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Research article

Some novel Kulisch-Miranker type inclusions for a generalized class of Godunova-Levin stochastic processes

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Abstract: Mathematical inequalities supporting interval-valued stochastic processes are rarely addressed. Recently, Afzal et al. introduced the notion of $\mathfrak{h}$-Godunova-Levin stochastic processes and developed Hermite-Hadamard and Jensen type inequalities in the setting of interval-valued functions. This note introduces a more generalized class of Godunova-Levin stochastic process that unifies several previously published results through the use of Kulisch-Miranker type order relations that are rarely discussed in relation to stochastic processes. Further, it is the first time that fractional version of Hermite-Hadamard inequality has been developed by using interval-valued stochastic processes in conjunction with a classical operator. Moreover, we give new modified forms for Ostrowski type results and present a new way to treat Jensen type inclusions under interval stochastic processes by using a discrete sequential form. We end with an open problem regarding Milne type results and discuss the importance of different types of order relations related to inequality terms in interval-valued settings.

Keywords: Hermite-Hadamard; Jensen; Ostrowski; stochastic process; Godunova-Levin; fractional operator; mathematical operators
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1. Introduction

In mathematics, stochastic processes are representations of random changes in systems. They can be described as random groups of variables by applying probability theory and other disciplines. Several academic fields, including mathematics, physics, economics, operational research, and finance, have given rise to interest in stochastic processes. Different random models have been used in reliability analysis to mathematically represent complex phenomena and systems that change in a stochastic way [1, 2]. Stochastic models are best suited to study such situations because they can be specified robustly and manipulated easily. Relativistic transforms are popular in this field, and they describe the lifespan of a component that is changed with another component of the same age, but with a different lifetime distribution at a random failure time. An overview of stochastic optimization under constraints is presented, including insurance, finance, and portfolios with a diverse set of investors [3]. Whenever there is an expectation over random states involved in a stochastic optimization problem, a constrained stochastic successive convex approximation algorithm is applied [4]. The following are some recent applications of stochastic processes in different disciplines [5–7].

In certain cases, interval analysis can be a useful method of assessing uncertainty. Among the various branches of mathematics and topology, interval analysis is concerned with the analysis of intervals. Today, it is also very important in a number of computing languages to reduce uncertainty, such as in Python, Mathematica, Javascript, and Matlab. This has resulted in an increase in interest in this subject recently [8–11]. In addition to being applied in many disciplines, it has also been connected to inequalities by using various interval order approaches including inclusion, the center-radius order relationship, fuzzy order relation, pseudo order relation, and the left right order relationship. In relation to interval analysis, each has its own characteristics and is calculated differently. Some are full-order relationships, while others are partial-order relationships.

A significant portion of linear and nonlinear optimization problems are affected by inequalities. Mathematicians use convex inequalities extensively to understand many different issues. Among the various inequalities, these three are most important and have significant meaning in various aspects. Hermite-Hadamard inequalities and Jensen inequalities are geometrically interpretable convex mappings that are utilized in a variety of results, whereas Ostrowski type inequalities, and their different variants, allow us to obtain a new estimate of a function based on its integral mean, which can be applied to the estimation of quadrature rules when performing numerical analysis. The relationship between convex inequalities and stochastic processes is a well-known one. Originally, in 1980 Nikodem first defined convex stochastic processes with some intriguing properties [12]. Skowronski later expanded his results and presented them in a more comprehensive manner [13]. Li and Hao [14] constructed some intriguing Hermite-Hadamard inequality with various properties by using h-convex stochastic processes. Budak and Sarikaya [15] took inspiration from Li and Hao's results and refined their results by using various improved variants of the Hermite-Hadamard inequality. Several academics have also developed proposed inequalities by combining different concepts of convex stochastic processes with different approaches [16–20]. Initially, Tunc utilized the concept of h-convexity and developed famous Ostrowski-type double disparties [21]. In [22], the authors used stochastic processes for convex mappings and developed an Ostrowski type inequality, among other interesting results. Due to the accuracy of its results, interval analysis has increasingly been applied in various fields of mathematics over the past few decades; thus, by using the concept of set-valued mappings in the context of intervals, authors have connected inequalities with interval inclusions in a variety of ways [23]. With the help of Hukuhara differentiability, Chalco-Cano et al. [24] developed Ostrowski-type disparties. Chen et al. [25] developed Ostrowski type inclusions for η_h-convex mappings. Budak et al. [26] developed Ostrowski-type results by using fractional integral operators. Bai et al. [27] developed a famous double inequality and Jensen-type inclusion by using interval-non-p-convex $(\mathtt{h}_1, \mathtt{h}_2)$ mappings. Agahi and Babakhani [28] developed inequalities by using fractional integral operators in a convex stochastic process. Hernandez [29] utilized the notion of a $(m, \mathtt{h}_1, \mathtt{h}_2)$ dominated G-convex stochastic process and developed a generalized form of Hermite-Hadamard inequalities. Vivas-Cortez and Garcıa [30] created some variants of Ostrowski type inequalities by using the idea of $(m, \mathtt{h}_1, \mathtt{h}_2)$-convex mappings. In 2023, Afzal and Botmart [31] developed interval stochastic processes in connection with Godunova-Levin functions and refined some previously published results. There are some other recent developments regarding Godunova-Levin type functions by using a variety of integral operators and order relations [32–37].

Recently, Afzal et al. [31, 38] formulated the Ostrowski-Hermite-Hadamard and Jensen-type inclusions based on the notions of h-convex and h-Godunova-Levin stochastic processes.

Theorem 1.1. *[31]. Suppose that* $\mathtt{h} : (0,1) \to \mathtt{R}^+$*, such that* $\mathtt{h} \neq 0$*. Then, the interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+_{\mathtt{I}}$ *where* $[\mathtt{a}, \mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* h*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGPX}(\mathtt{h}, [\mathtt{a}, \mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$*; if* $\forall\ \mathtt{a}, \mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$*; then, one has*

$$\frac{\mathtt{h}\left(\frac{1}{2}\right)}{2}\mathfrak{V}\left(\frac{\mathtt{a}+\mathtt{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \frac{1}{\mathtt{b}-\mathtt{a}} \int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{V}(\mathfrak{y}, \cdot) d\mathfrak{y} \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathtt{a}, \cdot) + \mathfrak{V}(\mathtt{b}, \cdot)] \int_0^1 \frac{d\mathfrak{y}}{\mathtt{h}(\mathfrak{y})}. \tag{1.1}$$

Theorem 1.2. *[31]. Let* $g_i \in \mathtt{R}^+$*. Consider that* $\mathtt{h} : (0,1) \to \mathtt{R}^+$*. An interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+_{\mathtt{I}}$ *where* $\mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* h*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGPX}(\mathtt{h}, [\mathtt{a}, \mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$ *and* $\mathfrak{y} \in (0,1)$*; then, one has*

$$\mathfrak{V}\left(\frac{1}{G_k}\sum_{i=1}^{k} g_i \mathfrak{y}_i, .\right) \supseteq_{\mathcal{K}_C} \sum_{i=1}^{k} \left[\frac{\mathfrak{V}(\mathfrak{y}_i, .)}{\mathtt{h}\left(\frac{g_i}{G_k}\right)}\right]. \tag{1.2}$$

Theorem 1.3. *[38]. Consider a non-negative function* $\mathtt{h} : (0,1)$ *to* $\mathtt{R}$ *with* $\mathfrak{y} \subseteq \frac{1}{\mathtt{h}(\mathfrak{y})}$ *for each* $\mathfrak{y} \in (0,1)$*. Let a differentiable mean square interval-valued stochastic process* $\mathfrak{V} : \mathtt{I} \times \mathtt{v} \subseteq \mathtt{R} \to \mathtt{R}^+_{\mathtt{I}}$ *on* $\mathtt{I}^o$ *with* $\mathfrak{V}'$ *as integrable in the mean square sense on* $[\mathtt{a}, \mathtt{b}]$*. If* $|\mathfrak{V}'|$ *is an* h*-convex stochastic process satisfying that* $|\mathfrak{V}'(\mathfrak{b}, \cdot)| \supseteq_{\mathcal{K}_C} \gamma$*, for each* $\mathfrak{b}$*, then one has*

$$\begin{aligned}\left\{\left|\underline{\mathfrak{V}}(\mathfrak{b}, \cdot) - \frac{1}{\mathtt{b}-\mathtt{a}} \int_{\mathtt{a}}^{\mathtt{b}} \underline{\mathfrak{V}}(\mathfrak{y}, \cdot) d\mathfrak{y}\right|, \left|\overline{\mathfrak{V}}(\mathfrak{b}, \cdot) - \frac{1}{\mathtt{b}-\mathtt{a}} \int_{\mathtt{a}}^{\mathtt{b}} \overline{\mathfrak{V}}(\mathfrak{y}, \cdot) d\mathfrak{y}\right|\right\} \\ \supseteq_{\mathcal{K}_C} \frac{\gamma\left[(\mathfrak{b}-\mathtt{a})^2 + (\mathtt{b}-\mathfrak{b})^2\right]}{\mathtt{b}-\mathtt{a}} \int_0^1 \left[\mathtt{h}(\mathfrak{y}^2) + \mathtt{h}(\mathfrak{y}-\mathfrak{y}^2)\right] d\mathfrak{y}\end{aligned}$$

$\forall\ \mathfrak{b} \in [\mathtt{a}, \mathtt{b}]$.

The main contribution of this study is the introduction of a more generalized and larger class of Goduona-Levin stochastic processes that generalizes the recently developed results in [31, 38] to obtain a more refined form, along with other findings that we unify with some remark. Moreover, we employ the Kulisch-Miranker type of order relations which is rarely discussed in conjunction with stochastic processes. Additionally, we have developed Hermite-Hadamard type inequalities for this class of generalized convexity for the first time by using set valued mappings for fractional integral operators. In addition, we have developed a new and improved form of Ostrowski and sequential variants of discrete Jensen type inequalities. The study of fractional integral inequalities is a very important and fascinating research topic. Various very recent research articles adopting fractional integral approaches are very closely related to the current topic. It would be interesting to develop these results by using fractional operators in a stochastic sense [39–42].

A review of the literature related to developed inequalities and various articles [29–31, 38] motivated us to develop an improved and modified version of Ostrowski-Jensen and Hermite-Hadamard type inclusions for a generalized class of Godunova-Levin stochastic processes. The main results are backed up with numerically significant examples to demonstrate their validity. The presentation style of this note is as follows. In Section 2, our primary focus is on discussing some essential elements associated with interval calculus. In Section 3, we primarily talk about stochastic processes and some of their characteristics, as well as stochastic convexities and the various pertinent classes to which they belong to. Section 5 presents a definition of a novel class of Godunova-Levin stochastic processes and uses fractional and classical integral operators to derive several variants of Hermite-Hadamard type inclusions. In Section 6, we created an enhanced and more improved version of Ostrowski type inclusions. In Section 7, we develop a more generalized form of discrete sequential Jensen type inclusions. Lastly we summarize our results by providing a brief conclusion in Section 8.

2. Preliminaries

Let $\mathtt{R}$ be the one-dimensional Euclidean space, and consider $\mathtt{R}_{\mathtt{I}}$ as the family of all non-empty compact convex subsets of $\mathtt{R}$, that is

$$\mathtt{R}_{\mathtt{I}} = \{[\rho, \eta] : \rho, \eta \in \mathtt{R} \textit{ and } \rho \leq \eta\}.$$

The Hausdorff metric on $\mathtt{R}_{\mathtt{I}}$ is defined as

$$\mathcal{D}(\rho, \eta) = \max\{d(\rho, \eta), d(\eta, \rho)\} \tag{2.1}$$

where $d(\rho, \eta) = \max_{\rho_1 \in \rho} d(\rho_1, \eta)$ and $d(\rho_1, \eta) = \min_{\eta_1 \in \eta} d(\rho_1, \eta_1) = \min_{\eta_1 \in \eta} |\rho_1 - \eta_1|$.

