AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 5110–5121.
DOI: 10.3934/math.2024248
Received: 10 September 2023
Revised: 16 January 2024
Accepted: 18 January 2024
Published: 25 January 2024

Research article

Browder spectra of closed upper triangular operator matrices

Qingmei Bai[1,2,3,*], Alatancang Chen[1,2,3] and Jingying Gao[1]

[1] School of Mathematical Sciences, Hohhot Minzu College, Hohhot, 010051, China
[2] Key Laboratory of Infinite-dimensional Hamiltonian System and Its Algorithm Application (Inner Mongolia Normal University), Ministry of Education, Hohhot, 010022, China
[3] Center for Applied Mathematics Inner Mongolia, Hohhot, 010022, China

* **Correspondence:** Email: baiqingmei@email.imnc.edu.cn.

Abstract: Let $T_B = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ be an unbounded upper operator matrix with diagonal domain, acting in $\mathcal{H} \oplus \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. In this paper, some sufficient and necessary conditions are characterized under which T_B is Browder (resp., invertible) for some closable operator B with $\mathcal{D}(B) \supset \mathcal{D}(D)$. Further, a sufficient and necessary condition is given under which the Browder spectrum (resp., spectrum) of T_B coincides with the union of the Browder spectrum (resp., spectrum) of its diagonal entries.

Keywords: closed operator; operator matrix; Browder spectrum; spectrum
Mathematics Subject Classification: 47A25, 47A55

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex infinite-dimensional separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ (resp., $C(\mathcal{H},\mathcal{K})$, $C^+(\mathcal{H},\mathcal{K})$) be the set of all bounded (resp., closed, closable) operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, we use $\mathcal{B}(\mathcal{H})$, $C(\mathcal{H})$, and $C^+(\mathcal{H})$ as usual. The domain of $T \in C(\mathcal{H})$ is denoted by $\mathcal{D}(T)$. A closed operator T is said to be left (resp., right) Fredholm if its range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), where $\alpha(T)$ and $\beta(T)$ denote the dimension of the null space $\mathcal{N}(T)$ and the quotient space $\mathcal{H}/\mathcal{R}(T)$, respectively. T is said to be Fredholm if it is both left and right Fredholm. If T is a left or right Fredholm operator, then we define the index of T by $ind(T) = \alpha(T) - \beta(T)$. T is called Weyl if it is a Fredholm operator of index zero. The left essential spectrum, right essential spectrum, essential spectrum, and Weyl spectrum of T are, respectively, defined by

$$\sigma_{le}(T) \quad = \quad \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Fredholm}\},$$

$$\begin{aligned}\sigma_{re}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Fredholm}\},\\ \sigma_{e}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},\\ \sigma_{w}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.\end{aligned}$$

Iterates $T^2, T^3, \cdots$ of $T \in C(\mathcal{H})$ are defined by $T^n x = T(T^{n-1}x)$ for $x \in \mathcal{D}(T^n)$ $(n = 2, 3, \cdots)$, where

$$\mathcal{D}(T^n) = \{x : x, Tx, \cdots, T^{n-1}x \in \mathcal{D}(T)\}.$$

It is easy to see that

$$\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1}), \quad \mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n).$$

If $n \geq 0$, we follow the convention that $T^0 = I$ (the identity operator on $\mathcal{H}$, with $\mathcal{D}(I) = \mathcal{H}$). Then $\mathcal{N}(T^0) = \{0\}$ and $\mathcal{R}(T^0) = \mathcal{H}$. It is also well known that if $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$, then $\mathcal{N}(T^n) = \mathcal{N}(T^k)$ for $n \geq k$. In this case, the smallest nonnegative integer k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$ is called the ascent of T, and is denoted by $asc(T)$. If no such k exists, we define $asc(T) = \infty$. Similarly, if $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$, then $\mathcal{R}(T^n) = \mathcal{R}(T^k)$ for $n \geq k$. Thus, we can analogously define the descent of T denoted by $des(T)$, and define $des(T) = \infty$ if $R(T^{n+1})$ is always a proper subset of $R(T^n)$. We call T left (resp., right) Browder if it is left (resp., right) Fredholm and $asc(T) < \infty$ (resp., $des(T) < \infty$). The left Browder spectrum, right Browder spectrum, and Browder spectrum of T are, respectively, defined by

$$\begin{aligned}\sigma_{lb}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Browder}\},\\ \sigma_{rb}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Browder}\},\\ \sigma_{b}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.\end{aligned}$$

We write $\sigma(T), \sigma_{ap}(T)$, and $\sigma_{\delta}(T)$ for the spectrum, approximate point spectrum, and defect spectrum of T, respectively.

