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Research article

Generalized high-order iterative methods for solutions of nonlinear systems and their applications

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Abstract: In this paper, we have constructed a family of three-step methods with sixth-order convergence and a novel approach to enhance the convergence order p of iterative methods for systems of nonlinear equations. Additionally, we propose a three-step scheme with convergence order $p + 3$ (for $p \geq 3$) and have extended it to a generalized $(m + 2)$-step scheme by merely incorporating one additional function evaluation, thus achieving convergence orders up to $p + 3m$, $m \in \mathbb{N}$. We also provide a thorough local convergence analysis in Banach spaces, including the convergence radius and uniqueness results, under the assumption of a Lipschitz-continuous Fréchet derivative. Theoretical findings have been validated through numerical experiments. Lastly, the performance of these methods is showcased through the analysis of their basins of attraction and their application to systems of nonlinear equations.

Keywords: iterative methods; systems of nonlinear equations; local convergence; Lipschitz condition; Banach space; basins of attraction
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1. Introduction

The pursuit of constructing fixed-point iterative techniques for the solution of nonlinear equations or systems of nonlinear equations stands as a compelling and formidable challenge within the domains of numerical analysis and various applied sciences. The significance of this topic has led to the development of numerous numerical techniques, often employing iterative approaches, to yield highly precise approximate solutions for systems of nonlinear equations, represented as follows:

$$\Omega(s) = 0, \tag{1.1}$$

where $\Omega : \mathbb{D} \subseteq \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X}$ and $\mathbb{Y}$ as Banach spaces, is continuously Fréchet-differentiable; and $\mathbb{D}$ is a non-empty open convex subset of $\mathbb{X}$.

One of the most prevalent iterative approaches for determining the solution α of (1.1) is the well-established quadratically convergent one-point Newton method [1], which is outlined below:

$$s_{n+1} = s_n - \Omega'(s_n)^{-1}\Omega(s_n),\ n = 0, 1, 2, ..., \tag{1.2}$$

where $\Omega'(s_n)^{-1}$ denotes the inverse of the first Fréchet derivative $\Omega'(s_n)$ of the function $\Omega(s_n)$. The method's convergence relies on two crucial prerequisites. First, the initial approximation s_0 must be in close proximity to the desired solution α, and second, the existence of the inverse $\Omega'(s_n)^{-1}$ of the derivative must be ensured within the neighborhood $\mathbb{D}$ centered around α.

In pursuit of achieving a higher order of convergence, scientific literature has introduced a range of modifications to Newton's method, referred to as Newton-like techniques. These strategies have been explored extensively in both univariate and multivariate scenarios, with comprehensive discussions available in References [2–12] and associated citations. One of the earliest and notable yet simple modifications to Newton's method is the cubically convergent Potra-Pták method (PPM3) [13], which is given below:

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ s_{n+1} &= y_n - \Omega'(s_n)^{-1}\Omega(y_n). \end{aligned} \tag{1.3}$$

However, achieving the enhanced convergence order in (1.3) involves incurring an additional computational cost in the form of an extra function evaluation, $\Omega(y_n)$.

An interesting new development in the field is the hybridization of iterative techniques and optimization algorithms. Several optimization algorithms, including the butterfly optimization algorithm [14] and the sperm swarm optimization algorithm [15], have been applied to solve problems involving systems of nonlinear equations. However, optimization algorithms often lack accurate solutions due to their limitations, including falling into local optima and divergence problems. Only a few researchers have attempted to combine iterative methods with optimization algorithms. Recently, Sihwail et al. [16] and Said Solaiman et al. [17] proposed new hybrid algorithms by combining iterative methods and optimization algorithms for the purpose of solving systems of nonlinear equations. These hybrid approaches leverage the benefits of both methods while overcoming their drawbacks.

Aiming to contribute to this evolving landscape, we propose multipoint iterative techniques that progressively increase convergence orders while minimizing function evaluations and inverse operators. Initially, we introduce a family of three-step schemes with sixth order of convergence and then generalize it to a scheme of convergence order $p + 3$. The first two steps are akin to Newton-like iterations with a convergence order of p (where $p \geq 3$). Building on this, we present a more generalized $(m + 2)$-step scheme with an increased convergence order of $p + 3m$, $m \in \mathbb{N}$. In fact, we can achieve a threefold increase in convergence order by adding only one more function evaluation for each additional step.

The subsequent sections of this paper are structured as follows. Section 2 introduces a three-step method and establishes its convergence, achieving a sixth order of convergence. Section 3 presents a generalized version of the family of methods with convergence order $p + 3$, which is then further extended to an $(m + 2)$-step scheme with a convergence order of $p + 3m$, $m \in \mathbb{N}$. Section 4 presents a comprehensive analysis of the local convergence properties. Section 5 offers numerical examples to validate our theoretical results. Again, Section 6 applies these methods to tackle systems of nonlinear equations. Section 7 presents a graphical analysis of the dynamical behaviours of our newly proposed methods, comparing them with existing methods through the lens of their basins of attraction on the Cartesian plane. Finally, Section 8 includes some concluding remarks.

2. Description of the proposed iterative scheme

In this section, we aim to develop new families of iterative methods of order six for the purpose of solving systems of nonlinear equations. First, we present a three-step scheme as follows:

$$\begin{aligned} y_n =& s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n =& y_n - \frac{1}{2}\Omega'(s_n)^{-1}[\Omega'(s_n) - \Omega'(y_n)]\Omega'(y_n)^{-1}\Omega(s_n), \\ s_{n+1} =& z_n - [k_1 I + k_2\Omega'(y_n)^{-1}\Omega'(s_n) + k_3\Omega'(s_n)^{-1}\Omega'(y_n) \\ &+ k_4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n), \end{aligned} \tag{2.1}$$

where k_1, k_2, k_3, and k_4 are free parameters to be determined in the sequel.

To obtain the convergence order of (2.1), we first recall the following result of the Taylor's expansion on vector functions (see [1]).

Lemma 2.1. *Let $\Omega\colon \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a p time Fréchet-differentiable function with $\mathbb{D}$ as a convex set; then, for any $s, h \in \mathbb{R}^n$, we have*

$$\Omega(s + h) = \Omega(s) + h\Omega'(s) + \frac{h^2}{2!}\Omega''(s) + \cdots + \frac{h^{p-1}}{(p-1)!}\Omega^{(p-1)}(s) + R_p, \tag{2.2}$$

where $\|R_p\| \leq \frac{1}{p!}\sup_{0\leq t\leq 1}\|\Omega^{(p)}(s + th)\|\|h\|^p$ and $h^p = (h, h, h, \overset{p}{\ldots}, h)$.

Theorem 2.1. *Suppose that the function $\Omega\colon \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently Fréchet-differentiable in a neighborhood $\mathbb{D} \subseteq \mathbb{R}^n$ which contains the root α of $\Omega(s) = 0$. Assuming that $\Omega'(\alpha)$ is nonsingular, the sequence $\{s_n\}_{n\geq 0}(s_0 \in \mathbb{D})$ produced by the family of methods given by (2.1) converges to the actual solution α with a convergence order of 6 if $k_2 = \frac{2+k_1}{4}, k_3 = \frac{2-k_1}{4}$, and $k_4 = -2k_1, k_1 \in \mathbb{R}$. And, it satisfies the error equation given by*

$$\varepsilon_{n+1} = -\frac{1}{4}C_2C_3\left(2(-4 + k_1)C_2^2 + 5C_3\right)\varepsilon_n^6 + O(\varepsilon_n^7). \tag{2.3}$$

Proof of Theorem 2.1. The expansion of $\Omega(s_n)$ by using Taylor's series expansion given by (2.2) near $s = \alpha$ gives

$$\Omega(s_n) = \Omega'(\alpha)\left[\varepsilon_n + C_2\varepsilon_n^2 + C_3\varepsilon_n^3 + C_4\varepsilon_n^4 + C_5\varepsilon_n^5 + C_6\varepsilon_n^6 + O(\varepsilon_n^7)\right], \tag{2.4}$$

where $C_j = \frac{1}{j!}\Omega'(\alpha)^{-1}\Omega^{(j)}(\alpha)$, $\varepsilon_n = s_n - \alpha$, and $\varepsilon_n^i = (\varepsilon_n, \varepsilon_n, \overset{i}{\ldots}, \varepsilon_n)$, $\varepsilon_n \in \mathbb{R}^n$.

Then, it is straightforward to obtain

$$\Omega'(s_n) = \Omega'(\alpha)\left[I + 2C_2\varepsilon_n + 3C_3\varepsilon_n^2 + \sum_{i=1}^{4}(i+3)C_{i+3}\varepsilon_n^{i+2} + O(\varepsilon_n^7)\right], \tag{2.5}$$

$$\Omega'(s_n)^{-1} = \Omega'(\alpha)^{-1}\left[I - 2C_2\varepsilon_n + (4C_2^2 - 3C_3)\varepsilon_n^2 + \sum_{i=1}^{4} K_i\varepsilon_n^{i+2} + O(\varepsilon_n^7)\right], \tag{2.6}$$

where K_i depends on $C_2, C_3, ..., C_7$, i.e., $K_1 = -4(2C_2^3 - 3C_2C_3 + C_4)$, $K_2 = 16C_2^4 - 36C_2^2C_3 + 9C_3^2$ $+16C_2C_4 - 5C_5$, etc.