Remark 2.1. *A parallel representation of the Hausdorff metric, as stated in (2.1) is given by*

$$\mathcal{D}([\underline{\rho_1}, \overline{\rho_1}], [\underline{\eta_1}, \overline{\eta_1}]) = \max\{|\underline{\rho_1} - \underline{\eta_1}|, |\overline{\rho_1} - \overline{\eta_1}|\}$$

which is referred to as the Moore metric in interval space.

As is commonly known for metric space, $(\mathtt{R}_{\mathtt{I}}, \mathcal{D})$ is complete. Throughout this paper, we will be using the following notations:

- ${\mathtt{R}_{\mathtt{I}}}^+$ is considered to be a family of all positive compact intervals of $\mathtt{R}$;
- ${\mathtt{R}_{\mathtt{I}}}^-$ is considered to be a family of all negative compact intervals of $\mathtt{R}$;
- $\mathtt{R}_{\mathtt{I}}$ is considered to be a family of all compact intervals of $\mathtt{R}$.

Now, we define the scalar multiplication and Minkowski sum on $\mathrm{R_I}$ by using

$$\rho + \eta = \{\rho_1 + \eta_1 \mid \rho_1 \in \rho, \eta_1 \in \eta\} \text{ and } \gamma\rho = \{\gamma\rho_1 \mid \rho_1 \in \rho\}.$$

Also, if $\rho = [\underline{\rho_1}, \overline{\rho_1}]$ and $\eta = [\underline{\eta_1}, \overline{\eta_1}]$ are two closed and bounded intervals, then we define the difference as follows:

$$\rho - \eta = [\underline{\rho_1} - \overline{\eta_1}, \overline{\rho_1} - \underline{\eta_1}]$$

the product

$$\rho \cdot \eta = [\min\{\underline{\rho_1\eta_1}, \underline{\rho_1}\overline{\eta_1}, \overline{\rho_1}\underline{\eta_1}, \overline{\rho_1\eta_1}\}, \max\{\underline{\rho_1\eta_1}, \underline{\rho_1}\overline{\eta_1}, \overline{\rho_1}\underline{\eta_1}, \overline{\rho_1\eta_1}\}]$$

and the division

$$\frac{\rho}{\eta} = \left[\min\left\{\frac{\underline{\rho_1}}{\underline{\eta_1}}, \frac{\underline{\rho_1}}{\overline{\eta_1}}, \frac{\overline{\rho_1}}{\underline{\eta_1}}, \frac{\overline{\rho_1}}{\overline{\eta_1}}\right\}, \max\left\{\frac{\underline{\rho_1}}{\overline{\eta_1}}, \frac{\underline{\rho_1}}{\overline{\eta_1}}, \frac{\overline{\rho_1}}{\underline{\eta_1}}, \frac{\overline{\rho_1}}{\overline{\eta_1}}\right\}\right]$$

whenever $0 \notin \eta$. The order relation "$\subseteq_{\mathcal{K}_C}$" was defined as follows by Kulisch and Miranker in 1981 [43].

$$[\underline{\rho_1}, \overline{\rho_1}] \subseteq_{\mathcal{K}_C} [\underline{\eta_1}, \overline{\eta_1}] \Leftrightarrow \underline{\eta_1} \leq \underline{\rho_1} \text{ and } \overline{\rho_1} \leq \overline{\eta_1}.$$

Next, we will describe how interval-valued functions are defined, followed by how these kinds of functions are integrated.

If $\mathcal{M} = [\rho_1, \eta_1]$ is a closed interval and $\mathfrak{Y} : \mathcal{M} \to \mathrm{R_I}$ is an interval set-valued mapping, then we will denote

$$\mathfrak{Y}(\eta_o) = [\underline{\mathfrak{s}}(\eta_o), \overline{\mathfrak{s}}(\eta_o)]$$

where $\underline{\mathfrak{s}}(\eta_o) \leq \overline{\mathfrak{s}}(\eta_o), \forall \eta_o \in \mathcal{M}$. The lower and upper endpoints of function $\mathfrak{Y}$ are denoted by the functions $\underline{\mathfrak{s}}(\eta_o)$ and $\overline{\mathfrak{s}}(\eta_o)$, respectively. For interval-valued function it is clear that $\mathfrak{Y} : \mathcal{M} \to \mathbb{K}_C$ is the continuous at $\eta_o \in \mathcal{M}$ if

$$\lim_{\eta \to \eta_o} \mathfrak{Y}(\eta) = \mathfrak{Y}(\eta_o)$$

where the limit is considered from the metric space $(\mathrm{R_I}, \mathcal{D})$. Consequently, $\mathfrak{Y}$ is continuous at $\eta_o \in \mathcal{M}$ if and only if its terminal functions $\underline{\mathfrak{s}}(\eta_o)$ and $\overline{\mathfrak{s}}(\eta_o)$ are continuous at any given point.

Theorem 2.1. *[38] Let $\mathfrak{Y} : [\mathsf{a}, \mathsf{b}] \to \mathrm{R_I}$ be an interval-valued function defined by $\mathfrak{Y}(\eta) = [\underline{\mathsf{S}}(\eta), \overline{\mathsf{S}}(\eta)]$. $\mathfrak{Y} \in \mathtt{IR}_{([\mathsf{a},\mathsf{b}])}$ iff $\underline{\mathsf{S}}(\eta), \overline{\mathsf{S}}(\eta) \in \mathrm{R}_{([\mathsf{a},\mathsf{b}])}$ and*

$$(\mathtt{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{Y}(\eta)\, d\eta = \left[(\mathrm{R}) \int_{\mathsf{a}}^{\mathsf{b}} \underline{\mathsf{S}}(\eta)\, d\eta, (\mathrm{R}) \int_{\mathsf{a}}^{\mathsf{b}} \overline{\mathsf{S}}(\eta)\, d\eta\right]$$

where $\mathrm{R}_{([\mathsf{a},\mathsf{b}])}$ is considered to be a pack of all interval-valued integrable functions. If $\mathfrak{Y}(\eta) \subseteq \mathfrak{V}(\eta)$ for all $\eta \in [\mathsf{a}, \mathsf{b}]$, then the following holds

$$(\mathtt{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{Y}(\eta)\, d\eta \subseteq (\mathtt{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\eta)\, d\eta.$$

3. Stochastic process

Definition 3.1. *Consider an arbitrary probability space* $(\mathtt{v}, \mathtt{A}, \mathtt{P})$. *A mapping* $\mathfrak{V} : \mathtt{v} \to \mathtt{R}$ *is considered to be a stochastic variable if it is* $\mathtt{A}$*-measurable. A mapping* $\mathfrak{V} : \mathtt{I} \times \mathtt{v} \to \mathtt{R}$ *where* $\mathtt{I} \subseteq \mathtt{R}$ *is a stochastic process; if* $\forall\ \mathtt{a} \in \mathtt{I}$ *the mapping* $\mathfrak{V}(\mathtt{a}, \cdot)$ *is considered to be stochastic variable.*

A stochastic process $\mathfrak{V}$ is said to adhere to the following conditions:

- Stochastically continous on $\mathtt{I}$ where, if $\forall\ \mathtt{a_o} \in \mathtt{I}$, then one has

$$\mathtt{p} - \lim_{\mathtt{a} \to \mathtt{a_o}} \mathfrak{V}(\mathtt{a}, .) = \mathfrak{V}(\mathtt{a_o}, .)$$

  where $\mathtt{p}$- lim denotes the limit in probability.
- In the mean square sense, stochastic continuity exists over $\mathtt{I}$; if $\forall\ \mathtt{a_o} \in \mathtt{I}$, then we have

$$\lim_{\mathtt{a} \to \mathtt{a_o}} \mathtt{E}\left[(\mathfrak{V}(\mathtt{a}, .) - \mathfrak{V}(\mathtt{a_o}, .))^2\right] = 0$$

  and the random variable's expected value is represented as $\mathtt{E}\left[\mathfrak{V}(\mathtt{a}, \cdot)\right]$.

- In the mean square sense, stochastic differentiability exists over $\mathtt{I}$; if $\forall\ \mathtt{a} \in \mathtt{I}$, if one has stochastic variable $\mathfrak{V}' : \mathtt{I} \times \mathtt{v} \to \mathtt{R}$, then

$$\mathfrak{V}'(\mathtt{a}, \cdot) = \mathtt{p} - \lim_{\mathtt{a} \to \mathtt{a_o}} \frac{\mathfrak{V}(\mathtt{a}, \cdot) - \mathfrak{V}(\mathtt{a_o}, \cdot)}{\mathtt{a} - \mathtt{a_o}}.$$

- In the mean square sense, stochastic integrability exists over $\mathtt{I}$, if $\forall\ \mathtt{a} \in \mathtt{I}$, with $\mathtt{E}\left[\mathfrak{V}(\mathtt{a}, \cdot)\right] < \infty$. Then, the stochastic variable $\mathtt{V} : \mathtt{v} \to \mathtt{R}$ with the partition of all convergence sequences of an interval $[\mathtt{a}, \mathtt{b}] \subseteq \mathtt{I}$, $\mathtt{a} = \mathfrak{b}_o < \mathfrak{b}_1 < \mathfrak{b}_2 ... < \mathfrak{b}_k = \mathtt{b}$; suppose that one has

$$\lim_{k \to \infty} \mathtt{E}\left[\left(\sum_{n=1}^{k} \mathfrak{V}(\mathfrak{u}_n, .)(\mathfrak{b}_n - \mathfrak{b}_{n-1}) - \mathtt{V}(\cdot,)\right)^2\right] = 0.$$

  In that case, it is written as

$$\mathtt{V}(\cdot,) = \int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{V}(\mathfrak{u}, \cdot) d\mathfrak{u} \qquad \text{(a.e)}.$$

To maximize efficiency, it is best to carry integrals and derivatives in fractional or non-integer orders. Authors in [44] defined stochastic mean-square fractional integral operators, which are represented as follows:

Definition 3.2. *[44]. Consider* $\mathfrak{V} : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+$ *to be a stochastic process; then, the mean-square fractional integral operators of order "*α*" are defined as follows:*

$$\mathtt{J}^{\alpha}_{\mathtt{a}^+} \mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)} \int_{\mathtt{a}}^{\mathtt{q}} (\mathtt{q} - \mathtt{w})^{(\alpha-1)} \mathfrak{V}(\mathtt{w}, \cdot)\, d\mathtt{w}, \quad \mathtt{q} > \mathtt{a}, \alpha > 0 \quad (a.e)$$

and

$$\mathtt{J}^{\alpha}_{\mathtt{b}^-} \mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)} \int_{\mathtt{q}}^{\mathtt{b}} (\mathtt{w} - \mathtt{q})^{(\alpha-1)} \mathfrak{V}(\mathtt{w}, \cdot)\, d\mathtt{w}, \quad \mathtt{q} < \mathtt{b}, \alpha > 0 \quad (a.e).$$

Definition 3.3. *[44]. Consider* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}_{\mathtt{I}}$ *to be an interval-valued stochastic process; then, the mean-square fractional integral operators of order "α" are defined as follows:*

$$\mathtt{J}^{\alpha}_{\mathtt{a}^+}\mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)}(\mathtt{IR})\int_{\mathtt{a}}^{\mathtt{q}}(\mathtt{q}-\mathtt{w})^{(\alpha-1)}\mathfrak{V}(\mathtt{w},\cdot)\,d\mathtt{w}, \quad \mathtt{q} > \mathtt{a}, \alpha > 0 \quad (a.e)$$

and

$$\mathtt{J}^{\alpha}_{\mathtt{b}^-}\mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)}(\mathtt{IR})\int_{\mathtt{q}}^{\mathtt{b}}(\mathtt{w}-\mathtt{q})^{(\alpha-1)}\mathfrak{V}(\mathtt{w},\cdot)\,d\mathtt{w}, \quad \mathtt{q} < \mathtt{b}, \alpha > 0 \quad (a.e)$$

where $\Gamma(\cdot)$ is the gamma function and $\mathtt{IR}_{([\mathtt{a},\mathtt{b}])}$ is a collection of all fractional integrals of interval order.

