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Research article

Some dynamic Hardy-type inequalities with negative parameters on time scales nabla calculus

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Abstract: In this paper, we establish some new dynamic Hardy-type inequalities with negative parameters on time scales nabla calculus by applying the reverse Hölder's inequality, integration by parts, and chain rule on time scales nabla calculus. As special cases of our results (when $\mathbb{T} = \mathbb{R}$), we get the continuous analouges of inequalities proven by Benaissa and Sarikaya, and when $\mathbb{T} = \mathbb{N}_0$, the results to the best of the authors' knowledge are essentially new.

Keywords: Hardy-type inequalities with negative parameters; time scales; reverse Hölder's inequality; chain rule
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1. Introduction

In 1920, Hardy [1] proved that

$$\sum_{\tau=1}^{\infty}\left(\frac{1}{\tau}\sum_{k=1}^{\tau}h(k)\right)^{\epsilon} \leq \left(\frac{\epsilon}{\epsilon-1}\right)^{\epsilon}\sum_{\tau=1}^{\infty}h^{\epsilon}(\tau), \tag{1.1}$$

where $\epsilon > 1$, $h(\tau) \geq 0$ for $\tau \geq 1$, and $\sum_{\tau=1}^{\infty} h^{\epsilon}(\tau) < \infty$. In 1925, Hardy [2, Theorem A] showed that if $\epsilon > 1$, $\varpi \geq 0$ such that $\int_0^{\infty} \varpi^{\epsilon}(\xi)d\xi < \infty$, then ϖ is integrable over any finite interval $(0, \xi)$, $\xi \in (0, \infty)$

and

$$\int_0^\infty \left(\frac{1}{\xi}\int_0^\xi \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{\epsilon-1}\right)^\epsilon \int_0^\infty \varpi^\epsilon(\xi)d\xi. \tag{1.2}$$

The constant $(\epsilon/(\epsilon-1))^\epsilon$ in (1.1) and (1.2) is the best possible.

In 1927, Hardy and Littlewood [3] proved that if $0 < \epsilon < 1$, $\varpi(\xi) \ge 0$ for $\xi \in (0,\infty)$, and $\int_0^\infty \varpi^\epsilon(\xi)d\xi < \infty$, then

$$\int_0^\infty \left(\frac{1}{\xi}\int_\xi^\infty \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \ge \left(\frac{\epsilon}{1-\epsilon}\right)^\epsilon \int_0^\infty \varpi^\epsilon(\xi)d\xi,\ 0<\epsilon<1,$$

where the constant $(\epsilon/\,(1-\epsilon))^\epsilon$ is the best possible.

The Hardy inequalities mentioned above are proved for a positive parameter $0 < \epsilon < 1$ and $\epsilon > 1$. Also, these inequalities depend on the power rule inequality in case of the positive parameter. So, some authors discovered new inequalities of Hardy type with negative parameters and noted that they proved these inequalities with a different technique which depends on the power rule inequality.

For example, in 2007, Bicheng [4] established the integral inequality of Hardy type with negative parameter and proved that if $\epsilon < 0$, $r > 1$, $\varpi(\vartheta) \ge 0$, and $0 < \int_0^\infty \xi^{-r}(\xi\varpi(\xi))^\epsilon d\xi < \infty$, then

$$\int_0^\infty \xi^{-r}\left(\int_\xi^\infty \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{1-r}\right)^\epsilon \int_0^\infty \xi^{-r}(\xi\varpi(\xi))^\epsilon d\xi, \tag{1.3}$$

where the constant factor $(\epsilon/\,(1-r))^\epsilon$ is the best possible. Also, in [4] it was proved that if $\epsilon < 0$, $r < 1$, $\varpi(\vartheta) \ge 0$, and $0 < \int_0^\infty \xi^{-r}(\xi\varpi(\xi))^\epsilon d\xi < \infty$, then

$$\int_0^\infty \xi^{-r}\left(\int_0^\xi \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{r-1}\right)^\epsilon \int_0^\infty \xi^{-r}(\xi\varpi(\xi))^\epsilon d\xi, \tag{1.4}$$

where the constant factor $(\epsilon/\,(r-1))^\epsilon$ is the best possible.

In 2020, Benaissa and Sarikaya [5] generalized (1.3), and proved that if $\epsilon < 0$, $r > 1$, and $\varpi, \Xi > 0$ such that the function $\xi/\Xi(\xi)$ is nondecreasing, then

$$\int_0^\infty \Xi^{-r}(\xi)\left(\int_\xi^\infty \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{1-r}\right)^\epsilon \int_0^\infty (\xi\varpi(\xi))^\epsilon\,\Xi^{-r}(\xi)d\xi, \tag{1.5}$$

and if $\epsilon < 0$, $0 \le r < 1$, and $\varpi, \Xi > 0$ such that the function $\xi/\Xi(\xi)$ is nonincreasing, then

$$\int_0^\infty \Xi^{-r}(\xi)\left(\int_0^\xi \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{r-1}\right)^\epsilon \int_0^\infty (\xi\varpi(\xi))^\epsilon\,\Xi^{-r}(\xi)d\xi. \tag{1.6}$$

Also, the authors of [5] proved that if $\epsilon < 0$, $r < 0$, and $\varpi, \Xi > 0$ such that the function $\xi/\Xi(\xi)$ is nondecreasing, then

$$\int_0^\infty \Xi^{-r}(\xi)\left(\int_0^\xi \varpi(\vartheta)d\vartheta\right)^\epsilon d\xi \le \left(\frac{\epsilon}{r-1}\right)^\epsilon \int_0^\infty (\xi\varpi(\xi))^\epsilon\,\Xi^{-r}(\xi)d\xi. \tag{1.7}$$

The time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. As an application on time scales, we can get the continuous and discrete forms of any inequality, i.e., $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. For more details about the dynamic inequalities on time scales, refer to [6–16], and for the applications of dynamic inequalities in the study of qualitative behavior of dynamic equations, refer to [17–24]. For including an exploration of advanced methodologies, we refer to the papers [25, 26].

The objective of this paper is to introduce novel generalizations of the continuous the inequalities (1.5)–(1.7) on nabla time scales. The proofs of these results rely on employing the reverse Hölder's inequality and the chain rule formula adapted to time scales.

This paper is divided into three sections: In Section 2, we present some lemmas on time scales needed in Section 3 where we prove our results. These results as special cases when $\mathbb{T} = \mathbb{R}$ give the inequalities (1.5)–(1.7), while for $\mathbb{T} = \mathbb{N}_0$, the results are fundamentally original.

2. Preliminaries and basic lemmas

In 2001, Bohner and Peterson [27] introduced the time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Also, they defined the backward jump operator by $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$. For any function $\varpi : \mathbb{T} \to \mathbb{R}$, the notation $\varpi^{\rho}(\tau)$ signifies $\varpi(\rho(\tau))$. The time scale interval $[\varrho, \delta]_{\mathbb{T}}$ is defined as $[\varrho, \delta]_{\mathbb{T}} := [\varrho, \delta] \cap \mathbb{T}$.

