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Research article

A cotangent fractional Gronwall inequality with applications

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Abstract: This article presents the cotangent fractional Gronwall inequality, a novel understanding of the Gronwall inequality within the context of the cotangent fractional derivative. We furnish an explanation of the cotangent fractional derivative and emphasize a selection of its distinct characteristics before delving into the primary findings. We present the cotangent fractional Gronwall inequality (Lemma 3.1) and a Corollary 3.2 using the Mittag-Leffler function, we establish singularity and compute an upper limit employing the Mittag-Leffler function for solutions in a nonlinear delayed cotangent fractional system, illustrating its practical utility. To underscore the real-world relevance of the theory, a tangible instance is given.

Keywords: cotangent fractional derivative; delay cotangent fractional equation; cotangent fractional Gronwall inequality; bound for the solution; uniqueness of solution
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1. Introduction

In references [1–6], the authors introduced novel fractional operators that incorporate kernels such as the exponential function or the Mittag-Leffler function. However, despite the introduction of these nonsingular kernels, they lack a semi-group property, complicating the resolution of certain intricate fractional systems. Concurrently, significant strides have been taken to define various fractional operators and integrals that involve Mittag-Leffler functions in their representations. Notably, the cotangent fractional operators in [7] introduce the exponential of the cotangent as a kernel and verify the semi-group property.

The cotangent fractional operators introduced a distinctive mathematical perspective by employing the cotangent function as a kernel. The novelty lies in the specific choice of the cotangent function, which is not a commonly used kernel in traditional fractional calculus. The main differences and novelties of these cotangent derivatives can be summarized as follows:

- Kernel choice: The cotangent fractional operators use the cotangent function as a kernel, setting them apart from more conventional fractional operators that often involve functions like the exponential or Mittag-Leffler functions.
- Semi-group property: One notable feature mentioned is that these cotangent fractional operators achieve a semi-group property. This property is significant in the context of solving fractional differential equations, as it can simplify the mathematical treatment and make certain computations more manageable.
- Distinctive representations: The use of the cotangent function in fractional operators leads to unique representations for fractional derivatives or integrals. This can have implications for solving specific types of problems or modeling certain phenomena in a novel way.
- Applicability: Depending on the context of the mathematical models being studied, the cotangent fractional operators may offer advantages or insights that are not readily achievable with other types of fractional operators.

Integral inequalities function as an extraordinary instrument for progressing both the qualitative and quantitative facets of differential equations. This realm of investigation has continuously broadened owing to its pertinence to a diverse range of applications that demand such forms of inequalities. Numerous scholars have immersed themselves in the examination of these inequalities, utilizing an assortment of methodologies to scrutinize and validate them [8, 9]. Among these inequalities, the renowned Gronwall inequality holds significant prominence [10–13].

The Gronwall inequality is a fundamental mathematical tool frequently used in the analysis of differential equations and integral inequalities. Named after the mathematician Thomas H. Gronwall, the inequality provides bounds on functions by establishing relationships involving integrals. It has wide applications in various branches of mathematics, physics, and engineering. The basic version is as follows: if u satisfies the integral inequality

$$u(\ell) \leq v(\ell) + \int_a^{\ell} u(s)\, ds,$$

then

$$u(\ell) \leq v(\ell) + \int_a^{\ell} v(s) \exp(\ell - s) ds.$$

Integral formulation: The Gronwall inequality is often extended to integral forms involving higher-order derivatives. For example, integral inequalities involving the second derivative may provide bounds for solutions to second-order differential equations. Systems of inequalities: Extensions to systems of differential inequalities exist, where the Gronwall inequality is applied to each component of a vector-valued function. Fractional Gronwall inequality: In fractional calculus, versions of the Gronwall inequality have been developed to address fractional differential equations. These inequalities involve fractional derivatives and integrals, and they play a crucial role in the analysis of fractional order systems. Stochastic Gronwall inequality: Variations of the Gronwall inequality have been developed to handle stochastic processes, introducing randomness into the framework. These versions are particularly relevant in the study of stochastic differential equations. The Gronwall inequality and its various versions play a vital role in stability analysis, control theory, and the study of differential equations across diverse mathematical disciplines. The extensions allow for a broader application to different types of problems and mathematical models. Amidst these developments, fractional calculus, an extension of classical calculus concerned with integration and differentiation of nonnegative integer orders, has conspicuously advanced as a swiftly evolving scholarly domain. This fascination can be ascribed to the enchanting results that emerge when fractional derivatives are utilized to simulate real-world issues [14–20]. The availability of many fractional operators in this field is a noteworthy feature that enables researchers to choose the best operator for issue modeling. Due to their easy application, recently discovered fractional operators without single kernels have lately attracted a lot of attention [1, 3], leading to an increase in papers devoted to these operator types.