For the spectrum and Browder spectrum, the perturbations of 2×2 bounded upper triangular operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, acting on a Hilbert or Banach space, have been studied by numerous authors (see, e.g., [5, 8, 9, 12, 13, 18]). However, the characterizations of the unbounded case are still unknown. In this paper, the unbounded operator matrix

$$T_B = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \subset \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}$$

is considered, where $A \in C(H)$, $D \in C(K)$, and $B \in C_D^+(\mathcal{K}, \mathcal{H}) = \{B \in C^+(\mathcal{K}, \mathcal{H}) : \mathcal{D}(B) \supset \mathcal{D}(D)\}$. It is obvious that T_B is a closed operator matrix. Our main goal is to present some sufficient and necessary conditions for T_B to be Browder (resp., invertible) for some closable operator $B \in C_D^+(\mathcal{K}, \mathcal{H})$ applying a space decomposition technique. Further, the sufficient and necessary condition, which is completely described by the diagonal operators A and D, for

$$\sigma_*(T_B) = \sigma_*(A) \cup \sigma_*(D) \tag{1.1}$$

is characterized, where σ_* is the Browder spectrum (resp., spectrum). This is an extension of the results from [3, 7, 10, 12]. In addition, this result is applied to a Hamiltonian operator matrix from elasticity theory.

Definition 1.1. *[4] Let $T : \mathcal{D}(T) \subset \mathcal{H} \longrightarrow \mathcal{K}$ be a densely defined closed operator. If there is an operator $T^{\dagger} : \mathcal{D}(T^{\dagger}) \subset \mathcal{K} \longrightarrow \mathcal{H}$ such that $\mathcal{D}(T^{\dagger}) = \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp}$, $\mathcal{N}(T^{\dagger}) = \mathcal{R}(T)^{\perp}$ and*

$$T^{\dagger}Tx = P_{\overline{\mathcal{R}(T^{\dagger})}}x, x \in \mathcal{D}(T), \quad TT^{\dagger}y = P_{\overline{\mathcal{R}(T)}}y, y \in \mathcal{D}(T^{\dagger})$$

then $T^{\dagger}$ is called the maximal Tseng inverse of T.

For the proof of the main results in the next sections, we need the following lemmas:

Lemma 1.1. *[2] Let $T_B = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \subset \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}$ be a closed operator matrix such that A, D are closed operators with dense domains and B is a closable operator. Then,*

(1) $\sigma_{le}(A) \cup \sigma_{re}(D) \subset \sigma_e(T_B) \subset \sigma_e(A) \cup \sigma_e(D)$.
(2) $\sigma_w(T_B) \subset \sigma_w(A) \cup \sigma_w(D)$.
(3) $\sigma_b(T_B) \subset \sigma_b(A) \cup \sigma_b(D)$.

Lemma 1.2. *[17] Let $T_B = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \subset \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}$ be a closed operator matrix such that A, D are closed operators with dense domains and B is a closable operator. Then,*

(1) $asc(A) \leq asc(T_B) \leq asc(A) + asc(D)$,
(2) $des(D) \leq des(T_B) \leq des(A) + des(D)$.

Lemma 1.3. *[16] Let $T \in C(H)$. Suppose that either $\alpha(T)$ or $\beta(T)$ is finite, and that $asc(T)$ is finite. Then, $\alpha(T) \leq \beta(T)$.*

Lemma 1.4. *[6] Let $T \in C(H)$. Then, the following inequalities hold for any non negative integer k:*

(1) $\alpha(T^k) \leq asc(T_B) \cdot \alpha(T)$;
(2) $\beta(T^k) \leq des(T_B) \cdot \beta(T)$.

Lemma 1.5. *[11] Suppose that the closed operator T is a Fredholm operator and B is T-compact. Then,*

(1) *$T + B$ is a Fredholm operator,*
(2) *$ind(T + B) = ind(T)$.*

Lemma 1.6. *[16] Let $T \in C(H)$. Suppose that $asc(T)$ is finite, and that $\alpha(T) = \beta(T) < \infty$. Then, $des(T) = asc(T)$.*

2. Browder spectra of operator matrix

In this section, some sufficient and necessary conditions for T_B to be Browder for some closable operator B with $\mathcal{D}(B) \supset \mathcal{D}(D)$ are given and the set $\bigcap_{B \in C_D^+(\mathcal{K},\mathcal{H})} \sigma_b(T_B)$ is estimated.