Definition 2.1. [28] A function $\lambda : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or *ld*-continuous provided that it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist at right-dense points in $\mathbb{T}$. The space of *ld*-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The set $\mathbb{T}^{\kappa}$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. In summary,

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.2. [28] A function $\psi : \mathbb{T} \to \mathbb{R}$ is said to be ∇-differentiable at $\vartheta \in \mathbb{T}^{\kappa}$ if ψ is defined in a neighbourhood U of ϑ and there exists a unique real number $\psi^{\nabla}(\vartheta)$, called the nabla derivative of ψ at ϑ, such that for each $\epsilon > 0$, there exists a neighbourhood N of ϑ with $N \subseteq U$ and

$$\left|\psi(\rho(\vartheta)) - \psi(s) - \psi^{\nabla}(\vartheta)[\rho(\vartheta) - s]\right| \leq \epsilon \left|\rho(\vartheta) - s\right|, \quad \forall s \in N.$$

Theorem 2.1. *[28] Assume $\psi, \Theta : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $\vartheta \in \mathbb{T}$. Then:*

(1) The product $\psi\Theta : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at ϑ, and we get the product rule

$$(\psi\Theta)^{\nabla}(\vartheta) = \psi^{\nabla}(\vartheta)\Theta(\vartheta) + \psi^{\rho}(\vartheta)\Theta^{\nabla}(\vartheta) = \psi(\vartheta)\Theta^{\nabla}(\vartheta) + \psi^{\nabla}(\vartheta)\Theta^{\rho}(\vartheta).$$

(2) If $\Theta(\vartheta)\Theta^{\rho}(\vartheta) \neq 0$, then ψ/Θ is nabla differentiable at ϑ, and we get the quotient rule

$$\left(\frac{\psi}{\Theta}\right)^{\nabla}(\vartheta) = \frac{\psi^{\nabla}(\vartheta)\Theta(\vartheta) - \psi(\vartheta)\Theta^{\nabla}(\vartheta)}{\Theta(\vartheta)\Theta^{\rho}(\vartheta)}.$$

Lemma 2.1. *[29] Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $\Theta : \mathbb{T} \to \mathbb{R}$ is continuous and nabla differentiable. Then, $\psi \circ \Theta : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and*

$$(\psi \circ \Theta)^{\nabla}(\vartheta) = \psi'(\Theta(d))\Theta^{\nabla}(\vartheta), \; d \in [\rho(\vartheta), \vartheta]. \tag{2.1}$$

Definition 2.3. [28] A function $\Lambda : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $\psi : \mathbb{T} \to \mathbb{R}$, if $\Lambda^{\nabla}(\vartheta) = \psi(\vartheta)\ \forall \vartheta \in \mathbb{T}$. We then define the nabla integral of ψ by

$$\int_{\varrho}^{\vartheta} \psi(s)\nabla s = \Lambda(\vartheta) - \Lambda(\varrho),\ \forall \vartheta \in \mathbb{T}.$$

Lemma 2.2. *[28] If $\varrho, \delta \in \mathbb{T}$ and $\varpi, \Xi : \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

$$\int_{\varrho}^{\delta} \varpi(\vartheta)\Xi^{\nabla}(\vartheta)\nabla\vartheta = \varpi(\vartheta)\Xi(\vartheta)|_{\varrho}^{\delta} - \int_{\varrho}^{\delta} \varpi^{\nabla}(\vartheta)\Xi^{\rho}(\vartheta)\nabla\vartheta. \tag{2.2}$$

Lemma 2.3. *[27] If $\varpi \in C_{ld}(\mathbb{T}, \mathbb{R})$ and $\vartheta \in \mathbb{T}$, then*

$$\int_{\rho(\vartheta)}^{\vartheta} \varpi(\xi)\nabla\xi = \nu(\vartheta)\varpi(\vartheta),$$

where $\nu(\vartheta) = \vartheta - \rho(\vartheta)$.

Lemma 2.4. *(Reverse Hölder's inequality [30]) If $\varrho, \delta \in \mathbb{T}$, $\alpha < 0$, $1/\alpha + 1/\beta = 1$, and $\phi, \omega \in C_{ld}([\varrho, \delta]_{\mathbb{T}}, \mathbb{R}^+)$, then*

$$\int_{\varrho}^{\delta} \phi(\vartheta)\omega(\vartheta)\nabla\vartheta \geq \left[\int_{\varrho}^{\delta} \phi^{\alpha}(\vartheta)\nabla\vartheta\right]^{\frac{1}{\alpha}} \left[\int_{\varrho}^{\delta} \omega^{\beta}(\vartheta)\nabla\vartheta\right]^{\frac{1}{\beta}}. \tag{2.3}$$

3. Main results

In this document, we will make the assumption that the functions are ld-continuous on the interval $[\varrho, \infty)_{\mathbb{T}}$ and we also assume the existence of the integrals under consideration. Additionally, we posit the existence of a positive constant K such that

$$\frac{\rho(\xi) - \varrho}{\xi - \varrho} \geq \frac{1}{K}, \quad \rho(\xi) > \varrho. \tag{3.1}$$

Now, we can state and prove our results.

Theorem 3.1. *Let $\varrho \in \mathbb{T}$, $\epsilon < 0, \epsilon^* = \epsilon/(\epsilon - 1), r > 1$, and $\varpi, \Xi \in C_{ld}([\varrho, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ where $(\xi - \varrho)/\Xi(\xi)$ is nondecreasing. If (3.1) holds, then*

$$\int_{\varrho}^{\infty} \Xi^{-r}(\xi)F^{\epsilon}(\xi)\nabla\xi \leq C \int_{\varrho}^{\infty} (\rho(\xi) - \varrho)^{\epsilon} \varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi, \tag{3.2}$$

where $F(\xi) = \int_{\xi}^{\infty} \varpi(\vartheta)\nabla\vartheta$ and

$$C = \begin{cases} \left(\frac{\epsilon}{1-r}\right)^{\epsilon} K^{\frac{r-1}{\epsilon^*}}, & 1 - r \leq \epsilon; \\ \left(\frac{\epsilon}{1-r}\right)^{\epsilon} K^{r}, & 1 - r \geq \epsilon. \end{cases}$$

Proof. Note that

$$\begin{aligned} F(\xi) &= \int_{\xi}^{\infty} \varpi(\vartheta)\nabla\vartheta \\ &= \int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon\epsilon^*}} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta. \end{aligned} \tag{3.3}$$

Applying Lemma 2.4 to

$$\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon\epsilon^*}} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta,$$

with

$$\epsilon < 0,\ \epsilon^* = \epsilon/(\epsilon-1),\ \phi(\vartheta) = (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \text{ and } \omega(\vartheta) = (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta),$$

we get

$$\begin{aligned} &\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon\epsilon^*}} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta \\ &\geq \left(\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon}} \nabla\vartheta\right)^{\frac{1}{\epsilon^*}} \left(\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \end{aligned} \tag{3.4}$$

Applying (2.1) to $(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}$, we see that

$$\frac{\epsilon}{r-1}\left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} = (d-\varrho)^{\frac{r-1-\epsilon}{\epsilon}},\ d \in [\rho(\vartheta), \vartheta]. \tag{3.5}$$

Since $\epsilon < 0$, $r > 1$, and $d \geq \rho(\vartheta)$, we have that $(r-1-\epsilon)/\epsilon < 0$ and

$$(d-\varrho)^{\frac{r-1-\epsilon}{\epsilon}} \leq (\rho(\vartheta)-\varrho)^{\frac{r-1-\epsilon}{\epsilon}}. \tag{3.6}$$

