1}}(u) \cdots \rho^{q_n}_{M_n}(u)du, \ \omega \subset \Omega_C.$$

Let $p_i = \frac{q_1+\cdots+q_k}{q_i}$, $1 \leqslant i \leqslant k$, then the Hölder inequality for μ shows that

$$\begin{aligned}
\tilde{V}^C_{q_1,\cdots,q_n}(M_1,\cdots,M_n) &= \frac{1}{n}\int_{\Omega_C}\rho^{q_1}_{M_1}(u)\cdots\rho^{q_n}_{M_n}(u)du \\
&= \frac{1}{n}\int_{\Omega_C}\rho^{q_1}_{M_1}(u)\cdots\rho^{q_k}_{M_k}(u)d\mu(u) \\
&\leqslant \prod_{i=1}^{k}\left(\frac{1}{n}\int_{\Omega_C}(\rho^{q_i}_{M_i}(u))^{p_i}d\mu(u)\right)^{\frac{1}{p_i}} \\
&= \prod_{i=1}^{k}\left(\frac{1}{n}\int_{\Omega_C}\rho^{q_1+\cdots+q_k}_{M_i}(u)d\mu(u)\right)^{\frac{q_i}{q_1+\cdots+q_k}} \\
&= \prod_{i=1}^{k}\left(\frac{1}{n}\int_{\Omega_C}\rho^{q_1+\cdots+q_k}_{M_i}(u)\rho^{q_{k+1}}_{M_{k+1}}(u)\cdots\rho^{q_n}_{M_n}(u)du\right)^{\frac{q_i}{q_1+\cdots+q_k}} \\
&= \prod_{i=1}^{k}\left(\frac{1}{n}\int_{\Omega_C}\rho^{q_1}_{M_i}(u)\cdots\rho^{q_k}_{M_i}(u)\rho^{q_{k+1}}_{M_{k+1}}(u)\cdots\rho^{q_n}_{M_n}(u)du\right)^{\frac{q_i}{q_1+\cdots+q_k}} \\
&= \prod_{i=1}^{k}\tilde{V}^C_{q_1,\cdots,q_n}(M_i,\cdots,M_i,M_{k+1},\cdots,M_n)^{\frac{q_i}{q_1+\cdots+q_k}},
\end{aligned}$$

that is,

$$\tilde{V}^C_{q_1,\cdots,q_n}(M_1,\cdots,M_n)^{q_1+\cdots+q_k} \leqslant \prod_{i=1}^{k}\tilde{V}^C_{q_1,\cdots,q_n}(M_i,\cdots,M_i,M_{k+1},\cdots,M_n)^{q_i}.$$

By the Hölder inequality for μ, equality holds if, and only if, the radial functions $\rho^{q_1+\cdots+q_k}_{M_1},\cdots,\rho^{q_1+\cdots+q_k}_{M_k}$ are dilates of each other with respect to μ on Ω_C. Since μ is absolutely continuous with respect to the standard spherical Lebesgue measure and has a positive continuous density, equality holds if, and only if, the radial functions $\rho_{M_1},\cdots,\rho_{M_k}$ are dilates of each other. □

Corollary 4.1. *(Dual Minkowski inequality for C-star bodies) Let $M,N \in \mathscr{S}^n_C$, $0 \leqslant q \leqslant n$, then*

$$\tilde{V}^C_q(M,N)^n \leqslant V(M)^{n-q}V(N)^q$$

with equality if, and only if, M,N are dilated.

Now, we provide the proof of our main result as follows.

Proof of Theorem 1.1. Let $M,N \in \mathscr{S}^n_C$ and $0<\lambda<1$. By Corollary 4.1, for any $L \in \mathscr{S}^n_C$, we have

$$\begin{aligned}
\tilde{V}^C_1(L, M\tilde{+}_cN) &= \frac{1}{n}\int_{\Omega_C}\rho^{n-1}_L(u)\rho_{M\tilde{+}_cN}(u)du \\
&= \frac{1}{n}\int_{\Omega_C}\rho^{n-1}_L(u)(\rho_M(u)+\rho_N(u))du \\
&= \tilde{V}^C_1(L,M)+\tilde{V}^C_1(L,N) \\
&\leqslant V(L)^{\frac{n-1}{n}}V(M)^{\frac{1}{n}}+V(L)^{\frac{n-1}{n}}V(N)^{\frac{1}{n}} \\
&= V(L)^{\frac{n-1}{n}}(V(M)^{\frac{1}{n}}+V(N)^{\frac{1}{n}}).
\end{aligned}$$

Let $L = M\tilde{+}_c N$, then
$$V(M\tilde{+}_c N)^{\frac{1}{n}} \leqslant V(M)^{\frac{1}{n}} + V(N)^{\frac{1}{n}}.$$
According to the homogeneity of the volume functional, we have
$$V((1-\lambda)M\tilde{+}_c \lambda N)^{\frac{1}{n}} \leqslant (1-\lambda)V(M)^{\frac{1}{n}} + \lambda V(N)^{\frac{1}{n}},$$
with equality if, and only if, M, N are dilates of each other.