Corollary 3.1. *[45]. Consider* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}_{\mathtt{I}}$ *to be an interval-valued stochastic process such that* $\mathfrak{V}(\mathtt{q}) = [\underline{\mathfrak{V}}(\mathtt{q}), \overline{\mathfrak{V}}(\mathtt{q})]$ *with* $\underline{\mathfrak{V}}(\mathtt{q}), \overline{\mathfrak{V}}(\mathtt{q}) \in \mathtt{IR}_{([\mathtt{a},\mathtt{b}])}$*; then, we have*

$$\mathtt{I}^{\alpha}_{\mathtt{a}^+}\mathfrak{V}(\mathtt{q}) = \left[\mathtt{J}^{\alpha}_{\mathtt{a}^+}\underline{\mathfrak{V}}(\mathtt{q}), \mathtt{J}^{\alpha}_{\mathtt{a}^+}\overline{\mathfrak{V}}(\mathtt{q})\right].$$

Definition 3.4. *[15]. Let* $\mathtt{h} : [0,1] \to \mathtt{R}^+$*, such that* $\mathtt{h} \neq 0$*. Then, the stochastic process* $\mathfrak{V} : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+$ *is considered to be an* h*-convex stochastic process; if* $\forall\ \mathtt{a},\mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$*, one has*

$$\mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b},\cdot) \leq \mathtt{h}(\mathfrak{y})\mathfrak{V}(\mathtt{a},\cdot) + \mathtt{h}(1-\mathfrak{y})\mathfrak{V}(\mathtt{b},\cdot). \tag{3.1}$$

Definition 3.5. *[31]. Let* $\mathtt{h} : (0,1) \to \mathtt{R}^+$ *such that* $\mathtt{h} \neq 0$*. Then, the interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+_{\mathtt{I}}$ *where* $[\mathtt{a},\mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* h*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGPX}(\mathtt{h}, [\mathtt{a},\mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$*; if* $\forall\ \mathtt{a},\mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$*, then one has*

$$\mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b},\cdot) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(\mathtt{a},\cdot)}{\mathtt{h}(\mathfrak{y})} + \frac{\mathfrak{V}(\mathtt{b},\cdot)}{\mathtt{h}(1-\mathfrak{y})}. \tag{3.2}$$

The set of all interval-valued h-Godunova-Levin convex stochastic processes is denoted by $\mathcal{SGPX}(\mathtt{h}, [\mathtt{a},\mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$.

Definition 3.6. *[38]. Let* $\mathtt{h} : [0,1] \to \mathtt{R}^+$ *such that* $\mathtt{h} \neq 0$*. Then, the interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+_{\mathtt{I}}$ *where* $[\mathtt{a},\mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* h*-convex stochastic process or* $\mathfrak{V} \in \mathcal{SPX}(\mathtt{h}, [\mathtt{a},\mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$*; if* $\forall\ \mathtt{a},\mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$*, then one has*

$$\mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b},\cdot) \supseteq_{\mathcal{K}_C} \mathtt{h}(\mathfrak{y})\mathfrak{V}(\mathtt{a},\cdot) + \mathtt{h}(1-\mathfrak{y})\mathfrak{V}(\mathtt{b},\cdot). \tag{3.3}$$

The set of all interval-valued h-convex stochastic processes is denoted by $\mathcal{SPX}(\mathtt{h}, [\mathtt{a},\mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$.

4. Main results

We can now define a new more general classes of Godunova-Levin stochastic processes by drawing ideas from the prior literature and definitions.

Definition 4.1. *Consider* $\mathtt{h}_1, \mathtt{h}_2 : [0,1] \to \mathtt{R}^+$*. An interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}^+_{\mathtt{I}}$ *where* $[\mathtt{a},\mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* $(\mathtt{h}_1,\mathtt{h}_2)$*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGPX}((\mathtt{h}_1,\mathtt{h}_2), [\mathtt{a},\mathtt{b}], \mathtt{R}^+_{\mathtt{I}})$*; if* $\forall\ \mathtt{a},\mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$*, we have*

$$\mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b},\cdot) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(\mathtt{a},\cdot)}{\mathtt{h}_1(\mathfrak{y})\mathtt{h}_2(1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathtt{b},\cdot)}{\mathtt{h}_1(1-\mathfrak{y})\mathtt{h}_2(\mathfrak{y})}. \tag{4.1}$$

The set of all interval-valued $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin convex stochastic processes is denoted by $\mathcal{SGPX}((\mathtt{h}_1, \mathtt{h}_2), [\mathtt{a}, \mathtt{b}], \mathtt{R}_{\mathtt{I}}^{+})$.

Remark 4.1. *(i) If* $\mathtt{h}_1(\mathfrak{y}) = \mathtt{h}(\mathfrak{y}), \mathtt{h}_2 = 1$ *in Definition 4.1, then the* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process turns into an h-Godunova-Levin stochastic process [31].*

(ii) If $\mathtt{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathtt{h}_2 = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *in Definition 4.1, then the* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process turns into a convex stochastic process [46].*

(iii) If $\mathtt{h}_1(\mathfrak{y}) = \frac{1}{\mathtt{h}(\mathfrak{y})}, \mathtt{h}_2 = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *in Definition 4.1, then the* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process turns into an h-convex stochastic [15].*

(iv) If $\mathtt{h}_1 = \frac{1}{\mathfrak{y}^s}, \mathtt{h}_2 = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *in Definition 4.1, then the* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process turns into an s-convex stochastic process [47].*

Definition 4.2. *Consider* $\mathtt{h}_1, \mathtt{h}_2 : [0, 1] \to \mathtt{R}^+$. *An interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}_{\mathtt{I}}^{+}$ *where* $[\mathtt{a}, \mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an harmonic* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGHPX}((\mathtt{h}_1, \mathtt{h}_2), [\mathtt{a}, \mathtt{b}], \mathtt{R}_{\mathtt{I}}^{+})$*; if* $\forall\ \mathtt{a}, \mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0, 1)$, *we have*

$$\mathfrak{V}\left(\frac{\mathtt{ab}}{\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b}}, \cdot\right) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(\mathtt{a}, \cdot)}{\mathtt{h}_1(\mathfrak{y})\mathtt{h}_2(1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathtt{b}, \cdot)}{\mathtt{h}_1(1-\mathfrak{y})\mathtt{h}_2(\mathfrak{y})}. \tag{4.2}$$

The set of all interval-valued harmonic $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin convex stochastic processes is denoted by $\mathcal{SGHPX}((\mathtt{h}_1, \mathtt{h}_2), [\mathtt{a}, \mathtt{b}], \mathtt{R}_{\mathtt{I}}^{+})$.

Using a fractional operator, we first construct the Hermite-Hadamard inequality. Next, we construct results for the products of two Godunova-Levin stochastic processes by using the standard Riemann integral. Lastly, we show that some results that have been published previously are generalized.

5. Fractional Hermite-Hadamard type inclusions for a generalized class of Godunova-Levin stochastic processes

Theorem 5.1. *Consider* $\mathtt{h}_1, \mathtt{h}_2 : [0, 1] \to \mathtt{R}^+$. *An interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}] : \mathtt{I} \times \mathtt{v} \to \mathtt{R}_{\mathtt{I}}^{+}$ *where* $[\mathtt{a}, \mathtt{b}] \subseteq \mathtt{I} \subseteq \mathtt{R}$ *is considered to be an* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGPX}((\mathtt{h}_1, \mathtt{h}_2), [\mathtt{a}, \mathtt{b}], \mathtt{R}_{\mathtt{I}}^{+})$*; if* $\forall\ \mathtt{a}, \mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0, 1)$, *then one has*

$$\begin{aligned}\frac{\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)}{\alpha}\mathfrak{V}\left(\frac{\mathtt{a}+\mathtt{b}}{2}, \cdot\right) &\supseteq_{\mathcal{K}_C} \frac{\Gamma(\alpha)}{\mathtt{b}-\mathtt{a}}\left[\mathtt{J}_{\mathtt{a}^+}^{\alpha}\mathfrak{V}(\mathtt{b}, \cdot) + \mathtt{J}_{\mathtt{b}^-}^{\alpha}\mathfrak{V}(\mathtt{a}, \cdot)\right] \\ &\supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathtt{a}, \cdot) + \mathfrak{V}(\mathtt{b}, \cdot)] \int_0^1 \mathfrak{y}^{\alpha-1}\left[\frac{1}{\mathtt{H}(\mathfrak{y}, 1-\mathfrak{y})} + \frac{1}{\mathtt{H}(1-\mathfrak{y}, \mathfrak{y})}\right] d\mathfrak{y}\end{aligned}$$

with $\mathtt{a} > 0$.

Proof. Since $\mathfrak{V} \in \mathcal{SGPX}((\mathtt{h}_1, \mathtt{h}_2), [\mathtt{a}, \mathtt{b}], \mathtt{R}_{\mathtt{I}}^{+})$, then one has

$$\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)\mathfrak{V}\left(\frac{\mathtt{a}+\mathtt{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b}, \cdot) + \mathfrak{V}((1-\mathfrak{y})\mathtt{a} + \mathfrak{y}\mathtt{b}, \cdot). \tag{5.1}$$

Multiplying (5.1) by $\mathfrak{y}^{\alpha-1}$ and integrating, we get

$$\frac{\mathtt{H}\left(\frac{1}{2}, \frac{1}{2}\right)}{\alpha}\mathfrak{V}\left(\frac{\mathtt{a}+\mathtt{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \left[\int_0^1 \mathfrak{y}^{\alpha-1}\mathfrak{V}(\mathfrak{y}\mathtt{a} + (1-\mathfrak{y})\mathtt{b}, \cdot)d\mathfrak{y} + \int_0^1 \mathfrak{y}^{\alpha-1}\mathfrak{V}((1-\mathfrak{y})\mathtt{a} + \mathfrak{y}\mathtt{b}, \cdot)d\mathfrak{y}\right]$$

$$= \Big[\int_0^1 \mathfrak{y}^{\alpha-1}\underline{\mathfrak{V}}(\mathfrak{y}\mathsf{a}+(1-\mathfrak{y})\mathsf{b},\cdot)d\mathfrak{y}+\int_0^1 \mathfrak{y}^{\alpha-1}\underline{\mathfrak{V}}((1-\mathfrak{y})\mathsf{a}+\mathfrak{y}\mathsf{b},\cdot)d\mathfrak{y},$$

$$\int_0^1 \mathfrak{y}^{\alpha-1}\overline{\mathfrak{V}}(\mathfrak{y}\mathsf{a}+(1-\mathfrak{y})\mathsf{b},\cdot)d\mathfrak{y}+\int_0^1 \mathfrak{y}^{\alpha-1}\overline{\mathfrak{V}}((1-\mathfrak{y})\mathsf{a}+\mathfrak{y}\mathsf{b},\cdot)d\mathfrak{y}\Big]$$