Substituting (3.6) into (3.5), we see that

$$\frac{\epsilon}{r-1}\left[(\vartheta-\varrho)^{-\frac{1+\epsilon-r}{\epsilon}+1}\right]^{\nabla} \leq (\rho(\vartheta)-\varrho)^{\frac{r-1-\epsilon}{\epsilon}}. \tag{3.7}$$

From (3.7), (note $\epsilon < 0$ and $\epsilon^* = \epsilon/(\epsilon-1) > 0$), we observe that

$$\begin{aligned} &\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon}} \nabla\vartheta \\ \geq\ & \frac{\epsilon}{r-1}\int_{\xi}^{\infty} \left[(\vartheta-\varrho)^{-\frac{1+\epsilon-r}{\epsilon}+1}\right]^{\nabla} \nabla\vartheta \\ =\ & \frac{\epsilon}{r-1}\int_{\xi}^{\infty} \left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} \nabla\vartheta = \frac{\epsilon}{1-r}(\xi-\varrho)^{\frac{r-1}{\epsilon}} \end{aligned}$$

and then

$$\left(\int_{\xi}^{\infty} (\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon}} \nabla\vartheta\right)^{\frac{1}{\epsilon^*}} \geq \left(\frac{\epsilon}{1-r}\right)^{\frac{1}{\epsilon^*}} (\xi-\varrho)^{\frac{r-1}{\epsilon\epsilon^*}}. \tag{3.8}$$

Substituting (3.8) into (3.4), we see that

$$\begin{aligned}&\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{-\frac{1+\epsilon-r}{\epsilon\epsilon^*}}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\varpi(\vartheta)\nabla\vartheta\\ \geq\;&\left(\frac{\epsilon}{1-r}\right)^{\frac{1}{\epsilon^*}}(\xi-\varrho)^{\frac{r-1}{\epsilon\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \qquad (3.9)\end{aligned}$$

From (3.3) and (3.9), we have for $\epsilon<0$ that

$$F^{\epsilon}(\xi)\leq\left(\frac{\epsilon}{1-r}\right)^{\epsilon-1}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right). \qquad (3.10)$$

Multiplying (3.10) by $\Xi^{-r}(\xi)$ and then integrating over ξ from ϱ to ∞, we get

$$\begin{aligned}&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)F^{\epsilon}(\xi)\nabla\xi\\ &\leq\left(\frac{\epsilon}{1-r}\right)^{\epsilon-1}\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi. \qquad (3.11)\end{aligned}$$

Applying (2.2) to

$$\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi,$$

with

$$u_1(\xi)=\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta \text{ and } v_1^{\nabla}(\xi)=\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}},$$

we have that

$$\begin{aligned}&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\ &=v_1(\xi)\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\Bigg|_{\varrho}^{\infty}+\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\xi)v_1^{\rho}(\xi)\nabla\xi,\end{aligned}$$

where

$$v_1(\xi)=\int_{\varrho}^{\xi}\Xi^{-r}(\vartheta)(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}}\nabla\vartheta.$$

Since $v_1(\varrho)=0$, we observe that

$$\begin{aligned}&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\ &=\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\xi)\left[\int_{\varrho}^{\rho(\xi)}\Xi^{-r}(\vartheta)(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}}\nabla\vartheta\right]\nabla\xi\\ &=\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\xi)\left[\int_{\varrho}^{\rho(\xi)}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right]\nabla\xi. \qquad (3.12)\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\varrho}^{\rho(\xi)}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta \\
&=\int_{\varrho}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta-\int_{\rho(\xi)}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta \\
&=\int_{\varrho}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta-\nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}. \qquad (3.13)
\end{aligned}$$

Since $(\vartheta-\varrho)/\Xi(\vartheta)$ is nondecreasing and $r>1$, we have for $\vartheta\le\xi$ that

$$\int_{\varrho}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta\le\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta. \qquad (3.14)$$

Substituting (3.14) into (3.13), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\rho(\xi)}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta \\
\le\;&\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta-\nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}-r} \\
=\;&\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left[\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta-\nu(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\right] \\
=\;&\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta-\int_{\rho(\xi)}^{\xi}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta\right) \\
=\;&\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left(\int_{\varrho}^{\rho(\xi)}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta\right). \qquad (3.15)
\end{aligned}$$

Substituting (3.15) into (3.12), we have

$$\begin{aligned}
&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
&\le\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}(\xi-\varrho)^{r}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\left(\int_{\varrho}^{\rho(\xi)}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}\nabla\vartheta\right)\nabla\xi. \qquad (3.16)
\end{aligned}$$

To complete the proof, we have two cases:
Case 1: For $1-r\le\epsilon$.

Applying (2.1) to the term $(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r+1}$, we see that

$$\frac{\epsilon}{1-r}\left((\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}-r+1}\right)^{\nabla}=\frac{\epsilon}{1-r}\left((\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right)^{\nabla}=(d-\varrho)^{\frac{1-r}{\epsilon}-1}, \qquad (3.17)$$

where $d\in[\rho(\vartheta),\vartheta]$. Since $\epsilon<0$ and $1-r\le\epsilon$ (note $(1-r)/\epsilon-1\ge0$), we have for $d\ge\rho(\vartheta)$ that

$$(d-\varrho)^{\frac{1-r}{\epsilon}-1}\ge(\rho(\vartheta)-\varrho)^{\frac{1-r}{\epsilon}-1}=(\rho(\vartheta)-\varrho)^{\frac{r-1}{\epsilon^{*}}-r}. \qquad (3.18)$$

From (3.17) and (3.18), we observe that

$$\frac{\epsilon}{1-r}\left((\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r+1}\right)^{\nabla}\geq(\rho(\vartheta)-\varrho)^{\frac{r-1}{\epsilon^*}-r}. \tag{3.19}$$

Integrating (3.19) over ϑ from ϱ to $\rho(\xi)$, since $r>1$ and $\epsilon<0$, we have that

$$\begin{aligned}
&\int_{\varrho}^{\rho(\xi)}(\rho(\vartheta)-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\\
\leq\ &\frac{\epsilon}{1-r}\int_{\varrho}^{\rho(\xi)}\left((\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r+1}\right)^{\nabla}\nabla\vartheta=\frac{\epsilon}{1-r}\int_{\varrho}^{\rho(\xi)}\left((\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right)^{\nabla}\nabla\vartheta\\
=\ &\frac{\epsilon}{1-r}(\rho(\xi)-\varrho)^{\frac{1-r}{\epsilon}}.
\end{aligned} \tag{3.20}$$

Using (3.1), since $\frac{r-1}{\epsilon^*}-r=\frac{1-r}{\epsilon}-1>0$, then (3.16) becomes

$$\begin{aligned}
&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
&\leq K^{\frac{r-1}{\epsilon^*}-r}\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}(\xi-\varrho)^{r}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\left(\int_{\varrho}^{\rho(\xi)}(\rho(\vartheta)-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)\nabla\xi.
\end{aligned} \tag{3.21}$$

Substituting (3.20) into (3.21) and using (3.1), we see that

$$\begin{aligned}
&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
\leq\ &K^{\frac{r-1}{\epsilon^*}-r}\left(\frac{\epsilon}{1-r}\right)\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\left(\frac{\xi-\varrho}{\rho(\xi)-\varrho}\right)^{r}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi\\
\leq\ &K^{\frac{r-1}{\epsilon^*}}\left(\frac{\epsilon}{1-r}\right)\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi.
\end{aligned} \tag{3.22}$$

From (3.11) and (3.22), we see that

$$\begin{aligned}
&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)F^{\epsilon}(\xi)\nabla\xi\\
\leq\ &\left(\frac{\epsilon}{1-r}\right)^{\epsilon}K^{\frac{r-1}{\epsilon^*}}\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi,
\end{aligned}$$

which is (3.2) with $C=(\epsilon/(1-r))^{\epsilon}K^{\frac{r-1}{\epsilon^*}}$.