Concurrent with the escalating enthusiasm surrounding fractional differential equations, scholars have broadened the scope of these mathematical inequalities to encompass fractional differential equations featuring both singular and nonsingular kernels. Here, we highlight a few of these results [21–28]. Aligning with this trend, we contribute a novel adaptation of the Gronwall inequality utilizing the cotangent fractional derivatives (CFD). Specifically, we establish the subsequent proposition: Given

$$u(\ell) \leq v(\ell) + \sin(\frac{\pi}{2}\sigma)^r \Gamma(r) f(\ell) \left({}_0 I^{r,\sigma} u\right)(\ell),$$

we then deduce that

$$u(\ell) \leq v(\ell) + \int_0^\ell \Big\{ \sum_{m=1}^{\infty} \frac{(f(\ell)\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)} (\ell-\jmath)^{mr-1} v(\jmath) \Big\} d\jmath, \quad \ell \in [0, L).$$

Here, ${}_0 I^{r,\sigma}$ denotes the cotangent fractional integral of degree r with $\sigma > 0$. In particular, u and v represent non-negative locally integrable functions over the range $[0, \ell)$, while $f(\ell)$ stands for a continuous, non-negative, monotonically increasing function defined for $\ell \in [0, L)$, adhering to the condition $f(\ell) \leq M$, where M stands as a constant. When compared to typical fractional operators, a distinctive characteristic of cotangent fractional operators is the inclusion of an exponential term in the kernel.

The outline of this essay is as follows: The CFD and integrals are introduced in Section 2, which also presents key concepts that will be discussed. The Gronwall inequality is presented inside the CFD framework in Section 3. Applications in the context of uniqueness and bound derivation for solutions in a delayed system are explored in Section 4. Section 5 provides an instructive case evaluating the theoretical conclusions. Section 6 of this paper concludes by summarizing the findings.

2. Preliminaries on cotangent fractional derivative and integral

This section is dedicated to introducing the terminology, definitions, and crucial lemmas that will underpin the subsequent segments of this paper. For thorough justifications and proofs, readers are referred to the paper by [7].

The concept of the conformable derivative was initially established in [29,30] through a limit-based definition, as follows:

$$D^r h(\ell) = \lim_{\epsilon\to 0} \frac{h\left(\ell + \epsilon \ell^{1-r}\right) - h(\ell)}{\epsilon}. \tag{2.1}$$

It is evident that for differentiable functions, the conformable derivative of h can be expressed as

$$D^r h(\ell) = \ell^{1-r} h'(\ell). \tag{2.2}$$

Nevertheless, a notable limitation of this derivative emerges when the derivative order is 0 or $r \to 0$, as it renders the function h inaccessible. Addressing this challenge, Anderson et al. devised an altered conformable derivative using the subsequent approach. Their aim was to employ the proportional derivative for controller output, which encompasses a duo of tuning parameters [31].

Definition 2.1. *For $\sigma \in [0, 1]$, let the functions $\kappa_0, \kappa_1 : [0, 1]\times\mathbb{R} \to [0, \infty)$ be continuous such that, for all $t \in \mathbb{R}$, we have*

$$\lim_{\sigma\to 0^+} \kappa_1(\sigma, \ell) = 1, \quad \lim_{\sigma\to 0^+} \kappa_0(\sigma, \ell) = 0, \quad \lim_{\sigma\to 1^-} \kappa_1(\sigma, \ell) = 0, \quad \lim_{\sigma\to 1^-} \kappa_0(\sigma, \ell) = 1,$$

and $\kappa_1(\sigma, \ell) \neq 0, \sigma \in [0, 1), \kappa_0(\sigma, \ell) \neq 0, \sigma \in (0, 1]$. Subsequently, the proportional derivative with an order of σ can be expressed as follows:

$$D^\sigma h(\ell) = \kappa_1(\sigma, \ell)h(\ell) + \kappa_0(\sigma, \ell)h'(\ell). \tag{2.3}$$

For a more thorough exploration of the control theory involving the proportional derivative and its associated functions κ_1 and κ_0. We recommend that the reader consult references [31, 32]. In our current discourse, we will confine our deliberations to situations where $\kappa_1(\sigma, \ell) = \cos(\frac{\pi}{2}\sigma)$ and $\kappa_0(\sigma, \ell) = \sin(\frac{\pi}{2}\sigma)$. Consequently, Eq (2.3) transforms into

$$D^\sigma h(\ell) = \cos(\frac{\pi}{2}\sigma)h(\ell) + \sin(\frac{\pi}{2}\sigma)h'(\ell). \tag{2.4}$$

It is simple to determine that $\lim_{\sigma\to 0^+} D^\sigma h(\ell) = h(\ell)$ and $\lim_{\sigma\to 1^-} D^\sigma h(\ell) = h'(\ell)$. Because the conformable derivative clearly fails to approach the original functions as σ approaches 0, the derivative Eq (2.4) is occasionally seen to have a wider range of applications.

The following definitions apply to the CFD integral and derivative.

Definition 2.2. *[7] The cotangent fractional integral of a function h of order r for $0 < \sigma \leq 1, r \in \mathbb{C}$, and $\mathrm{Re}(r) > 0$ may be expressed as follows*

$$\begin{aligned}\left({}_aI^{r,\sigma} h\right)(\ell) &= \frac{1}{\sin(\frac{\pi}{2}\sigma)^r\Gamma(r)} \int_a^J e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\psi)}(\ell - \psi)^{r-1} h(\psi)d\psi \\ &= \sin(\frac{\pi}{2}\sigma)^{-r} e^{-\cot(\frac{\pi}{2}\sigma)J}\left({}_aI^r\left(e^{\frac{1-\sigma}{\sigma}\ell}h(\ell)\right)\right),\end{aligned} \tag{2.5}$$

where

$$ {}_aI^r y(\ell) = \frac{1}{\Gamma(r)} \int_a^\ell (\ell - s)^{r-1} y(s) ds, $$

and

$$ \Gamma(r) = \int_0^\infty e^{-\mu} \mu^{r-1} d\mu. $$

Example 2.1. *[7] Let* $h(\ell) = e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)}(\ell - a)^{\sigma_2 - 1}$. *We have*

$$ {}_aI^{r,\sigma}\Big(h(\ell)\Big) = \frac{\Gamma(\sigma_2)}{\sin(\frac{\pi}{2}\sigma)^r \Gamma(r + \sigma_2)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)} (\ell - a)^{r+\sigma_2-1}. $$

