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Research article

Stability analysis through the Bielecki metric to nonlinear fractional integral equations of n-product operators

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Abstract: This work is devoted to the analysis of Hyers, Ulam, and Rassias types of stabilities for nonlinear fractional integral equations with n-product operators. In some special cases, our considered integral equation is related to an integral equation which arises in the study of the spread of an infectious disease that does not induce permanent immunity. n-product operators are described here in the sense of Riemann-Liouville fractional integrals of order $\sigma_i \in (0, 1]$ for $i \in \{1, 2, \ldots, n\}$. Sufficient conditions are provided to ensure Hyers-Ulam, λ-semi-Hyers-Ulam, and Hyers-Ulam-Rassias stabilities in the space of continuous real-valued functions defined on the interval $[0, a]$, where $0 < a < \infty$. Those conditions are established by applying the concept of fixed-point arguments within the framework of the Bielecki metric and its generalizations. Two examples are discussed to illustrate the established results.

Keywords: Hyers-Ulam stability; λ-semi-Hyers-Ulam stability; Hyers-Ulam-Rassias stability; fractional integral equation; Bielecki metric
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1. Introduction

Integral equations represent a significant area of applied mathematics because they are effective tools for modeling a wide range of issues that arise in various branches of science [1–7]. In several references, the authors have discussed the existence, stability, or other qualitative characteristics of solutions to different kinds of integral equations [7–13]. For instance, in [7], Gripenberg described an integral equation that arises in the study of the spread of an infectious disease that does not induce permanent immunity and is of the following form:

$$\omega(\varkappa) = k\left(p(\varkappa) - \int_0^{\varkappa} A(\varkappa - \ell)\omega(\ell)d\ell\right)\left(f(\varkappa) + \int_0^{\varkappa} \tilde{a}(\varkappa - \ell)\omega(\ell)d\ell\right), \quad \varkappa \in [0, \infty). \tag{1.1}$$

In establishing Eq (1.1), the main consideration was that the rate at which susceptibles become infected is proportional to the number of susceptibles and the total infectivity. For this purpose, the author made the assumption that the population is of constant size $\mathcal{P}$ and that the average infectivity of an individual infected at time ℓ is proportional to $\tilde{a}(\varkappa - \ell)$ at time $\varkappa$. If the rate at which individuals susceptible to the disease have become infected up to time $\varkappa$ is $\omega(\ell)$, $\ell < \varkappa$, then $\int_{-\infty}^{\varkappa} \tilde{a}(\varkappa - \ell)\omega(\ell)d\ell$ will be approximately proportional to the total infectivity. If at time ℓ, the cumulative probability function for the loss of immunity of an individual infected is $1 - A(\varkappa - \ell)$, $\varkappa \geq \ell$, then $\mathcal{P} - \int_{-\infty}^{\varkappa} A(\varkappa - \ell)\omega(\ell)d\ell$ will approximate the number of susceptibles. In Eq (1.1), $k > 0$ is a constant and the effects of the infection before $\varkappa = 0$ are considered by the functions p and f.

Later, in [8], Brestovanská studied some existence and convergence results to the following generalized Gripenberg-type integral equation:

$$\omega(\varkappa) = \left(V_1(\varkappa) + \int_0^{\varkappa} A_1(\varkappa - \ell)\omega(\ell)d\ell\right)\ldots\left(V_n(\varkappa) + \int_0^{\varkappa} A_n(\varkappa - \ell)\omega(\ell)d\ell\right), \quad \varkappa \geq 0.$$

In [9], Olaru studied some results on solvability for the following integral equation:

$$\omega(\varkappa) = \prod_{i=1}^{n}\left(V_i(\varkappa) + \int_a^{\varkappa} K_i(\varkappa, \ell, \omega(\ell))d\ell\right), \quad \varkappa \in [a, b].$$

Recently, in [10], Metwali and Cichoń studied the existence results for the following integral equation of n-product type:

$$\omega(\varkappa) = \prod_{i=1}^{n}\left(V_i(\varkappa) + \lambda_i \cdot h_i(\varkappa, \omega(\varkappa)) \cdot \int_a^{b} \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right), \quad \varkappa \in [a, b].$$

The theory of fractional integrals, which deals with integrals of arbitrary order by using the gamma function, is one of the most significant tools for physical investigation, including in fields such as computer networking, image processing, signals, biology, viscoelastic theory, and several others [14–24]. In [24], Jleli and Samet studied the solvability of the following q-fractional integral equation of product type:

$$\omega(\varkappa) = \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{g_i(\varkappa, \omega(\varkappa))}{\Gamma_q(\sigma_i)} \int_0^{\varkappa} (\varkappa - q\ell)^{(\sigma_i - 1)} u_i(\ell, \omega(\ell))d_q\ell\right), \quad \varkappa \in [0, 1],$$

where $q \in (0, 1)$ and $\sigma_i > 1$.

Motivated by the above literature on this significant and interesting topic, we consider here a nonlinear fractional integral equation of n-product type that contains the Riemann-Liouville fractional integral operators as follows:

$$\omega(\varkappa) = \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right), \quad \varkappa \in [0, a], \tag{1.2}$$

where $0 < a < \infty$, $0 < \sigma_i \leq 1$, $V_i, G_i : [0, a] \to \mathbb{R}$, $\mathcal{F}_i : [0, a] \times \mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers and $i = 1, 2, \ldots, n$).

Remark 1. *In some special cases, when* $n = 2$, $G_1(\varkappa) = G_2(\varkappa) = 1$ *and* $\sigma_1 = \sigma_2 = 1$*; then, Eq* (1.2) *is related to Eq* (1.1)*.*

In this paper, we discuss some results on the stability of solutions to Eq (1.2). In order to achieve these aims, we use the concepts of the fixed-point theorem to establish the uniqueness of solutions and analyze some stabilities, namely, Hyers-Ulam (H-U), λ-semi-Hyers-Ulam, and Hyers-Ulam-Rassias (H-U-R) stabilities through the use of the Bielecki metric. Two examples are discussed to illustrate the established results.

This paper is structured as follows: Notations and supporting information are included in Section 2. Some results on H-U-R stability are discussed in Section 3. In Section 4, we discuss some results on λ-semi-Hyers-Ulam and H-U stabilities. Section 5 includes two examples to illustrate the established results. Conclusions and suggestions for further research are given in Section 6.

2. Notations and auxiliary facts

This section includes some notations, definitions and supporting information which are useful to establish the main results.

Let $\delta > 0$ be a constant, and $C_\delta([0, a])$ denotes the space of real-valued continuous functions on $[0, a]$, equipped with the Bielecki metric as follows:

$$d_\delta(\omega, \varphi) = \sup_{\varkappa\in[0,a]} \frac{|\omega(\varkappa) - \varphi(\varkappa)|}{e^{\delta\varkappa}}.$$

In general, we consider the space $C_g([0, a])$ of real-valued continuous functions on $[0, a]$, equipped with the Bielecki metric as follows:

$$d_g(\omega, \varphi) = \sup_{\varkappa\in[0,a]} \frac{|\omega(\varkappa) - \varphi(\varkappa)|}{\lambda(\varkappa)},$$

where $\lambda : [0, a] \to (0, \infty)$ is a nondecreasing continuous function. Then, the metric spaces $(C_\delta([0, a]), d_\delta)$ and $\left(C_g([0, a]), d_g\right)$ are complete [25–28].

The following definitions of stability are stated in the sense of the paper given in reference [25].