Case 2: For $1-r\geq\epsilon$.

Applying (2.1) to $(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}$, we see that

$$\frac{\epsilon}{1-r}\left((\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right)^{\nabla}=(d-\varrho)^{\frac{1-r}{\epsilon}-1},\ d\in[\rho(\vartheta),\vartheta]. \tag{3.23}$$

Since $\epsilon<0$, and $1-r\geq\epsilon$ (note that $((1-r)/\epsilon)-1\leq 0$), we have for $d\leq\vartheta$ that

$$(d-\varrho)^{\frac{1-r}{\epsilon}-1}\geq(\vartheta-\varrho)^{\frac{1-r}{\epsilon}-1}=(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}. \tag{3.24}$$

From (3.23) and (3.24), we observe that

$$\frac{\epsilon}{1-r}\left((\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right)^{\nabla} \geq (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}. \tag{3.25}$$

Integrating (3.25) over ϑ from ϱ to $\rho(\xi)$, we have that

$$\begin{aligned}
&\int_{\varrho}^{\rho(\xi)} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r} \nabla\vartheta \\
\leq\ & \frac{\epsilon}{1-r}\int_{\varrho}^{\rho(\xi)} \left((\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right)^{\nabla} \nabla\vartheta = \frac{\epsilon}{1-r}(\rho(\xi)-\varrho)^{\frac{1-r}{\epsilon}}.
\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.16), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\infty} \Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
\leq\ & \left(\frac{\epsilon}{1-r}\right)\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\left(\frac{\xi-\varrho}{\rho(\xi)-\varrho}\right)^{r}\nabla\xi.
\end{aligned} \tag{3.27}$$

Using (3.1), (3.27) then becomes

$$\begin{aligned}
&\int_{\varrho}^{\infty} \Xi^{-r}(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\left(\int_{\xi}^{\infty}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
\leq\ & \left(\frac{\epsilon}{1-r}\right)K^r\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi.
\end{aligned} \tag{3.28}$$

From (3.11) and (3.28), we see that

$$\int_{\varrho}^{\infty}\Xi^{-r}(\xi)F^{\epsilon}(\xi)\nabla\xi \leq \left(\frac{\epsilon}{1-r}\right)^{\epsilon}K^r\int_{\varrho}^{\infty}(\rho(\xi)-\varrho)^{\epsilon}\varpi^{\epsilon}(\xi)\Xi^{-r}(\xi)\nabla\xi,$$

which is (3.2) with $C = (\epsilon/(1-r))^{\epsilon}K^r$. □

Remark 3.1. In Theorem 3.1, if $\mathbb{T} = \mathbb{R}$, and $\varrho = 0$, then $\rho(\xi) = \xi$, and we observe that (3.1) holds with $K = 1$. Then, we get (1.5), and for $\Xi(\xi) = \xi$, we get (1.3).

Corollary 3.1. *If $\mathbb{T} = \mathbb{N}_0$, $\varrho = 0$, and ϖ, Ξ are positive sequences such that $\tau/\Xi(\tau)$ is nondecreasing, then*

$$\sum_{\tau=1}^{\infty}\Xi^{-r}(\tau)\left(\sum_{k=\tau+1}^{\infty}\varpi(k)\right)^{\epsilon} \leq C\sum_{\tau=2}^{\infty}(\tau-1)^{\epsilon}\,\Xi^{-r}(\tau)\varpi^{\epsilon}(\tau),$$

where

$$C = \begin{cases} 2^{\frac{r-1}{\epsilon^*}}\left(\frac{\epsilon}{1-r}\right)^{\epsilon}, & 1-r \leq \epsilon; \\ 2^{r}\left(\frac{\epsilon}{1-r}\right)^{\epsilon}, & 1-r \geq \epsilon. \end{cases}$$

Here, we used that for $\rho(\tau) > \varrho$, we have for $\tau \geq 2$, and

$$\frac{\rho(\tau)-\varrho}{\tau-\varrho} = \frac{\tau-1}{\tau} = 1-\frac{1}{\tau} \geq \frac{1}{2}$$

and thus inequality (3.1) holds with $K = 2$.

Theorem 3.2. *Let* $\varrho \in \mathbb{T}$, $\epsilon < 0$, $0 \leq r < 1$, *and* ϖ, $\Xi \in C_{ld}\left([\varrho, \infty)_{\mathbb{T}}, \mathbb{R}^{+}\right)$ *where* $(\vartheta - \varrho)/\Xi(\vartheta)$ *is nonincreasing. If (3.1) holds, then*

$$\int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \leq D\int_{\varrho}^{\infty} (\rho(\xi) - \varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi, \tag{3.29}$$

where $\Omega(\xi) = \int_{\varrho}^{\xi} \varpi(\vartheta)\nabla\vartheta$ *and*

$$D = \begin{cases} \left(\frac{\epsilon}{r-1}\right)^{\epsilon} K^{\frac{r-1}{\epsilon}}, & (r-1)/\epsilon \geq 1; \\ K^{r}\left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \leq 1. \end{cases}$$

Proof. To establish this theorem, we have two cases:
Case 1: For $(r-1)/\epsilon \geq 1$.

Note that

$$\begin{aligned} \Omega(\xi) &= \int_{\varrho}^{\xi} \varpi(\vartheta)\nabla\vartheta \\ &= \int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^{*}}} (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^{*}}} \varpi(\vartheta)\nabla\vartheta, \end{aligned} \tag{3.30}$$

where $\epsilon^{*} = \epsilon/(\epsilon - 1)$. Applying Lemma 2.4 to $\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^{*}}} (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^{*}}} \varpi(\vartheta)\nabla\vartheta$, with

$$\epsilon < 0,\ \epsilon^{*} = \epsilon/(\epsilon - 1),\ \phi(\vartheta) = (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^{*}}} \text{ and } \omega(\vartheta) = (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^{*}}} \varpi(\vartheta),$$

we get

$$\begin{aligned} &\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^{*}}} (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^{*}}} \varpi(\vartheta)\nabla\vartheta \\ \geq\ &\left(\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta\right)^{\frac{1}{\epsilon^{*}}} \left(\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \end{aligned} \tag{3.31}$$

From (3.30), and (3.31), we see that

$$\Omega(\xi) \geq \left(\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta\right)^{\frac{1}{\epsilon^{*}}} \left(\int_{\varrho}^{\xi} (\vartheta - \varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \tag{3.32}$$