Definition 2.3. *[7] The cotangent fractional derivative of h of order r is thus for* $0 < \sigma \leq 1, r \in \mathbb{C}, \mathrm{Re}(r) \geq 0$, *and* $n = [\mathrm{Re}(r)] + 1$.

$$ \begin{aligned} ({}_aD^{r,\sigma} h)(\ell) &= (D^{n,\sigma}\, {}_aI^{n-r,\sigma} h)(\ell) \\ &= \frac{D_\ell^{n,\sigma}}{\sin(\frac{\pi}{2}\sigma)^{n-r}\Gamma(n - r)} \int_a^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\psi)} (\ell - \psi)^{n-r-1} h(\psi) d\psi. \end{aligned} \tag{2.6} $$

Putting $\sigma = 1$ into Definition 2.3 yields the left Riemann-Liouville fractional derivative [16, 18]. Furthermore the following is evident:

$$ \lim_{r \to 0} (D^{r,\sigma} h)(\ell) = h(\ell) \quad \text{and} \quad \lim_{r \to 1} (D^{r,\sigma} h)(\ell) = (D^\sigma h)(\ell). $$

Example 2.2. *[7] Let* $h(\ell) = e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)}(\ell - a)^{\sigma_2 - 1}$. *We have*

$$ {}_aD^{r,\sigma}\Big(h(\ell)\Big) = \frac{\Gamma(\sigma_2)}{\sin(\frac{\pi}{2}\sigma)^r \Gamma(\sigma_2 - r)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)} (\ell - a)^{\sigma_2-1-r}. $$

Proposition 2.1. *[7] Let* $r, \rho \in \mathbb{C}$ *be such that* $\mathrm{Re}(r) \geq 0$ *and* $\mathrm{Re}(\rho) > 0$. *In this context, for any* $\sigma > 0$, *the following relationship holds:*

- $\left({}_aI^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\ell}(\ell - a)^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+r)\sin(\frac{\pi}{2}\sigma)^r} e^{-\cot(\frac{\pi}{2}\sigma)x}(x - a)^{r+\rho-1}, \mathrm{Re}(r) > 0.$
- $\left({}_aD^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\ell}(\ell - a)^{\rho-1}\right)(x) = \frac{\sin(\frac{\pi}{2}\sigma)^r \Gamma(\rho)}{\Gamma(\rho-r)} e^{-\cot(\frac{\pi}{2}\sigma)x}(x - a)^{\rho-1-r}, \mathrm{Re}(r) \geq 0.$

We describe a few aspects of the cotangent fractional operator in the following lemmas.

Lemma 2.1. *[7] Assuming* $\sigma > 0, \mathrm{Re}(r) > 0$, *and* $\mathrm{Re}(\rho) > 0$, *and considering h to be a continuous function defined for* $\ell \geq a$, *we obtain:*

$$ ({}_aI^{r,\sigma}\, {}_aI^{\rho,\sigma} h)(\ell) = ({}_aI^{\rho,\sigma}\, {}_aI^{r,\sigma} h)(\ell) = ({}_aI^{r+\rho,\sigma} h)(\ell). \tag{2.7} $$

Lemma 2.2. *[7] Given* $0 \leq m < [\mathrm{Re}(r)] + 1$ *and assuming that h is integrable within each interval* $[a, L]$ *for* $\ell > a$, *we have:*

$$ ({}_aD^{m,\sigma}\, {}_aI^{r,\sigma} h)(\ell) = ({}_aI^{r-m,\sigma} h)(\ell). \tag{2.8} $$

Corollary 2.1. *[7] Assuming* $0 < \mathrm{Re}(\rho) < \mathrm{Re}(r)$ *and* $m - 1 < \mathrm{Re}(\rho) \leq m$, *we can conclude:*

$$ ({}_aD^{\rho,\sigma}\, {}_aI^{r,\sigma} h)(\ell) = ({}_aI^{r-\rho,\sigma} h)(\ell). $$

Lemma 2.3. *[7] Suppose h is integrable for $\ell \geq a$ and* $\mathrm{Re}(r) > 0, \sigma > 0, n = [\mathrm{Re}(r)] + 1$. *We can then state*

$$({}_aD^{a,\sigma} I^{r,\sigma} h)(\ell) = h(\ell).$$

Lemma 2.4. *[7] Assuming* $\mathrm{Re}(r) > 0, n = [\mathrm{Re}(r)], h \in L_1(a, b)$, *and* $({}_aI^{r,\sigma} h)(\ell) \in AC^n[a, b]$. *Then, considering the given conditions and context, we can deduce:*

$$({}_aI^{r,\sigma}_a D^{r,\sigma} h)(\ell) = h(\ell) - e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)} \sum_{b=1}^{n} \left({}_aI^{b-r,\sigma} h\right)(a^+) \frac{(\ell - a)^{r-b}}{\sin(\frac{\pi}{2}\sigma)^{r-b}\Gamma(r + 1 - b)}. \tag{2.9}$$

Definition 2.4. *[7] Let* $0 < \sigma \leq 1$ *and* $r \in \mathbb{C}$ *with* $\mathrm{Re}(r) \geq 0$. *Using a as an initial point, we define the cotangent fractional derivative of the Caputo type as*

$$\begin{aligned}\left({}^C_aD^{r,\sigma} h\right)(\ell) &= ({}_aI^{n-r,\sigma} D^{n,\sigma} h)(\ell) \\ &= \frac{1}{\sin(\frac{\pi}{2}\sigma)^{n-r}\Gamma(n - r)} \int_a^J e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\psi)} (\ell - \psi)^{n-r-1} (D^{n,\sigma} h)(\psi) d\psi,\end{aligned} \tag{2.10}$$

where $n = [\mathrm{Re}(r)] + 1$.