Using (2.4) and (2.6), we have

$$y_n - \alpha = C_2\varepsilon_n^2 + \left(2C_3 - 2C_2^2\right)\varepsilon_n^3 + \sum_{i=1}^{3} M_i\varepsilon_n^{i+3} + O(\varepsilon_n^7), \tag{2.7}$$

where M_i depends on $C_2, C_3, ..., C_6$, i.e., $M_1 = 4C_2^3 + 3C_4 - 7C_2C_3$, $M_2 = -8C_2^4 + 20C_2^2C_3 - 6C_3^2$ $-10C_2C_4 + 4C_5$, etc.

By using (2.2) and (2.7), the expansion of $\Omega(y_n)$ gives

$$\Omega(y_n) = \Omega'(\alpha)\left[C_2\varepsilon_n^2 + 2\left(C_3 - C_2^2\right)\varepsilon_n^3 + \sum_{i=1}^{3} N_i\varepsilon_n^{i+3} + O(\varepsilon_n^7)\right], \tag{2.8}$$

where N_i depends on $C_2, C_3, ..., C_6$. Then, from (2.8), it follows that

$$\Omega'(y_n) = \Omega'(\alpha)\left[I + 2C_2^2\varepsilon_n^2 + 4C_2\left(C_3 - C_2^2\right)\varepsilon_n^3 + \sum_{i=1}^{3} P_i\varepsilon_n^{i+3} + O(\varepsilon_n^7)\right], \tag{2.9}$$

where P_i depends on $C_2, C_3, ..., C_6$. Also,

$$\Omega'(y_n)^{-1} = \Omega'(\alpha)^{-1}\left[I - 2C_2^2\varepsilon_n^2 + \sum_{i=1}^{4} Q_i\varepsilon_n^{i+2} + O(\varepsilon_n^7)\right], \tag{2.10}$$

where Q_i depends on $C_2, C_3, ..., C_6$.

Now, replacing the values of (2.4)–(2.7), (2.9), and (2.10) in the second step of (2.1), we get

$$z_n - \alpha = \frac{1}{2}C_3\varepsilon_n^3 + \left(C_2^3 - \frac{3C_2C_3}{2} + C_4\right)\varepsilon_n^4 + \sum_{i=1}^{2} R_i\varepsilon_n^{i+4} + O(\varepsilon_n^7), \tag{2.11}$$

where R_i depends on $C_2, C_3, ..., C_6$. Then, by using (2.2), (2.11) becomes as follows:

$$\Omega(z_n) = \Omega'(\alpha)\left[\frac{1}{2}C_3\varepsilon_n^3 + \left(C_2^3 - \frac{3C_2C_3}{2} + C_4\right)\varepsilon_n^4 + \sum_{i=1}^{2} S_i\varepsilon_n^{i+4} + O(\varepsilon_n^7)\right], \tag{2.12}$$

where S_i depends on $C_2, C_3, ..., C_6$.

Incorporating the values from (2.5), (2.6), and (2.9)–(2.12) in the concluding step of (2.1), we arrive at the error equation as follows:

$$\varepsilon_{n+1} = -\frac{1}{4}\left(2(-1 + k_1 + k_2 + k_3) + k_4\right) C_3\varepsilon_n^3 + \sum_{i=1}^{3} T_i\varepsilon_n^{i+3} + O(\varepsilon_n^7), \tag{2.13}$$

where T_i depends on $C_2, C_3, ..., C_6$, i.e.,
$T_1 = \frac{1}{2}(-(2(-1 + k_1 + k_2 + k_3) + k_4) + (-3 + 3k_1 + k_2 + 5k_3 + k_4)C_2C_3 - (2(-1 + k_1 + k_2 + k_3) + k_4)C_4)$,
$T_2 = \frac{1}{8}\Big(4(-8 + 8k_1 + 4k_2 + 12k_3 + 3k_4)C_2^4 - 4(13k_1 + 5(-3 + k_2 + 5k_3) + 4k_4)C_2^2C_3 + 3(-8 + 8k_1 + 4k_2 + 12k_3 + 3k_4)C_3^2 + 4(-4 + 4k_1 + 8k_3 + k_4)C_2C_4 - 6(2(-1 + k_1 + k_2 + k_3) + k_4)C_5\Big)$, etc.

Finally, by substituting $k_2 = \frac{2+k_1}{4}, k_3 = \frac{2-k_1}{4}$, and $k_4 = -2k_1$ in the above error (2.13), we get

$$\varepsilon_{n+1} = -\frac{1}{4}C_2C_3\left(2(-4 + k_1)C_2^2 + 5C_3\right)\varepsilon_n^6 + O(\varepsilon_n^7).$$

As a result, the proof has been successfully established. □

Now, upon substituting the values of the parameters k_1, k_2, k_3, and k_4, the proposed sixth-order family of methods derived from (2.1) can be formulated as follows:

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n &= y_n - \frac{1}{2}\Omega'(s_n)^{-1}[\Omega'(s_n) - \Omega'(y_n)]\Omega'(y_n)^{-1}\Omega(s_n), \\ s_{n+1} &= z_n - \Big[kI + \left(\frac{2+k}{4}\right)\Omega'(y_n)^{-1}\Omega'(s_n) + \left(\frac{2-k}{4}\right)\Omega'(s_n)^{-1}\Omega'(y_n) \\ &\quad - 2k\left(\Omega'(s_n) + \Omega'(y_n)\right)^{-1}\Omega'(s_n)\Big]\Omega'(y_n)^{-1}\Omega(z_n), \end{aligned} \tag{2.14}$$

where the free parameter $k \in \mathbb{R}$. We shall denote it by PFM6.

3. Generalization of the proposed scheme

3.1. *The iterative methods with a* $p + 3$ *order of convergence*

Our objective here is to generalize the proposed family of methods given by (2.14) by establishing a universal principle that is capable of enhancing lower-order methods with convergence order $p \geq 3$. Through this approach, we aim to achieve an improvement of order $p + 3$.

The generalized method is characterized by the following construction:

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n &= \mu_p(s_n, y_n), \\ s_{n+1} &= z_n - [k_1I + k_2\Omega'(y_n)^{-1}\Omega'(s_n) + k_3\Omega'(s_n)^{-1}\Omega'(y_n) \\ &\quad + k_4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n). \end{aligned} \tag{3.1}$$

Here, the parameters k_1, k_2, k_3, and k_4 will be determined later. It is worth noting that $z_n = \mu_p(s_n, y_n)$ represents the iteration function with a convergence order of $p \geq 3$.

Theorem 3.1. *Suppose that $\Omega : \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently Fréchet-differentiable function in a neighborhood $\mathbb{D} \subseteq \mathbb{R}^n$ which contains the root α of $\Omega(s) = 0$. Assuming that $\Omega'(\alpha)$ is nonsingular, the sequence $\{s_n\}$ generated by method (3.1) for $s_0 \in \mathbb{D}$ converges to α with order $p+3$ for $p \geq 3$, provided that $k_2 = \frac{2+k_1}{4}, k_3 = \frac{2-k_1}{4}$, and $k_4 = -2k_1$.*

Proof of Theorem 3.1. We consider all of the assumptions made in Theorem 2.1; using (2.5), (2.6), (2.9), and (2.10), we have

$$\begin{aligned}
&[k_1 I + k_2\Omega'(y_n)^{-1}\Omega'(s_n) + k_3\Omega'(s_n)^{-1}\Omega'(y_n) + k_4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\\
&\quad=\Big[(k_1 + k_2 + k_3 + \frac{k_4}{2})I + (2k_2 - 2k_3 + \frac{k_4}{2})C_2\varepsilon_n + (- 2(k_1 + 2k_2 - 2k_3 + k_4)C_2^2\\
&\qquad + 3(k_2 - k_3 + \frac{k_4}{4})C_3)\varepsilon_n^2 + ((4k_1 - 8k_2 + \frac{5}{2}k_4)C_2^3 - (4k_1 + 8k_2 - 12k_3 + \frac{9}{2}k_4)C_2C_3\\
&\qquad + (4k_2 - 4k_3 + k_4)C_4)\varepsilon_n^3 + O(\varepsilon_n^4)\Big]\Omega'(\alpha)^{-1}.
\end{aligned} \tag{3.2}$$

Furthermore, by considering the iteration function $z_n = \mu_p(s_n, y_n)$ with convergence order of p, we can introduce the following definition:

$$\widetilde{\varepsilon}_n = z_n - \alpha = O(\varepsilon_n^p). \tag{3.3}$$

Then, using (2.2) and (3.3), the expansion of $\Omega(z_n)$ about α is obtained as follows:

$$\Omega(z_n) = \Omega'(\alpha)\left[\widetilde{\varepsilon}_n + O(\widetilde{\varepsilon}_n^2)\right]. \tag{3.4}$$