$$= \Big[\int_{\mathsf{b}}^{\mathsf{a}}\left(\frac{\mathsf{b}-\mathfrak{u}}{\mathsf{b}-\mathsf{a}}\right)^{\alpha-1}\underline{\mathfrak{V}}(\mathfrak{u},\cdot)\frac{d\mathfrak{u}}{\mathsf{a}-\mathsf{b}}+\int_{\mathsf{a}}^{\mathsf{b}}\left(1-\frac{\mathsf{b}-\mathfrak{u}}{\mathsf{b}-\mathsf{a}}\right)^{\alpha-1}\underline{\mathfrak{V}}(\mathfrak{u},\cdot)\frac{d\mathfrak{u}}{\mathsf{b}-\mathsf{a}},$$

$$\int_{\mathsf{b}}^{\mathsf{a}}\left(\frac{\mathsf{b}-\mathfrak{u}}{\mathsf{b}-\mathsf{a}}\right)^{\alpha-1}\overline{\mathfrak{V}}(\mathfrak{u},\cdot)\frac{d\mathfrak{u}}{\mathsf{a}-\mathsf{b}}+\int_{\mathsf{a}}^{\mathsf{b}}\left(1-\frac{\mathsf{b}-\mathfrak{u}}{\mathsf{b}-\mathsf{a}}\right)^{\alpha-1}\overline{\mathfrak{V}}(\mathfrak{u},\cdot)\frac{d\mathfrak{u}}{\mathsf{b}-\mathsf{a}}\Big]$$

$$= \frac{\Gamma(\alpha)}{\mathsf{b}-\mathsf{a}}\left[\mathtt{J}^{\alpha}_{\mathsf{a}^+}\underline{\mathfrak{V}}(\mathsf{b},\cdot),\mathtt{J}^{\alpha}_{\mathsf{b}^-}\underline{\mathfrak{V}}(\mathsf{a},\cdot),\mathtt{J}^{\alpha}_{\mathsf{a}^+}\overline{\mathfrak{V}}(\mathsf{b},\cdot)+\mathtt{J}^{\alpha}_{\mathsf{b}^-}\overline{\mathfrak{V}}(\mathsf{a},\cdot)\right]$$

$$= \frac{\Gamma(\alpha)}{\mathsf{b}-\mathsf{a}}\left[\mathtt{J}^{\alpha}_{\mathsf{a}^+}\mathfrak{V}(\mathsf{b},\cdot),\mathtt{J}^{\alpha}_{\mathsf{b}^-}\mathfrak{V}(\mathsf{a},\cdot))\right]. \tag{5.2}$$

Similarly since $\mathfrak{V}\in\mathcal{SGPX}((\mathsf{h}_1,\mathsf{h}_2),[\mathsf{a},\mathsf{b}],\mathsf{R}^+_{\mathtt{I}})$, we have

$$\mathfrak{V}(\mathfrak{y}\mathsf{a}+(1-\mathfrak{y})\mathsf{b},\cdot)+\mathfrak{V}((1-\mathfrak{y})\mathsf{a}+\mathfrak{y}\mathsf{b},\cdot)\supseteq_{\mathcal{K}_C}\left[\frac{1}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right][\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)]. \tag{5.3}$$

Multiplying (5.3) by $\mathfrak{y}^{\alpha-1}$ and integrating on $[0,1]$, we have

$$\frac{\Gamma(\alpha)}{\mathsf{b}-\mathsf{a}}\left[\mathtt{J}^{\alpha}_{\mathsf{a}^+}f(\mathsf{b},\cdot)+\mathtt{J}^{\alpha}_{\mathsf{b}^-}f(\mathsf{a},\cdot)\right]\supseteq_{\mathcal{K}_C}[\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)]\int_0^1\mathfrak{y}^{\alpha-1}\left[\frac{1}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}. \tag{5.4}$$

Take into account (5.2) with (5.4), and the result follows. □

Example 5.1. *Consider that* $[\mathsf{a},\mathsf{b}]=[1,2]$. *Let* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}},\mathsf{h}_2(\mathfrak{y})=1,\forall\ \mathfrak{y}\in\ (0,1)$ *and* $\alpha=\frac{1}{2}$. *Suppose that a stochastic process* $\mathfrak{V}$ *is defined as*

$$\mathfrak{V}(\mathfrak{u},\cdot)=[-\mathfrak{u}^{\frac{1}{2}}+2,\mathfrak{u}^{\frac{1}{2}}+2].$$

Then,

$$\frac{\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)}{\alpha}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)$$
$$=\left[2(4-\sqrt{6}),2(4+\sqrt{6})\right],$$
$$\frac{\Gamma(\alpha)}{\mathsf{b}-\mathsf{a}}\left[\mathtt{J}^{\alpha}_{\mathsf{a}^+}\mathfrak{V}(\mathsf{b},\cdot)+\mathtt{J}^{\alpha}_{\mathsf{b}^-}\mathfrak{V}(\mathsf{a},\cdot)\right]$$
$$=\left[\frac{14-2\sqrt{2}-\pi}{2}-0.38277,\frac{18+2\sqrt{2}+\pi}{2}+0.38277\right],$$
$$[\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)]\int_0^1\mathfrak{y}^{\alpha-1}\left[\frac{1}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$=\left[2(3-\sqrt{2}),2(3+\sqrt{2})\right].$$

As a result,

$$\left[2(4-\sqrt{6}),2(4+\sqrt{6})\right]$$

$$\supseteq_{\mathcal{K}_C} \left[\frac{14-2\sqrt{2}-\pi}{2} - 0.38277, \frac{18+2\sqrt{2}+\pi}{2} + 0.38277\right]$$
$$\supseteq_{\mathcal{K}_C} \left[2(3-\sqrt{2}), 2(3+\sqrt{2})\right].$$

As a result, Theorem 5.1 is true.

Remark 5.1. *(i) If $\alpha = 1$, $\mathrm{h}_1(\mathfrak{y}) = \frac{1}{\mathrm{h}(\mathfrak{y})}$ and $\mathrm{h}_2(\mathfrak{y}) = 1$ with $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$, then Theorem 5.1 turns into an h-convex stochastic process [48].*

Theorem 5.2. *Based on the same hypotheses in Theorem 5.1, the successive inclusion relation can be defined as follows:*

$$\frac{\left[\mathrm{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\mathfrak{u}, \cdot)d\mathfrak{u} \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathsf{a}, \cdot) + \mathfrak{V}(\mathsf{b}, \cdot)]\int_0^1 \frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y}, 1-\mathfrak{y})}. \tag{5.5}$$

Proof. Since $\mathfrak{V} \in \mathcal{SGPX}((\mathrm{h}_1, \mathrm{h}_2), [\mathsf{a}, \mathsf{b}], \mathrm{R}_{\mathrm{I}}^{+})$, we have

$$\begin{aligned}
&\left[\mathrm{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot) + \mathfrak{V}((1-\mathfrak{y})\mathsf{a} + \mathfrak{y}\mathsf{b}, \cdot)\\
&\left[\mathrm{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2}, \cdot\right) \supseteq_{\mathcal{K}_C} \left[\int_0^1 \mathfrak{V}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot)d\mathfrak{y} + \int_0^1 \mathfrak{V}((1-\mathfrak{y})\mathsf{a} + \mathfrak{y}\mathsf{b}, \cdot)d\mathfrak{y}\right]\\
&= \left[\int_0^1 \underline{\mathfrak{V}}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot)d\mathfrak{y} + \int_0^1 \underline{\mathfrak{V}}((1-\mathfrak{y})\mathsf{a} + \mathfrak{y}\mathsf{b}, \cdot)d\mathfrak{y},\right.\\
&\qquad \left.\int_0^1 \overline{\mathfrak{V}}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot)d\mathfrak{y} + \int_0^1 \overline{\mathfrak{V}}((1-\mathfrak{y})\mathsf{a} + \mathfrak{y}\mathsf{b}, \cdot)d\mathfrak{y}\right]\\
&= \left[\frac{2}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \underline{\mathfrak{V}}(\mathfrak{u}, \cdot)d\mathfrak{u}, \frac{2}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \overline{\mathfrak{V}}(\mathfrak{u}, \cdot)d\mathfrak{u}\right]\\
&= \frac{2}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\mathfrak{u}, \cdot)d\mathfrak{u}.
\end{aligned} \tag{5.6}$$

By Definition 4.1, one has

$$\mathfrak{V}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(\mathsf{a}, \cdot)}{\mathrm{H}(\mathfrak{y}, 1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathsf{b}, \cdot)}{\mathrm{H}(1-\mathfrak{y}, \mathfrak{y})}.$$

Following integration, one has

$$\int_0^1 \mathfrak{V}(\mathfrak{y}\mathsf{a} + (1-\mathfrak{y})\mathsf{b}, \cdot)d\mathfrak{y} \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathsf{a}, \cdot)\int_0^1 \frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y}, 1-\mathfrak{y})} + \mathfrak{V}(\mathsf{b}, \cdot)\int_0^1 \frac{d\mathfrak{y}}{\mathrm{H}(1-\mathfrak{y}, \mathfrak{y})}$$

Accordingly,

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\mathfrak{u}, \cdot)d\mathfrak{u} \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathsf{a}, \cdot) + \mathfrak{V}(\mathsf{b}, \cdot)]\int_0^1 \frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y}, 1-\mathfrak{y})}. \tag{5.7}$$

Now, combining (5.6) and (5.7), we achieve the desired outcome.

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\supseteq_{\mathcal{K}_C}[\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)]\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}.$$

□

Remark 5.2. *(i) If* $\mathtt{h}_1(\mathfrak{y})=\mathtt{h}(\mathfrak{y})$ *and* $\mathtt{h}_2(\mathfrak{y})=1$, *then Theorem 5.2 turns into an h-Godunova-Levin stochastic process [31]:*

$$\frac{\mathtt{h}\left(\frac{1}{2}\right)}{2}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\supseteq_{\mathcal{K}_C}\int_0^1\frac{d\mathfrak{y}}{\mathtt{h}(\mathfrak{y})}.$$

(ii) If $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathtt{h}(\mathfrak{y})}$ *and* $\mathtt{h}_2(\mathfrak{y})=1$ *with* $\underline{\mathfrak{V}}=\overline{\mathfrak{V}}$, *then Theorem 5.2 turns into an h-convex stochastic process [15]:*

$$\frac{1}{2\mathtt{h}\left(\frac{1}{2}\right)}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)\leq\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\leq\int_0^1\mathtt{h}(\mathfrak{y})d\mathfrak{y}.$$

(iii) If $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}}$ *and* $\mathtt{h}_2(\mathfrak{y})=1$ *with* $\underline{\mathfrak{V}}=\overline{\mathfrak{V}}$, *then Theorem 5.2 turns into a convex stochastic proces [46]:*

$$\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)\leq\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\leq\frac{\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)}{2}.$$

(iv) If $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}^s}$, $\mathtt{h}_2(\mathfrak{y})=1$ *with* $\underline{\mathfrak{V}}=\overline{\mathfrak{V}}$, *then Theorem 5.2 turns into an s-convex stochastic process [47]:*

$$2^{s-1}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)\leq\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\leq\frac{\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)}{s+1}.$$

Example 5.2. *Consider that* $[\mathsf{a},\mathsf{b}]=[-1,1]$ *with* $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}},\mathtt{h}_2=1$, $\forall\ \mathfrak{y}\in(0,1)$. *Suppose that a stochastic process* $\mathfrak{V}$ *is defined as*

$$\mathfrak{V}(\mathfrak{u},\cdot)=[\mathfrak{u}^2,4-e^{\mathfrak{u}}].$$

Then,

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2},\cdot\right)=[0,3],$$

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u}\approx[0.3333,2.82479],$$

$$[\mathfrak{V}(\mathsf{a},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)]\int_0^1\mathtt{H}(\mathfrak{y},1-\mathfrak{y})d\mathfrak{y}\approx[1,2.45691].$$

As a result,

$$[0,3]\supseteq_{\mathcal{K}_C}[0.3333,2.82479]\supseteq_{\mathcal{K}_C}[1,2.45691].$$

This verifies Theorem 5.2.