Applying (2.1) to $(\vartheta - \varrho)^{\frac{r-1}{\epsilon}}$, we observe that

$$\frac{\epsilon}{r-1}\left[(\vartheta - \varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} = (d - \varrho)^{\frac{r-\epsilon-1}{\epsilon}}, \tag{3.33}$$

where $d \in [\rho(\vartheta), \vartheta]$. Since $d \leq \vartheta$, and $(r-1)/\epsilon \geq 1$, we have from (3.33) that

$$\frac{\epsilon}{r-1}\left[(\vartheta - \varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} \leq (\vartheta - \varrho)^{\frac{r-\epsilon-1}{\epsilon}}. \tag{3.34}$$

By integrating (3.34) over ϑ from ϱ to ξ, we get

$$\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \geq \frac{\epsilon}{r-1} \int_{\varrho}^{\xi} \left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} \nabla\vartheta = \frac{\epsilon}{r-1} (\xi-\varrho)^{\frac{r-1}{\epsilon}} . \tag{3.35}$$

Substituting (3.35) into (3.32), since $\epsilon^* > 0$, we observe that

$$\Omega(\xi) \geq \left(\frac{\epsilon}{r-1}\right)^{\frac{1}{\epsilon^*}} (\xi-\varrho)^{\frac{r-1}{\epsilon\epsilon^*}} \left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}$$

and then we have for $\epsilon < 0$, that

$$\Omega^{\epsilon}(\xi) \leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta. \tag{3.36}$$

Multiplying (3.36) with $\Xi^{-r}(\xi)$ and then integrating over ξ from ϱ to ∞, we see that

$$\begin{aligned} &\int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \\ &\leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1} \int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi. \end{aligned} \tag{3.37}$$

Applying (2.2) to

$$\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi,$$

with

$$u_3(\xi) = \int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \text{ and } v_3^{\nabla}(\xi) = (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi),$$

we get

$$\begin{aligned} &\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\ &= v_3(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\Bigg|_{\varrho}^{\infty} \\ &- \int_{\varrho}^{\infty} v_3^{\rho}(\xi) (\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi, \end{aligned} \tag{3.38}$$

where

$$v_3(\xi) = -\int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\vartheta)\nabla\vartheta.$$

Since $\lim_{\xi\to\infty} v_3(\xi) = 0$, we have from (3.38) that

$$\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi$$

$$
\begin{aligned}
&= \int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
&= \int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\left(\int_{\rho(\xi)}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)\nabla\xi. \qquad (3.39)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{\rho(\xi)}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
&= \int_{\rho(\xi)}^{\xi} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta + \int_{\xi}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
&= \nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}-r} + \int_{\xi}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta. \qquad (3.40)
\end{aligned}
$$

Since $(\vartheta-\varrho)/\Xi(\vartheta)$ is nonincreasing, $0 \le r < 1$, and $\vartheta \ge \xi$, we observe that

$$
\int_{\xi}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \le \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} \int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta
$$

and then we have from (3.40) that

$$
\begin{aligned}
&\int_{\rho(\xi)}^{\infty} \left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
&\le \nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}-r} + \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} \int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
&= \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} \left[\int_{\rho(\xi)}^{\xi} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta + \int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right] \\
&= \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} \int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta. \qquad (3.41)
\end{aligned}
$$

Substituting (3.41) into (3.39), we observe that

$$
\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
&\le \int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)\nabla\xi. \qquad (3.42)
\end{aligned}
$$

Using (3.23), since $(1-r)/\epsilon < 0$ and $d \le \vartheta$, we have that

$$
\frac{\epsilon}{1-r}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla} \ge (\vartheta-\varrho)^{\frac{1-r}{\epsilon}-1} = (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}
$$

and then

$$
\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta
$$

$$\leq \frac{\epsilon}{1-r}\int_{\rho(\xi)}^{\infty}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla}\nabla\vartheta=\frac{\epsilon}{r-1}\left(\rho\left(\xi\right)-\varrho\right)^{\frac{1-r}{\epsilon}}. \tag{3.43}$$

Substituting (3.43) into (3.42), and then using (3.1), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\,\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
\leq\;& \frac{\epsilon}{r-1}\int_{\varrho}^{\infty}\left(\frac{\xi-\varrho}{\rho\left(\xi\right)-\varrho}\right)^{\frac{r-1}{\epsilon}}(\xi-\varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi\\
\leq\;& \frac{\epsilon}{r-1}K^{\frac{r-1}{\epsilon}}\int_{\varrho}^{\infty}(\xi-\varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi.
\end{aligned} \tag{3.44}$$

Substituting (3.44) into (3.37), we see that

$$\begin{aligned}
&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi\\
\leq\;& \left(\frac{\epsilon}{r-1}\right)^{\epsilon}K^{\frac{r-1}{\epsilon}}\int_{\varrho}^{\infty}(\xi-\varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi\\
\leq\;& \left(\frac{\epsilon}{r-1}\right)^{\epsilon}K^{\frac{r-1}{\epsilon}}\int_{\varrho}^{\infty}\left(\rho\left(\xi\right)-\varrho\right)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi,
\end{aligned}$$

which is (3.29) with $D=(\epsilon/\left(r-1\right))^{\epsilon}K^{\frac{r-1}{\epsilon}}$.
Case 2: For $\left(r-1\right)/\epsilon\leq 1$.

Note that

$$\begin{aligned}
\Omega(\xi) &= \int_{\varrho}^{\xi}\varpi(\vartheta)\nabla\vartheta\\
&= \int_{\varrho}^{\xi}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\varpi(\vartheta)\nabla\vartheta.
\end{aligned} \tag{3.45}$$

Applying Lemma 2.4 to the term

$$\int_{\varrho}^{\xi}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\varpi(\vartheta)\nabla\vartheta,$$

with

$$\epsilon<0,\ \epsilon^*=\epsilon/\left(\epsilon-1\right),\ \phi(\vartheta)=\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\ \text{and}\ \omega(\vartheta)=\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\varpi(\vartheta),$$

we get

$$\begin{aligned}
&\int_{\varrho}^{\xi}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\varpi(\vartheta)\nabla\vartheta\\
\geq\;& \left(\int_{\varrho}^{\xi}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{r-\epsilon-1}{\epsilon}}\nabla\vartheta\right)^{\frac{1}{\epsilon^*}}\left(\int_{\varrho}^{\xi}\left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}.
\end{aligned} \tag{3.46}$$

From (3.45), and (3.46), we see that

$$\Omega(\xi) \geq \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \right)^{\frac{1}{\epsilon^*}} \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right)^{\frac{1}{\epsilon}}. \tag{3.47}$$

Using (3.5), since $d \geq \rho(\vartheta)$, and $0 < (r-1)/\epsilon \leq 1$, we have that

$$\frac{\epsilon}{r-1} \left[(\vartheta - \varrho)^{\frac{r-1}{\epsilon}} \right]^{\nabla} \leq (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon}}$$

and then

$$\begin{aligned} & \int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \\ \geq \quad & \frac{\epsilon}{r-1} \int_{\varrho}^{\xi} \left[(\vartheta - \varrho)^{\frac{r-1}{\epsilon}} \right]^{\nabla} \nabla\vartheta = \frac{\epsilon}{r-1} (\xi - \varrho)^{\frac{r-1}{\epsilon}}. \end{aligned} \tag{3.48}$$