Theorem 2.1. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. If $\beta(A) = \infty$, then T_B is left Browder for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if A is left Browder.*

Proof. The necessity is obvious by Lemmas 1.1–1.3.

Now we verify the sufficiency. Let $p = asc(A) < \infty$, then $\alpha(A^p) \leq p \cdot \alpha(A) < \infty$ from Lemma 1.4. Thus, we get $\dim(\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp} = \infty$, since $\beta(A) = \infty$. It follows that there are two infinite dimensional subspaces Δ_1 and Δ_2 of $(\mathcal{R}(A)+\mathcal{N}(A^p))^\perp$ such that $(\mathcal{R}(A)+\mathcal{N}(A^p))^\perp = \Delta_1 \oplus \Delta_2$. Define an operator $B : \mathcal{K} \to \mathcal{H}$ by

$$B = T_B = \begin{bmatrix} 0 \\ U \\ 0 \end{bmatrix} : \mathcal{K} \to \begin{bmatrix} \mathcal{R}(A)+\mathcal{N}(A^p) \\ \Delta_1 \\ \Delta_2 \end{bmatrix}$$

where $U : \mathcal{K} \to \Delta_1$ is a unitary operator. Then, we can claim that T_B is left Browder.

First we prove that $\mathcal{N}(T_B) = \mathcal{N}(A) \oplus \{0\}$, which implies $\alpha(T_B) < \infty$. It is enough to verify that $\mathcal{N}(T_B) \subset \mathcal{N}(A) \oplus \{0\}$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(T_B)$, then $Ax + By = 0$ and $Dy = 0$. Thus, $Ax = -By \in \mathcal{R}(A) \cap \Delta_1 = \{0\}$, and hence $x \in \mathcal{N}(A)$ and $By = 0$. From the definition of B, we get $y = 0$, which means $\mathcal{N}(T_B) \subset \mathcal{N}(A) \oplus \{0\}$.

Next, we check that $\mathcal{R}(T_B)$ is closed. We only need to check that the range of $\begin{bmatrix} U \\ 0 \\ D \end{bmatrix}$ is closed since $\mathcal{R}(A)$ is closed. Suppose that

$$\begin{bmatrix} U \\ 0 \\ D \end{bmatrix} y_n \to \begin{pmatrix} y_1 \\ 0 \\ y_3 \end{pmatrix} (n \to \infty)$$

where $y_n \in \mathcal{D}(D)$. Then, $Uy_n \to y_1 (n \to \infty)$ and $Dy_n \to y_3 (n \to \infty)$. Thus, we have that $y_n \to U^{-1}y_1 (n \to \infty)$ from U is a unitary operator. We also obtain $U^{-1}y_1 \in \mathcal{D}(D)$ and $DU^{-1}y_1 = y_3$ since D is a closed operator. Hence,

$$\begin{pmatrix} y_1 \\ 0 \\ y_3 \end{pmatrix} = \begin{bmatrix} U \\ 0 \\ D \end{bmatrix} U^{-1} y_1$$

which implies $\mathcal{R}(T_B)$ is closed.

Last, we verify that $\mathcal{N}(T_B^p) = \mathcal{N}(T_{B_0}^{p+1})$, which induces $asc(T_B) < \infty$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(T_B^{p+1})$, then

$$\begin{cases} A^{p+1}x + A^pBy + A^{p-1}BDy + \cdots + ABD^{p-1}y + BD^py = 0, \\ D^{p+1}y = 0. \end{cases}$$

Thus, $D^py \in \mathcal{N}(D) \cap \mathcal{R}(D^p)$ and $BD^py \in (\mathcal{R}(A)+\mathcal{N}(A^p))^\perp \subset \mathcal{R}(A)^\perp$. This implies that $A^{p+1}x + A^pBy + A^{p-1}BDy + \cdots + ABD^{p-1}y = -BD^py \in \mathcal{R}(A) \cap (\mathcal{R}(A)+\mathcal{R}(A^p))^\perp \subset \mathcal{R}(A) \cap \mathcal{R}(A)^\perp = \{0\}$, and then

$$\begin{cases} A^{p+1}x + A^pBy + A^{p-1}BDy + \cdots + ABD^{p-1}y = 0, \\ BD^py = 0. \end{cases}$$