Proposition 2.2. *[7] Given* $r, \rho \in \mathbb{C}$ *with* $\mathrm{Re}(r) > 0$ *and* $\mathrm{Re}(\rho) > 0$, *and considering* $\sigma \in (0, 1]$ *and* $n = [\mathrm{Re}(r)] + 1$, *it can be concluded that:*

$$\left({}^C_aD^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\ell} (\ell - a)^{\rho-1}\right)(x) = \frac{\sin(\frac{\pi}{2}\sigma)^r\Gamma(\rho)}{\Gamma(\rho - r)} e^{-\cot(\frac{\pi}{2}\sigma)x} (x - a)^{\rho-1-r}, \quad \mathrm{Re}(r) \geq n, \tag{2.11}$$

$k = 0, 1, \cdots, n - 1$, *we get* $\left({}^C_aD^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\ell} (\ell - a)^k\right)(x) = 0$.

Lemma 2.5. *[7] For* $\sigma \in (0, 1]$ *and* $n = [\mathrm{Re}(r)] + 1$, *we get*

$$\left({}_aI^{r,\sigma C}{}_aD^{r,\sigma} h\right)(\ell) = h(\ell) - \sum_{k=0}^{n-1} \frac{\left(D^{k,\sigma} h\right)(a)}{\sin(\frac{\pi}{2}\sigma)^k\Gamma(k + 1)} (\ell - a)^k e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)}. \tag{2.12}$$

The relationship between the Caputo and Riemann-Liouville cotangent fractional derivatives is demonstrated by the example below:

Proposition 2.3. *[7] For every* $r \in \mathbb{C}$ *with* $\mathrm{Re}(r) > 0$ *and* $\sigma \in (0, 1], n = [\mathrm{Re}(r)] + 1$ *and the following holds:*

$$\left({}^C_aD^{r,\sigma} h\right)(\ell) = ({}_aD^{r,\sigma} h)(\ell) - \sum_{k=0}^{n-1} \frac{\sin(\frac{\pi}{2}\sigma)^{r-k}}{\Gamma(k + 1 - r)} (\ell - a)^{k-r} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-a)} \left(D^{k,\sigma} h\right)(a). \tag{2.13}$$

3. Cotangent fractional Gronwall inequality

For the cotangent fractional derivative, we establish a Gronwall inequality in this section. Using Mittag-Leffler functions, we provide a thorough explanation of this inequality as well.

Lemma 3.1. *(Cotangent fractional Gronwall inequality) Taking into consideration r and σ as positive parameters, let $u(\ell)$ and $v(\ell)$ be nonnegative functions possessing local integrability over the interval $[0, L)$, and let $f(\ell)$ be a non-negative, monotonically increasing, and continuous function defined for $\ell \in [0, L)$, subject to the condition that $f(\ell) \leq M$ where M remains constant. In the case the inequality*

$$u(\ell) \leq v(\ell) + \sin(\frac{\pi}{2}\sigma)^r \Gamma(r) f(\ell) \left({}_0 I^{r,\sigma} u\right)(\ell), \tag{3.1}$$

is satisfied, then it follows that

$$u(\ell) \leq v(\ell) + \int_0^\ell \Big\{ \sum_{m=1}^{\infty} \frac{(f(\ell)\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} (\ell - \jmath)^{mr-1} v(\jmath) \Big\} d\jmath, \quad \ell \in [0, L). \tag{3.2}$$

Proof. Define

$$G\phi(\ell) = f(\ell) \int_0^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} (\ell - \jmath)^{r-1} \phi(\jmath), \quad \ell \in [0, L).$$

This leads to the conclusion that $u(\ell) \leq v(\ell) + Bu(\ell)$, which, in turn, implies $u(\ell) \leq \sum_{k=0}^{m-1} G^k v(\ell) + G^m u(\ell)$. We claim that

$$G^m u(\ell) \leq \int_0^\ell \frac{(f(\ell)\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} (\ell - \jmath)^{mr-1} u(\jmath) d\jmath, \tag{3.3}$$

and $G^m u(\ell) \to 0$ as $m \to \infty$ for $\ell \in [0, L)$. For $m = 1$ Eq (3.3) is correct. Now suppose that applies to $m = k$, which means

$$G^k u(\ell) \leq \int_0^\ell \frac{(f(\ell)\Gamma(r))^k}{\Gamma(kr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} (\ell - \jmath)^{kr-1} u(\jmath) d\jmath.$$

and we have for $m = k + 1$, then

$$\begin{aligned} G^{k+1} u(\ell) &= G\left(G^k u(\ell)\right) \\ &\leq f(\ell) \int_0^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} (\ell - \jmath)^{r-1} \left[\int_0^s \frac{(w(\jmath)\Gamma(r))^k}{\Gamma(kr)} e^{-\cot(\frac{\pi}{2}\sigma)(\jmath - v)} (\jmath - v)^{kr-1} u(v) dv \right] d\jmath \\ &= \frac{w^{k+1}(\ell)\Gamma^k(r)}{\Gamma(kr)} \int_0^\ell \left[\int_v^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell - \jmath)} e^{-\cot(\frac{\pi}{2}\sigma)(\jmath - v)} (\ell - \jmath)^{r-1} (\jmath - v)^{kr-1} d\jmath \right] u(v) dv. \end{aligned} \tag{3.4}$$