Definition 1. *Let* $\lambda(\varkappa)$ *be a non-negative function on* $[0, a]$*. If for each function* $\omega(\varkappa)$ *satisfying*

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa - \ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right)\right| \le \lambda(\varkappa), \quad \forall\, \varkappa \in [0, a],$$

there is a solution $\omega_0(\varkappa)$ of Eq (1.2) and a constant $\aleph > 0$ such that

$$\left|\omega(\varkappa) - \omega_0(\varkappa)\right| \le \aleph\lambda(\varkappa), \quad \forall\, \varkappa \in [0, a],$$

then we say that Eq (1.2) possesses H-U-R stability, where $\aleph$ is independent of $\omega(\varkappa)$ and $\omega_0(\varkappa)$.

Definition 2. *Let* ε *be a non-negative number. If for each function* $\omega(\varkappa)$ *satisfying*

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| \le \varepsilon, \quad \forall\,\varkappa \in [0,a],$$

there is a solution $\omega_0(\varkappa)$ of Eq (1.2) and a constant $\aleph > 0$ such that

$$\left|\omega(\varkappa) - \omega_0(\varkappa)\right| \le \aleph\varepsilon, \quad \forall\,\varkappa \in [0,a],$$

then we say that Eq (1.2) possesses H-U stability, where $\aleph$ is independent of $\omega(\varkappa)$ and $\omega_0(\varkappa)$.

Definition 3. *Let $\lambda(\varkappa)$ be a nondecreasing function on $[0,a]$ and $\varepsilon \ge 0$. Then, Eq (1.2) possesses λ-semi-Hyers-Ulam stability if for each function $\omega(\varkappa)$ satisfying*

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| \le \varepsilon, \quad \forall\,\varkappa \in [0,a],$$

there is a solution $\omega_0(\varkappa)$ of Eq (1.2) with

$$\left|\omega(\varkappa) - \omega_0(\varkappa)\right| \le \aleph\lambda(\varkappa), \quad \forall\,\varkappa \in [0,a],$$

where $\aleph > 0$ is a constant that is independent of $\omega(\varkappa)$ and $\omega_0(\varkappa)$.

Definition 4. *[29, 30] The Riemann-Liouville fractional integral of order $\sigma > 0$ of a function $f(\varkappa)$ is described as follows:*

$$^{\sigma}\mathcal{J}_0^{\varkappa}f(\varkappa) = \frac{1}{\Gamma(\sigma)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma-1}f(\ell)d\ell,$$

where $\Gamma(\sigma) = \int_0^{\infty} e^{-t}t^{\sigma-1}dt$, provided that the right-hand side is point-wise defined on $[0,\infty)$.

Theorem 1. *[31, 32] Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{Y} : \mathcal{X} \to \mathcal{X}$. If there exists a nonnegative constant $\eta \in [0,1)$ such that $d(\mathcal{Y}y, \mathcal{Y}z) \le \eta d(y,z)$, for all $y, z \in \mathcal{X}$, then $\mathcal{Y}$ has a unique fixed point.*

To establish the main results, we define an operator $\mathcal{Y}$ as

$$(\mathcal{Y}\omega)(\varkappa) = \prod_{i=1}^{n}(\mathcal{Y}_i\omega)(\varkappa), \tag{2.1}$$

where

$$(\mathcal{Y}_i\omega)(\varkappa) = V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell, \quad \varkappa \in [0,a],\ i = 1,2,3,\ldots,n. \tag{2.2}$$

Lemma 1. *Let us take $\omega \in C_g([0,a])$. Assume that, for every $i \in \{1,2,\ldots,n\}$, the functions $V_i : [0,a] \to \mathbb{R}$, $G_i : [0,a] \to \mathbb{R}$, $\mathcal{F}_i : [0,a]\times\mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0,a]\times[0,a] \to \mathbb{R}$ are all continuous, and that there exist constants $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, and that $F_i^0 \ge 0$, such that*

$$\left|V_i(\varkappa)\right| \le \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \le \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa,\ell)\right| \le \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell,\omega_1)\right| \le F_i^0, \quad \forall\, \varkappa,\ell \in [0,a],\ \omega_1 \in \mathbb{R}.$$

Then, $\mathcal{Y}\omega \in C_g([0,a])$.

Proof. To prove this, it is enough to show that if $\omega \in C_g([0,a])$, then the operators denoted by $\mathcal{T}_i\omega$ are continuous on $[0,a]$, where

$$(\mathcal{T}_i\omega)(\varkappa) = \frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell, \quad i = 1,2,\ldots,n.$$

When $\sigma_i = 1$, the result is obvious. So, we prove this for $0 < \sigma_i < 1$. To do this, fix $i \in \{1,2,\ldots,n\}$, suppose that $\omega \in C_g([0,a])$, $\varkappa_1,\varkappa_2 \in [0,a]$ with $\varkappa_2 > \varkappa_1$ and fix $\epsilon > 0$ such that $|\varkappa_2-\varkappa_1| \le \epsilon$; then, we get

$$\begin{aligned}
|(\mathcal{T}_i\omega)(\varkappa_2) - (\mathcal{T}_i\omega)(\varkappa_1)| &= \left|\frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_2,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right.\\
&\quad \left.-\frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa_1} (\varkappa_1-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right|\\
&\le \frac{1}{\Gamma(\sigma_i)}\left|\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_2,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right.\\
&\quad \left.-\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right|\\
&\quad +\frac{1}{\Gamma(\sigma_i)}\left|\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right.\\
&\quad \left.-\int_0^{\varkappa_1} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right|\\
&\quad +\frac{1}{\Gamma(\sigma_i)}\left|\int_0^{\varkappa_1} (\varkappa_2-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right.\\
&\quad \left.-\int_0^{\varkappa_1} (\varkappa_1-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right|\\
&\le \frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,|\mathcal{K}_i(\varkappa_2,\ell)-\mathcal{K}_i(\varkappa_1,\ell)|\left|\mathcal{F}_i(\ell,\omega(\ell))\right|d\ell\\
&\quad +\frac{1}{\Gamma(\sigma_i)}\int_{\varkappa_1}^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,|\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))|\,d\ell\\
&\quad +\frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa_1} \left|(\varkappa_2-\ell)^{\sigma_i-1}-(\varkappa_1-\ell)^{\sigma_i-1}\right|\,|\mathcal{K}_i(\varkappa_1,\ell)\mathcal{F}_i(\ell,\omega(\ell))|\,d\ell.
\end{aligned}$$

Let $U(\mathcal{K}_i,\epsilon) = \sup\{|\mathcal{K}_i(\varkappa_2,\ell)-\mathcal{K}_i(\varkappa_1,\ell)| : \varkappa_1,\varkappa_2,\ell \in [0,a], |\varkappa_2-\varkappa_1| \le \epsilon\}$. Then,

$$\begin{aligned}
|(\mathcal{T}_i\omega)(\varkappa_2) - (\mathcal{T}_i\omega)(\varkappa_1)| &\le \frac{U(\mathcal{K}_i,\epsilon)F_i^0}{\Gamma(\sigma_i)}\int_0^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,d\ell + \frac{\widehat{K}_iF_i^0}{\Gamma(\sigma_i)}\int_{\varkappa_1}^{\varkappa_2} (\varkappa_2-\ell)^{\sigma_i-1}\,d\ell\\
&\quad + \frac{\widehat{K}_iF_i^0}{\Gamma(\sigma_i)}\int_0^{\varkappa_1} \left|(\varkappa_2-\ell)^{\sigma_i-1}-(\varkappa_1-\ell)^{\sigma_i-1}\right|d\ell
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{U(\mathcal{K}_i,\epsilon)F_i^0 a^{\sigma_i}}{\Gamma(\sigma_i+1)} + \frac{\widehat{K}_i F_i^0}{\Gamma(\sigma_i+1)}(\varkappa_2-\varkappa_1)^{\sigma_i} \\
&\quad + \frac{\widehat{K}_i F_i^0}{\Gamma(\sigma_i)}\int_0^{\varkappa_1}\left((\varkappa_1-\ell)^{\sigma_i-1}-(\varkappa_2-\ell)^{\sigma_i-1}\right)d\ell \\
&\leq \frac{U(\mathcal{K}_i,\epsilon)F_i^0 a^{\sigma_i}}{\Gamma(\sigma_i+1)} + \frac{\widehat{K}_i F_i^0}{\Gamma(\sigma_i+1)}(\varkappa_2-\varkappa_1)^{\sigma_i} \\
&\quad + \frac{\widehat{K}_i F_i^0}{\Gamma(\sigma_i+1)}\left[(\varkappa_2-\varkappa_1)^{\sigma_i}+\varkappa_1^{\sigma_i}-\varkappa_2^{\sigma_i}\right].
\end{aligned}
$$