Now, using (3.2)–(3.4) in the last step of (3.1), we obtain the error equation as follows:

$$\begin{aligned}
\varepsilon_{n+1} =&(1 - k_1 - k_2 - k_3 - \frac{k_4}{2})\widetilde{\varepsilon}_n - (2k_2 - 2k_3 + \frac{k_4}{2})C_2\varepsilon_n\widetilde{\varepsilon}_n - (- 2(k_1 + 2k_2 - 2k_3 + k_4)C_2^2\\
&+ 3(k_2 - k_3 + \frac{k_4}{4})C_3)\varepsilon_n^2\widetilde{\varepsilon}_n - ((4k_1 - 8k_3 + \frac{5}{2}k_4)C_2^3 - (4k_1 + 8k_2 - 12k_3 + \frac{9}{2}k_4)C_2C_3\\
&+ (4k_2 - 4k_3 + k_4)C_4)\varepsilon_n^3\widetilde{\varepsilon}_n + O(\varepsilon_n^4\widetilde{\varepsilon}_n) + O(\widetilde{\varepsilon}_n^2).
\end{aligned} \tag{3.5}$$

For $p \geq 3$, the method (3.1) exhibits convergence to the root α with an order of $p + 3$ if and only if the constants k_1, k_2, and k_3 satisfy the following system:

$$\begin{aligned}
1 - k_1 - k_2 - k_3 - \frac{1}{2}k_4 &= 0,\\
2k_2 - 2k_3 + \frac{1}{2}k_4 &= 0,\\
k_1 + 2k_2 - 2k_3 + k_4 &= 0,\\
k_2 - k_3 + \frac{1}{4}k_4 &= 0.
\end{aligned} \tag{3.6}$$

The only solution of the above system (3.6) is as follows: $k_2 = \frac{2+k_1}{4}, k_3 = \frac{2-k_1}{4}$, and $k_4 = -2k_1$. By substituting $k_1 = k$, the error Eq (3.5) reduces to

$$\varepsilon_{n+1} = -((k - 4)C_2^3 + 2C_2C_3)\varepsilon_n^{p+3} + O(\varepsilon_n^{p+4}). \tag{3.7}$$

Hence, the theorem is proved. □

Here, it is worth highlighting that, based on the proof of Theorem 3.1, we can readily derive the following significant results:

Corollary 1. *When considering a special case of* $k = k_1 = 0$*, solving the system* (3.6) *yields that* $k_2 = k_3 = \frac{1}{2}$*. Thus, the proposed approach given by* (3.1) *is reduced to the following construction given by*

$$s_{n+1} = z_n - \frac{1}{2}[\Omega'(y_n)^{-1}\Omega'(s_n) + \Omega'(s_n)^{-1}\Omega'(y_n)]\Omega'(y_n)^{-1}\Omega(z_n). \tag{3.8}$$

where y_n, z_n *are defined as given in* (3.1)*. In fact, the above construction* (3.8) *was the technique introduced by the authors of [18] in the year 2018.*

Remark 1. *A particular case of the above approach given by* (3.8) *is the following sixth-order method developed by Sharma et al. [19] in the year 2019. We shall call it the Sharma-Sharma-Karla method (SSKM6).*

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n &= s_n - \frac{1}{2}\left[\Omega'(s_n)^{-1} + \Omega'(y_n)^{-1}\right]\Omega(s_n), \\ s_{n+1} &= z_n - \frac{1}{2}[\Omega'(s_n)^{-1} + \Omega'(y_n)^{-1}\Omega'(s_n)\Omega'(y_n)^{-1}]\Omega(z_n). \end{aligned} \tag{3.9}$$

Corollary 2. *When considering the special case of* $k = k_1 = 2$*, solving the system* (3.6) *yields that* $k_2 = 1, k_3 = 0$*, and* $k_4 = -4$*. Thus, we obtain the following approach given by*

$$s_{n+1} = z_n - [2I + \Omega'(y_n)^{-1}\Omega'(s_n) - 4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n). \tag{3.10}$$

Also, by considering the special case of $k = k_1 = -2$*, solving the system* (3.6) *yields that* $k_2 = 0$*,* $k_3 = 1$*, and* $k_4 = 4$*. Then, we obtain the approach given by*

$$s_{n+1} = z_n - [-2I + \Omega'(s_n)^{-1}\Omega'(y_n) + 4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n). \tag{3.11}$$

Remark 2. *Applying* (3.10) *to the third-order method of Liu et al. [20], developed in 2016, gives the following new sixth-order method, which is denoted by PFM6.1:*

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n &= s_n - 2\left[\Omega'(u_n - v_n) + \Omega'(u_n + v_n)\right]^{-1}\Omega(s_n), \\ s_{n+1} &= z_n - [2I + \Omega'(y_n)^{-1}\Omega'(s_n) - 4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n), \end{aligned} \tag{3.12}$$

where $u_n = \frac{s_n + y_n}{2}$ *and* $v_n = \frac{y_n - s_n}{2\sqrt{3}}$.

Also, applying (3.11) *to the third-order method of Sharma and Gupta [21], developed in 2013, gives the following new sixth-order method denoted by PFM6.2.*

$$\begin{aligned} y_n &= s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n &= s_n - \frac{1}{2}\left(3I - \Omega'(s_n)^{-1}\Omega'(y_n)\right)\Omega'(s_n)^{-1}\Omega(s_n), \\ s_{n+1} &= z_n - [-2I + \Omega'(s_n)^{-1}\Omega'(y_n) + 4(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)]\Omega'(y_n)^{-1}\Omega(z_n). \end{aligned} \tag{3.13}$$

3.2. The extended method consisting of $(m+2)$ *steps*

The extended form of the method given by (3.1), involving $(m+2)$ steps, can be represented as:

$$
\begin{aligned}
y_n =& s_n - \Omega'(s_n)^{-1}\Omega(s_n),\\
z_n =& \mu_p(s_n, y_n),\\
z_n^{(1)} =& z_n - \Psi(s_n, y_n)\Omega(z_n),\\
z_n^{(2)} =& z_n^{(1)} - \Psi(s_n, y_n)\Omega(z_n^{(1)}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
z_n^{(m-1)} =& z_n^{(m-2)} - \Psi(s_n, y_n)\Omega(z_n^{(m-2)}),\\
z_n^{(m)} =& s_{n+1} = z_n^{(m-1)} - \Psi(s_n, y_n)\Omega(z_n^{(m-1)}),
\end{aligned}
\tag{3.14}
$$

where $m \in \mathbb{N}$, $z_n^{(0)} = z_n$ and $\Psi(s_n, y_n) = \left(kI + \left(\frac{2+k}{4}\right)\Omega'(y_n)^{-1}\Omega'(s_n) + \left(\frac{2-k}{4}\right)\Omega'(s_n)^{-1}\Omega'(y_n) - 2k(\Omega'(s_n) + \Omega'(y_n))^{-1}\Omega'(s_n)\right)\Omega'(y_n)^{-1}$.

Theorem 3.2. *Given the assumptions of Theorem 3.1, the sequence* $\{s_n\}$ *generated by employing the method given by* (3.14) *with an initial value of* $s_0 \in \mathbb{D}$ *achieves convergence toward* α *with a convergence order of* $p + 3m$ *for cases in which* $p \geq 3$ *and* $m \in \mathbb{N}$.

Proof of Theorem 3.2. Considering the expression given in (3.2), it follows that

$$\Psi(s_n, y_n) = \left(I + ((k-4)C_2^3 + 2C_2C_3)\varepsilon_n^3 + \cdots\right)\Omega'(\alpha)^{-1}. \tag{3.15}$$

Utilizing Taylor's series, we can express the expansion of $\Omega(z_n^{(m-1)})$ around α as follows:

$$\Omega(z_n^{(m-1)}) = \Omega'(\alpha)\left(\left(z_n^{(m-1)} - \alpha\right) + C_2\left(z_n^{(m-1)} - \alpha\right)^2 + \cdots\right). \tag{3.16}$$

Then, it follows from (3.15) and (3.16) that

$$\Psi(s_n, y_n)\Omega(z_n^{(m-1)}) = \left(z_n^{(m-1)} - \alpha\right) + ((k-4)C_2^3 + 2C_2C_3)\varepsilon_n^3\left(z_n^{(m-1)} - \alpha\right) + C_2\left(z_n^{(m-1)} - \alpha\right)^2 + \cdots. \tag{3.17}$$

By applying (3.17) in the concluding step of (3.14), we acquire the following:

$$z_n^{(m)} - \alpha = ((4-k)C_2^3 - 2C_2C_3)\varepsilon_n^3\left(z_n^{(m-1)} - \alpha\right) - C_2\left(z_n^{(m-1)} - \alpha\right)^2 + \cdots. \tag{3.18}$$

Referring to (3.7), it is evident that $z_n^{(1)} - \alpha = ((4-k)C_2^3 - 2C_2C_3)\varepsilon_n^{p+3} + O(\varepsilon_n^{p+4})$. Consequently, applying (3.18) with $m = 2, 3$, we derive the following:

$$z_n^{(2)} - \alpha = ((4-k)C_2^3 - 2C_2C_3)\varepsilon_n^3\left(z_n^{(1)} - \alpha\right) + \cdots = ((4-k)C_2^3 - 2C_2C_3)^2\varepsilon_n^{p+6} + O(\varepsilon_n^{p+7}). \tag{3.19}$$

$$z_n^{(3)} - \alpha = ((4-k)C_2^3 - 2C_2C_3)\varepsilon_n^3\left(z_n^{(2)} - \alpha\right) + \cdots = ((4-k)C_2^3 - 2C_2C_3)^3\varepsilon_n^{p+9} + O(\varepsilon_n^{p+10}). \tag{3.20}$$