Next, we will discuss the relationship between the co-sum and the Lutwak sum. Let A_1 and A_2 be C-coconvex sets. Also note that $A_1\tilde{+}_c A_2$ is a C-star body and $C \setminus (A_1\tilde{+}_c A_2)$ is a closed set in C. That is, any point in $C \setminus (A_1\tilde{+}_c A_2)$ can be represented as λu, where $\lambda \geqslant \rho_{A_1\tilde{+}_c A_2}(u)$ and $u \in \Omega_C$. Since
$$\rho_{A_1}(u)u \in C \setminus A_1 \text{ and } \rho_{A_2}(u)u \in C \setminus A_2,$$
we have
$$\rho_{A_1\tilde{+}_c A_2}(u)u = \rho_{A_1}(u)u + \rho_{A_2}(u)u \in C \setminus A_1 + C \setminus A_2.$$
Note that $C \setminus A_1 + C \setminus A_2$ is also a pseudo-cone; hence, for any $\lambda \geqslant \rho_{A_1\tilde{+}_c A_2}(u)$, there is
$$\lambda u \in C \setminus A_1 + C \setminus A_2.$$
Therefore,
$$C \setminus (A_1\tilde{+}_c A_2) \subset C \setminus A_1 + C \setminus A_2,$$
which leads to
$$A_1 \oplus A_2 = C \setminus (C \setminus A_1 + C \setminus A_2) \subset A_1\tilde{+}_c A_2.$$
This, together with the monotonicity of the volume functional, gives
$$V(A_1 \oplus A_2) \leqslant V(A_1\tilde{+}_c A_2).$$
Therefore,
$$V((1-\lambda)A_1 \oplus \lambda A_2)^{\frac{1}{n}} \leqslant V((1-\lambda)A_1\tilde{+}_c \lambda A_2)^{\frac{1}{n}} \leqslant (1-\lambda)V(A_1)^{\frac{1}{n}} + \lambda V(A_2)^{\frac{1}{n}}.$$

□

Remark 4.1. *Let A_1 and A_2 be C-coconvex sets, then $A_1 \oplus A_2 \subset A_1\tilde{+}_c A_2$. However, there is a counterexample that shows $A_1\tilde{+}_c A_2 \not\subset A_1 \oplus A_2$. For example, let C be the closure of the first quadrant in plane $\mathbb{R}^2$, We consider two C-coconvex sets*
$$A = C \cap \{(x, y) \in \mathbb{R}^2 \,|\, 2x + y - 2 < 0\},$$
$$B = C \cap \{(x, y) \in \mathbb{R}^2 \,|\, x + 2y - 2 < 0\}.$$
For the direction $u = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we have $\rho_A(u)u = (\frac{2}{3}, \frac{2}{3})$ and $\rho_B(u)u = (\frac{2}{3}, \frac{2}{3})$. Thus, $\rho_{A\tilde{+}_c B}(u)u = (\frac{4}{3}, \frac{4}{3})$, which also shows $(1, 1) \in A\tilde{+}_c B$. On the other hand, since $(1, 0) \in C \setminus A$ and $(0, 1) \in C \setminus B$, we have
$$(1, 1) = (1, 0) + (0, 1) \in (C \setminus A + C \setminus B).$$
Hence, $(1, 1) \notin C \setminus (C \setminus A + C \setminus B) = A \oplus B$.

Theorem 4.2. *(Dual Brunn-Minkowski inequality of (C, q)-close sets) Let $M_1, M_2 \in \mathscr{S}_C^n$ and $0 \leqslant q \leqslant n$, then M_1, M_2 are (C, q)-close sets and*

$$\begin{cases} \tilde{V}_q(M_1 \tilde{+}_c M_2)^{\frac{1}{q}} \leqslant \tilde{V}_q(M_1)^{\frac{1}{q}} + \tilde{V}_q(M_2)^{\frac{1}{q}}, \ \text{if } q \geqslant 1, \\ \tilde{V}_q(M_1 \tilde{+}_c M_2)^{\frac{1}{q}} \geqslant \tilde{V}_q(M_1)^{\frac{1}{q}} + \tilde{V}_q(M_2)^{\frac{1}{q}}, \ \text{if } q < 1. \end{cases}$$

with equality if, and only if, M_1 and M_2 are dilates of each other.