Substituting (3.48) into (3.47), we see that

$$\Omega(\xi) \geq \left(\frac{\epsilon}{r-1} \right)^{\frac{1}{\epsilon^*}} (\xi - \varrho)^{\frac{r-1}{\epsilon\epsilon^*}} \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right)^{\frac{1}{\epsilon}}$$

and then (note $\epsilon < 0$)

$$\Omega^{\epsilon}(\xi) \leq \left(\frac{\epsilon}{r-1} \right)^{\epsilon-1} (\xi - \varrho)^{\frac{r-1}{\epsilon^*}} \int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta. \tag{3.49}$$

Multiplying (3.49) with $\Xi^{-r}(\xi)$ and then integrating over ξ from ϱ to ∞, we observe that

$$\begin{aligned} & \int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \\ \leq & \left(\frac{\epsilon}{r-1} \right)^{\epsilon-1} \int_{\varrho}^{\infty} (\xi - \varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right) \nabla\xi. \end{aligned} \tag{3.50}$$

Applying (2.2) to

$$\int_{\varrho}^{\infty} (\xi - \varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right) \nabla\xi$$

with

$$u_4(\xi) = \int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \text{ and } v_4^{\nabla}(\xi) = (\xi - \varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi),$$

we see that

$$\begin{aligned} & \int_{\varrho}^{\infty} (\xi - \varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right) \nabla\xi \\ = \; & v_4(\xi) \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right) \Bigg|_{\varrho}^{\infty} \end{aligned} \tag{3.51}$$

$$-\int_{\varrho}^{\infty} v_4^{\varrho}(\xi)\left(\rho(\xi)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi,$$

where

$$v_4(\xi) = -\int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\vartheta)\nabla\vartheta.$$

Since $\lim_{\xi\to\infty} v_4(\xi) = 0$, we have from (3.51) that

$$\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
= &\int_{\varrho}^{\infty} \left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\vartheta)\nabla\vartheta\right)(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi \\
= &\int_{\varrho}^{\infty} (\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\left(\int_{\rho(\xi)}^{\infty} \left[\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right]^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\,\nabla\vartheta\right)\nabla\xi. \qquad (3.52)
\end{aligned}$$

Using (3.41) and (3.52), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
\leq &\int_{\varrho}^{\infty} (\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\,\nabla\vartheta\right)\nabla\xi. \qquad (3.53)
\end{aligned}$$

Using (3.23), since $(1-r)/\epsilon < 0$ and $d \leq \vartheta$, we have that

$$\frac{\epsilon}{1-r}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla} \geq (\vartheta-\varrho)^{\frac{1-r}{\epsilon}-1} = (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}$$

and then

$$\begin{aligned}
&\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\,\nabla\vartheta \\
\leq &\frac{\epsilon}{1-r}\int_{\rho(\xi)}^{\infty} \left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla}\nabla\vartheta = \frac{\epsilon}{r-1}\left(\rho(\xi)-\varrho\right)^{\frac{1-r}{\epsilon}}. \qquad (3.54)
\end{aligned}$$

Substituting (3.54) into (3.53), and then using (3.1), we obtain

$$\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\,\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
\leq &\frac{\epsilon}{r-1}\int_{\varrho}^{\infty}\left(\frac{\xi-\varrho}{\rho(\xi)-\varrho}\right)^{r}\Xi^{-r}(\xi)\left(\rho(\xi)-\varrho\right)^{\epsilon}\varpi^{\epsilon}(\xi)\nabla\xi \\
\leq &\frac{\epsilon}{r-1}K^{r}\int_{\varrho}^{\infty}\Xi^{-r}(\xi)\left(\rho(\xi)-\varrho\right)^{\epsilon}\varpi^{\epsilon}(\xi)\nabla\xi. \qquad (3.55)
\end{aligned}$$

Substituting (3.55) into (3.50), we observe that

$$\int_{\varrho}^{\infty}\Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon}K^{r}\int_{\varrho}^{\infty}\Xi^{-r}(\xi)\left(\rho(\xi)-\varrho\right)^{\epsilon}\varpi^{\epsilon}(\xi)\nabla\xi,$$

which is (3.29) with $D = (\epsilon/(r-1))^{\epsilon}K^{r}$. □

Remark 3.2. In Theorem 3.2, if $\mathbb{T} = \mathbb{R}$ and $\varrho = 0$, then $\rho(\xi) = \xi$ and we see that (3.1) holds with $K = 1$. Then, we get (1.6), and for $\Xi(\xi) = \xi$, we get (1.4).

Corollary 3.2. *If $\mathbb{T} = \mathbb{N}_0$, $\varrho = 0$, and ϖ, Ξ are positive sequences such that $\tau/\Xi(\tau)$ is nonincreasing, then*

$$\sum_{\tau=1}^{\infty} \Xi^{-r}(\tau) \left[\sum_{k=1}^{\tau} \varpi(k) \right]^{\epsilon} \leq D^* \sum_{\tau=1}^{\infty} (\tau - 1)^{\epsilon} \Xi^{-r}(\tau) \varpi^{\epsilon}(\tau), \tag{3.56}$$

where

$$D^* = \begin{cases} 2^{\frac{r-1}{\epsilon}} \left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \geq 1; \\ 2^{r} \left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \leq 1. \end{cases}$$

Here, inequality (3.1) holds with $K = 2$.

Theorem 3.3. *Assume that $\varrho \in \mathbb{T}$, $\epsilon < 0$, $r < 0$, and $\varpi, \Xi \in C_{ld}([\varrho, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that the function $(\vartheta - \varrho)/\Xi(\vartheta)$ is nondecreasing. Then,*

$$\int_{\varrho}^{\infty} \Xi^{-r}(\xi) \Omega^{\epsilon}(\xi) \nabla\xi \leq E \int_{\varrho}^{\infty} (\rho(\xi) - \varrho)^{\epsilon} \Xi^{-r}(\xi) \varpi^{\epsilon}(\xi) \nabla\xi, \tag{3.57}$$

where $\Omega(\xi) = \int_{\varrho}^{\xi} \varpi(\vartheta)\nabla\vartheta$ and

$$E = \begin{cases} \left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \leq 1; \\ \left(\frac{\epsilon}{r-1}\right)^{\epsilon} K^{\frac{r-1}{\epsilon}}, & (r-1)/\epsilon \geq 1. \end{cases}$$

Proof. To prove this theorem, we have two cases:
Case 1: For $(r-1)/\epsilon \leq 1$.