By utilizing the uniform continuity of the function $\mathcal{K}_i$ on $[0,a]\times[0,a]$, we have that $U(\mathcal{K}_i,\epsilon)\to 0$ as $\epsilon\to 0$; thus, it follows that the right side of the above inequality tends to zero as $\varkappa_2\to\varkappa_1$. Hence, the operators denoted by $\mathcal{Y}_i\omega$ are continuous on $[0,a]$ for $i\in\{1,2,\ldots,n\}$, and consequently, $\mathcal{Y}\omega\in C_g([0,a])$.

Remark 2. *By the above conditions of Lemma 1, one can easily conclude that if $\omega\in C_\delta([0,a])$, then $\mathcal{Y}\omega\in C_\delta([0,a])$.*

Lemma 2. *Assume that, for every $i\in\{1,2,\ldots,n\}$, the functions $V_i:[0,a]\to\mathbb{R}$, $G_i:[0,a]\to\mathbb{R}$, $\mathcal{F}_i:[0,a]\times\mathbb{R}\to\mathbb{R}$, and $\mathcal{K}_i:[0,a]\times[0,a]\to\mathbb{R}$ are all continuous, and that there exist constants $\widehat{V}_i>0$, $\widehat{G}_i>0$, $\widehat{K}_i>0$, and $F_i^0\geq 0$ such that*

$$
\left|V_i(\varkappa)\right|\leq\widehat{V}_i,\quad \left|G_i(\varkappa)\right|\leq\widehat{G}_i,\quad \left|\mathcal{K}_i(\varkappa,\ell)\right|\leq\widehat{K}_i,\quad \left|\mathcal{F}_i(\ell,\omega_1)\right|\leq F_i^0,\quad \forall\,\varkappa,\ell\in[0,a],\ \omega_1\in\mathbb{R}.
$$

Then for $\omega,\varphi\in C_g([0,a])$, we get

$$
|(\mathcal{Y}\omega)(\varkappa)-(\mathcal{Y}\varphi)(\varkappa)|\leq\mathcal{M}^{n-1}\sum_{i=1}^{n}|(\mathcal{Y}_i\omega)(\varkappa)-(\mathcal{Y}_i\varphi)(\varkappa)|,
$$

where $\mathcal{M}=\max\left\{\widehat{V}_i+\frac{\widehat{G}_i\widehat{K}_iF_i^0a^{\sigma_i}}{\Gamma(\sigma_i+1)}:i=1,2,\ldots,n\right\}$.

Proof. For any $\omega\in C_g([0,a])$, we obtain

$$
\begin{aligned}
|(\mathcal{Y}_i\omega)(\varkappa)| &= \left|V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right| \\
&\leq |V_i(\varkappa)|+\frac{|G_i(\varkappa)|}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}|\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))|\,d\ell \\
&\leq \widehat{V}_i+\frac{\widehat{G}_i\widehat{K}_iF_i^0a^{\sigma_i}}{\Gamma(\sigma_i+1)},\quad i=1,2,\ldots,n.
\end{aligned}
$$

Let $\mathcal{M}=\max\left\{\widehat{V}_i+\frac{\widehat{G}_i\widehat{K}_iF_i^0a^{\sigma_i}}{\Gamma(\sigma_i+1)}:i=1,2,\ldots,n\right\}$.
This gives

$$
|(\mathcal{Y}_i\omega)(\varkappa)|\leq\mathcal{M},\quad \text{for any }\omega\in C_g([0,a]),\ i=1,2,\ldots,n. \tag{2.3}
$$

Now, let $\omega, \varphi \in C_g([0,a])$; then, by using the inequality (2.3), we obtain

$$\begin{aligned}
|(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\varphi)(\varkappa)| &= \left| \prod_{i=1}^{n} (\mathcal{Y}_i\omega)(\varkappa) - \prod_{i=1}^{n} (\mathcal{Y}_i\varphi)(\varkappa) \right| \\
&= |(\mathcal{Y}_1\omega)(\varkappa)(\mathcal{Y}_2\omega)(\varkappa)\dots(\mathcal{Y}_n\omega)(\varkappa) - (\mathcal{Y}_1\varphi)(\varkappa)(\mathcal{Y}_2\varphi)(\varkappa)\dots(\mathcal{Y}_n\varphi)(\varkappa)| \\
&= \big| [(\mathcal{Y}_1\omega)(\varkappa)(\mathcal{Y}_2\omega)(\varkappa)\dots(\mathcal{Y}_n\omega)(\varkappa) - (\mathcal{Y}_1\varphi)(\varkappa)(\mathcal{Y}_2\omega)(\varkappa)\dots(\mathcal{Y}_n\omega)(\varkappa)] \\
&\quad + [(\mathcal{Y}_1\varphi)(\varkappa)(\mathcal{Y}_2\omega)(\varkappa)\dots(\mathcal{Y}_n\omega)(\varkappa) - (\mathcal{Y}_1\varphi)(\varkappa)(\mathcal{Y}_2\varphi)(\varkappa)\dots(\mathcal{Y}_n\omega)(\varkappa)] \\
&\quad + \dots + [(\mathcal{Y}_1\varphi)(\varkappa)\dots(\mathcal{Y}_{n-1}\varphi)(\varkappa)(\mathcal{Y}_n\omega)(\varkappa) - (\mathcal{Y}_1\varphi)(\varkappa)(\mathcal{Y}_2\varphi)(\varkappa)\dots(\mathcal{Y}_n\varphi)(\varkappa)] \big| \\
&\le \mathcal{M}^{n-1} \sum_{i=1}^{n} |(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\varphi)(\varkappa)| .
\end{aligned}$$

3. Results for H-U-R stability

Some results on H-U-R stability are discussed in this section through the application of the Bielecki metric on the interval $[0,a]$. All of the theorems are as follows:

Theorem 2. *Let $0 < p < \sigma_i \le 1$ for $i \in \{1,2,\dots,n\}$. Let $\beta > 0$ and $\lambda : [0,a] \to (0,\infty)$ be a nondecreasing function such that*

$$\left(\int_0^{\varkappa} (\lambda(\ell))^{\frac{1}{p}} \, d\ell \right)^p \le \beta, \quad \forall \varkappa \in [0,a].$$

Moreover, let, for every $i \in \{1,2,\dots,n\}$, the functions $V_i : [0,a] \to \mathbb{R}$, $G_i : [0,a] \to \mathbb{R}$, $\mathcal{F}_i : [0,a]\times\mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0,a] \times [0,a] \to \mathbb{R}$ be continuous and there exist constants $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, $\widehat{F}_i > 0$, and $F_i^0 \ge 0$ such that

$$\left|V_i(\varkappa)\right| \le \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \le \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa,\ell)\right| \le \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell,\omega_1)\right| \le F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell,\omega_2) - \mathcal{F}_i(\ell,\omega_1)| \le \widehat{F}_i \, |\omega_2 - \omega_1| , \quad \forall\, \omega_2, \omega_1 \in \mathbb{R},\ \ell, \varkappa \in [0,a].$$

If $\omega \in C_g([0,a])$ is such that

$$\left| \omega(\varkappa) - \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell \right) \right| \le \lambda(\varkappa), \quad \forall\, \varkappa \in [0,a],$$

and $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i - p} \right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_g([0,a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \le \frac{1}{1 - \left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i - p} \right)} \lambda(\varkappa), \quad \varkappa \in [0,a]. \tag{3.1}$$

This means that Eq (1.2) possesses H-U-R stability.