Proof. Applying the Minkowski inequality for the q-norm, the relevant results can be obtained directly. □

5. Equality case in the Brunn-Minkowski inequality for C-coconvex sets

In this section, we provide a detail on the equality case in the Brunn-Minkowski inequality for C-coconvex sets ([25, p. 209]). The symbols of the below proof are the same as the symbols in [25]. For example, for C-close set $A_i^\bullet$, $H^-_{z_i(\sigma)}$ represents the negative half-space with a fixed normal such that $V(A_i^\bullet \cap H^-_{z_i(\sigma)}) = \sigma$, $i = 0, 1$.

Proposition 5.1 (The equality case in the Brunn-Minkowski inequality for C-coconvex sets)**.** *The translating vector is independent of the volume parameter.*

Proof. Assume that $0 < \sigma < \tau$, We have

$$A_0^\bullet \cap H^-_{z_0(\sigma)} + x(\sigma) = A_1^\bullet \cap H^-_{z_1(\sigma)}, \tag{5.1}$$

and

$$A_0^\bullet \cap H^-_{z_0(\tau)} + x(\tau) = A_1^\bullet \cap H^-_{z_1(\tau)}. \tag{5.2}$$

Denote $M = A_1^\bullet \cap H^-_{z_1(\tau)} + x(\sigma)$, $E = A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\sigma)$, and $F = A_1^\bullet \cap H^+_{z_1(\sigma)} \cap H^-_{z_1(\tau)} + x(\sigma)$, then there exists the following decomposition

$$\begin{aligned} M &= A_1^\bullet \cap H^-_{z_1(\tau)} + x(\sigma) \\ &= (A_1^\bullet \cap H^-_{z_1(\sigma)}) \cup (A_1^\bullet \cap H^+_{z_1(\sigma)} \cap H^-_{z_1(\tau)}) + x(\sigma) \\ &= (A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\sigma)) \cup (A_1^\bullet \cap H^+_{z_1(\sigma)} \cap H^-_{z_1(\tau)} + x(\sigma)) \\ &= E \cup F. \end{aligned}$$

Since $V(E) = V(A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\sigma)) = \sigma$, we have

$$E = M \cap H^-_{z_M(\sigma)}. \tag{5.3}$$

On the other hand, combined with (5.1) and (5.2), there exists

$$\begin{aligned} M &= A_0^\bullet \cap H^-_{z_0(\tau)} + x(\tau) + x(\sigma) \\ &= (A_0^\bullet \cap H^-_{z_0(\sigma)}) \cup (A_0^\bullet \cap H^+_{z_0(\sigma)} \cap H^-_{z_0(\tau)}) + x(\tau) + x(\sigma) \\ &= (A_0^\bullet \cap H^-_{z_0(\sigma)} + x(\tau) + x(\sigma)) \cup (A_0^\bullet \cap H^+_{z_0(\sigma)} \cap H^-_{z_0(\tau)} + x(\tau) + x(\sigma)) \\ &= (A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\tau)) \cup (A_0^\bullet \cap H^+_{z_0(\sigma)} \cap H^-_{z_0(\tau)} + x(\tau) + x(\sigma)) \\ &\triangleq G \cup H, \end{aligned}$$

where $V(G) = V(A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\tau)) = \sigma$, so we have

$$G = M \cap H^-_{z_M(\sigma)}. \tag{5.4}$$

Combining (5.3) and (5.4), there exists $E = G$, namely,

$$A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\sigma) = A_1^\bullet \cap H^-_{z_1(\sigma)} + x(\tau),$$

which shows

$$x(\sigma) = x(\tau).$$

Therefore, the translating vector x is independent of the volume parameter τ. □

6. Application: Some inequalities for volume difference and volume sum

As applications of the dual Brunn-Minkowski inequality for C-star bodies, we will get the following inequalities of volume difference and volume sum for C-star bodies. At first, we need the following general Bellman inequality as a critical tool.