Note that

$$\Omega(\xi) = \int_{\varrho}^{\xi} \varpi(\vartheta)\nabla\vartheta = \int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta, \tag{3.58}$$

where $\epsilon^* = \epsilon/(\epsilon - 1)$. Applying Lemma 2.4 to $\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta$ with

$$\epsilon < 0, \ \epsilon^* = \epsilon/(\epsilon - 1), \ \phi(\vartheta) = (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}, \text{ and } \omega(\vartheta) = (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta),$$

we get

$$\begin{aligned} &\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}} \varpi(\vartheta)\nabla\vartheta \\ \geq \ &\left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \right)^{\frac{1}{\epsilon^*}} \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right)^{\frac{1}{\epsilon}}. \end{aligned} \tag{3.59}$$

From (3.58) and (3.59), we see that

$$\Omega(\xi) \geq \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \right)^{\frac{1}{\epsilon^*}} \left(\int_{\varrho}^{\xi} (\rho(\vartheta) - \varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \right)^{\frac{1}{\epsilon}}. \tag{3.60}$$

Using (3.5), since $d \geq \rho(\vartheta)$, and $0 < (r-1)/\epsilon \leq 1$, we have that

$$\frac{\epsilon}{r-1}\left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} \leq (\rho(\vartheta)-\varrho)^{\frac{r-\epsilon-1}{\epsilon}}$$

and then

$$\begin{aligned}&\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{r-\epsilon-1}{\epsilon}}\nabla\vartheta\\ \geq\ & \frac{\epsilon}{r-1}\int_{\varrho}^{\xi}\left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla}\nabla\vartheta=\frac{\epsilon}{r-1}(\xi-\varrho)^{\frac{r-1}{\epsilon}}.\end{aligned} \tag{3.61}$$

Substituting (3.61) into (3.60), since $\epsilon^{*} > 0$, we see that

$$\Omega(\xi) \geq \left(\frac{\epsilon}{r-1}\right)^{\frac{1}{\epsilon^{*}}}(\xi-\varrho)^{\frac{r-1}{\epsilon\epsilon^{*}}}\left(\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}$$

and then (note $\epsilon < 0$)

$$\Omega^{\epsilon}(\xi) \leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta. \tag{3.62}$$

Multiplying (3.62) with $\Xi^{-r}(\xi)$ and then integrating over ξ from ϱ to ∞, we observe that

$$\begin{aligned}&\int_{\varrho}^{\infty}\Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi\\ \leq\ & \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1}\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi.\end{aligned} \tag{3.63}$$

Applying (2.2) to $\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi$, with

$$u_5(\xi)=\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta \text{ and } v_5^{\nabla}(\xi)=(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\Xi^{-r}(\xi),$$

we see that

$$\begin{aligned}&\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^{*}}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\ =\ & v_5(\xi)\left(\int_{\varrho}^{\xi}(\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\Bigg|_{\varrho}^{\infty}\\ &-\int_{\varrho}^{\infty}v_5^{\rho}(\xi)(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^{*}}}\varpi^{\epsilon}(\xi)\nabla\xi,\end{aligned} \tag{3.64}$$

where

$$v_5(\xi)=-\int_{\xi}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^{*}}}\Xi^{-r}(\vartheta)\nabla\vartheta.$$

Since $\lim_{\xi\to\infty} v_5(\xi) = 0$, we have from (3.64) that

$$\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
=\ & \int_{\varrho}^{\infty}\left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\vartheta)\nabla\vartheta\right)(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi \\
=\ & \int_{\varrho}^{\infty}\left(\int_{\rho(\xi)}^{\infty}\left[\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right]^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi. \qquad (3.65)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\rho(\xi)}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
=\ & \int_{\rho(\xi)}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta + \int_{\xi}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
=\ & \nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}-r} + \int_{\xi}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta. \qquad (3.66)
\end{aligned}$$

Since $(\vartheta-\varrho)/\Xi(\vartheta)$ is nondecreasing, $r<0$, and $\vartheta\geq\xi$, we see that

$$\int_{\xi}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \leq \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta.$$

Substituting the last inequality into (3.66), we get

$$\begin{aligned}
&\int_{\rho(\xi)}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta \\
\leq\ & \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left[\nu(\xi)(\xi-\varrho)^{\frac{r-1}{\epsilon^*}-r} + \int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right] \\
=\ & \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\left[\int_{\rho(\xi)}^{\xi} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta + \int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right] \\
=\ & \left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta. \qquad (3.67)
\end{aligned}$$

Substituting (3.67) into (3.65), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\rho(\vartheta)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\
\leq\ & \int_{\varrho}^{\infty}\left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)(\xi-\varrho)^{r}(\rho(\xi)-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi. \qquad (3.68)
\end{aligned}$$

Using (3.23), since $(1-r)/\epsilon<0$, and $d\leq\vartheta$, we have that

$$\frac{\epsilon}{1-r}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla} \geq (\vartheta-\varrho)^{\frac{1-r}{\epsilon}-1} = (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}$$

and then

$$\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r} \nabla\vartheta \le \frac{\epsilon}{1-r}\int_{\rho(\xi)}^{\infty} \left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla} \nabla\vartheta = \frac{\epsilon}{r-1}\left(\rho\left(\xi\right)-\varrho\right)^{\frac{1-r}{\epsilon}}. \tag{3.69}$$

Substituting (3.69) into (3.68), we obtain

$$\begin{aligned} &\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\, \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} \left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\ \le\ & \frac{\epsilon}{r-1}\int_{\varrho}^{\infty} \left(\frac{\xi-\varrho}{\rho\left(\xi\right)-\varrho}\right)^{r} (\rho(\xi)-\varrho)^{\epsilon}\, \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi. \end{aligned} \tag{3.70}$$

Since $r<0$, and $\xi \ge \rho\left(\xi\right)$, inequality (3.70) becomes

$$\begin{aligned} &\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\, \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} \left(\rho\left(\vartheta\right)-\varrho\right)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\ \le\ & \frac{\epsilon}{r-1}\int_{\varrho}^{\infty} (\rho(\xi)-\varrho)^{\epsilon}\, \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi. \end{aligned} \tag{3.71}$$

Substituting (3.71) into (3.63), we observe that

$$\int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \le \left(\frac{\epsilon}{r-1}\right)^{\epsilon}\int_{\varrho}^{\infty} (\rho(\xi)-\varrho)^{\epsilon}\, \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi,$$

which is (3.57) with $E=\left(\epsilon/\left(r-1\right)\right)^{\epsilon}$.

Case 2: For $\left(r-1\right)/\epsilon \ge 1$.

Note that

$$\Omega(\xi)=\int_{\varrho}^{\xi}\varpi(\vartheta)\nabla\vartheta=\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\,(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\,\varpi(\vartheta)\nabla\vartheta. \tag{3.72}$$

Applying Lemma 2.4 to $\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\,(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\,\varpi(\vartheta)\nabla\vartheta$, with

$$\epsilon<0,\ \epsilon^*=\epsilon/\left(\epsilon-1\right),\ \phi(\vartheta)=(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}} \text{ and } \omega(\vartheta)=(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\,\varpi(\vartheta),$$

we get

$$\begin{aligned} &\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon\epsilon^*}}\,(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon\epsilon^*}}\,\varpi(\vartheta)\nabla\vartheta \\ \ge\ & \left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon}}\nabla\vartheta\right)^{\frac{1}{\epsilon^*}}\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\,\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \end{aligned} \tag{3.73}$$

From (3.72) and (3.73), we see that

$$\Omega(\xi)\ge\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon}}\nabla\vartheta\right)^{\frac{1}{\epsilon^*}}\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\,\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}. \tag{3.74}$$

Using (3.5), since $d\le\vartheta$ and $\left(r-1\right)/\epsilon\ge 1$, we have that

$$\frac{\epsilon}{r-1}\left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla}\le(\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon}}. \tag{3.75}$$

By integrating (3.75) over ϑ from ϱ to ξ, we get

$$\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{r-\epsilon-1}{\epsilon}} \nabla\vartheta \geq \frac{\epsilon}{r-1} \int_{\varrho}^{\xi} \left[(\vartheta-\varrho)^{\frac{r-1}{\epsilon}}\right]^{\nabla} \nabla\vartheta = \frac{\epsilon}{r-1} (\xi-\varrho)^{\frac{r-1}{\epsilon}}. \tag{3.76}$$