Proof. Let us define an operator $\mathcal{Y} : C_g([0,a]) \to C_g([0,a])$ by

$$(\mathcal{Y}\omega)(\varkappa) = \prod_{i=1}^{n} (\mathcal{Y}_i\omega)(\varkappa), \tag{3.2}$$

where

$$(\mathcal{Y}_i\omega)(\varkappa) = V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell) \mathcal{F}_i(\ell, \omega(\ell)) d\ell, \quad \varkappa \in [0,a],\ i = 1,2,3,\ldots,n. \tag{3.3}$$

Now, to fulfill the criteria of Theorem 1, we take $\omega, \varphi \in C_g([0,a])$; then,

$$\begin{aligned}
|(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\varphi)(\varkappa)| &= \Big| V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell) \mathcal{F}_i(\ell, \omega(\ell)) d\ell \\
&\quad - V_i(\varkappa) - \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell) \mathcal{F}_i(\ell, \varphi(\ell)) d\ell \Big| \\
&\le \frac{|G_i(\varkappa)|}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} |\mathcal{K}_i(\varkappa, \ell)| \, |\mathcal{F}_i(\ell, \omega(\ell)) - \mathcal{F}_i(\ell, \varphi(\ell))| \, d\ell \\
&\le \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} |\omega(\ell) - \varphi(\ell)| \, d\ell \\
&= \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \lambda(\ell) \frac{|\omega(\ell) - \varphi(\ell)|}{\lambda(\ell)} d\ell \\
&\le \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i d_g(\omega, \varphi)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \lambda(\ell) d\ell \\
&\le \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i d_g(\omega, \varphi)}{\Gamma(\sigma_i)} \left(\int_0^{\varkappa} (\varkappa - \ell)^{\frac{\sigma_i - 1}{1-p}} \, d\ell \right)^{1-p} \left(\int_0^{\varkappa} (\lambda(\ell))^{\frac{1}{p}} \, d\ell \right)^{p} \\
&\le \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i d_g(\omega, \varphi) \beta}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p}, \quad i = 1,2,\ldots,n.
\end{aligned} \tag{3.4}$$

Then by using Lemma 2 and inequality (3.4), we get

$$\begin{aligned}
|(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\varphi)(\varkappa)| &\le \mathcal{M}^{n-1} \sum_{i=1}^{n} |(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\varphi)(\varkappa)| \\
&\le \mathcal{M}^{n-1} \sum_{i=1}^{n} \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i d_g(\omega, \varphi) \beta}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p}.
\end{aligned} \tag{3.5}$$

Now,

$$\begin{aligned}
d_g(\mathcal{Y}\omega, \mathcal{Y}\varphi) &= \sup_{\varkappa \in [0,a]} \frac{|(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\varphi)(\varkappa)|}{\lambda(\varkappa)} \\
&\le \sup_{\varkappa \in [0,a]} \frac{1}{\lambda(\varkappa)} \left\{ \mathcal{M}^{n-1} \sum_{i=1}^{n} \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i d_g(\omega, \varphi) \beta}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p} \right\} \\
&\le \left(\frac{\mathcal{M}^{n-1} \beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p} \right) d_g(\omega, \varphi).
\end{aligned}$$

From the condition $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)}\sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i-p}\right)^{1-p}a^{\sigma_i-p}\right)<1$ and Theorem 1, it follows that $\mathcal{Y}$ has a unique fixed point and hence, Eq (1.2) has a unique solution.

Let $\omega_0(\varkappa)\in\mathcal{C}_g([0,a])$ be a unique solution of Eq (1.2) and let $\omega\in\mathcal{C}_g([0,a])$ be such that

$$\left|\omega(\varkappa)-\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right|\leq\lambda(\varkappa),\quad\forall\,\varkappa\in[0,a].$$

Then,

$$\begin{aligned}
d_g(\omega,\omega_0)&=\sup_{\varkappa\in[0,a]}\frac{|\omega(\varkappa)-\omega_0(\varkappa)|}{\lambda(\varkappa)}\\
&=\sup_{\varkappa\in[0,a]}\frac{1}{\lambda(\varkappa)}\left|\omega(\varkappa)-\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\right|\\
&\leq\sup_{\varkappa\in[0,a]}\frac{1}{\lambda(\varkappa)}\Bigg\{\left|\omega(\varkappa)-\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right|\\
&\quad+\Bigg|\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\\
&\quad-\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\Bigg|\Bigg\}\\
&\leq\sup_{\varkappa\in[0,a]}\frac{1}{\lambda(\varkappa)}\Bigg\{\lambda(\varkappa)+\Bigg|\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\\
&\quad-\prod_{i=1}^{n}\left(V_i(\varkappa)+\frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\Bigg|\Bigg\}\\
&=\sup_{\varkappa\in[0,a]}\frac{1}{\lambda(\varkappa)}\Big\{\lambda(\varkappa)+|(\mathcal{Y}\omega)(\varkappa)-(\mathcal{Y}\omega_0)(\varkappa)|\Big\}.
\end{aligned}$$

By using the inequality (3.5), we get

$$\begin{aligned}
d_g(\omega,\omega_0)&\leq\sup_{\varkappa\in[0,a]}\frac{1}{\lambda(\varkappa)}\left\{\lambda(\varkappa)+\mathcal{M}^{n-1}\sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega,\omega_0)\beta}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i-p}\right)^{1-p}a^{\sigma_i-p}\right\}\\
&\leq 1+\frac{\mathcal{M}^{n-1}}{\lambda(0)}\sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega,\omega_0)\beta}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i-p}\right)^{1-p}a^{\sigma_i-p},
\end{aligned}$$

i.e.,

$$d_g(\omega,\omega_0)\leq\frac{1}{1-\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)}\sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i-p}\right)^{1-p}a^{\sigma_i-p}\right)},\tag{3.6}$$

which implies that

$$\sup_{\varkappa\in[0,a]}\frac{|\omega(\varkappa)-\omega_0(\varkappa)|}{\lambda(\varkappa)}\leq\frac{1}{1-\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)}\sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i-p}\right)^{1-p}a^{\sigma_i-p}\right)},\tag{3.7}$$

and consequently the inequality (3.1) holds. This ensures the H-U-R stability for Eq (1.2).

Corollary 1. *Let* $0 < p < \sigma_i \leq 1$*, for* $i \in \{1, 2, \ldots, n\}$ *and* $\delta > 0$*. Moreover, let, for every* $i \in \{1, 2, \ldots, n\}$*, the functions* $V_i : [0, a] \to \mathbb{R}$*,* $G_i : [0, a] \to \mathbb{R}$*,* $\mathcal{F}_i : [0, a] \times \mathbb{R} \to \mathbb{R}$*, and* $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ *be continuous and there exist constants* $\widehat{V}_i > 0$*,* $\widehat{G}_i > 0$*,* $\widehat{K}_i > 0$*,* $\widehat{F}_i > 0$*, and* $F_i^0 \geq 0$ *such that*

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa, \ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell, \omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell, \omega_2) - \mathcal{F}_i(\ell, \omega_1)| \leq \widehat{F}_i \left|\omega_2 - \omega_1\right|, \quad \forall\, \omega_2, \omega_1 \in \mathbb{R},\ \ell, \varkappa \in [0, a].$$

If $\omega \in C_\delta([0, a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa - \ell)^{\sigma_i - 1}\mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right)\right| \leq e^{\delta\varkappa}, \quad \forall\, \varkappa \in [0, a],$$

and $\left(\mathcal{M}^{n-1}\left(\frac{p}{\delta}\right)^p\left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i - p}\right)^{1-p} a^{\sigma_i - p}\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_\delta([0, a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{e^{\delta\varkappa}}{1 - \left(\mathcal{M}^{n-1}\left(\frac{p}{\delta}\right)^p\left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^{n}\frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)}\left(\frac{1-p}{\sigma_i - p}\right)^{1-p} a^{\sigma_i - p}\right)}, \quad \varkappa \in [0, a]. \tag{3.8}$$

This means that Eq (1.2) possesses H-U-R stability.