Continuing by the process of induction, we obtain

$$z_n^{(m)} - \alpha = ((4-k)C_2^3 - 2C_2C_3)^m\varepsilon_n^{p+3m} + O(\varepsilon_n^{p+3m+1}). \tag{3.21}$$

This concludes the proof of Theorem 3.2. □

Remark 3. *Applying the construction given by* (3.14) *for $m = 2$ to the Potra-Pták method described by* (1.3) *[13] of third order yields the following newly extended family of ninth-order method (PFM9).*

$$
\begin{aligned}
y_n =& s_n - \Omega'(s_n)^{-1}\Omega(s_n),\\
z_n =& y_n - \Omega'(s_n)^{-1}\Omega(y_n),\\
\hat{z}_n =& z_n - \Big[kI + \left(\frac{2+k}{4}\right)\Omega'(y_n)^{-1}\Omega'(s_n) + \left(\frac{2-k}{4}\right)\Omega'(s_n)^{-1}\Omega'(y_n)\\
&- 2k\left(\Omega'(s_n) + \Omega'(y_n)\right)^{-1}\Omega'(s_n)\Big]\Omega'(y_n)^{-1}\Omega(z_n),\\
s_{n+1} =& \hat{z}_n - \Big[kI + \left(\frac{2+k}{4}\right)\Omega'(y_n)^{-1}\Omega'(s_n) + \left(\frac{2-k}{4}\right)\Omega'(s_n)^{-1}\Omega'(y_n)\\
&- 2k\left(\Omega'(s_n) + \Omega'(y_n)\right)^{-1}\Omega'(s_n)\Big]\Omega'(y_n)^{-1}\Omega(\hat{z}_n).
\end{aligned} \tag{3.22}
$$

4. Local convergence analysis

In this section, we extend the local convergence analysis of (2.14) (PFM6) from Section 2 to the Banach space setting. To analyze this local convergence under the Lipschitz continuity condition, we make the following assumptions on all real numbers $\kappa_0 > 0$, $\kappa > 0$, and all points $s, y \in \mathbb{D}$:

$$\Omega(\alpha) = 0, \Omega'(\alpha)^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X}),$$

$$||\Omega'(\alpha)^{-1}(\Omega'(s) - \Omega'(\alpha))|| \le \kappa_0||s - \alpha||, \tag{4.1}$$

$$||\Omega'(\alpha)^{-1}(\Omega'(s) - \Omega'(y))|| \le \kappa||s - y||, \tag{4.2}$$

where $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ represents the set of linear bounded operators from $\mathbb{Y}$ to $\mathbb{X}$.

Lemma 4.1. *Under the assumption that operator Ω satisfies conditions* (4.1) *and* (4.2)*, the following inequalities hold for all points $s \in \mathbb{D}$:*

$$||\Omega'(\alpha)^{-1}\Omega'(s)|| \le 1 + \kappa_0||s - \alpha||, \tag{4.3}$$

$$||\Omega'(\alpha)^{-1}\Omega(s)|| \le (1 + \kappa_0||s - \alpha||)||s - \alpha||. \tag{4.4}$$

Proof of Lemma 4.1. By applying condition (4.1), we obtain:

$$||\Omega'(\alpha)^{-1}\Omega'(s)|| = ||\Omega'(\alpha)^{-1}(\Omega'(s) - \Omega'(\alpha)) + I|| \le 1 + ||\Omega'(\alpha)^{-1}(\Omega'(s) - \Omega'(\alpha))|| \le 1 + \kappa_0||s - \alpha||.$$

Again, by virtue of the mean value theorem, it follows that:

$$||\Omega'(\alpha)^{-1}\Omega(s)|| = ||\Omega'(\alpha)^{-1}(\Omega(s) - \Omega(\alpha))|| \le (1 + \kappa_0||s - \alpha||)||s - \alpha||.$$

□

The following Theorem 4.1 provides the local convergence analysis of the considered scheme (2.14) (PFM6) under the Lipschitz continuity condition.

Theorem 4.1. *Let $\Omega : \mathbb{D} \subseteq \mathbb{X} \to \mathbb{Y}$ be a Fréchet differentiable operator. Suppose that there exist $\alpha \in \mathbb{D}$ and $k \in (-\infty, +\infty)$ such that conditions* (4.1) *and* (4.2) *are satisfied and $\bar{B}(\alpha, \rho) \subseteq \mathbb{D}$, where ρ is to be determined. Then, the iterative sequence $\{s_n\}$ produced by PFM6* (2.14)*, starting from $s_0 \in B(\alpha, \rho)$, remains in $B(\alpha, \rho)$ $\forall$ $n \geq 0$, and converges to α. Futhermore, the following inequalities hold for all $n \geq 0$:*

$$||y_n - \alpha|| \leq \eta_1(||s_n - \alpha||)||s_n - \alpha|| < ||s_n - \alpha|| < \rho, \tag{4.5}$$

$$||z_n - \alpha|| \leq \eta_2(||s_n - \alpha||)||s_n - \alpha|| < ||s_n - \alpha|| < \rho, \tag{4.6}$$

$$||s_{n+1} - \alpha|| \leq \eta_3(||s_n - \alpha||)||s_n - \alpha|| < ||s_n - \alpha|| < \rho, \tag{4.7}$$

where the functions represented by η_i are to be defined. Additionally, if there exists $R \in [\rho, \frac{1}{\kappa_0})$ such that $\bar{B}(\alpha, R) \in \mathbb{D}$, then the limit point α is the unique solution in $\bar{B}(\alpha, R)$.

Proof of Theorem 4.1. Given that $s_0 \in \mathbb{D}$, and by assuming that $||s_0 - \alpha|| < \frac{1}{\kappa_0}$, (4.1) gives

$$||\Omega'(\alpha)^{-1}(\Omega'(s_0) - \Omega'(\alpha))|| \leq \kappa_0||s_0 - \alpha|| < 1.$$

Consequently, by applying the Banach lemma on invertible operators, $\Omega'(s_0)^{-1}$ exists, and it follows that

$$||\Omega'(s_0)^{-1}\Omega'(\alpha)|| \leq \frac{1}{1 - \kappa_0||s_0 - \alpha||}. \tag{4.8}$$

Therefore, y_0 is well defined. Now, considering the first sub-step of (2.14) for $n = 0$, we obtain

$$\begin{aligned} y_0 - \alpha &= s_0 - \Omega'(s_0)^{-1}\Omega(s_0) \\ &= \Omega'(s_0)^{-1}\Omega'(\alpha) \int_0^1 \Omega'(\alpha)^{-1} \left[\Omega'(s_0) - \Omega'(\alpha + t(s_0 - \alpha))\right](s_0 - \alpha)\, dt. \end{aligned}$$

Taking the norm on both sides and using (4.2) and (4.8), we obtain

$$\begin{aligned} ||y_0 - \alpha|| &\leq ||\Omega'(s_0)^{-1}\Omega'(\alpha)|| \left\| \int_0^1 \Omega'(\alpha)^{-1} \left[\Omega'(s_0) - \Omega'(\alpha + t(s_0 - \alpha))\right](s_0 - \alpha)\, dt \right\| \\ &\leq \frac{\kappa||s_0 - \alpha||}{2(1 - \kappa_0||s_0 - \alpha||)}||s_0 - \alpha|| \\ &= \eta_1(||s_0 - \alpha||)||s_0 - \alpha||, \end{aligned} \tag{4.9}$$

where

$$\eta_1(\vartheta) = \frac{\kappa\vartheta}{2(1 - \kappa_0\vartheta)}.$$

We define the function $\omega_1(\vartheta) = \eta_1(\vartheta) - 1$. Since $\omega_1(0) < 0$ and $\omega_1(\frac{1}{\kappa_0}) \to +\infty$, the intermediate value theorem guarantees that at least one root of $\omega_1(\vartheta)$ exists in the interval $(0, \frac{1}{\kappa_0})$. Let ρ_1 represent the smallest root of $\omega_1(\vartheta)$ within this interval. Then, we obtain

$$0 < \rho_1 < \frac{1}{\kappa_0}, \text{ and } 0 \leq \eta_1(\vartheta) < 1, \forall\, \vartheta \in [0, \rho_1). \tag{4.10}$$

Applying (4.9) and (4.10), we arrive at the following result:

$$\|y_0 - \alpha\| \le \eta_1(\|s_0 - \alpha\|)\|s_0 - \alpha\| < \|s_0 - \alpha\|.$$

Since $y_0 \in \mathbb{D}$, by using the assumption (4.1), we can deduce that

$$\|\Omega'(\alpha)^{-1}(\Omega'(y_0) - \Omega'(\alpha))\| \le \kappa_0\|y_0 - \alpha\| \le \kappa_0\|s_0 - \alpha\| < 1.$$