When change of variables $\jmath = v + z(\ell - v)$ is made, however, we achieve

$$\begin{aligned} \int_v^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell - v)} (\ell - \jmath)^{r-1} (\jmath - v)^{kr-1} dt &= (\ell - v)^{kr+r-1} e^{-\cot(\frac{\pi}{2}\sigma)(\ell - v)} \int_0^1 (1 - z)^{r-1} z^{kr-1} dz \\ &= \frac{\Gamma(r)\Gamma(kr)}{\Gamma((k+1)r)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell - v)} (\ell - v)^{(k+1)r-1}. \end{aligned}$$

Therefore, Eq (3.4) becomes

$$G^{k+1} u(\ell) \leq \frac{f^{k+1}(\ell)\Gamma^{k+1}(r)}{\Gamma((k+1)r)} \int_0^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell - v)} (\ell - v)^{(k+1)r-1} u(v) dv. \tag{3.5}$$

Furthermore, one can figure out that

$$G^m u(\ell) \leq \int_0^\ell \frac{(M\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)}(\ell-\jmath)^{mr-1}u(\jmath)d\jmath \to 0 \quad \text{as } m\to\infty, \ell\in[0,L).$$

To conclude the proof, we consider the limit as $m \to \infty$ in

$$u(\ell) \leq \sum_{k=0}^{m-1} G^k v(\ell) + G^m u(\ell) \leq v(\ell) + \sum_{k=1}^{m-1} G^k v(\ell) + G^m u(\ell),$$

to achieve $u(\ell) \leq v(\ell) + \sum_{k=1}^{\infty} G^k v(\ell)$. Exploiting the property of the semigroup and the definition of G, we come to the expression Eq (3.2). This completes the proof. □

The following is a corollary that may be derived when $f(\ell) \equiv \alpha$ is in Lemma 3.1:

Corollary 3.1. *Let r, σ, and α be a positive constant. Suppose that $u(\ell)$ and $v(\ell)$ are non-negative functions that are locally integrable over the interval $[0, L)$, while $f(\ell)$ remains constant at $\alpha \geq 0$. Given these conditions, if the criterion*

$$u(\ell) \leq v(\ell) + \sin(\frac{\pi}{2}\sigma)^r \Gamma(r)\alpha \left({}_0I^{r,\sigma}u\right)(\ell), \tag{3.6}$$

is satisfied, then

$$u(\ell) \leq v(\ell) + \int_0^\ell \left\{ \sum_{m=1}^{\infty} \frac{(\alpha\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)}(\ell-\jmath)^{mr-1}v(\jmath)\right\} d\jmath, \quad \ell\in[0,L). \tag{3.7}$$

Let

$$E_r(\lambda, z) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{kr}}{\Gamma(rk+1)},$$

represent the single-parameter Mittag-Leffler function that was first described in [16]. In our subsequent examination, the straight-forward result of Lemma 3.1 that follows is significant.

Corollary 3.2. *Extending the premises of Lemma 3.1, and introducing an added assumption that $v(\ell)$ is nondecreasing for $\ell \in [0, L)$, it follows that*

$$u(\ell) \leq v(\ell) E_r(f(\ell)\Gamma(r), \ell), \quad \ell\in[0,L). \tag{3.8}$$

Proof. Referring to Eq (3.2) and based on the provided assumption that $v(\ell)$ exhibits non-decreasing behavior for $\ell \in [0, L)$, we can formulate the following expression:

$$u(\ell) \leq v(\ell)\left[1 + \int_0^\ell \left\{ \sum_{m=1}^{\infty} \frac{(f(\ell)\Gamma(r))^m}{\Gamma(mr)} e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)}(\ell-\jmath)^{mr-1}\right\} d\jmath\right],$$

or

$$u(\ell) \leq v(\ell)\left[1 + \sum_{m=1}^{\infty} \left(\sin(\frac{\pi}{2}\sigma)^r f(\ell)\Gamma(r)\right)^m \frac{1}{\sin(\frac{\pi}{2}\sigma)^{mr}\Gamma(mr)} \int_0^\ell e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)}(\ell-\jmath)^{mr-1}d\jmath\right].$$

Using Proposition 2.1 and $e^{-\cot(\frac{\pi}{2}\sigma)(\ell-\jmath)} \leq 1$, it follows that

$$\begin{aligned}
u(\ell) &\leq v(\ell)\left[1+\sum_{m=1}^{\infty}\left(\sin(\frac{\pi}{2}\sigma)^r f(\ell)\Gamma(r)\right)^m \frac{\ell^{mr}}{\sin(\frac{\pi}{2}\sigma)^{mr}\Gamma(mr+1)}\right] \\
&= v(\ell)\left[1+\sum_{m=1}^{\infty}\frac{(f(\ell)\Gamma(r))^m\ell^{mr}}{\Gamma(mr+1)}\right] \\
&= v(\ell)\sum_{m=0}^{\infty}\frac{(f(\ell)\Gamma(r))^m\ell^{mr}}{\Gamma(mr+1)} \\
&= v(\ell)E_r(f(\ell)\Gamma(r),\ell).
\end{aligned}$$