Theorem 3. *Let* $0 < \sigma_i \leq 1$ *and* $\beta_i > 0$ *for* $i \in \{1, 2, \ldots, n\}$ *and* $\lambda : [0, a] \to (0, \infty)$ *be a nondecreasing function, such that*

$$\frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa - \ell)^{\sigma_i - 1}\lambda(\ell)d\ell \leq \beta_i\lambda(\varkappa), \quad \forall \varkappa \in [0, a],\ i = 1, 2, \ldots, n.$$

Moreover, let, for every $i \in \{1, 2, \ldots, n\}$, the functions $V_i : [0, a] \to \mathbb{R}$, $G_i : [0, a] \to \mathbb{R}$, $\mathcal{F}_i : [0, a] \times \mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ be continuous and there exist constants $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, $\widehat{F}_i > 0$, and $F_i^0 \geq 0$ such that

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa, \ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell, \omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell, \omega_2) - \mathcal{F}_i(\ell, \omega_1)| \leq \widehat{F}_i \left|\omega_2 - \omega_1\right|, \quad \forall\, \omega_2, \omega_1 \in \mathbb{R},\ \ell, \varkappa \in [0, a].$$

If $\omega \in C_g([0, a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa - \ell)^{\sigma_i - 1}\mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right)\right| \leq \lambda(\varkappa), \quad \forall\, \varkappa \in [0, a],$$

and $\left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_g([0, a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{1}{1 - \left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right)}\lambda(\varkappa), \quad \varkappa \in [0, a]. \tag{3.9}$$

This means that Eq (1.2) possesses H-U-R stability.

Proof. Let us define an operator $\mathcal{Y} : C_g([0,a]) \to C_g([0,a])$ by

$$(\mathcal{Y}\omega)(\varkappa) = \prod_{i=1}^{n} (\mathcal{Y}_i\omega)(\varkappa), \tag{3.10}$$

where

$$(\mathcal{Y}_i\omega)(\varkappa) = V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell, \quad \varkappa \in [0,a],\ i = 1,2,3,\ldots,n. \tag{3.11}$$

Now, to fulfill the criteria of Theorem 1, we take $\omega, \varphi \in C_g([0,a])$; then,

$$\begin{aligned}
|(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\varphi)(\varkappa)| &= \Big| V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell \\
&\quad - V_i(\varkappa) - \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \varphi(\ell))d\ell \Big| \\
&\le \frac{|G_i(\varkappa)|}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} |\mathcal{K}_i(\varkappa, \ell)| \, |\mathcal{F}_i(\ell, \omega(\ell)) - \mathcal{F}_i(\ell, \varphi(\ell))| \, d\ell \\
&\le \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} |\omega(\ell) - \varphi(\ell)| \, d\ell \\
&= \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \lambda(\ell) \frac{|\omega(\ell) - \varphi(\ell)|}{\lambda(\ell)} d\ell \\
&\le \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega, \varphi)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \lambda(\ell) d\ell \\
&\le \widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega, \varphi)\beta_i\lambda(\varkappa), \quad i = 1,2,\ldots,n.
\end{aligned} \tag{3.12}$$

Then, by using Lemma 2 and inequality (3.12), we obtain

$$\begin{aligned}
|(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\varphi)(\varkappa)| &\le \mathcal{M}^{n-1} \sum_{i=1}^{n} |(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\varphi)(\varkappa)| \\
&\le \mathcal{M}^{n-1} \sum_{i=1}^{n} \widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega, \varphi)\beta_i\lambda(\varkappa).
\end{aligned} \tag{3.13}$$

Now,

$$\begin{aligned}
d_g(\mathcal{Y}\omega, \mathcal{Y}\varphi) &= \sup_{\varkappa \in [0,a]} \frac{|(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\varphi)(\varkappa)|}{\lambda(\varkappa)} \\
&\le \sup_{\varkappa \in [0,a]} \frac{1}{\lambda(\varkappa)} \left\{ \mathcal{M}^{n-1} \sum_{i=1}^{n} \widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega, \varphi)\beta_i\lambda(\varkappa) \right\},
\end{aligned}$$

i.e.,

$$d_g(\mathcal{Y}\omega, \mathcal{Y}\varphi) \le \left(\mathcal{M}^{n-1} \sum_{i=1}^{n} \widehat{G}_i\widehat{K}_i\widehat{F}_i \beta_i \right) d_g(\omega, \varphi).$$

From the condition $\left(\mathcal{M}^{n-1} \sum_{i=1}^{n} \widehat{G}_i\widehat{K}_i\widehat{F}_i \beta_i\right) < 1$ and Theorem 1, it follows that $\mathcal{Y}$ has a unique fixed point and hence, Eq (1.2) has a unique solution.

Let $\omega_0(\varkappa) \in C_g([0,a])$ be a unique solution of Eq (1.2), and let $\omega \in C_g([0,a])$ be such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| \le \lambda(\varkappa), \quad \forall\, \varkappa \in [0,a].$$

Then,

$$\begin{aligned}
d_g(\omega,\omega_0) &= \sup_{\varkappa\in[0,a]} \frac{|\omega(\varkappa)-\omega_0(\varkappa)|}{\lambda(\varkappa)} \\
&= \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)}\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\right| \\
&\le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)}\Bigg\{\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| \\
&\quad + \Bigg|\prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right) \\
&\quad - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\Bigg|\Bigg\} \\
&\le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)}\Bigg\{\lambda(\varkappa) + \Bigg|\prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right) \\
&\quad - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell\right)\Bigg|\Bigg\}.
\end{aligned}$$

By using the inequality (3.13), we get

$$d_g(\omega,\omega_0) \le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)}\left\{\lambda(\varkappa) + \mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega,\omega_0)\beta_i\lambda(\varkappa)\right\},$$

i.e.,

$$d_g(\omega,\omega_0) \le 1 + \mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i d_g(\omega,\omega_0)\beta_i,$$

or,

$$d_g(\omega,\omega_0) \le \frac{1}{1-\left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right)},$$

which implies that

$$\sup_{\varkappa\in[0,a]} \frac{|\omega(\varkappa)-\omega_0(\varkappa)|}{\lambda(\varkappa)} \le \frac{1}{1-\left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right)}, \tag{3.14}$$

consequently, the inequality (3.9) holds. This ensures the H-U-R stability for Eq (1.2).