Lemma 6.1. *(see [18]) Let $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ be two series of nonnegative real numbers such that $a_1^q - \sum_{i=2}^n a_i^q \geq 0$ and $b_1^q - \sum_{i=2}^n b_i^q \geq 0$ for $0 \neq q \in \mathbb{R}$, respectively, then the following inequalities hold:*
(i) If $q \geq 1$, then

$$(a_1^q - \sum_{i=2}^n a_i^q)^{1/q} + (b_1^q - \sum_{i=2}^n b_i^q)^{1/q} \leq \left((a_1 + b_1)^q - \sum_{i=2}^n (a_i + b_i)^q\right)^{1/q}.$$

(ii) If $0 < q < 1$ or $q < 0$, then

$$(a_1^q - \sum_{i=2}^n a_i^q)^{1/q} + (b_1^q - \sum_{i=2}^n b_i^q)^{1/q} \geq \left((a_1 + b_1)^q - \sum_{i=2}^n (a_i + b_i)^q\right)^{1/q}.$$

Equality holds if, and only if, $a = cb$ for $c \geqslant 0$.

By the above inequalities and the method in [17], we obtain the following corollary.

Corollary 6.1. *(Dual Brunn-Minkowski inequality for volume difference) Let C-star bodies $K_1 \subseteq L_1$, $K_2 \subseteq L_2$. If L_1 is a dilate of L_2, then*

$$\left(V(L_1 \tilde{+}_c L_2) - V(K_1 \tilde{+}_c K_2)\right)^{\frac{1}{n}} \geq \left(V(L_1) - V(K_1)\right)^{\frac{1}{n}} + \left(V(L_2) - V(K_2)\right)^{\frac{1}{n}},$$

with equality if, and only if, K_1, K_2 are dilates of each other and $(V(L_1), V(K_1)) = c(V(L_2), V(K_2))$ for $c \geqslant 0$.

Proof. Since K_1, L_1, K_2, L_2 are C-star bodies, by the dual Brunn-Minkowski inequality for C-star bodies, we have

$$V(K_1 \tilde{+}_c K_2)^{\frac{1}{n}} \leq V(K_1)^{\frac{1}{n}} + V(K_2)^{\frac{1}{n}}, V(L_1 \tilde{+}_c L_2)^{\frac{1}{n}} = V(L_1)^{\frac{1}{n}} + V(L_2)^{\frac{1}{n}}.$$

Taking n-power and subtracting from both sides of the above formulas, there is

$$(V(L_1\tilde{+}_cL_2)-V(K_1\tilde{+}_cK_2))^{\frac{1}{n}}\geq\left((V(L_1)^{\frac{1}{n}}+V(L_2)^{\frac{1}{n}})^n-(V(K_1)^{\frac{1}{n}}+V(K_2)^{\frac{1}{n}})^n\right)^{\frac{1}{n}}.$$

Due to the general Bellman inequality for $q\geq 1$, this further yields

$$(V(L_1\tilde{+}_cL_2)-V(K_1\tilde{+}_cK_2))^{\frac{1}{n}}\geq(V(L_1)-V(K_1))^{\frac{1}{n}}+(V(L_2)-V(K_2))^{\frac{1}{n}}.$$

□

Corollary 6.2. *(Dual Brunn-Minkowski inequality for volume sum) Let C-star bodies $K_1\subseteq L_1$, $K_2\subseteq L_2$. If L_1 is a dilate of L_2, then*

$$(V(L_1\tilde{+}_cL_2)+V(K_1\tilde{+}_cK_2))^{\frac{1}{n}}\leq(V(L_1)+V(K_1))^{\frac{1}{n}}+(V(L_2)+V(K_2))^{\frac{1}{n}},$$

with equality if, and only if, K_1,K_2 are dilates of each other and $(V(L_1),V(K_1))=c(V(L_2),\,V(K_2))$ for $c\geqslant 0$.

Proof. By the Minkowski inequality, it is obvious that

$$\begin{aligned}(V(L_1\tilde{+}_cL_2)+V(K_1\tilde{+}_cK_2))^{\frac{1}{n}}&\leq\left((V(L_1)^{\frac{1}{n}}+V(L_2)^{\frac{1}{n}})^n+(V(K_1)^{\frac{1}{n}}+V(K_2)^{\frac{1}{n}})^n\right)^{\frac{1}{n}}\\&\leq(V(L_1)+V(K_1))^{\frac{1}{n}}+(V(L_2)+V(K_2))^{\frac{1}{n}}.\end{aligned}$$

□

7. Conclusions

In this work, we establish the dual Brunn-Minkowski inequality for C-star bodies. Importantly, this dual Brunn-Minkowski inequality for C-star bodies can strengthen the previous Brunn-Minkowski inequality for C-coconvex sets.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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