□

4. Applications of the cotangent fractional Gronwall inequality

Consider the m-dimensional Euclidean space denoted by $\mathbb{R}^m$. Our primary emphasis is placed on the ensuing system:

$$\begin{cases} \left({}_0^C D^{r,\sigma}x\right)(\ell) = e^{-\cot(\frac{\pi}{2}\sigma)\ell}\left[A_0x(\ell)+A_1x(\ell-\psi)+h(\ell,x(\ell),x(\ell-\psi))\right], & \ell\in[0,L], \\ x(\ell)=\varphi(\ell), \quad \ell\in[-\psi,0]. \end{cases} \tag{4.1}$$

Here, the notation ${}_0^C D^{r,\sigma}$ stands for the cotangent Caputo fractional derivative, where the order is denoted by $r \in (0, 1)$. The function describing the state vector is represented by $x : [-\psi, L] \to \mathbb{R}^m$. Additionally, the constant matrices A_0 and A_1 have dimensions that are appropriate for the given context. The nonlinearity h is represented by $h : [0, L] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and the function $\varphi : [-\psi, 0] \to \mathbb{R}^m$ represents the initial condition. Our primary focus is on establishing the uniqueness of solutions to Eq (4.1), and deriving estimates for these solutions by drawing on insights obtained from the previous sections. Furthermore, we offer a numerical example to illustrate the practical application of the fundamental results in practical scenarios.

The norm induced by this vector in terms of matrices is denoted as $|\cdot|$, while $|\cdot|$ signifies any Euclidean norm. The ensemble encompassing all continuous functions is designated as $C := C([-\psi, 0], \mathbb{R}^m)$. The fact that C forms a Banach space is clear when employing the norm $|z|_C := \sup_{\ell\in[-\psi,0]} |z(\ell)|$. We use the following presumptions for the remainder of the paper:

(H1) According to the Lipschitz condition, the nonlinearity $h \in C([0, L] \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ is satisfied. In particular, there is a positive constant $L_1 > 0$ for which

$$\|h(\ell,x(\ell),x(\ell-\psi)) - h(\ell,y(\ell),y(\ell-\psi))\| \leq L_1(\|x(\ell)-y(\ell)\| + \|x(\ell-\psi)-y(\ell-\psi)\|),$$

for $\ell \in [0, L]$.

(H2) There is a positive constant L_2 where $|h(\ell, x(\ell), x(\ell-\psi))| \leq L_2$.

In the forthcoming sections, we unveil an expression for the solutions of the system (4.1). This representation is expected to possess considerable significance in the forthcoming analysis.

Lemma 4.1. *The function $x : [-\psi, 0] \to \mathbb{R}^m$ constitutes a solution to Eq (4.1) if and only if*

$$\begin{cases} x(\ell) = \varphi(0) e^{-\cot(\frac{\pi}{2}\sigma)\ell} \\ \qquad + \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 x(\jmath) + A_1 x(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi)) \right] \right)(\ell), \quad \ell \in [0, L], \\ x(\ell) = \varphi(\ell), \quad \ell \in [-\psi, 0]. \end{cases} \tag{4.2}$$

Proof. For $\ell \in [-\psi, 0]$, it is clear that $x(\ell) = \varphi(\ell)$ serves as the solution to Eq (4.1). Next, we employ the operator ${}_0D^{r,\sigma}$ to both sides of Eq (4.2), utilizing Lemma 2.3 and Proposition 2.1. This leads to the following equation for $\ell \in [0, L]$:

$$({}_0D^{r,\sigma} x)(\ell) = \varphi(0) \frac{\sin(\frac{\pi}{2}\sigma)^r e^{-\cot(\frac{\pi}{2}\sigma)\ell} \ell^{-r}}{\Gamma(1-r)} + e^{-\cot(\frac{\pi}{2}\sigma)\ell} \left[A_0 x(\ell) + A_1 x(\ell - \psi) + h(\ell, x(\ell), x(\ell - \psi)) \right].$$

This is deduced from Proposition 2.3, which employs the connection between the cotangent fractional derivatives of Caputo and Riemann-Liouville.

$$\left({}_0^C D^{r,\sigma} x \right)(\ell) = e^{-\cot(\frac{\pi}{2}\sigma)\ell} \left[A_0 x(\ell) + A_1 x(\ell - \psi) + h(\ell, x(\ell), x(\ell - \psi)) \right].$$

Regarding Eq (4.1), it is evident that $x(\ell) = \varphi(\ell)$ for $\ell \in [-\psi, 0]$. Then, for $\ell \in [0, L]$, we utilize the operator ${}_0I^{r,\sigma}$ on both sides of Eq (4.1) to obtain:

$$\left({}_0I^{r,\sigma C} D^{r,\sigma} x \right)(\ell) = \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 x(\jmath) + A_1 x(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi)) \right] \right)(\ell).$$

Considering Lemma 2.5, it becomes apparent that:

$$x(\ell) = \varphi(0) e^{-\cot(\frac{\pi}{2}\sigma)\ell} + \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 x(\jmath) + A_1 x(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi)) \right] \right)(\ell).$$

□

4.1. Uniqueness of solutions

The subsequent theorem constitutes the primary application within this study.