4. Results for λ-semi-Hyers-Ulam and H-U stabilities

Theorem 4. *Let $0 < p < \sigma_i \le 1$, for $i \in \{1, 2, \ldots, n\}$. Let $\beta > 0$ and $\lambda : [0, a] \to (0, \infty)$ be a nondecreasing function, such that*

$$\left(\int_0^{\varkappa} (\lambda(\ell))^{\frac{1}{p}}\, d\ell\right)^{p} \le \beta, \quad \forall \varkappa \in [0, a].$$

Moreover, let, for every $i \in \{1, 2, \ldots, n\}$, the functions $V_i : [0, a] \to \mathbb{R}$, $G_i : [0, a] \to \mathbb{R}$, $\mathcal{F}_i : [0, a]\times\mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ be continuous and there exist constants $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, $\widehat{F}_i > 0$, and $F_i^0 \ge 0$ such that

$$\left|V_i(\varkappa)\right| \le \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \le \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa, \ell)\right| \le \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell, \omega_1)\right| \le F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell, \omega_2) - \mathcal{F}_i(\ell, \omega_1)| \le \widehat{F}_i\, |\omega_2 - \omega_1|, \quad \forall\, \omega_2, \omega_1 \in \mathbb{R},\ \ell, \varkappa \in [0, a].$$

If $\omega \in C_g([0, a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1}\, \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right)\right| \le \varepsilon, \quad \forall\, \varkappa \in [0, a],$$

where $\varepsilon > 0$ and $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i - p}\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_g([0, a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \le \frac{\varepsilon}{\lambda(0) - \left(\mathcal{M}^{n-1}\beta \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right)}\lambda(\varkappa), \quad \varkappa \in [0, a]. \tag{4.1}$$

This means that Eq (1.2) possesses λ-semi-Hyers-Ulam stability.

Proof. We define the operator $\mathcal{Y} : C_g([0, a]) \to C_g([0, a])$ by

$$(\mathcal{Y}\omega)(\varkappa) = \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1}\, \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right), \quad \varkappa \in [0, a]. \tag{4.2}$$

Given that $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i - p}\right) < 1$, similar to Theorem 2, we have that Eq (1.2) has a unique solution. To establish the λ-semi-Hyers-Ulam stability, let $\omega_0(\varkappa) \in C_g([0, a])$ be a unique solution of Eq (1.2) and let $\omega \in C_g([0, a])$ be such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1}\, \mathcal{K}_i(\varkappa, \ell)\mathcal{F}_i(\ell, \omega(\ell))d\ell\right)\right| \le \varepsilon, \quad \forall\, \varkappa \in [0, a].$$

Then,

$$d_g(\omega, \omega_0) = \sup_{\varkappa \in [0,a]} \frac{|\omega(\varkappa) - \omega_0(\varkappa)|}{\lambda(\varkappa)}$$

$$= \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)} \left| \omega(\varkappa) - \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell \right) \right|$$

$$\le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)} \Bigg\{ \left| \omega(\varkappa) - \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell \right) \right|$$

$$+ \Bigg| \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell \right)$$

$$- \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell \right) \Bigg| \Bigg\}$$

$$\le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)} \Bigg\{ \varepsilon + \Bigg| \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell \right)$$

$$- \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell \right) \Bigg| \Bigg\}.$$

By using Lemma 2 and following a procedure similar to that for inequality (3.4), we get

$$\Bigg| \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell \right)$$

$$- \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1} \mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega_0(\ell))d\ell \right) \Bigg|$$

$$= |(\mathcal{Y}\omega)(\varkappa) - (\mathcal{Y}\omega_0)(\varkappa)|$$

$$\le \mathcal{M}^{n-1} \sum_{i=1}^{n} |(\mathcal{Y}_i\omega)(\varkappa) - (\mathcal{Y}_i\omega_0)(\varkappa)|$$

$$\le \mathcal{M}^{n-1} \sum_{i=1}^{n} \frac{\widehat{G_i}\widehat{K_i}\widehat{F_i} d_g(\omega,\omega_0)\beta}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p}. \tag{4.3}$$

By using the inequality (4.3), we get

$$d_g(\omega,\omega_0) \le \sup_{\varkappa\in[0,a]} \frac{1}{\lambda(\varkappa)} \Bigg\{ \varepsilon + \mathcal{M}^{n-1} \sum_{i=1}^{n} \frac{\widehat{G_i}\widehat{K_i}\widehat{F_i} d_g(\omega,\omega_0)\beta}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p} \Bigg\},$$

$$\le \frac{\varepsilon}{\lambda(0)} + \mathcal{M}^{n-1} \sum_{i=1}^{n} \frac{\widehat{G_i}\widehat{K_i}\widehat{F_i} d_g(\omega,\omega_0)\beta}{\lambda(0)\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p} \right)^{1-p} a^{\sigma_i - p},$$

i.e.,

$$d_g(\omega,\omega_0) \le \frac{\varepsilon}{\lambda(0) - \left(\mathcal{M}^{n-1}\beta \sum_{i=1}^{n} \frac{\widehat{G_i}\widehat{K_i}\widehat{F_i}}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p} \right)^{1-p} a^{\sigma_i-p} \right)}, \tag{4.4}$$

which implies that

$$\sup_{\varkappa\in[0,a]} \frac{|\omega(\varkappa) - \omega_0(\varkappa)|}{\lambda(\varkappa)} \le \frac{\varepsilon}{\lambda(0) - \left(\mathcal{M}^{n-1}\beta \sum_{i=1}^{n} \frac{\widehat{G_i}\widehat{K_i}\widehat{F_i}}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p} \right)^{1-p} a^{\sigma_i-p} \right)}, \tag{4.5}$$

consequently, the inequality (4.1) holds. This ensures the λ-semi-Hyers-Ulam stability for Eq (1.2).

Corollary 2. *Let* $0 < p < \sigma_i \leq 1$, *for* $i \in \{1, 2, \ldots, n\}$ *and* $\delta > 0$. *Moreover, let, for every* $i \in \{1, 2, \ldots, n\}$, *the functions* $V_i : [0, a] \to \mathbb{R}$, $G_i : [0, a] \to \mathbb{R}$, $\mathcal{F}_i : [0, a] \times \mathbb{R} \to \mathbb{R}$, *and* $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ *be continuous and there exist constants* $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, $\widehat{F}_i > 0$, *and* $F_i^0 \geq 0$ *such that*

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa, \ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell, \omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell, \omega_2) - \mathcal{F}_i(\ell, \omega_1)| \leq \widehat{F}_i \, |\omega_2 - \omega_1| \,, \quad \forall\, \omega_2, \omega_1 \in \mathbb{R}, \ \ell, \varkappa \in [0, a].$$

If $\omega \in C_\delta([0, a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell) \mathcal{F}_i(\ell, \omega(\ell)) d\ell\right)\right| \leq \varepsilon, \quad \forall\, \varkappa \in [0, a],$$

where $\varepsilon > 0$ and $\left(\mathcal{M}^{n-1} \left(\frac{p}{\delta}\right)^p \left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^n \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p}\right)^{1-p} a^{\sigma_i - p}\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_\delta([0, a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{\varepsilon e^{\delta \varkappa}}{1 - \left(\mathcal{M}^{n-1} \left(\frac{p}{\delta}\right)^p \left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^n \frac{\widehat{G}_i \widehat{K}_i \widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i - p}\right)^{1-p} a^{\sigma_i - p}\right)}, \quad \varkappa \in [0, a]. \tag{4.6}$$

This means that Eq (1.2) possesses λ-semi-Hyers-Ulam stability.

Corollary 3. *Let* $0 < p < \sigma_i \leq 1$ *for* $i \in \{1, 2, \ldots, n\}$. *Let* $\beta > 0$ *and* $\lambda : [0, a] \to (0, \infty)$ *be a nondecreasing function such that*

$$\left(\int_0^{\varkappa} (\lambda(\ell))^{\frac{1}{p}} \, d\ell\right)^p \leq \beta, \quad \forall \varkappa \in [0, a].$$

Moreover, let, for every $i \in \{1, 2, \ldots, n\}$, the functions $V_i : [0, a] \to \mathbb{R}$, $G_i : [0, a] \to \mathbb{R}$, $\mathcal{F}_i : [0, a] \times \mathbb{R} \to \mathbb{R}$, and $\mathcal{K}_i : [0, a] \times [0, a] \to \mathbb{R}$ be continuous and there exist constants $\widehat{V}_i > 0$, $\widehat{G}_i > 0$, $\widehat{K}_i > 0$, $\widehat{F}_i > 0$, and $F_i^0 \geq 0$ such that

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa, \ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell, \omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell, \omega_2) - \mathcal{F}_i(\ell, \omega_1)| \leq \widehat{F}_i \, |\omega_2 - \omega_1| \,, \quad \forall\, \omega_2, \omega_1 \in \mathbb{R}, \ \ell, \varkappa \in [0, a].$$

If $\omega \in C_g([0, a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n} \left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa - \ell)^{\sigma_i - 1} \mathcal{K}_i(\varkappa, \ell) \mathcal{F}_i(\ell, \omega(\ell)) d\ell\right)\right| \leq \varepsilon, \quad \forall\, \varkappa \in [0, a],$$

where $\varepsilon > 0$ and $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_g([0,a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{\lambda(a)}{\lambda(0) - \left(\mathcal{M}^{n-1}\beta \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right)}\varepsilon, \quad \varkappa \in [0,a]. \tag{4.7}$$

This means that Eq (1.2) possesses H-U stability.