Theorem 5.3. *Based on the same hypotheses as in Theorem 5.1, the successive inclusion relation can be defined*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathfrak{V}\left(\frac{a+b}{2},\cdot\right) \supseteq_{\mathcal{K}_C} \triangle_1 \supseteq_{\mathcal{K}_C} \frac{1}{b-a}\int_a^b \mathfrak{V}(\mathfrak{u},\cdot)d\mathfrak{u} \supseteq_{\mathcal{K}_C} \triangle_2$$
$$\supseteq_{\mathcal{K}_C} \left\{[\mathfrak{V}(a,\cdot)+\mathfrak{V}(b,\cdot)]\left[\frac{1}{2}+\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1 H(\mathfrak{y},1-\mathfrak{y})d\mathfrak{y}$$

where

$$\triangle_1 = \frac{H\left(\frac{1}{2},\frac{1}{2}\right)}{4}\left[\mathfrak{V}\left(\frac{3a+b}{4},\cdot\right)+\mathfrak{V}\left(\frac{3b+a}{4},\cdot\right)\right]$$

$$\triangle_2 = \left[\mathfrak{V}\left(\frac{a+b}{2},\cdot\right)+\frac{\mathfrak{V}(a,\cdot)+\mathfrak{V}(b,\cdot)}{2}\right]\int_0^1 H(\mathfrak{y},1-\mathfrak{y})d\mathfrak{y}.$$

Proof. We get the required result by taking into account Definition 4.2 and using the same technique as Afzal and Botmart [31]. □

Example 5.3. *From Example 5.2, one has*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathfrak{V}\left(\frac{a+b}{2},\cdot\right) = [0,3]$$
$$\triangle_1 \approx [0.25, 2.87237]$$
$$\triangle_2 \approx [0.5, 1.95691]$$

and

$$\left\{[\mathfrak{V}(a,\cdot)+\mathfrak{V}(b,\cdot)]\left[\frac{1}{2}+\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1 H(\mathfrak{y},1-\mathfrak{y})d\mathfrak{y} \approx [1, 2.45691].$$

Thus, we obtain

$$[0,3] \supseteq_{\mathcal{K}_C} [0.25, 2.87237] \supseteq_{\mathcal{K}_C} [0.3333, 2.82479] \supseteq_{\mathcal{K}_C} [0.5, 1.95691] \supseteq_{\mathcal{K}_C} [1, 2.45691]$$

This verifies Theorem 5.3.

Theorem 5.4. *Based on the same hypotheses as in Theorem 5.1, the successive inclusion relation can be defined*

$$\frac{1}{b-a}\int_a^b \mathfrak{V}(\mathfrak{u},\cdot)S(\mathfrak{u},\cdot)d\mathfrak{u} \supseteq_{\mathcal{K}_C} \mathcal{T}(a,b)\int_0^1 \frac{d\mathfrak{y}}{H^2(\mathfrak{y},1-\mathfrak{y})} + \mathcal{U}(a,b)\int_0^1 \frac{d\mathfrak{y}}{H(\mathfrak{y},\mathfrak{y})H(1-\mathfrak{y},1-\mathfrak{y})}.$$

Proof. Since $\mathfrak{V}, S \in \mathcal{SGPX}((h_1,h_2),[a,b],R_I^+)$, we have

$$\mathfrak{V}(a\mathfrak{y}+(1-\mathfrak{y})b,\cdot) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(a,\cdot)}{H(\mathfrak{y},1-\mathfrak{y})}+\frac{\mathfrak{V}(b,\cdot)}{H(1-\mathfrak{y},\mathfrak{y})}$$

$$\mathsf{S}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{\mathsf{S}(\mathsf{a},\cdot)}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}+\frac{\mathsf{S}(\mathsf{b},\cdot)}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}.$$

Then,

$$\begin{aligned}&\mathfrak{V}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\mathsf{S}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\\ &\supseteq_{\mathcal{K}_C}\frac{\mathfrak{V}(\mathsf{a},\cdot)\mathsf{S}(\mathsf{a},\cdot)}{\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})}+\frac{[\mathfrak{V}(\mathsf{a},\cdot)\mathsf{S}(\mathsf{b},\cdot)+\mathfrak{V}(\mathsf{b},\cdot)\mathsf{S}(\mathsf{a},\cdot)]}{\mathtt{H}^2(1-\mathfrak{y},\mathfrak{y})}+\frac{\mathfrak{V}(\mathsf{b},\cdot)\mathsf{S}(\mathsf{b},\cdot)}{\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y})}\end{aligned}$$

Following integration, one has

$$\begin{aligned}&\int_0^1\mathfrak{V}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\mathsf{S}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)d\mathfrak{y}\\ &=\left[\int_0^1\underline{\mathfrak{V}}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\underline{\mathsf{S}}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)d\mathfrak{y}\right.,\\ &\quad\left.\int_0^1\overline{\mathfrak{V}}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)\overline{\mathsf{S}}\left(\mathsf{a}\mathfrak{y}+(1-\mathfrak{y})\mathsf{b},\cdot\right)d\mathfrak{y}\right]\\ &=\left[\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\underline{\mathfrak{V}}(\mathfrak{u},\cdot)\underline{\mathsf{S}}(\mathfrak{u},\cdot)d\mathfrak{u},\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\overline{\mathfrak{V}}(\mathfrak{u},\cdot)\overline{\mathsf{S}}(\mathfrak{u},\cdot)d\mathfrak{u}\right]\\ &=\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)\mathsf{S}(\mathfrak{u},\cdot)d\mathfrak{u}\\ &\supseteq_{\mathcal{K}_C}\mathcal{T}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})}+\mathcal{U}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y})}.\end{aligned}$$

Thus, it follows

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)\mathsf{S}(\mathfrak{u},\cdot)d\mathfrak{u}\supseteq_{\mathcal{K}_C}\mathcal{T}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})}+\mathcal{U}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y})}.$$

□

Example 5.4. *Let* $[\mathsf{a},\mathsf{b}]=[0,1]$ *with* $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}}$, $\mathtt{h}_2(\mathfrak{y})=1$ *for all* $\mathfrak{y}\in(0,1)$. *Suppose that* $\mathfrak{V},\mathsf{S}$ *are two stochastic process mappings that are defined as follows:*

$$\mathfrak{V}(\mathfrak{u},\cdot)=[\mathfrak{u}^2,4-e^{\mathfrak{u}}]\quad\textit{and}\quad\mathfrak{y}(\mathfrak{u},\cdot)=[\mathfrak{u},3-\mathfrak{u}^2].$$

Then, we have

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\mathfrak{u},\cdot)\mathsf{S}(\mathfrak{u},\cdot)d\mathfrak{u}\approx[0.25,6.23010]$$

$$\mathcal{T}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})}=[0.3333,3.85447]$$

and

$$\mathcal{U}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y})}=[0,1.64085].$$

Since

$$[0.25,6.23010]\supseteq_{\mathcal{K}_C}[0.3333,5.49533].$$

Consequently, Theorem 5.4 is verified.

Theorem 5.5. *Based on the same hypotheses as in Theorem 5.1, the successive inclusion relation can be defined*

$$\frac{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{B}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\mathtt{S}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{B}(\mathfrak{u},\cdot)\mathtt{S}(\mathfrak{u},\cdot)d\mathfrak{u}$$
$$+\mathcal{T}(\mathtt{a},\mathtt{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y})}+\mathcal{U}(\mathtt{a},\mathtt{b})\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})}.$$

Proof. Since $\mathfrak{B},\mathtt{S}\in\mathcal{SGPX}((\mathtt{h}_1,\mathtt{h}_2),[\mathtt{a},\mathtt{b}],\mathtt{R}_{\mathtt{I}}^{+})$, then one has

$$\mathfrak{B}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{1}{\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{B}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)+\frac{1}{\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)}\mathfrak{B}\left(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot\right),$$
$$\mathtt{S}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\supseteq_{\mathcal{K}_C}\frac{1}{\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)}\mathtt{S}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)+\frac{1}{\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)}\mathtt{S}\left(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot\right).\tag{5.8}$$

$$\mathfrak{B}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\mathtt{S}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)$$
$$\supseteq_{\mathcal{K}_C}\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathfrak{B}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)\mathtt{S}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)+\mathfrak{B}\left(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot\right)\mathtt{S}(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot)\right]$$
$$+\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathfrak{B}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)\mathtt{S}(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot)+\mathfrak{B}(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot)\mathtt{S}(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot)\right]$$
$$\supseteq_{\mathcal{K}_C}\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathfrak{B}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)\mathtt{S}(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot)+\mathfrak{B}\left(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot\right)\mathtt{S}(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot)\right]$$
$$+\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{\mathfrak{B}(\mathtt{a},\cdot)}{\mathtt{h}_1(\mathfrak{y})\mathtt{h}_2(1-\mathfrak{y})}+\frac{\mathfrak{B}(\mathtt{b},\cdot)}{\mathtt{h}_1(1-\mathfrak{y})\mathtt{h}_2(\mathfrak{y})}+\frac{\mathtt{S}(\mathtt{a},\cdot)}{\mathtt{h}_1(1-\mathfrak{y})\mathtt{h}_2(1-\mathfrak{y})}+\frac{\mathtt{S}(\mathtt{b},\cdot)}{\mathtt{h}_1(1-\mathfrak{y})\mathtt{h}_2(\mathfrak{y})}\right]$$
$$+\left[\left(\frac{\mathfrak{B}(\mathtt{a},\cdot)}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}+\frac{\mathfrak{B}(\mathtt{b},\cdot)}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}\right)+\left(\frac{\mathtt{S}(\mathtt{a},\cdot)}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\mathtt{S}(\mathtt{b},\cdot)}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)\right]$$
$$\supseteq_{\mathcal{K}_C}\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathfrak{B}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)\mathtt{S}\left(\mathtt{a}\mathfrak{y}+(1-\mathfrak{y})\mathtt{b},\cdot\right)+\mathfrak{B}\left(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot\right)\mathtt{S}(\mathtt{a}(1-\mathfrak{y})+\mathfrak{y}\mathtt{b},\cdot)\right]$$
$$+\frac{1}{\left[\mathtt{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{2\mathcal{T}(\mathtt{a},\mathtt{b})}{(\mathtt{H}(\mathfrak{y},\mathfrak{y})\mathtt{H}(1-\mathfrak{y},1-\mathfrak{y}))}+\frac{\mathcal{U}(\mathtt{a},\mathtt{b})}{(\mathtt{H}^2(\mathfrak{y},1-\mathfrak{y})+\mathtt{H}^2(1-\mathfrak{y},\mathfrak{y}))}\right].$$

Integration over $(0,1)$ yields that

$$\int_0^1\mathfrak{B}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\mathtt{S}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)d\mathfrak{y}$$
$$=\left[\int_0^1\underline{\mathfrak{B}}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\underline{\mathtt{S}}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)d\mathfrak{y},\int_0^1\overline{\mathfrak{B}}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)\overline{\mathtt{S}}\left(\frac{\mathtt{a}+\mathtt{b}}{2},\cdot\right)d\mathfrak{y}\right]$$

$$\supseteq_{\mathcal{K}_C} \frac{2}{\left[\mathrm{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathfrak{V}(\mathfrak{u},\cdot)\mathrm{S}(\mathfrak{u},\cdot)d\mathfrak{u}\right]+\frac{2}{\left[\mathrm{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{T}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y},\mathfrak{y})\mathrm{H}(1-\mathfrak{y},1-\mathfrak{y})}\right.$$
$$\left.+\mathcal{U}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}^2(\mathfrak{y},1-\mathfrak{y})}\right].$$

Multiplying both sides by $\frac{\left[\mathrm{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}$ in the above inclusion, we have

$$\frac{\left[\mathrm{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{V}\left(\frac{\mathrm{a}+\mathrm{b}}{2},\cdot\right)\mathrm{S}\left(\frac{\mathrm{a}+\mathrm{b}}{2},\cdot\right)$$
$$\supseteq_{\mathcal{K}_C}\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathfrak{V}(\mathfrak{u},\cdot)\mathrm{S}(\mathfrak{u},\cdot)d\mathfrak{u}$$
$$+\mathcal{T}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y},\mathfrak{y})\mathrm{H}(1-\mathfrak{y},1-\mathfrak{y})}+\mathcal{U}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}^2(\mathfrak{y},1-\mathfrak{y})}.$$

Accordingly, the above theorem can be proved.