Theorem 4.1. *Given the adherence to condition (H1), if Eq (4.1) possesses two distinct solutions, denoted as x and y, then it holds that $x = y$.*

Proof. Consider two solutions x and y of Eq (4.1). Let $z = x - y$, it is straightforward to observe that $z(\ell) = 0$ for $\ell \in [-\psi, 0]$. Consequently, Eq (4.1) exhibits a distinctive solution for $\ell \in [-\psi, 0]$.

Nonetheless, when $\ell \in [0, L]$, we encounter the situation

$$z(\ell) = \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 z(\jmath) + A_1 z(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi)) - h(\jmath, y(\jmath), y(\jmath - \psi)) \right] \right)(\ell).$$

If $\ell \in [0, \psi]$, then $z(\ell - \psi) = 0$. Hence, it follows that:

$$z(\ell) = \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 z(\jmath) + h(\jmath, x(\jmath), x(\jmath - \psi)) - h(\jmath, y(\jmath), y(\jmath - \psi)) \right] \right)(\ell). \tag{4.3}$$

This suggests that

$$\begin{aligned}\|z(\ell)\| &\le \left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[\|A_0\|\,\|z(\jmath)\| + \|h(\jmath,x(\jmath),x(\jmath-\psi)) - h(\jmath,y(\jmath),y(\jmath-\psi))\|\right]\right)(\ell)\\ &\le {}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[\|A_0\|\,\|z(\jmath)\| + L_1(\|x(\jmath)-y(\jmath)\| + \|x(\jmath-\psi)-y(\jmath-\psi)\|\right])(\ell)\\ &= \left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[(\|A_0\|+L_1)\,\|z(\jmath)\| + L_1\|z(\jmath-\psi)\|\right]\right)(\ell)\\ &= (\|A_0\|+L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|z(\jmath)\|\right)(\ell).\end{aligned} \tag{4.4}$$

By using the Corollary 3.2 result, we have

$$\|z(\ell)\| \le (0)\times E_r\left(\|A_0\|+L_1,\ell\right), \tag{4.5}$$

and thus concludes that $x(\ell)=y(\ell)$ for $\ell\in I_\psi$. For $\ell\in[\psi,L]$, we obtain

$$\begin{aligned}z(\ell) =&\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[A_0z(\jmath) + h(\jmath,x(\jmath),x(\jmath-\psi)) - h(\jmath,y(\jmath),y(\jmath-\psi))\right]\right)(\ell)\\ &+\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[A_1z(\jmath-\psi)\right]\right)(\ell).\end{aligned} \tag{4.6}$$

It follows that

$$\begin{aligned}\|z(\ell)\| \le&\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[\|A_0\|\,\|z(\jmath)\| + \|h(\jmath,x(\jmath),x(\jmath-\psi)) - h(\jmath,y(\jmath),y(\jmath-\psi))\|\right]\right)(\ell)\\ &+\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\left[\|A_1\|\,\|z(\jmath-\psi)\|\right]\right)(\ell)\\ \le&(\|A_0\|+L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|z(\jmath)\|\right)(\ell) + (\|A_1\|+L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|z(\jmath-\psi)\|\right)(\ell).\end{aligned}$$

Let $\bar{z}(\ell) = \sup\limits_{\beta\in[-\psi,0]}\|z(\ell+\beta)\|$, then we get

$$\begin{aligned}\bar{z}(\ell) &\le (\|A_0\|+L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\bar{z}(\jmath)\right)(\ell) + (\|A_1\|+L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\bar{z}(\jmath)\right)(\ell)\\ &\le (\|A_0\|+\|A_1\|+2L_1)\left({}_0I^{r,\sigma}e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\bar{z}(\jmath)\right)(\ell).\end{aligned} \tag{4.7}$$

By using the Corollary 3.2 result, we obtain

$$\|z(\ell)\| \le \bar{z}(\ell) \le (0)\times E_r\left(\|A_0\|+\|A_1\|+2L_1,\ell\right). \tag{4.8}$$

Consequently, we obtain $x(\ell)=y(\ell)$ for $\ell\in[-\psi,L]$. □

4.2. Bound for solutions

In this part, we give a bound on the solution to Eq (4.1).

Theorem 4.2. *Given the adherence to condition (H2), the following estimation for the solution $x(\ell)$ of Eq (4.1) is valid:*

$$\|x(\ell)\| \le \left[\|\varphi\| + (L_2 + (\|A_0\|+\|A_1\|)\,\|\varphi\|)\,\frac{\ell^r}{\sin(\frac{\pi}{2}\sigma)^r\Gamma(r+1)}\right]E_r\left((\|A_0\|+\|A_1\|)\,\Gamma(r),\ell\right). \tag{4.9}$$

Proof. For $\ell \in [0, L]$, the solution of Eq (4.1) is expressed as follows:

$$x(\ell) = \varphi(0)e^{-\cot(\frac{\pi}{2}\sigma)\ell} + \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[A_0 x(\jmath) + A_1 x(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi))\right]\right)(\ell). \tag{4.10}$$