Corollary 4. *Let* $0 < p < \sigma_i \leq 1$ *for* $i \in \{1,2,\ldots,n\}$ *and* $\delta > 0$*. Moreover, let, for every* $i \in \{1,2,\ldots,n\}$*, the functions* $V_i : [0,a] \to \mathbb{R}$*,* $G_i : [0,a] \to \mathbb{R}$*,* $\mathcal{F}_i : [0,a] \times \mathbb{R} \to \mathbb{R}$*, and* $\mathcal{K}_i : [0,a] \times [0,a] \to \mathbb{R}$ *be continuous and there exist constants* $\widehat{V}_i > 0$*,* $\widehat{G}_i > 0$*,* $\widehat{K}_i > 0$*,* $\widehat{F}_i > 0$*, and* $F_i^0 \geq 0$ *such that*

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa,\ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell,\omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell,\omega_2) - \mathcal{F}_i(\ell,\omega_1)| \leq \widehat{F}_i\,|\omega_2 - \omega_1|\,, \quad \forall\, \omega_2,\omega_1 \in \mathbb{R},\ \ell,\varkappa \in [0,a].$$

If $\omega \in C_\delta([0,a])$ is such that

$$\left|\omega(\varkappa) - \prod_{i=1}^{n}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)} \int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| \leq \varepsilon, \quad \forall\,\varkappa \in [0,a],$$

where $\varepsilon > 0$ and $\left(\mathcal{M}^{n-1}\left(\frac{p}{\delta}\right)^p \left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right) < 1$, then there is a unique solution $\omega_0(\varkappa) \in C_\delta([0,a])$ of Eq (1.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{e^{\delta a}}{1 - \left(\mathcal{M}^{n-1}\left(\frac{p}{\delta}\right)^p \left(e^{\frac{\delta a}{p}} - 1\right)^p \sum_{i=1}^{n} \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right)}\varepsilon, \quad \varkappa \in [0,a]. \tag{4.8}$$

This means that Eq (1.2) possesses H-U stability.

5. Examples

We will discuss two examples in this section to illustrate the established results.

Example 1. *Consider the following integral equation:*

$$\omega(\varkappa) = \left(V_1(\varkappa) + \frac{G_1(\varkappa)}{\Gamma(\frac{1}{2})} \int_0^{\varkappa} (\varkappa-\ell)^{-\frac{1}{2}} (\varkappa+\ell)\sin(\omega(\ell))d\ell\right)\left(V_2(\varkappa) + \frac{G_2(\varkappa)}{\Gamma(\frac{1}{2})} \int_0^{\varkappa} (\varkappa-\ell)^{-\frac{1}{2}}\,\ell(\ell+\cos(\omega(\ell)))d\ell\right),$$
$$\varkappa \in [0,1], \tag{5.1}$$

where $V_1(\varkappa) = 2\pi$, $V_2(\varkappa) = 1 - \frac{(16\varkappa^{\frac{5}{2}}+20\varkappa^{\frac{3}{2}})\sin(\varkappa)}{1800\,\Gamma(\frac{1}{2})}$, $G_1(\varkappa) = \frac{\varkappa}{264}$, and $G_2(\varkappa) = \frac{\sin(\varkappa)}{120}$. Comparing Eq (5.1) with Eq (1.2), we have that $n = 2$, $a = 1$, $\mathcal{K}_1(\varkappa,\ell) = \varkappa + \ell$, $\mathcal{F}_1(\ell,\omega(\ell)) = \sin(\omega(\ell))$, $\mathcal{K}_2(\varkappa,\ell) = \ell$, $\mathcal{F}_2(\ell,\omega(\ell)) = \ell + \cos(\omega(\ell))$, and $\sigma_1 = \sigma_2 = \frac{1}{2}$.

It can be observed that the functions V_1, G_1, $\mathcal{K}_1$, $\mathcal{F}_1$, V_2, G_2, $\mathcal{K}_2$, and $\mathcal{F}_2$ are all continuous and satisfy the following conditions:

$$\left|V_i(\varkappa)\right| \leq \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \leq \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa,\ell)\right| \leq \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell,\omega_1)\right| \leq F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell,\omega_2) - \mathcal{F}_i(\ell,\omega_1)| \leq \widehat{F}_i\,|\omega_2 - \omega_1|, \quad \forall\,\omega_2,\omega_1 \in \mathbb{R},\ \ell,\varkappa \in [0,1], \quad \text{for } i = 1,2;$$

where $\widehat{V}_1 = 2\pi$, $\widehat{G}_1 = \frac{1}{264}$, $\widehat{K}_1 = 2$, $F_1^0 = 1$, $\widehat{F}_1 = 1$, $\widehat{V}_2 = 1$, $\widehat{G}_2 = \frac{1}{120}$, $\widehat{K}_2 = 1$, $F_2^0 = 2$, and $\widehat{F}_2 = 1$.
Thus, all conditions of Lemmas 1 and 2 are satisfied and we get
$\mathcal{M} = \max\left\{\widehat{V}_i + \frac{\widehat{G}_i\widehat{K}_iF_i^0a^{\sigma_i}}{\Gamma(\sigma_i+1)} : i = 1,2\right\} \approx 6.292.$

We choose $p = \frac{1}{3}$ such that $0 < p < \sigma_i \leq 1$ holds for $i = 1,2$, and we consider the nondecreasing function $\lambda : [0,1] \to (0,\infty)$ given by $\lambda(\varkappa) = \varkappa + 2\pi$. Then, the following condition

$$\left(\int_0^{\varkappa} (\lambda(\ell))^{\frac{1}{p}}\, d\ell\right)^p \leq \beta, \quad \forall \varkappa \in [0,1],$$

is satisfied by $\beta = (313.8010)^{\frac{1}{3}}$.

Now, $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^n \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right) \approx 0.1539 < 1.$
If we take $\omega(\varkappa) = 0$, then

$$\left|\omega(\varkappa) - \prod_{i=1}^{2}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| = |0 - 2\pi| \leq \varkappa + 2\pi = \lambda(\varkappa), \quad \forall\,\varkappa \in [0,1].$$

Thus, by Theorem 2, there exists a unique solution $\omega_0(\varkappa) \in C_g([0,a])$ of Eq (5.1) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{1}{1 - \left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^n \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right)}\lambda(\varkappa), \quad \varkappa \in [0,1].$$

This ensures the H-U-R stability for Eq (5.1).
Again, since $\left(\frac{\mathcal{M}^{n-1}\beta}{\lambda(0)} \sum_{i=1}^n \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right) \approx 0.1539 < 1$, and if we take $\omega(\varkappa) = 0$, then, for $\varepsilon \geq 2\pi$, we get

$$\left|\omega(\varkappa) - \prod_{i=1}^{2}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1}\,\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| = |0 - 2\pi| \leq \varepsilon, \quad \forall\,\varkappa \in [0,1].$$

Hence, by Theorem 4, there exists a unique solution $\omega_0(\varkappa) \in C_g([0,a])$ of Eq (5.1) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \leq \frac{\varepsilon}{\lambda(0) - \left(\mathcal{M}^{n-1}\beta \sum_{i=1}^n \frac{\widehat{G}_i\widehat{K}_i\widehat{F}_i}{\Gamma(\sigma_i)} \left(\frac{1-p}{\sigma_i-p}\right)^{1-p} a^{\sigma_i-p}\right)}\lambda(\varkappa), \quad \varkappa \in [0,a],$$

which ensures the λ-semi-Hyers-Ulam stability for Eq (5.1); also, by Corollary 3, we can conclude the H-U stability for Eq (5.1).