As a result, by virtue of the Banach lemma about invertible operators, $\Omega'(y_0)^{-1}$ exists and

$$\|\Omega'(y_0)^{-1}\Omega'(\alpha)\| \le \frac{1}{1 - \kappa_0\|y_0 - \alpha\|}. \tag{4.11}$$

Hence, z_0 is well-defined. As such, from (2.14) for $n = 0$, we have

$$\begin{aligned}
\|z_0 - \alpha\| \le & \|y_0 - \alpha\| + \frac{1}{2}\|\Omega'(s_0)^{-1}\left(\Omega'(s_0) - \Omega'(y_0)\right)\|\|\Omega'(y_0)^{-1}\Omega(s_0)\| \\
\le & \|y_0 - \alpha\| + \frac{1}{2}\|\Omega'(s_0)^{-1}\Omega'(\alpha)\|[\|\Omega'(\alpha)^{-1}\left(\Omega'(s_0) - \Omega'(\alpha)\right)\| \\
& + \|\Omega'(\alpha)^{-1}\left(\Omega'(y_0) - \Omega'(\alpha)\right)\|]\|\Omega'(y_0)^{-1}\Omega'(\alpha)\|\|\Omega'(\alpha)^{-1}\Omega(s_0)\| \\
\le & \left[\eta_1(\|s_0 - \alpha\|) + \frac{1}{2}\frac{\kappa_0(1 + \eta_1(\|s_0 - \alpha\|))\|s_0 - \alpha\|}{1 - \kappa_0\|s_0 - \alpha\|}\frac{1 + \kappa_0\|s_0 - \alpha\|}{1 - \kappa_0\eta_1(\|s_0 - \alpha\|)\|s_0 - \alpha\|}\right]\|s_0 - \alpha\| \\
= & \eta_2(\|s_0 - \alpha\|)\|s_0 - \alpha\|,
\end{aligned} \tag{4.12}$$

where

$$\eta_2(\vartheta) = \eta_1(\vartheta) + \frac{1}{2}\frac{\kappa_0(1 + \eta_1(\vartheta))(1 + \kappa_0\vartheta)}{(1 - \kappa_0\vartheta)(1 - \kappa_0\eta_1(\vartheta)\vartheta)}\vartheta.$$

We define the function $\omega_2(\vartheta) = \eta_2(\vartheta) - 1$. Clearly, there is at least one root of $\omega_2(\vartheta)$ in the interval $(0, \rho_1)$ since $\omega_2(0) < 0$ and $\omega_2(\rho_1) > 0$. Let ρ_2 denote the smallest root of $\omega_2(\vartheta)$ within this interval. Then, we obtain

$$0 < \rho_2 < \rho_1, \text{ and } 0 \le \eta_2(\vartheta) < 1, \forall\, \vartheta \in [0, \rho_2). \tag{4.13}$$

Now, applying (4.12) and (4.13), we arrive at the following result:

$$\|z_0 - \alpha\| \le \eta_2(\|s_0 - \alpha\|)\|s_0 - \alpha\| < \|s_0 - \alpha\|.$$

Moreover, by using (4.1) and (4.10), we have

$$\begin{aligned}
&\left\|(2\Omega'(\alpha))^{-1}\left[(\Omega'(s_0) - \Omega'(\alpha)) + (\Omega'(y_0) - \Omega'(\alpha))\right]\right\| \\
&\quad \le \frac{1}{2}\left[\|\Omega'(\alpha)^{-1}(\Omega'(s_0) - \Omega'(\alpha))\| + \|\Omega'(\alpha)^{-1}(\Omega'(y_0) - \Omega'(\alpha))\|\right] \\
&\quad \le \frac{\kappa_0}{2}\left[1 + \frac{\kappa\|s_0 - \alpha\|}{2(1 - \kappa_0\|s_0 - \alpha\|)}\right]\|s_0 - \alpha\| \\
&\quad \le \phi(\|s_0 - \alpha\|)\|s_0 - \alpha\| < 1.
\end{aligned}$$

Consequently, by applying the Banach lemma on invertible operators, $(\Omega'(s_0)+\Omega'(y_0))^{-1}$ exists and

$$\|(\Omega'(s_0)+\Omega'(y_0))^{-1}\Omega'(\alpha)\| \leq \frac{1}{2(1-\phi(\|s_0-\alpha\|)\|s_0-\alpha\|)}. \tag{4.14}$$

Accordingly, s_1 is well-defined. As such, from (2.14) with $n = 0$, we have

$$\begin{aligned}
\|s_1-\alpha\| \leq& \|z_0-\alpha\| + \Bigg\|k + \left(\frac{2+k}{4}\right)\Omega'(y_0)^{-1}\Omega'(s_0) + \left(\frac{2-k}{4}\right)\Omega'(s_0)^{-1}\Omega'(y_0) \\
& -2k(\Omega'(s_0)+\Omega'(y_0))^{-1}\Omega'(s_0)\Bigg\| \left\|\Omega'(y_0)^{-1}\Omega(z_0)\right\| \\
\leq& \|z_0-\alpha\| + \Big[\left\|k\left(\Omega'(s_0)+\Omega'(y_0)\right)^{-1}\Omega'(\alpha)\right\| \Big(\left\|\Omega'(\alpha)^{-1}\left(\Omega'(y_0)-\Omega'(\alpha)\right)\right\| \\
& + \left\|\Omega'(\alpha)^{-1}\left(\Omega'(s_0)-\Omega'(\alpha)\right)\right\|\Big) + \left\|\left(\frac{2+k}{4}\right)\Omega'(y_0)^{-1}\Omega'(\alpha)\right\| \left\|\Omega'(\alpha)^{-1}\Omega'(s_0)\right\| \\
& + \left\|\left(\frac{2-k}{4}\right)\Omega'(s_0)^{-1}\Omega'(\alpha)\right\| \left\|\Omega'(\alpha)^{-1}\Omega'(y_0)\right\|\Big] \left\|\Omega'(y_0)^{-1}\Omega'(\alpha)\right\| \left\|\Omega'(\alpha)^{-1}\Omega(z_0)\right\| \\
\leq& \|z_0-\alpha\| + \left[|k|\frac{\kappa_0\|y_0-\alpha\| + \kappa_0\|s_0-\alpha\|}{2(1-\phi(\|s_0-\alpha\|)\|s_0-\alpha\|)} + \left|\frac{2+k}{4}\right| \frac{1+\kappa_0\|s_0-\alpha\|}{1-\kappa_0\|y_0-\alpha\|}\right. \\
& \left. + \left|\frac{2-k}{4}\right| \frac{1+\kappa_0\|y_0-\alpha\|}{1-\kappa_0\|s_0-\alpha\|}\right] \frac{(1+\kappa_0\|z_0-\alpha\|)\|z_0-\alpha\|}{1-\kappa_0\|y_0-\alpha\|} \\
\leq& \left[1 + \left(|k|\frac{\kappa_0\|s_0-\alpha\|(1+\eta_1(\|s_0-\alpha\|))}{2(1-\phi(\|s_0-\alpha\|)\|s_0-\alpha\|)} + \left|\frac{2+k}{4}\right| \frac{1+\kappa_0\|s_0-\alpha\|}{1-\kappa_0\eta_1(\|s_0-\alpha\|)\|s_0-\alpha\|}\right.\right. \\
& \left.\left. + \left|\frac{2-k}{4}\right| \frac{1+\kappa_0\eta_1(\|s_0-\alpha\|)\|s_0-\alpha\|}{1-\kappa_0\|s_0-\alpha\|}\right) \frac{1+\kappa_0\eta_2(\|s_0-\alpha\|)\|s_0-\alpha\|}{1-\kappa_0\eta_1(\|s_0-\alpha\|)\|s_0-\alpha\|}\right]\eta_2(\|s_0-\alpha\|)\|s_0-\alpha\| \\
=& \eta_3(\|s_0-\alpha\|)\|s_0-\alpha\|,
\end{aligned} \tag{4.15}$$

where

$$\eta_3(\vartheta) = \left[1 + \left(|k|\frac{\kappa_0\vartheta(1+\eta_1(\vartheta))}{2(1-\phi(\vartheta)\vartheta)} + \left|\frac{2+k}{4}\right| \frac{1+\kappa_0\vartheta}{1-\kappa_0\eta_1(\vartheta)\vartheta} + \left|\frac{2-k}{4}\right| \frac{1+\kappa_0\eta_1(\vartheta)\vartheta}{1-\kappa_0\vartheta}\right) \frac{1+\kappa_0\eta_2(\vartheta)\vartheta}{1-\kappa_0\eta_1(\vartheta)\vartheta}\right]\eta_2(\vartheta).$$

Let us define the function $\omega_3(\vartheta) = \eta_3(\vartheta) - 1$. It is evident that, with $\omega_3(0) < 0$ and $\omega_3(\rho_2) > 0$, there exists at least one root of $\omega_3(\vartheta)$ in the interval $(0, \rho_2)$. Let ρ represent the smallest root of $\omega_3(\vartheta)$ within this interval. Then, we obtain

$$\rho < \rho_2 < \rho_1 < \frac{1}{\kappa_0}, \text{ and } 0 \leq \eta_3(\vartheta) < 1, \forall\, \vartheta \in [0, \rho). \tag{4.16}$$