Using the fact $e^{-\cot(\frac{\pi}{2}\sigma)\ell} \leq 1$ for all $\ell \in [0, L]$, this leads to

$$\begin{aligned}\|x(\ell)\| \leq& \|\varphi(0)\| + \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \|A_0 x(\jmath) + A_1 x(\jmath - \psi) + h(\jmath, x(\jmath), x(\jmath - \psi))\|\right)(\ell)\\ \leq& \|\varphi\| + \|A_0\| \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|x(\jmath)\|\right)(\ell) + \|A_1\| \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|x(\jmath - \psi)\|\right)(\ell)\\ &+ \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath}\|h(\jmath, x(\jmath), x(\jmath - \psi))\|\right)(\ell).\end{aligned}$$

The above-mentioned inequality can be expressed as follows using assumption (H2) Proposition 2.1:

$$\begin{aligned}\|x(\ell)\| \leq& \|\varphi\| + (\|A_0\| + \|A_1\|) \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[\sup_{\beta \in [-\psi, 0]} \|x(\jmath + \beta)\| + \|\varphi\|\right]\right)(\ell)\\ &+ L_2 \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath}(1)\right)(\ell)\\ =& \|\varphi\| + (L_2 + (\|A_0\| + \|A_1\|)\,\|\varphi\|) \frac{\ell^r}{\sin(\frac{\pi}{2}\sigma)^r \Gamma(r+1)}\\ &+ (\|A_0\| + \|A_1\|) \left({}_0I^{r,\sigma} e^{-\cot(\frac{\pi}{2}\sigma)\jmath} \left[\sup_{\beta \in [-\psi, 0]} \|x(\jmath + \beta)\|\right]\right)(\ell).\end{aligned}$$

Consider $v(\ell) = |\varphi| + (L_2 + (|A_0| + |A_1|)\,|\varphi|) \frac{L^r}{\sin(\frac{\pi}{2}\sigma)^r \Gamma(r+1)}$. As a consequence, v demonstrates a non-decreasing behavior. Therefore, through the utilization of Corollary 3.2 with $f(\ell) = |A_0| + |A_1|$, we deduce that:

$$\|x(\ell)\| \leq \sup_{\beta \in [-\psi, 0]} \|x(\ell + \beta)\| \leq v(\ell) E_r\left((\|A_0\| + \|A_1\|)\,\Gamma(r), \ell\right). \tag{4.11}$$

Hence, the solution x to Eq (4.1) adheres to the determinant

$$\|x(\ell)\| \leq \left[\|\varphi\| + (L_2 + (\|A_0\| + \|A_1\|)\,\|\varphi\|) \frac{\ell^r}{\sin(\frac{\pi}{2}\sigma)^r \Gamma(r+1)}\right] E_r\left((\|A_0\| + \|A_1\|)\,\Gamma(r), \ell\right). \tag{4.12}$$

□

5. Example

Think about the cotangent fractional nonlinear delay equation of the following form:

$$\begin{cases} \left({}_0^C D^{\frac{1}{2}, \frac{1}{3}} x\right)(\ell) = e^{-\cot(\frac{\pi}{6})\ell}[3x(\ell) + x(\ell - 2) + 2\cos x(\ell) - \cos x(\ell - 2)], & \ell \in [0, 1],\\ x(\ell) = \sin(2\ell), & \ell \in [-2, 0]. \end{cases} \tag{5.1}$$

This corresponds to Eq (4.1) with $r = 1/2, \sigma = 1/3, A_0 = 3, A_1 = 1, L = 1$, and $\psi = 2$. The nonlinearity is expressed in the following manner:

$$h(\ell, x(\ell), x(\ell - \psi)) = 2\cos x(\ell) - \cos x(\ell - 2).$$

Hence, we can deduce that

$$\begin{aligned}\|h(\ell, x(\ell), x(\ell-\psi)) - h(\ell, y(\ell), y(\ell-\psi))\| &= \|2\cos x(\ell) - \cos x(\ell-2) - 2\cos y(\ell) + \cos y(\ell-2)\| \\ &\leq 2(\|\cos x(\ell) - \cos y(\ell)\| + \|\cos x(\ell-2) - \cos y(\ell-2)\|).\end{aligned}$$

As a result, assumption (H1) is satisfied with $L_1 = 2$. According to the implication of the Lemma 4.1, Eq (5.1) possesses a distinct solution. Furthermore,

$$\|h(\ell, x(\ell), x(\ell-\psi))\| = \|2\cos x(\ell) - \cos x(\ell-2)\| \leq 3.$$

This suggests that the claim (H2) about $L_2 = 3$ is accurate. According to Theorem 4.2, the solution x of Eq (5.1) is subject to the subsequent estimation:

$$\|x(\ell)\| \leq \left[1 + \frac{24\sqrt{2}}{\sqrt{\pi}}\ell^{\frac{1}{2}}\right] \sum_{k=0}^{\infty} \frac{(4\sqrt{\pi})^k \ell^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}+1\right)}.$$

6. Conclusions

Examining the qualitative characteristics of differential equations holds paramount importance in the realm of differential equation theory. Integral equations serve as indispensable tools for delving into these traits, providing valuable insights into various properties. This paper makes a significant contribution to this field by introducing the Gronwall inequality within the context of cotangent fractional operators. This innovative inequality assumes a critical role in establishing solution uniqueness for delay differential equations that incorporate cotangent fractional derivatives, as well as in deriving upper bounds for these solutions. Expanding beyond the scope of this paper, the derived Gronwall inequality can be employed to delve into further qualitative aspects of these solutions, including their stability. In future work, we will use the fractional Gronwall inequality to conduct a theoretical study on the existence and uniqueness of the cotangent fractional Cauchy problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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