Example 2. *Consider the following integral equation:*

$$\omega(\varkappa) = \left(V_1(\varkappa) + \frac{G_1(\varkappa)}{\Gamma(\frac{1}{3})}\int_0^{\varkappa} (\varkappa-\ell)^{-\frac{2}{3}} \frac{(1+\ell)}{1+|\omega(\ell)|e^{-\ell}} d\ell\right)\left(V_2(\varkappa) + \frac{G_2(\varkappa)}{\Gamma(\frac{1}{3})}\int_0^{\varkappa} (\varkappa-\ell)^{-\frac{2}{3}} \cos\left(\frac{\pi}{2}\omega(\ell)e^{-\ell}\right) d\ell\right), \quad \varkappa \in [0,1], \tag{5.2}$$

where $V_1(\varkappa) = 1 - \frac{e^{\varkappa}\left(9\varkappa^{\frac{4}{3}}+12\varkappa^{\frac{1}{3}}\right)}{2592\,\Gamma(\frac{1}{3})}$, $V_2(\varkappa) = e^{\varkappa}$, $G_1(\varkappa) = \frac{e^{\varkappa}}{324}$, and $G_2(\varkappa) = \frac{e^{\varkappa}}{224}$. Comparing Eq (5.2) with Eq (1.2), we have that $n = 2$, $a = 1$, $\mathcal{K}_1(\varkappa,\ell) = 1+\ell$, $\mathcal{F}_1(\ell,\omega(\ell)) = \frac{1}{1+|\omega(\ell)|e^{-\ell}}$, $\mathcal{K}_2(\varkappa,\ell) = 1$, $\mathcal{F}_2(\ell,\omega(\ell)) = \cos\left(\frac{\pi}{2}\omega(\ell)e^{-\ell}\right)$, and $\sigma_1 = \sigma_2 = \frac{1}{3}$.

It can be observed that the functions V_1, G_1, $\mathcal{K}_1$, $\mathcal{F}_1$, V_2, G_2, $\mathcal{K}_2$, and $\mathcal{F}_2$ are all continuous and satisfy the following conditions:

$$\left|V_i(\varkappa)\right| \le \widehat{V}_i, \quad \left|G_i(\varkappa)\right| \le \widehat{G}_i, \quad \left|\mathcal{K}_i(\varkappa,\ell)\right| \le \widehat{K}_i, \quad \left|\mathcal{F}_i(\ell,\omega_1)\right| \le F_i^0,$$

$$\text{and} \quad |\mathcal{F}_i(\ell,\omega_2) - \mathcal{F}_i(\ell,\omega_1)| \le \widehat{F}_i\,|\omega_2-\omega_1|, \quad \forall\, \omega_2,\omega_1 \in \mathbb{R},\ \ell,\varkappa \in [0,1], \quad \text{for } i = 1,2;$$

where $\widehat{V}_1 = 1$, $\widehat{G}_1 = 0.0084$, $\widehat{K}_1 = 2$, $F_1^0 = 1$, $\widehat{F}_1 = 1$, $\widehat{V}_2 = 2.7183$, $\widehat{G}_2 = 0.0121$, $\widehat{K}_2 = 1$, $F_2^0 = 1$, and $\widehat{F}_2 = \frac{\pi}{2}$.
Thus, all conditions of Lemmas 1 and 2 are satisfied and we get the following:
$\mathcal{M} = \max\left\{\widehat{V}_i + \frac{\widehat{G}_i\widehat{K}_iF_i^0a^{\sigma_i}}{\Gamma(\sigma_i+1)} : i = 1,2\right\} \approx 2.732.$

We consider the nondecreasing function $\lambda : [0,1] \to (0,\infty)$ given by $\lambda(\varkappa) = 4\varkappa + 2$. Then, the condition

$$\frac{1}{\Gamma(\sigma_i)}\int_0^{\varkappa} (\varkappa-\ell)^{\sigma_i-1}\lambda(\ell)d\ell \le \beta_i\lambda(\varkappa), \quad \forall\varkappa \in [0,1],\ i = 1,2,$$

is satisfied by $\beta_1 = \beta_2 = 2.7996$.

Now, $\left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right) \approx 0.274 < 1$.
If we take $\omega(\varkappa) = 3e^{\varkappa}$, then

$$\left|\omega(\varkappa) - \prod_{i=1}^{2}\left(V_i(\varkappa) + \frac{G_i(\varkappa)}{\Gamma(\sigma_i)}\int_0^{\varkappa}(\varkappa-\ell)^{\sigma_i-1}\mathcal{K}_i(\varkappa,\ell)\mathcal{F}_i(\ell,\omega(\ell))d\ell\right)\right| = \left|3e^{\varkappa} - \left(1 - \frac{e^{\varkappa}\left(9\varkappa^{\frac{4}{3}}+12\varkappa^{\frac{1}{3}}\right)}{5184\,\Gamma(\frac{1}{3})}\right)e^{\varkappa}\right|$$
$$\le 4\varkappa + 2 = \lambda(\varkappa), \qquad \forall\,\varkappa \in [0,1].$$

Thus, by Theorem 3, there exists a unique solution $\omega_0(\varkappa) \in C_g([0,1])$ of Eq (5.2) such that

$$|\omega(\varkappa) - \omega_0(\varkappa)| \le \frac{1}{1-\left(\mathcal{M}^{n-1}\sum_{i=1}^{n}\widehat{G}_i\widehat{K}_i\widehat{F}_i\beta_i\right)}\lambda(\varkappa), \quad \varkappa \in [0,1].$$

This ensures the H-U-R stability for Eq (5.2).

6. Conclusions and future tasks

Three types of stabilities, namely, H-U, λ-semi-Hyers-Ulam, and H-U-R stabilities, have been analyzed in this paper for Eq (1.2) through the application of the Bielecki metric in the space of continuous real-valued functions defined on the finite interval $[0, a]$. In Theorem 2, conditions for H-U-R stability have been established in the space $C_g([0, a])$ through the application of the metric d_g. In Corollary 1, we stated the conditions for H-U-R stability in the space $C_\delta([0, a])$ through the application of the metric d_δ. Some easily checked conditions for H-U-R stability have been provided in Theorem 3. In Theorem 4, conditions for λ-semi-Hyers-Ulam stability have been discussed in the space $C_g([0, a])$ through the application of the metric d_g. In Corollary 2, we stated the conditions for λ-semi-Hyers-Ulam stability in the space $C_\delta([0, a])$ through the application of the metric d_δ. In Corollary 3, conditions for H-U stability have been discussed in the space $C_g([0, a])$ through the application of the metric d_g, and in Corollary 4, we stated the conditions for H-U stability in the space $C_\delta([0, a])$ through the application of the metric d_δ. These results indicate that there is a close analytic solution of Eq (1.2) that is stable in the sense of the above stabilities. Two examples have been discussed on the interval $[0, 1]$ to illustrate the established results. In the future, one can extend the concept presented here to the system of fractional integral equations of n-product type. Also, new results can be obtained by considering more generalized kernels. Subsequently, interested researchers can extend this concept to two-dimensional integral equations of fractional order.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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