Substituting (3.76) into (3.74), we observe that

$$\Omega(\xi) \geq \left(\frac{\epsilon}{r-1}\right)^{\frac{1}{\epsilon^*}} (\xi-\varrho)^{\frac{r-1}{\epsilon\epsilon^*}} \left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)^{\frac{1}{\epsilon}}$$

and then we have for $\epsilon < 0$ that

$$\Omega^{\epsilon}(\xi) \leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta. \tag{3.77}$$

Multiplying (3.77) with $\Xi^{-r}(\xi)$ and then integrating over ξ from ϱ to ∞, we see that

$$\begin{aligned} &\int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \\ &\leq \left(\frac{\epsilon}{r-1}\right)^{\epsilon-1} \int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi. \end{aligned} \tag{3.78}$$

Applying (2.2) to $\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi$, with

$$u_6(\xi) = \int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta \text{ and } v_6^{\nabla}(\xi) = (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi),$$

we get

$$\begin{aligned} &\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\ = \;& v_6(\xi)\left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\Bigg|_{\varrho}^{\infty} \\ &-\int_{\varrho}^{\infty} v_6^{\rho}(\xi)\,(\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi, \end{aligned} \tag{3.79}$$

where

$$v_6(\xi) = -\int_{\xi}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\vartheta)\nabla\vartheta.$$

Since $\lim_{\xi\to\infty} v_6(\xi) = 0$, we have from (3.79) that

$$\begin{aligned} &\int_{\varrho}^{\infty} (\xi-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\xi) \left(\int_{\varrho}^{\xi} (\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi \\ = \;& \int_{\varrho}^{\infty} \left(\int_{\rho(\xi)}^{\infty} (\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}} \Xi^{-r}(\vartheta)\nabla\vartheta\right)(\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}} \varpi^{\epsilon}(\xi)\nabla\xi \end{aligned}$$

$$= \int_{\varrho}^{\infty}\left(\int_{\rho(\xi)}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}\nabla\vartheta\right)(\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\xi)\nabla\xi. \tag{3.80}$$

Since $(\vartheta-\varrho)/\Xi(\vartheta)$ is nondecreasing and $r<0$, we have for $\vartheta\geq\xi$ that

$$\int_{\xi}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\leq\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\xi}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta$$

and then

$$\begin{aligned}
&\int_{\rho(\xi)}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\\
=&\int_{\rho(\xi)}^{\xi}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta+\int_{\xi}^{\infty}\left(\frac{\vartheta-\varrho}{\Xi(\vartheta)}\right)^{r}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\\
\leq&\ \nu(\xi)\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}-r}+\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\xi}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\\
=&\left(\frac{\xi-\varrho}{\Xi(\xi)}\right)^{r}\int_{\rho(\xi)}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta.
\end{aligned} \tag{3.81}$$

Substituting (3.81) into (3.80), we see that

$$\begin{aligned}
&\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
\leq&\int_{\varrho}^{\infty}\left(\int_{\rho(\xi)}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\right)(\xi-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}+r}\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi.
\end{aligned} \tag{3.82}$$

Using (3.23), since $(1-r)/\epsilon<0$, and $d\leq\vartheta$, we have that

$$\frac{\epsilon}{1-r}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla}\geq(\vartheta-\varrho)^{\frac{1-r}{\epsilon}-1}=(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}$$

and then

$$\int_{\rho(\xi)}^{\infty}(\vartheta-\varrho)^{\frac{r-1}{\epsilon^*}-r}\nabla\vartheta\leq\frac{\epsilon}{1-r}\int_{\rho(\xi)}^{\infty}\left[(\vartheta-\varrho)^{\frac{1-r}{\epsilon}}\right]^{\nabla}\nabla\vartheta=\frac{\epsilon}{r-1}\left(\rho(\xi)-\varrho\right)^{\frac{1-r}{\epsilon}}. \tag{3.83}$$

Substituting (3.83) into (3.82), we observe that

$$\begin{aligned}
&\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
\leq&\ \frac{\epsilon}{r-1}\int_{\varrho}^{\infty}\left(\frac{\xi-\varrho}{\rho(\xi)-\varrho}\right)^{\frac{r-1}{\epsilon}}(\xi-\varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi
\end{aligned}$$

and then we have from (3.1) that

$$\begin{aligned}
&\int_{\varrho}^{\infty}(\xi-\varrho)^{\frac{r-1}{\epsilon^*}}\Xi^{-r}(\xi)\left(\int_{\varrho}^{\xi}(\vartheta-\varrho)^{\frac{1+\epsilon-r}{\epsilon^*}}\varpi^{\epsilon}(\vartheta)\nabla\vartheta\right)\nabla\xi\\
\leq&\ \frac{\epsilon}{r-1}K^{\frac{r-1}{\epsilon}}\int_{\varrho}^{\infty}(\xi-\varrho)^{\epsilon}\,\Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi.
\end{aligned} \tag{3.84}$$

Substituting (3.84) into (3.78), we see that

$$\begin{aligned}
&\int_{\varrho}^{\infty} \Xi^{-r}(\xi)\Omega^{\epsilon}(\xi)\nabla\xi \\
\leq\ & \left(\frac{\epsilon}{r-1}\right)^{\epsilon} K^{\frac{r-1}{\epsilon}} \int_{\varrho}^{\infty} (\xi-\varrho)^{\epsilon}\, \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi \\
\leq\ & \left(\frac{\epsilon}{r-1}\right)^{\epsilon} K^{\frac{r-1}{\epsilon}} \int_{\varrho}^{\infty} (\rho(\xi)-\varrho)^{\epsilon}\, \Xi^{-r}(\xi)\varpi^{\epsilon}(\xi)\nabla\xi,
\end{aligned}$$

which is (3.57) with $E = (\epsilon/\,(r-1))^{\epsilon}\, K^{\frac{r-1}{\epsilon}}$. □

Remark 3.3. In Theorem 3.3, if $\mathbb{T} = \mathbb{R}$ and $\varrho = 0$, then $\rho(\xi) = \xi$ and we see that (3.1) holds with $K = 1$, and thus we get (1.7). In addition, if $\Xi(\xi) = \xi$, then we get (1.4).

Corollary 3.3. *If $\mathbb{T} = \mathbb{N}_0$, $\varrho = 0$, and ϖ, Ξ are positive sequences such that $\tau/\Xi(\tau)$ is nondecreasing, then we see that (3.1) holds with $K = 2$ and then*

$$\sum_{\tau=1}^{\infty} \Xi^{-r}(\tau)\left[\sum_{k=1}^{\tau} \varpi(k)\right]^{\epsilon} \leq E \sum_{\tau=1}^{\infty} (\tau-1)^{\epsilon}\, \Xi^{-r}(\tau)\varpi^{\epsilon}(\tau), \tag{3.85}$$

where

$$E = \begin{cases} \left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \leq 1; \\ 2^{\frac{r-1}{\epsilon}}\left(\frac{\epsilon}{r-1}\right)^{\epsilon}, & (r-1)/\epsilon \geq 1. \end{cases}$$

4. Conclusions

In this paper, we established some new generalizations of the continuous inequalities on nabla calculus time scales. These inequalities were proved by employing the reverse Hölder's inequality and the chain rule formula adapted to time scales.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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