Now, applying (4.15) and (4.16), we arrive at the following result:

$$\|s_1-\alpha\| \leq \eta_3(\|s_0-\alpha\|)\|s_0-\alpha\| < \|s_0-\alpha\| < \rho.$$

It follows that the theorem is valid for $n = 0$. By repeating the computations above with s_n, y_n, z_n, s_{n+1} respectively replacing s_0, y_0, z_0, s_1, the induction will be completed and we can establish the inequalities detailed in (4.5)–(4.7). Additionally, based on the estimate $\|s_{n+1}-\alpha\| \leq \|s_n-\alpha\| < \rho$, we conclude that $s_{n+1} \in B(\alpha, \rho)$. Evidently, the function η_3 is increasing in its domain; so, we have

$$\|s_{n+1}-\alpha\| \leq \eta_3(\vartheta)\|s_n-\alpha\| \leq \eta_3(\vartheta)\eta_3(\|s_{n-1}-\alpha\|)\|s_{n-1}-\alpha\|$$

$$\leq \eta_3(\vartheta)^2\eta_3(\|s_{n-2}-\alpha\|)\|s_{n-2}-\alpha\| \leq \cdots \leq \eta_3(\vartheta)^{n+1}\|s_0-\alpha\|.$$

Using $\lim_{n\to\infty}\eta_3(\vartheta)^{n+1}=0$, we obtained that $\lim_{n\to\infty}s_n=\alpha$; hence, the method tends to the solution.

To establish the uniqueness aspect, consider that $\alpha^*\in B(\alpha,\rho)$, where $\alpha^*\neq\alpha$ and satisfies the condition that $\Omega(\alpha^*)=0$. Let $T=\int_0^1\Omega'(\alpha^*+\vartheta(\alpha-\alpha^*))d\vartheta$. Then, by using (4.1), we have

$$\|\Omega'(\alpha)^{-1}(T-\Omega'(\alpha))\| \leq \int_0^1 \kappa_0\|\alpha^*+\vartheta(\alpha-\alpha^*)-\alpha\|d\vartheta \leq \frac{\kappa_0}{2}\|\alpha-\alpha^*\| = \frac{\kappa_0}{2}R<1.$$

Hence, T^{-1} exists, and by utilizing the following identity:

$$0=\Omega(\alpha)-\Omega(\alpha^*=T(\alpha-\alpha^*), \tag{4.17}$$

we deduce that $\alpha=\alpha^*$. □

5. Numerical results

In this section, we will use a series of numerical examples to show how well our local convergence analysis works for the proposed method PFM6 (2.14).

To begin with, we will determine the radius of convergence for each example and subsequently compare our method with several established alternatives found in the literature. Specifically, we will focus on two sixth-order convergence schemes (schemes (1.2) and (1.3) from [22]) and one fifth-order convergence scheme (scheme from [23]). We will refer to these schemes as M1, M2, and M3, respectively.

Example 1. *[24] Let us consider τ defined over the interval $\mathbb{D}=[-\frac{1}{2},\frac{5}{2}]$ by*

$$\tau(s)=\begin{cases} s^3\log s^2+s^5-s^4, & s\neq 0,\\ 0, & s=0.\end{cases}$$

The zero of τ is $\alpha=1$. Also, we have

$$\tau'(s)=3s^2\log s^2+5s^4-4s^3+2s^2,$$
$$\tau''(s)=6s\log s^2+20s^3-12s^2+10s,$$
$$\tau(s)=6\log s^2+60s^2-24s+22.$$

Although the third derivative of τ is unbounded on $\mathbb{D}$, the iterative method given by (2.14) *still converges, according to Theorem 4.1 with $\alpha=1$. We found that $\kappa_0=\kappa=96.662907$. By setting $k=0.1$, we calculate the radius of convergence as follows:*

$$\rho=0.00249028<\rho_2=0.0040526<\rho_1=0.00689682.$$

The comparison results for the radii of convergence are displayed in Table 1.

Example 2. *[24] Consider a system of differential equations governing the dynamics of an object. These equations are given as follows:*

$$\tau_1'(\omega_1)=e^{\omega_1}, \tau_2'(\omega_2)=(e-1)\omega_2+1, \tau_3'(\omega_3)=1,$$

with the initial conditions $\tau_1(0) = \tau_2(0) = \tau_3(0) = 0$. *These equations can be collectively represented as the vector* $\tau = (\tau_1, \tau_2, \tau_3)$. *Let* $\mathbb{X} = \mathbb{Y} = \mathbb{R}^3$ *and* $\mathbb{D} = \bar{\mathbb{B}}(0, 1)$. *Define* τ *on* $\mathbb{D}$ *for* $\nu = (\omega_1, \omega_2, \omega_3)^T$ *by*

$$\tau(\nu) = \left(e^{\omega_1} - 1, \frac{e-1}{2}\omega_2^2 + \omega_2, \omega_3\right)^T.$$

The Fréchet derivative is given by

$$\tau'(\nu) = \begin{pmatrix} e^{\omega_1} & 0 & 0 \\ 0 & (e-1)\omega_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $\alpha = (0, 0, 0)^T$, $\tau'(\alpha) = \tau'(\alpha)^{-1} = diag\{1, 1, 1\}$, *and* $\kappa_0 = e - 1$, $\kappa = e^{\frac{1}{e-1}}$, *and taking* $k = 0.1$, *we get*

$$\rho = 0.13843 < \rho_2 = 0.225675 < \rho_1 = 0.382692.$$

The comparison results for the radii of convergence are displayed in Table 1.

Example 3. *[24] (Also, see [23] for details) Let* $\mathbb{X} = \mathbb{Y} = C[0, 1]$ *and the space of continuous functions defined on* $[0, 1]$ *be equipped with the max norm, and let* $\mathbb{D} = B(0, 1)$. *Consider the nonlinear integral equation of the Hammerstein type* τ *defined on* $\mathbb{D}$ *by*

$$\tau(\nu)(s) = \nu(s) - 5\int_0^1 su\nu(u)^3 du,$$

with $\nu(s) \in C[0, 1]$. *The first derivative of* τ *is given by*

$$\tau'(\nu(\varphi))(s) = \varphi(s) - 15\int_0^1 su\nu(u)^2\varphi(u)du, \text{ for each } \varphi \in \mathbb{D}.$$

For the solution $\alpha = 0$, *we obtain that* $\kappa_0 = 7.5$ *and* $\kappa = 15$. *Then, using the iterative method given by* (2.14) *for* $k = 0.1$, *we get the radii of convergence as follows:*

$$\rho = 0.0251314 < \rho_2 = 0424972 < \rho_1 = 0.0666667.$$

The comparison results for the radii of convergence are displayed in Table 1.

Table 1. Comparison of convergence radii.

Examples	M1	M2	M3	PFM6 (2.14)
1	0.001141	0.001215	0.000916634	0.00249028
2	0.064030	0.068190	0.0500153	0.13843
3	0.013823	0.014756	0.0067881	0.0251314

As evident from the data presented in Table 1, the proposed family of methods given by PFM6 (2.14) exhibits a significantly broad convergence radius. Moreover, across all three examples, it is consistently observed that PFM6 (2.14) outperforms the other three methods by exhibiting a notably larger convergence radius.

6. Applications

In this section, we apply the proposed method PFM6 (2.14), along with PFM6.1 (3.12), PFM6.2 (3.13), and PFM9 (3.22) to solve systems of nonlinear equations in $\mathbb{R}^n$; we also compare their performance to the existing methods given by PPM3 (1.3) and SSKM6 (3.9). Also, the following two methods are considered for the comparison.

The sixth-order method established by Lotfi et al. in [25] (TLM6):

$$\begin{aligned} y_n =& s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n =& s_n - 2\left(\Omega'(s_n) + \Omega'(y_n)\right)^{-1}\Omega(s_n), \\ s_{n+1} =& z_n - \Big[\frac{7}{2}I - 4\Omega'(s_n)^{-1}\Omega'(y_n) + \frac{3}{2}\left(\Omega'(s_n)^{-1}\Omega'(y_n)\right)^2\Big]\Omega'(s_n)^{-1}\Omega(z_n). \end{aligned} \tag{6.1}$$

The ninth-order method established by Lotfi et al. in [25] (TLM9):

$$\begin{aligned} y_n =& s_n - \Omega'(s_n)^{-1}\Omega(s_n), \\ z_n =& s_n - 2\left(\Omega'(s_n) + \Omega'(y_n)\right)^{-1}\Omega(s_n), \\ \hat{z}_n =& z_n - \Big[\frac{7}{2}I - 4\Omega'(s_n)^{-1}\Omega'(y_n) + \frac{3}{2}\left(\Omega'(s_n)^{-1}\Omega'(y_n)\right)^2\Big]\Omega'(s_n)^{-1}\Omega(z_n), \\ s_{n+1} =& \hat{z}_n - \Big[\frac{7}{2}I - 4\Omega'(s_n)^{-1}\Omega'(y_n) + \frac{3}{2}\left(\Omega'(s_n)^{-1}\Omega'(y_n)\right)^2\Big]\Omega'(s_n)^{-1}\Omega(\hat{z}_n). \end{aligned} \tag{6.2}$$

All computations were performed in the Mathematica 12.2 programming package by using multiple-precision arithmetic with 1000 significant digits. For each problem, we recorded the number of iterations (n) needed to converge to the root such that the stopping criterion $\|\Omega(s_n)\| < 10^{-100}$ is satisfied. Numerical tests were conducted for the following set of problems:

Problem 1. A nonlinear system with two unknowns is given by

$$\Omega_1(s) = \left(s_1 + e^{-s_2} - 1, 3s_1 + \cos s_2 - 2\right)^T.$$

By taking $s_0 = (-1, -1)^T$ as the initial approximation, we arrive at the root:
$\alpha = (0.367758471822148, 0.458483793003288)^T$.