□

Example 5.5. *By virtue of Example 6.1, one has*

$$\frac{\left[\mathrm{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{V}\left(\frac{\mathrm{a}+\mathrm{b}}{2},\cdot\right)\mathrm{S}\left(\frac{\mathrm{a}+\mathrm{b}}{2},\cdot\right)\approx[0.25,12.93203]$$
$$\mathcal{T}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}^2(\mathfrak{y},1-\mathfrak{y})}\approx[0.3333,3.85447]$$

and

$$\mathcal{U}(\mathrm{a},\mathrm{b})\int_0^1\frac{d\mathfrak{y}}{\mathrm{H}(\mathfrak{y},\mathfrak{y})\mathrm{H}(1-\mathfrak{y},1-\mathfrak{y})}\approx[0,1.64085].$$

This implies

$$[0.25,12.93203]\supseteq_{\mathcal{K}_C}[0.41666,11.43906].$$

This verifies Theorem 5.5.

6. Ostrowski type inequality for Godunova-Levin stochastic processes

An Ostrowski type inequality s developed here along with some examples for Godunova-Levin functions with a more generalized class. The lemma that follows helps us to accomplish our objective [22].

Lemma 6.1. *Consider a differentiable mean square stochastic process* $\mathfrak{V}:\mathrm{I}\times\mathrm{v}\subseteq\mathrm{R}\to\mathrm{R}$ *on* I^o. *Likewise, if* $\mathfrak{V}'$ *is integrable in the mean square sense on* $[\mathrm{a},\mathrm{b}]$, *then one has*

$$\mathfrak{V}(\mathfrak{b},\cdot)-\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathfrak{V}(\mathfrak{y},\cdot)d\mathfrak{y}$$
$$=\frac{(\mathfrak{b}-\mathrm{a})^2}{\mathrm{b}-\mathrm{a}}\int_0^1\mathfrak{y}\mathfrak{V}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathrm{a},\cdot)d\mathfrak{y}-\frac{(\mathrm{b}-\mathfrak{b})^2}{\mathrm{b}-\mathrm{a}}\int_0^1\mathfrak{y}\mathfrak{V}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathrm{b},\cdot)d\mathfrak{y},\ \forall\,\mathfrak{b}\in[\mathrm{a},\mathrm{b}].$$

Theorem 6.1. *Consider three non-negative functions* $\mathtt{h}, \mathtt{h}_1, \mathtt{h}_2 : (0,1) \to \mathtt{R}$ *with* $\mathfrak{y} \subseteq \frac{1}{\mathtt{h}(\mathfrak{y})}$ *for each* $\mathfrak{y} \in (0,1)$. *Let a differentiable mean square interval-valued stochastic process* $\mathfrak{V} : \mathtt{I} \times \mathtt{v} \subseteq \mathtt{R} \to \mathtt{R}_{\mathtt{I}}^{+}$ *on* $\mathtt{I}^{o}$ *with* $\mathfrak{V}'$ *as integrable in the mean square sense on* $[\mathtt{a}, \mathtt{b}]$. *If* $|\mathfrak{V}'|$ *is an* $(\mathtt{h}_1, \mathtt{h}_2)$*-Godunova-Levin stochastic process and satisfying that* $|\mathfrak{V}'(\mathfrak{b}, \cdot)| \supseteq_{\mathcal{K}_C} \gamma$ *for each* $\mathfrak{b}$*, then one has*

$$\begin{aligned}
&\left\{\left|\underline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\underline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|,\left|\overline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\overline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|\right\}\\
&\supseteq_{\mathcal{K}_C}\frac{\gamma\left[(\mathfrak{b}-\mathtt{a})^2+(\mathtt{b}-\mathfrak{b})^2\right]}{\mathtt{b}-\mathtt{a}}\int_0^1\left[\frac{1}{\mathtt{h}(\mathfrak{y})\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{h}(\mathfrak{y})\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}
\end{aligned}$$

$\forall\ \mathfrak{b} \in [\mathtt{a}, \mathtt{b}]$.

Proof. By virtue of Lemma 6.1 and the fact that $|\mathfrak{V}'|$ is an $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova-Levin stochastic process, we have

$$\begin{aligned}
&\left\{\left|\underline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\underline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|,\left|\overline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\overline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|\right\}\\
&\supseteq_{\mathcal{K}_C}\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\,|\mathfrak{V}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{a},\cdot)|\,d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\,|\mathfrak{V}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{b},\cdot)|\,d\mathfrak{y}.
\end{aligned}$$

Utilizing the interval order inclusion relation, one has

$$\begin{aligned}
&\left\{\left|\underline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\underline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|\right\}\\
&\leq\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\underline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{a},\cdot)\right|d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\underline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{b},\cdot)\right|d\mathfrak{y}
\end{aligned}$$

$$\begin{aligned}
&\left\{\left|\overline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\overline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|\right\}\\
&\geq\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\overline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{a},\cdot)\right|d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\overline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{b},\cdot)\right|d\mathfrak{y}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\underline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{a},\cdot)\right|d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\underline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{b},\cdot)\right|d\mathfrak{y}\\
&\leq\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left[\frac{|\underline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\underline{\mathfrak{V}}'(\mathtt{a},\cdot)|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left[\frac{|\underline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\underline{\mathfrak{V}}'(\mathtt{b},\cdot)|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}.
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\overline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{a},\cdot)\right|d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left|\overline{\mathfrak{V}}'(\mathfrak{y}\mathfrak{b}+(1-\mathfrak{y})\mathtt{b},\cdot)\right|d\mathfrak{y}\\
&\geq\frac{(\mathfrak{b}-\mathtt{a})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left[\frac{|\overline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\overline{\mathfrak{V}}'(\mathtt{a},\cdot)|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}+\frac{(\mathtt{b}-\mathfrak{b})^2}{\mathtt{b}-\mathtt{a}}\int_0^1\mathfrak{y}\left[\frac{|\overline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\overline{\mathfrak{V}}'(\mathtt{b},\cdot)|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}.
\end{aligned}$$

Consequently, we have

$$\frac{(\mathfrak{b}-\mathsf{a})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \mathfrak{y}\left[\frac{|\underline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\underline{\mathfrak{V}}'(\mathsf{a},\cdot)|}{\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$+\frac{(\mathsf{b}-\mathfrak{b})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \mathfrak{y}\left[\frac{|\underline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\underline{\mathfrak{V}}'(\mathsf{b},\cdot)|}{\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$\leq \frac{\gamma(\mathfrak{b}-\mathsf{a})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$+\frac{(\mathsf{b}-\mathfrak{b})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$

and

$$\frac{(\mathfrak{b}-\mathsf{a})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \mathfrak{y}\left[\frac{|\overline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\overline{\mathfrak{V}}'(\mathsf{a},\cdot)|}{\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$+\frac{(\mathsf{b}-\mathfrak{b})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \mathfrak{y}\left[\frac{|\overline{\mathfrak{V}}'(\mathfrak{b},\cdot)|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\overline{\mathfrak{V}}'(\mathsf{b},\cdot)|}{\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$\geq \frac{\gamma(\mathfrak{b}-\mathsf{a})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}$$
$$+\frac{(\mathsf{b}-\mathfrak{b})^2}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}.$$

This implies

$$\left|\underline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\underline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|$$
$$\leq \frac{\gamma\left[(\mathfrak{b}-\mathsf{a})^2+(\mathsf{b}-\mathfrak{b})^2\right]}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}. \tag{6.1}$$

Similarily

$$\left|\overline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\overline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|$$
$$\geq \frac{\gamma\left[(\mathfrak{b}-\mathsf{a})^2+(\mathsf{b}-\mathfrak{b})^2\right]}{\mathsf{b}-\mathsf{a}}\int_0^1 \left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}. \tag{6.2}$$

The proof is completed. □

Example 6.1. *Let* $[\mathsf{a},\mathsf{b}]=[0,1]$, $\mathrm{h}(\mathfrak{y})=\frac{1}{\mathfrak{y}}$, $\mathrm{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}}$ *and* $\mathrm{h}_2(\mathfrak{y})=1$ *for all* $\mathfrak{y}\in(0,1)$. *Suppose that a stochastic process* $\mathfrak{V}$ *is defined as*

$$\mathfrak{V}(\mathfrak{u},\cdot)=[\mathfrak{u}^2,3-e^{\mathfrak{u}}]$$

Choose $\mathfrak{b}=1$*; then, we have*

$$\left|\underline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}}\underline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right|=\frac{2}{3}. \tag{6.3}$$

Since $\left|\underline{\mathfrak{V}}'(\mathfrak{b},\cdot)\right| \leq \gamma = 2$, *we have*

$$\frac{\gamma\left[(\mathfrak{b}-\mathtt{a})^2+(\mathtt{b}-\mathfrak{b})^2\right]}{\mathtt{b}-\mathtt{a}}\int_0^1\left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y} = 1. \tag{6.4}$$

Similarily

$$\left|\overline{\mathfrak{V}}(\mathfrak{b},\cdot)-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\overline{\mathfrak{V}}(\mathfrak{y},\cdot)d\mathfrak{y}\right| = 2. \tag{6.5}$$

Since $\left|\overline{\mathfrak{V}}'(\mathfrak{b},\cdot)\right| \leq \gamma = e$, *we have*

$$\frac{\gamma\left[(\mathfrak{b}-\mathtt{a})^2+(\mathtt{b}-\mathfrak{b})^2\right]}{\mathtt{b}-\mathtt{a}}\int_0^1\left[\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathrm{h}(\mathfrak{y})\mathrm{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y} = \frac{e}{2}. \tag{6.6}$$

Consequently,

$$\left[\frac{2}{3},2\right] \supseteq_{\mathcal{K}_C} \left[1,\frac{e}{2}\right].$$

This verifies Theorem 6.1.

Remark 6.1.
If $\mathrm{h}(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathrm{h}_1(\mathfrak{y}) = \frac{1}{\mathrm{h}(\mathfrak{y})}$ *and* $\mathrm{h}_2(\mathfrak{y}) = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$, *then Theorem 6 has a similar result for the* h*-convex-function [22].*

7. Generalized sequential variants of Jensen type inclusions for Godunova-Levin stochastic processes

In this section, we develop the Jensen type inclusion for the $(\mathrm{h}_1,\mathrm{h}_2)$-Godunova-Levin stochastic process, and with some remarks we show that this is a more generalized class. Throughout we make use of supermultiplicative and submultiplicative type mappings; regarding that concern, please see [49].