We can obtain $D^py = 0$ by the definition of B. According to $A^{p+1}x + A^pBy + \cdots + ABD^{p-1}y = A(A^px + A^{p-1}By + \cdots + BD^{p-1}y) = 0$, we have

$$A^px + A^{p-1}By + \cdots + BD^{p-1}y \in \mathcal{N}(A).$$

Let $x_1 := A^p x + A^{p-1}By + \cdots + BD^{p-1}y$, then

$$A^p x + A^{p-1}By + \cdots + ABD^{p-2}y - x_1 + BD^{p-1}y = 0.$$

It follows that $A^p x + A^{p-1}By + \cdots + ABD^{p-2}y - x_1 = -BD^{p-1}y \in (\mathcal{R}(A) + \mathcal{N}(A)) \cap (\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp} \subset (\mathcal{R}(A) + \mathcal{N}(A^p)) \cap (\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp} = \{0\}$. Then, $A^p x + A^{p-1}By + \cdots + ABD^{p-2}y - x_1 = -BD^{p-1}y = 0$, i.e.,

$$\begin{cases} A^p x + A^{p-1}By + A^{p-1}BDy + \cdots + ABD^{p-2}y = x_1, \\ BD^{p-1}y = 0. \end{cases}$$

Let $x_2 := A^{p-1}x + A^{p-2}By + A^{p-1}BDy + \cdots + BD^{p-2}y$, then $x_2 \in \mathcal{N}(A^2)$, as $Ax_2 = x_1$ and $x_1 \in \mathcal{N}(A)$. This implies $A^{p-1}x + A^{p-2}By + A^{p-3}BDy + \cdots + BD^{p-3}y - x_2 = -BD^{p-2}y \in (\mathcal{R}(A) + \mathcal{N}(A^2)) \cap (\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp} \subset (\mathcal{R}(A) + \mathcal{N}(A^p)) \cap (\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp} = \{0\}$, and then

$$\begin{cases} A^{p-1}x + A^{p-2}By + A^{p-3}BDy + \cdots + BD^{p-3}y = x_2, \\ BD^{p-2}y = 0. \end{cases}$$

Continuing this process, there is $x_p \in \mathcal{N}(A^p)$ such that

$$\begin{cases} Ax + By = x_p, \\ By = 0. \end{cases}$$

Thus, $x \in \mathcal{N}(A^{p+1}) = \mathcal{N}(A^p)$. This induces that $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(T_B^p)$, which means that $\mathcal{N}(T_B^p) = \mathcal{N}(T_B^{p+1})$. □

Theorem 2.2. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. If $\alpha(D) = \infty$, then T_B is right Browder for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if D is right Browder.*

Proof. If T_B is right Browder for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$, then D is right Fredholm and $des(D) < \infty$ by Lemmas 1.1 and 1.2. According to the assumption $\alpha(D) = \infty$, we can obtain $ind(D) \geq 0$, which means D is right Browder.

Now, we verify the reverse implication. Denote $q = des(D)$. By Lemma 1.4, we have $\beta(D^q) \leq q \cdot \beta(D) < \infty$, and then $\mathcal{R}(D^q)$ is closed. Thus, $\mathcal{K} = \mathcal{R}(D^q) \oplus \mathcal{R}(D^q)^{\perp}$. This includes $\mathcal{N}(D) = [\mathcal{N}(D) \cap \mathcal{R}(D^q)] \oplus [\mathcal{N}(D) \cap \mathcal{R}(D^q)^{\perp}]$. According to the assumption $\dim(\mathcal{N}(D)) = \infty$ and $\dim(\mathcal{N}(D) \cap \mathcal{R}(D^q)^{\perp}) < \infty$, we know that $\dim(\mathcal{N}(D) \cap \mathcal{R}(D^q)) = \infty$. Then, there exist two infinite dimensional subspaces Ω_1 and Ω_2 such that $\mathcal{N}(D) \cap \mathcal{R}(D^q) = \Omega_1 \oplus \Omega_2$. Define the operator B_0 by

$$B = \begin{bmatrix} U & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \mathcal{N}(D) \cap \mathcal{R}(D^q)^{\perp} \\ \mathcal{N}(D)^{\perp} \end{bmatrix} \to \mathcal{H}$$

where $U : \Omega_1 \to \mathcal{H}$ is a unitary operator. Then, we obtain that $\mathcal{R}(T_B) = \begin{bmatrix} \mathcal{H} \\ \mathcal{R}(D) \end{bmatrix}$ is closed, $\beta(T_B) = \beta(D) < \infty$, and $\alpha(T_B) = \alpha(A) + \dim \Omega_2 + \dim(\mathcal{N}(D) \cap \mathcal{R}(D^q)^{\perp}) = \infty$. Thus, we also get $des(T_B) < \infty$ by $des(D) = q < \infty$. This means T_B is right Browder. □