Problem 2. A nonlinear system with three unknowns is given by

$$\Omega_2(s) = \left(s_1 + s_2 - e^{-s_3}, s_1 - e^{-s_2} + s_3, -e^{-s_1} + s_2 + s_3\right)^T.$$

By choosing $s_0 = (1, 1, 1)^T$ as the initial approximation, we get the root:
$\alpha = (0.351733711249196, 0.351733711249196, 0.351733711249196)^T$.

Table 2 provides a comprehensive overview of the comparison results for the various methods applied to the test problems. The table shows the residual error (i.e., $\|\Omega(s_n)\|$), the error in the consecutive iterations $\|s_n - s_{n-1}\|$, and the approximated computational order of convergence (COC) after the methods satisfy the stopping criterion. The COC is calculated as follows [26]:

$$COC \approx \frac{\log\left(\|s_n - s_{n-1}\|/\|s_{n-1} - s_{n-2}\|\right)}{\log\left(\|s_{n-1} - s_{n-2}\|/\|s_{n-2} - s_{n-3}\|\right)}. \tag{6.3}$$

Table 2. Comparison of the methods on the test Problems 1 and 2.

Method	$\Omega(s_n)$	n	$\|\|\Omega(s_n)\|\|$	$\|\|s_n - s_{n-1}\|\|$	μ	COC
PPM3	$\Omega_1(s_n)$	7	1.3604×10^{-256}	5.2810×10^{-86}	1.6899	3.000
SSKM6	$\Omega_1(s_n)$	4	1.6132×10^{-218}	6.7182×10^{-37}	0.3041	6.000
TLM6	$\Omega_1(s_n)$	5	8.0714×10^{-436}	6.3215×10^{-74}	4020.6	6.000
TLM9	$\Omega_1(s_n)$	4	2.1109×10^{-355}	8.4589×10^{-41}	6.1887×10^{6}	9.036
PFM6 ($k = 3$)	$\Omega_1(s_n)$	4	2.0440×10^{-268}	4.7518×10^{-45}	0.0238	6.024
PFM6.1	$\Omega_1(s_n)$	4	1.0271×10^{-219}	4.2262×10^{-37}	0.2486	6.015
PFM6.2	$\Omega_1(s_n)$	4	8.3154×10^{-147}	3.8408×10^{-25}	4.4963	6.000
PFM9 ($k = 1$)	$\Omega_1(s_n)$	4	1.1198×10^{-876}	4.9987×10^{-98}	0.9799	9.000
PPM3	$\Omega_2(s_n)$	5	1.6768×10^{-238}	1.7648×10^{-79}	1.1285×10^{-2}	3.000
SSKM6	$\Omega_2(s_n)$	3	5.3603×10^{-215}	1.1794×10^{-35}	7.3680×10^{-6}	6.000
TLM6	$\Omega_2(s_n)$	4	1.8557×10^{-140}	2.0466×10^{-35}	1.7105×10^{15}	4.000
TLM9	$\Omega_2(s_n)$	4	7.7971×10^{-231}	1.1938×10^{-46}	1.1856×10^{35}	5.000
PFM6 ($k = -3$)	$\Omega_2(s_n)$	3	2.7939×10^{-252}	9.0979×10^{-42}	1.8225×10^{-6}	6.000
PFM6.1	$\Omega_2(s_n)$	3	6.9583×10^{-217}	5.4463×10^{-36}	9.8625×10^{-6}	6.000
PFM6.2	$\Omega_2(s_n)$	3	3.7879×10^{-196}	1.3967×10^{-32}	1.8871×10^{-5}	6.000
PFM9 ($k = 1$)	$\Omega_2(s_n)$	3	4.0965×10^{-706}	2.7339×10^{-78}	1.7758×10^{-8}	9.000

Moreover, the order of convergence can be confirmed through the analysis of the asymptotic behavior of the convergence rate, i.e., the asymptotic error constant (μ), by using the following formula [26]:

$$\mu = \frac{||s_n - s_{n-1}||}{||s_{n-1} - s_{n-2}||^p}, \text{for } p = 3, 6, 9. \tag{6.4}$$

Furthermore, we provide in Table 3 a comparison of the CPU time (measured in seconds) required by each method to meet the stopping criterion. We averaged the CPU running time over three trials to improve accuracy. All computations were carried out on a computer equipped with an Intel(R) Core(TM) i5-10210U CPU @ 2.11 GHz and 8 GB of RAM running Windows 11.

Table 3. CPU time comparison for the methods on Problems 1 and 2.

Method	$\Omega_1(s)$	$\Omega_2(s)$
PPM3	0.02144	0.02281
SSKM6	0.03867	0.02352
TLM6	0.02836	0.05682
TLM9	0.02969	0.07172
PFM6	0.01791	0.02568
PFM6.1	0.02076	0.02128
PFM6.2	0.02185	0.07753
PFM9	0.02490	0.03720

Our numerical results, as presented in Tables 2 and 3, demonstrate that the proposed methods exhibit highly competitive performance. They converge rapidly toward the root in the smallest number of iterations (n), achieve better accuracy in terms of minimal residual error ($||\Omega(s_n)||$) and the error in the consecutive iterations ($||s_n - s_{n-1}||$), and consume less CPU time than well-known existing methods. Additionally, the computed COC aligns with the theoretical convergence order for the newly proposed methods.

7. Basins of attraction

In this section, we offer a graphical analysis of our newly proposed scheme, PFM6 ($k = 0.001$) (2.14), along with PFM6.1 (3.12), PFM6.2 (3.13), and PFM9 ($k = 0.001$) (3.22), against the established methods of PPM3 (1.3), SSKM6 (3.9), TLM6 (6.1), and TLM9 (6.2). This comparison is facilitated by an analysis of their dynamical behaviors on the Cartesian plane, specifically through the lens of their basins of attraction. These basins not only serve as a visual comparative tool, they also shed light on the convergence and stability attributes of each method. We focus on the following systems of nonlinear polynomial equations for the analysis:

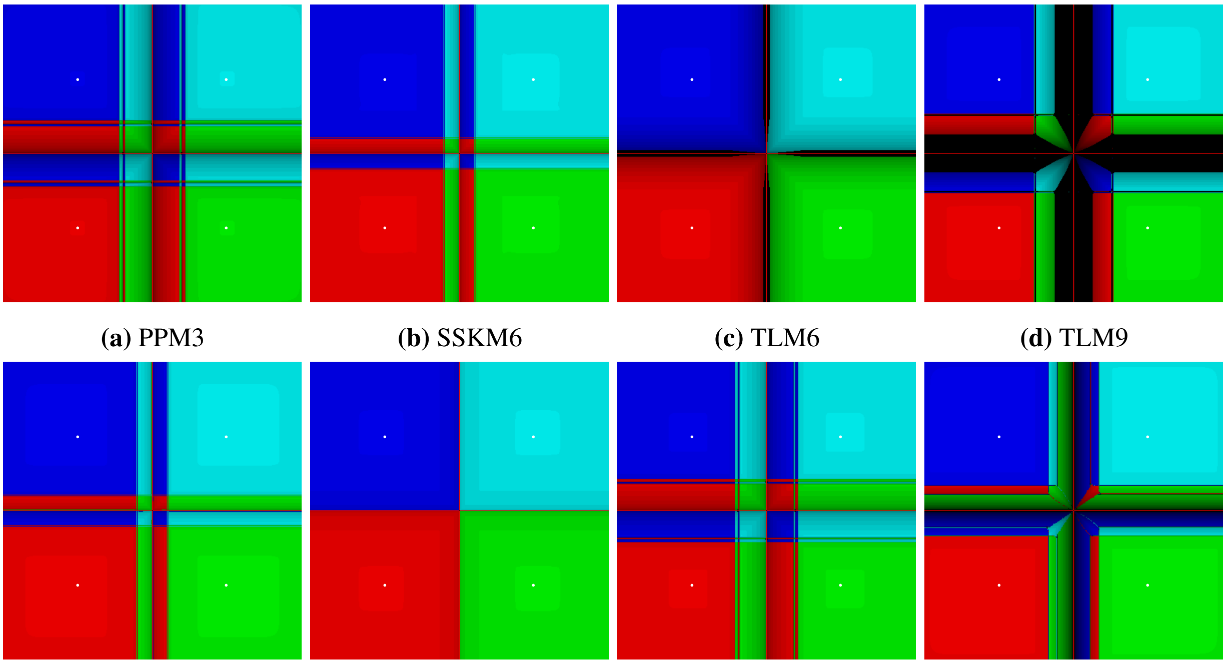
(i) $P_1(s) = (s_1^2 - 1, s_2^2 - 1)^T$.