Theorem 7.1. *Let* $g_i, \mathfrak{y}_i \in \mathrm{R}^+$. *Consider that* $\mathrm{h}_1,\mathrm{h}_2 : [0,1] \to \mathrm{R}^+$. *An interval-valued stochastic process* $\mathfrak{V} = [\underline{\mathfrak{V}},\overline{\mathfrak{V}}] : \mathtt{I}\times\mathtt{v} \to \mathrm{R}_{\mathtt{I}}^+$ *where* $[\mathtt{a},\mathtt{b}] \subseteq \mathtt{I} \subseteq \mathrm{R}$ *is considered to be an* $(\mathrm{h}_1,\mathrm{h}_2)$*-Godunova-Levin stochastic process or* $\mathfrak{V} \in \mathcal{SGHPX}((\mathrm{h}_1,\mathrm{h}_2),[\mathtt{a},\mathtt{b}],\mathrm{R}_{\mathtt{I}}^+)$*; if* $\forall\ \mathtt{a},\mathtt{b} \in \mathtt{I}$ *and* $\mathfrak{y} \in (0,1)$, *then one has*

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}},\cdot\right) \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathfrak{y}_1,\cdot)+\mathfrak{V}(\mathfrak{y}_d,\cdot)-\sum_{i=1}^{d}\left[\frac{\mathfrak{V}(\mathfrak{y}_i,\cdot)}{\mathrm{H}\left(\frac{g_i}{G_d},\frac{G_{i-1}}{G_d}\right)}\right]. \tag{7.1}$$

Proof. Since $G_d = \sum_{i=1}^{d} g_i$ and $\mathfrak{V}$ is a harmonic $(\mathrm{h}_1,\mathrm{h}_2)$-Godunova-Levin stochastic process, taking into account [31], [Theorem 3.5], we have

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}},\cdot\right) = \mathfrak{V}\left(\frac{1}{\sum_{i=1}^{d}\frac{g_i}{G_d}\left(\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{\mathfrak{y}_i}\right)},\cdot\right) \supseteq_{\mathcal{K}_C} \sum_{i=1}^{d}\left[\frac{\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{\mathfrak{y}_k}},\cdot\right)}{\mathrm{h}_1\left(\frac{g_i}{G_d}\right)\mathrm{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right].$$

By virtue of the Kulisch-Miranker order relation, if $(\mathtt{h}_1, \mathtt{h}_2)$ denotes supermultiplicative type mappings, $\sum_{i=1}^{d} \mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right) \leq 1$, then we have

$$\begin{aligned}
\underline{\mathfrak{V}}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}}, \cdot\right) &\leq \sum_{i=1}^{d}\left[\frac{\underline{\mathfrak{V}}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{\mathfrak{y}_k}}, \cdot\right)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\leq \sum_{i=1}^{d}\left[\frac{\underline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\underline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\underline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\leq \sum_{i=1}^{d}\left[\frac{\underline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\underline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)} - \frac{\underline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\leq \underline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\underline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\sum_{i=1}^{d}\left[\frac{\underline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&= \underline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\underline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\sum_{i=1}^{d}\left[\frac{\underline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{H}\left(\frac{g_i}{G_d}, \frac{G_{i-1}}{G_d}\right)}\right].
\end{aligned}$$

Similarly, if $(\mathtt{h}_1, \mathtt{h}_2)$ denotes submultiplicative type mappings, $\sum_{i=1}^{d} \mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right) \geq 1$, then we have

$$\begin{aligned}
\overline{\mathfrak{V}}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}}, \cdot\right) &\geq \sum_{i=1}^{d}\left[\frac{\overline{\mathfrak{V}}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{\mathfrak{y}_k}}, \cdot\right)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\geq \sum_{i=1}^{d}\left[\frac{\overline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\overline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\overline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\geq \sum_{i=1}^{d}\left[\frac{\overline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\overline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)} - \frac{\overline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&\geq \overline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\overline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\sum_{i=1}^{d}\left[\frac{\overline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{h}_1\left(\frac{g_i}{G_d}\right)\mathtt{h}_2\left(\frac{G_{i-1}}{G_d}\right)}\right] \\
&= \overline{\mathfrak{V}}(\mathfrak{y}_1, \cdot)+\overline{\mathfrak{V}}(\mathfrak{y}_d, \cdot)-\sum_{i=1}^{d}\left[\frac{\overline{\mathfrak{V}}(\mathfrak{y}_i, \cdot)}{\mathtt{H}\left(\frac{g_i}{G_d}, \frac{G_{i-1}}{G_d}\right)}\right].
\end{aligned}$$

Take into account results related to submultiplicative- and supermultiplicative-type mappings, we have

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}}, \cdot\right) \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathfrak{y}_1, \cdot)+\mathfrak{V}(\mathfrak{y}_d, \cdot)-\sum_{i=1}^{d}\left[\frac{\mathfrak{V}(\mathfrak{y}_i, \cdot)}{\mathtt{H}\left(\frac{g_i}{G_d}, \frac{G_{i-1}}{G_d}\right)}\right].$$

$\square$

Remark 7.1. *(i) If* $\mathtt{h}_1(\mathfrak{y}) = \mathtt{h}(\mathfrak{y})$ *and* $\mathtt{h}_2(\mathfrak{y}) = 1$*, then Theorem 7.1 has a similar result for harmonic* $\mathtt{h}$*-Godunova-Levin functions, which is new as well.*

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}},\cdot\right) \supseteq_{\mathcal{K}_C} \mathfrak{V}(\mathfrak{y}_1,\cdot)+\mathfrak{V}(\mathfrak{y}_d,\cdot)-\sum_{i=1}^{d}\left[\frac{\mathfrak{V}(\mathfrak{y}_i,\cdot)}{\mathtt{h}\left(\frac{g_i}{G_d}\right)}\right]. \tag{7.2}$$

(ii) If $\mathtt{h}_1(\mathfrak{y}) = \frac{1}{\mathtt{h}(\mathfrak{y})}$ *and* $\mathtt{h}_2(\mathfrak{y}) = 1$ *with* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$*, then Theorem 7.1 has a similar result for harmonic* $\mathtt{h}$*-convex functions, which is new as well.*

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}},\cdot\right) \leq \mathfrak{V}(\mathfrak{y}_1,\cdot)+\mathfrak{V}(\mathfrak{y}_d,\cdot)-\sum_{i=1}^{d} h\left(\frac{g_i}{G_{\mathfrak{y}}}\right)\mathfrak{V}(\mathfrak{y}_i,\cdot). \tag{7.3}$$

(iii) If $\mathtt{h}_1(\mathfrak{y}) = \mathtt{h}_2(\mathfrak{y}) = 1$*, Theorem 7.1 has a similar result for* $\mathtt{P}$*-functions, which is new as well.*

$$\mathfrak{V}\left(\frac{1}{\frac{1}{\mathfrak{y}_1}+\frac{1}{\mathfrak{y}_d}-\frac{1}{G_d}\sum_{i=1}^{d}\frac{g_i}{\mathfrak{y}_i}},\cdot\right) \leq \mathfrak{V}(\mathfrak{y}_1,\cdot)+\mathfrak{V}(\mathfrak{y}_d,\cdot)-\mathfrak{V}(\mathfrak{y}_i,\cdot). \tag{7.4}$$

8. Conclusions and open problems

As part of this note, we use Kulisch-Miranker types of inclusions in conjunction with stochastic processes, and we have refined and improved three well known inequalities, known as Hermite-Hadamard, Ostrowski, and Jensen types. Additionally, we have generalized the work in some recent articles related to stochastic convexity. To prove the Hermite-Hadamard type results, we use two types of integral operators: classical and generalized fractional integral operators. Moreover, we present a new way to treat Jensen type inclusions under interval stochastic processes by using a discrete sequential form. For further development of these results, we recommend that interested researchers use fractional operators based on the stochastic version defined in that [44]:

$$\mathtt{J}_{\mathtt{a}^+}^{\alpha}\mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{a}}^{\mathtt{q}} e^{-\frac{1-\alpha}{\alpha}(\mathtt{q}-\mathtt{w})}\mathfrak{V}(\mathtt{w},\cdot)\,d\mathtt{w}, \quad \mathtt{q} > \mathtt{a}, \alpha > 0 \quad (a.e)$$

and

$$\mathtt{J}_{\mathtt{b}^-}^{\alpha}\mathfrak{V}(\mathtt{q}) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{q}}^{\mathtt{b}} e^{-\frac{1-\alpha}{\alpha}(\mathtt{w}-\mathtt{q})}\mathfrak{V}(\mathtt{w},\cdot)\,d\mathtt{w}, \quad \mathtt{q} < \mathtt{b}, \alpha > 0 \quad (a.e).$$

According to inequality theory, there are various types of order relations, including total order relations, inclusions, pseudo-order relations, fuzzy order relations, standard partial order relations, and various others [50–56]. This paper demonstrates that some results, more specifically Theorem 11, do not apply to Milne type inequalites in the inclusion order setting [57]. In the context of the center and radius order relation, Abbas et al. [58] recently developed a number of inequalities that are of full order. Therefore, interested researchers can apply the above to test whether Theorem 11 holds with this type of order relation when using a fractional operator defined with an exponential kernel for Milne type results.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. N. Malik, M. Raza, Q. Xin, J. Sokol, R. Manzoor, S. Zainab, On convex functions associated with symmetric cardioid domain, *Symmetry*, **13** (2021), 2321. https://doi.org/10.3390/sym13122321
2. H. N. Shi, W. S. Du, Schur-power convexity of a completely symmetric function dual, *Symmetry* **11** (2019), 897. https://doi.org/10.3390/sym11070897
3. M. Mnif, H. Pham, Stochastic optimization under constraints, *Stoch. Proc. Appl.*, **93** (2001), 149–180. https://doi.org/10.1016/S0304-4149(00)00089-2
4. A. Liu, V. K. Lau, B. Kananian, Stochastic successive convex approximation for non-convex constrained stochastic optimization, *IEEE Trans. Signal Proc.*, **67** (2019), 4189–4203. https://doi.org/10.1109/TSP.2019.2925601
5. W. Afzal, M. Abbas, J. E. Macias-Diaz, S. Treanţă, Some $\mathtt{H}$-Godunova-Levin Function inequalities using center radius (Cr) order relation, *Fractal Fract.*, **6** (2022), 518. https://doi.org/10.3390/fractalfract6090518
6. V. Preda, L. I. Catana, Tsallis Log-Scale-Location models. moments, Gini index and some stochastic orders, *Mathematics*, **9** (2021), 1216. https://doi.org/10.3390/math9111216
7. W. Afzal, A. A. Lupaş, K. Shabbir, Hermite-Hadamard and Jensen-type inequalities for harmonical $(\mathtt{h}_1, \mathtt{h}_2)$-Godunova Levin interval-valued functions, *Mathematics*, **10** (2022), 2970. https://doi.org/10.3390/math10162970
8. T. Saeed, W. Afzal, K. Shabbir, S. Treanţă, M. De la Sen, Some Novel Estimates of Hermite-Hadamard and Jensen Type Inequalities for (h_1, h_2)-convex functions pertaining to total order relation, *Mathematics*, **10** (2022), 4777. https://doi.org/10.3390/math10244777
9. W. Afzal, W. Nazeer, T. Botmart, S. Treanţă, Some properties and inequalities for generalized class of harmonical Godunova-Levin function via center radius order relation, *AIMS Math.*, **8** (2023), 1696–1712. http://dx.doi.org/10.3934/math.2023087

10. R. E. Moore, *Method and Applications of Interval Analysis*, Philadelphia: Society for Industrial and Applied Mathematics, 1979.

11. W. Afzal, M. Abbas, W. Hamali, A. M. Mahnashi, M. D. Sen, Hermite-Hadamard-Type inequalities via Caputo-Fabrizio fractional integral for h-Godunova-Levin and (h_1, h_2)-Convex functions, *Fractal Fract.*, **7** (2023), 687. https://doi.org/10.3390/fractalfract7090687

12. K. Nikodem, On convex stochastic processes, *Aequat. Math.*, **20** (1980), 184–197.

13. A. Skowroński, On some properties of J-convex stochastic processes, *Aequat. Math.*, **44** (1992), 249–258.

14. L. Li, Z. Hao, On Hermite-Hadmard inequlity for h-convex stochastic processes, *Aequat. Math.*, **91** (2017), 909–920. https://doi.org/10.1007/s00010-017-0488-5

15. H. Budak, M. Z. Sarikaya, A new Hermite-Hadamard inequality for h-convex stochastic processes, *RGMIA Res. Rep. Collect.*, **19** (2016), 30.

16. N. Okur, ˙I. ˙Işcan, E. Y. Dizdar, Hermite-Hadmard type inequalities for harmonically convex stochastic processes, *Int. Econ. Adm. Stud.*, **18** (2018), 281–292.

17. L. González, D. Kotrys, K. Nikodem, Separation by convex and strongly convex stochastic processes, *Publ. Math. Debrecen*, **3** (2016), 365–372.

18. O. Almutairi, A. Kiliçman, Generalized Fejer-Hermite-Hadamard type via generalized $(h - m)$-convexity on fractal sets and applications, *Chaos Solit. Fract.*, **147** (2021), 110938. https://doi.org/10.1016/j.chaos.2021.110938

19. H. Zhou, M. S. Saleem, M. Ghafoor, J. Li, Generalization of h-convex stochastic processes and some classical inequalities, *Math. Probl. Eng.*, **2020** (2020), 1583807. https://doi.org/10.1155/2020/1583807

20. H. Fu, M. S. Saleem, W. Nazeer, M. Ghafoor, P. Li, On Hermite-Hadamard type inequalities for η-polynomial convex stochastic processes, *AIMS Math.*, **6** (2021), 6322–6339. http://dx.doi.org/10.3934/math.2021371

21. M. Tunc, Ostrowski-type inequalities via h-convex functions with applications to special means, *J. Inequal. Appl.*, **2013** (2013), 326. http://dx.doi.org/10.1186/1029-242X-2013-326

22. L. Gonzales, J. Materano, M. V. Lopez, Ostrowski-Type inequalities via h-convex stochastic processes, *JP J. Math. Sci.*, **15** (2016), 15–29.

23. Y. An, G. Ye, D. Zhao, W. Liu, Hermite-Hadamard type inequalities for interval $(\mathrm{h}_1, \mathrm{h}_2)$-Convex functions, *Mathematics*, **7** (2019), 436. http://dx.doi.org/10.3390/math7050436

24. Y. Chalco-Cano, A. Flores-Franulič, H. Román-Flores, Ostrowski-type inequalities for interval-valued functions using generalized Hukuhara derivative, *Comput. Appl. Math.*, **31** (2012), 457–472. http://dx.doi.org/10.1590/S1807-03022012000300002

25. L. Chen, M. S. Saleem, M. S. Zahoor, R. Bano, Some inequalities related to Interval-Valued η_h-Convex functions, *J. Math.*, **2021** (2021), 6617074. http://dx.doi.org/10.1155/2021/6617074

26. H. Budak, A. Kashuri, S. Butt, Fractional Ostrowski type inequalities for interval valued functions, *Filomat*, **36** (2022), 2531–2540. https://doi.org/10.2298/FIL2208531B

27. H. Bai, M. S. Saleem, W. Nazeer, M. S. Zahoor, T. Zhao, Hermite-Hadamard- and Jensen-type inequalities for interval $(\mathtt{h}_1, \mathtt{h}_2)$ nonconvex function, *J. Math.*, **2020** (2020), 3945384. https://doi.org/10.1155/2020/3945384

28. H. Agahi, A. Babakhani, On fractional stochastic inequalities related to Hermite-Hadamard and Jensen types for convex stochastic processes, *Aequat. Math.*, **90** (2016), 1035–1043. https://doi.org/10.1007/s00010-016-0425-z

29. J. E. H. Hernandez, On $(m, \mathtt{h}_1, \mathtt{h}_2)$-G-convex dominated stochastic processes, *Kragujev. J. Math.*, **46** (2022), 215–227.

30. M. Vivas-Cortez, C. Garcıa, Ostrowski type inequalities for functions whose derivatives are $(m, \mathtt{h}_1, \mathtt{h}_2)$-convex, *Appl. Math. Inf. Sci.*, **11** (2017), 79–86. https://doi.org/10.18576/amis/110110

31. W. Afzal, T. Botmart, Some novel estimates of Jensen and Hermite–Hadamard inequalities for h-Godunova-Levin stochastic processes, *AIMS Math.*, **8** (2023), 7277–7291. http://dx.doi.org/10.3934/math.2023366

32. W. Afzal, K. Shabbir, T. Botmart, Generalized version of Jensen and Hermite-Hadamard inequalities for interval-valued (h_1, h_2)-Godunova-Levin functions, *AIMS Math.*, **8** (2023), 13793–13794. http://dx.doi.org/10.3934/math.20221064

33. X. Zhang, K. Shabbir, W. Afzal, H. Xiao, D. Lin, Hermite-hadamard and jensen-type inequalities via Riemann integral operator for a generalized class of godunova–levin functions, *J. Math.*, **2022** (2022), 3830324. https://doi.org/10.1155/2022/3830324

34. W. Afzal, M. Abbas, S. M. Eldin, Z. A. Khan, Some well known inequalities for (h_1, h_2)-convex stochastic process via interval set inclusion relation, *AIMS Math.*, **8** (2023), 19913–19932. http://dx.doi.org/10.3934/math.20231015

35. T. Saeed, W. Afzal, M. Abbas, S. Treanţă, M. De la Sen, Some new generalizations of integral inequalities for Harmonical Cr-(h_1, h_2)-Godunova-Levin functions and applications, *Mathematics*, **10** (2022), 4540. https://doi.org/10.3390/math10234540

36. W. Afzal, S. M. Eldin, W. Nazeer, A. M. Galal, Some integral inequalities for Harmonical *Cr-h*-Godunova-Levin stochastic processes, *AIMS Math.*, **8** (2023), 13473–13491. http://dx.doi.org/10.3934/math.2023683

37. W. Afzal, K. Shabbir, T. Botmart, S. Treanţă, Some new estimates of well known inequalities for (h_1, h_2)-Godunova-Levin functions by means of center-radius order relation, *AIMS Math.*, **8** (2023), 3101–3119. http://dx.doi.org/10.3934/math.2023160

38. W. Afzal, E. Prosviryakov, S. M. El-Deeb, Y. Almalki, Some new estimates of Hermite-Hadamard, Ostrowski and Jensen-type inclusions for h-convex stochastic process via interval-valued functions, *Symmetry*, **15** (2023), 831. https://doi.org/10.3390/sym15040831

39. T. Du, Y. Peng, Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Comput. Appl. Math.*, **440** (2014), 115582. https://doi.org/10.1016/j.cam.2023.115582

40. T. Du, T. Zhou, On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings, *Chaos Solit. Fract.*, **156** (2022), 111846. https://doi.org/10.1016/j.chaos.2022.111846

41. T. Zhou, Z. Yuan, T. Du, On the fractional integral inclusions having exponential kernels for interval-valued convex functions, *Math. Sci.*, **17** (2023), 107–120. https://doi.org/10.1007/s40096-021-00445-x
42. T. Du, C. Luo, Z. Cao, On the Bullen-type inequalities via generalized fractional integrals and their application, *Fractals*, **29** (2021), 2150188. https://doi.org/10.1142/S0218348X21501887
43. U. Kulish, W. Miranker, *Computer Arithmetic in Theory and Practice*, New York: Academic Press, 2014.
44. F. Jarad, S. K. Sahoo, K. S. Nisar, S. Treanţă, H. Emadifar, T. Botmart, New stochastic fractional integral and related inequalities of Jensen-Mercer and Hermite-Hadamard-Mercer type for convex stochastic processes, *J. Inequal. Appl.*, **2023** (2023), 51. https://doi.org/10.1186/s13660-023-02944-y
45. X. Liu, G. Ye, D. Zhao, W. Liu, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *J. Inequal. Appl.*, **2019** (2019), 266. https://doi.org/10.1186/s13660-019-2217-1
46. L. González, N. Merentes. M. Valera-López, Some estimates on the Hermite-Hadamard inequality through convex and quasi-convex stochastic processes, *Math. Aeterna*, **5** (2015), 745–767.
47. E. Set, M. Tomar, S. Maden, Hermite-Hadamard type inequalities for s-convex stochastic processes in the second sense, *Turkish J. Anal. Number Theory*, **2** (2014), 202–207. https://doi.org/10.12691/tjant-2-6-3
48. D. Barráez, L. González, N. Merentes, A. Moros, On h-convex stochastic processes, *Math. Aeterna*, **5** (2015), 571–581.
49. B. Feng, M. Ghafoor, Y. M. Chu, M. I. Qureshi, X. Feng, C. Yao, et al., Hermite-Hadamard and Jensen's type inequalities for modified (p,h)-convex functions, *AIMS Math.*, **5** (2020), 6959–6971. http://dx.doi.org/10.3934/math.2020446
50. M. B. Khan, H. M. Srivastava, P. O. Mohammed, K. Nonlaopon, Y. S. Hamed, Some new estimates on coordinates of left and right convex interval-valued functions based on pseudo order relation, *Symmetry*, **14** (2022), 473. https://doi.org/10.3390/sym14030473
51. M. B. Khan, M. A. Noor, K. I. Noor, Y. M. Chu, New Hermite-Hadamard-type inequalities for-convex fuzzy-interval-valued functions, *Adv. Differ. Equ.*, **2021** (2021), 149. https://doi.org/10.1186/s13662-021-03245-8
52. Y. Almalki, W. Afzal, Some new estimates of Hermite-Hadamard inequalities for harmonical Cr-h-convex functions via generalized fractional integral operator on set-valued mappings, *Mathematics*, **11** (2023), 4041. https://doi.org/10.3390/math11194041
53. M. B. Khan, M. A. Noor, M. Noor, M. Al-Shomrani, L. Abdullah, Some novel inequalities for LR-h-convex interval-valued functions by means of pseudo-order relation, *Math. Meth. App Sci.*, **45** (2022), 1310–1340. https://doi.org/10.1002/mma.7855
54. W. Liu, F. Shi, G. Ye, D. Zhao, Some inequalities for Cr-Log-h- convex functions, *J. Inequal. Appl.*, **2022** (2022), 160. https://doi.org/10.1186/s13660-022-02900-2
55. W. Afzal, K. Shabbir, M. Arshad, J. K. K. Asamoah, A. M. Galal, Some novel estimates of integral inequalities for a generalized class of harmonical convex mappings by means of center-radius order relation, *J. Math.*, **2023** (2023), 8865992. https://doi.org/10.1155/2023/8865992

56. M. B. Khan, G. Santos-García, M. A. Noor, M. S. Soliman, Some new concepts related to fuzzy fractional calculus for up and down convex fuzzy-number valued functions and inequalities, *Chaos Solit. Fract.*, **164** (2022), 112692. https://doi.org/10.1016/j.chaos.2022.112692

57. H. Román-Flores, V. Ayala, A. Flores-Franulič, Milne type inequality and interval orders, *Comput. Appl. Math.*, **40** (2021), 130. https://doi.org/10.1007/s40314-021-01500-y

58. M. Abbas, W. Afzal, T. Botmart, A. M. Galal, Ostrowski and Hermite-Hadamard type inequalities for h-convex stochastic processes by means of center-radius order relation, *AIMS Math.*, **8** (2023), 16013–16030. http://dx.doi.org/10.3934/math.2023817