(ii) $P_2(s) = (s_1^2 + s_2^2 - 1, \frac{1}{4}s_1^2 + 4s_2^2 - 1)^T$.

(iii) $P_3(s) = (3s_1^2 s_2 - s_2^3, s_1^3 - 3s_1 s_2^2 - 1)^T$.

For this graphical analysis, we employed a precisely defined rectangular grid $\mathbb{S} = [-2, 2] \times [-2, 2] \in \mathbb{R}^2$ within the Cartesian plane that has been subdivided into a 401×401 matrix of grid points. Each point on this grid serves as an initial point for iterations. These points were each assigned a unique color to indicate the specific real root to which the iterative method will converge when initialized from that point. The roots are visually demarcated by small white circles in the plot. Points that fail to converge within the set tolerance of 10^{-3} or before reaching a maximum of 80 iterations have been marked in black, denoting them as divergent points. Additionally, the brightness of the colour within each basin serves as an indicator of the speed of convergence, with brighter hues representing faster convergence and darker shades indicating a slower rate. The graphical illustrations of the basins of attraction for the methods under evaluation are displayed in Figures 1–3. To supplement this, Table 4 enumerates the instances of divergence, sourced from a 401×401 matrix of initial points, when the methods are applied to $P_i(s), i = 1, 2, 3$.

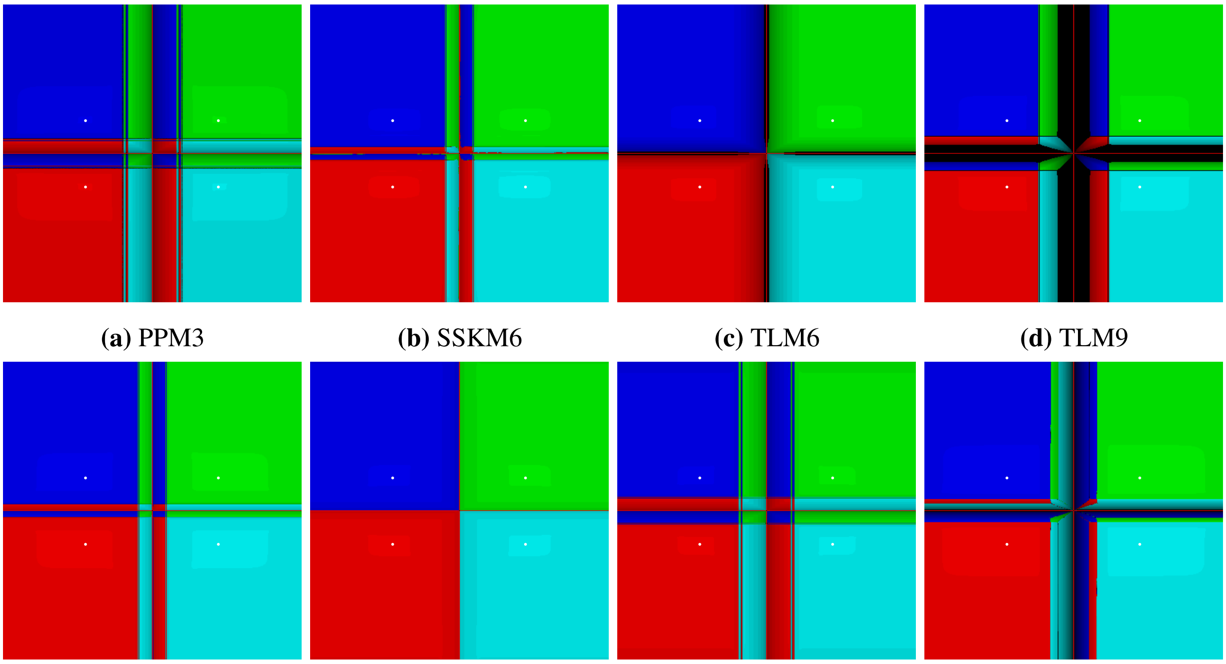
Figures 1–3 reveal key insights into the performance of the methods based on their basins of attraction. Our proposed methods exhibit strong performance with large basins, demonstrating their robustness by yielding a notably low count of divergent points. PFM6 emerges as the top performer, with SSKM6 following closely behind and offering a strong challenge to our proposed approach. On the other hand, PFM9 and TLM9 underperform, indicating that higher order does not guarantee better convergence or stability, as evidenced by their small basins and large number of divergent points. Intriguingly, while PFM6.1 excels and matches the performance of PFM6 and SSKM6 for $P_1(s)$ and $P_2(s)$, it struggles significantly with $P_3(s)$, lagging behind the other methods and delivering the poorest performance in this specific scenario. These observations are corroborated by the divergent point data in Table 4.



(a) PPM3 (b) SSKM6 (c) TLM6 (d) TLM9

(e) PFM6 (f) PFM6.1 (g) PFM6.2 (h) PFM9

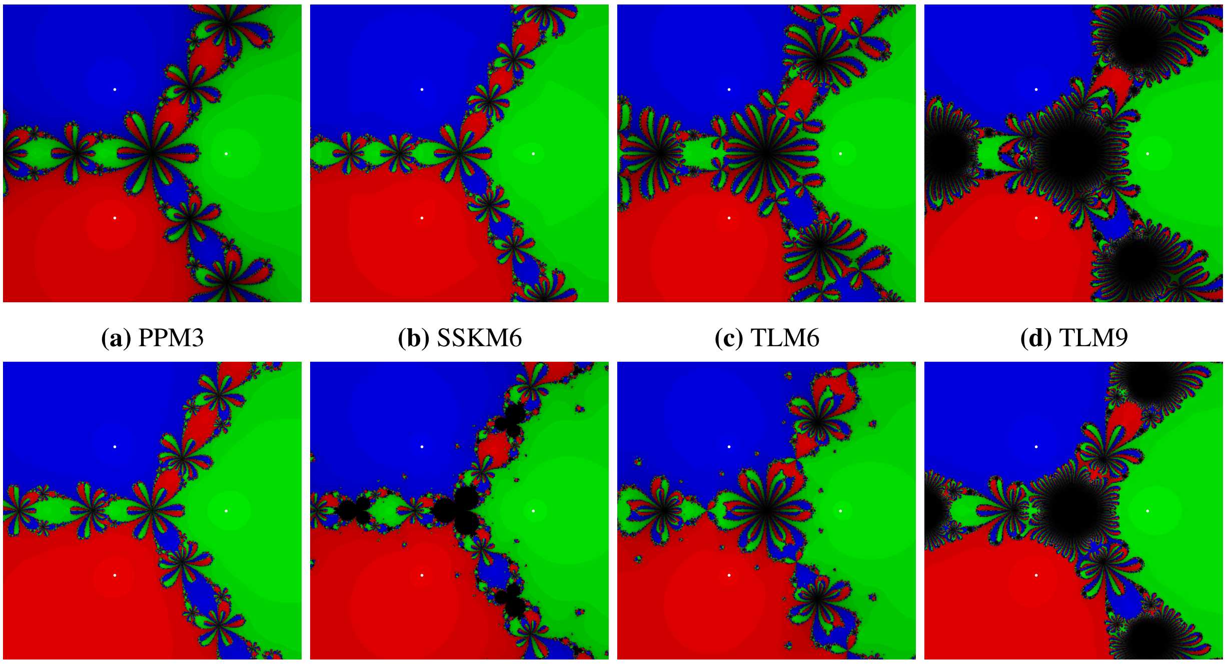
Figure 1. Basins of attraction on $P_1(s)$.



(a) PPM3 (b) SSKM6 (c) TLM6 (d) TLM9

(e) PFM6 (f) PFM6.1 (g) PFM6.2 (h) PFM9

Figure 2. Basins of attraction on $P_2(s)$.



(a) PPM3 (b) SSKM6 (c) TLM6 (d) TLM9 (e) PFM6 (f) PFM6.1 (g) PFM6.2 (h) PFM9

Figure 3. Basins of attraction on $P_3(s)$.

Table 4. Number of divergent points for the compared methods when applied to $P_1(s)$, $P_2(s)$ and $P_3(s)$.

$P(s)$	PPM3	SSKM6	TLM6	TLM9	PFM6	PFM6.1	PFM6.2	PFM9
$P_1(s)$	17	1	5425	37081	1	1	17	1601
$P_2(s)$	7	1	3593	25057	1	1	5	875
$P_3(s)$	77	20	44	5974	1	7685	18	4473

8. Concluding remarks

In this paper, we have presented a family of three-step iterative methods with sixth-order convergence for the purpose of solving systems of nonlinear equations. The proposed methods are based on a novel approach to enhance the convergence order of iterative methods. We have also proposed a three-step scheme with convergence order $p + 3$ (for $p \geq 3$) and extended it to a generalized $(m + 2)$-step scheme by merely incorporating one additional function evaluation, thus achieving convergence orders up to $p + 3m$, $m \in \mathbb{N}$. We have provided thorough local convergence analysis and numerical experiments to validate the theoretical findings. Lastly, we have showcased the performance of these methods through the analysis of their basins of attraction and their application to systems of nonlinear equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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