Remark 2.1. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. If T_B is Browder for some $B \in C_D^+(\mathcal{K},\mathcal{H})$, then $\alpha(D) < \infty$ if and only if $\beta(A) < \infty$. In fact, if T_B is Browder for some $B \in C_D^+(\mathcal{K},\mathcal{H})$, then A is left Fredholm with finite ascent and D is right Fredholm with finite descent, by Lemmas 1.1 and 1.2. It follows that T_B admits the following representation:*

$$T_B = \begin{bmatrix} A_1 & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \\ 0 & 0 & D_1 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(D) \\ \mathcal{N}(D)^\perp \cap \mathcal{D}(A) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(D) \\ \mathcal{R}(D)^\perp \end{bmatrix}$$

where $D_1 : \mathcal{D}(D) \to \mathcal{R}(D)$ is invertible and $A_1 : \mathcal{D}(A) \to \mathcal{R}(A_1)(= \mathcal{R}(A))$ is an operator with closed range. Let $A_1^\dagger : \mathcal{R}(A) \to \mathcal{H}$ be the maximal Tseng inverse of A_1, then $A_1 A_1^\dagger = I_{\mathcal{R}(A)}$. Set $P = \begin{bmatrix} I & 0 & -B_{12}D_1^{-1} & 0 \\ 0 & I & -B_{22}D_1^{-1} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, $Q = \begin{bmatrix} I & -A_1^\dagger B_{11} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, then

$$PT_BQ = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_{21} & 0 \\ 0 & 0 & D_1 \\ 0 & 0 & 0 \end{bmatrix} := T_{B_{21}}.$$

Thus, $ind(T_B) = ind(T_{B_{21}})$ since P and Q are both injective. Assume that $\alpha(D) < \infty$ (resp., $\beta(A) < \infty$), then B_{21} is a compact operator. By Lemma 1.5, we also obtain $ind(T_B) = ind(A) + ind(D) = 0$. Therefore, $\alpha(D) < \infty$ if and only if $\beta(A) < \infty$.

The next result is given under the hypothesis that $\beta(A)$ and $\alpha(D)$ are both finite.

Theorem 2.3. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. Suppose that $\beta(A) < \infty$ and $\alpha(D) < \infty$. Then, T_B is Browder for some $B \in C_D^+(\mathcal{K},\mathcal{H})$ if and only if A is left Browder, D is right Browder, and $\alpha(A) + \alpha(D) = \beta(A) + \beta(D)$.*

Proof. Necessity. If T_B is Browder for some $B \in C_D^+(\mathcal{K},\mathcal{H})$, then A is left Fredholm, D is right Fredholm, $asc(A) < \infty$, and $des(D) < \infty$ by Lemmas 1.1 and 1.2. This means A is left Browder and D is right Browder. We can also get $\alpha(A) + \alpha(D) = \beta(A) + \beta(D)$ by Remark 2.1.

Sufficiency. There are two cases to consider.

Case I: Assume that $\alpha(D) \le \beta(A)$, then we can define a unitary operator $J_1 : \mathcal{N}(D) \to \mathcal{M}_1$, where $\mathcal{M}_1 \subset \mathcal{R}(A)^\perp$ and $\dim(\mathcal{M}_1) = \alpha(D)$. Set

$$B = \begin{bmatrix} 0 & 0 \\ J_1 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D) \\ \mathcal{N}(D)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

We have $\alpha(T_B) = \alpha(A)$ and $\beta(T_B) = \beta(D) + \beta(A) - \alpha(D)$. Then, $asc(T_B) = asc(A) < \infty$ and $\alpha(T_B) = \beta(T_B) < \infty$ by the assumption $\alpha(A) + \alpha(D) = \beta(A) + \beta(D)$. Thus, $des(T_B) < \infty$ from Lemma 1.6, which means T_B is Browder.

Case II: Suppose that $\alpha(D) > \beta(A)$. Define a unitary operator $J_2 : \mathcal{M}_2 \to \mathcal{R}(A)^\perp$, where $\mathcal{M}_2 \subset \mathcal{N}(D)$ and $\dim(\mathcal{M}_2) = \beta(A)$. Let

$$B = \begin{bmatrix} 0 & 0 \\ J_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M}_2 \\ \mathcal{M}_2^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

We get $\alpha(T_B) = \alpha(A) + \alpha(D) - \beta(A)$ and $\beta(T_B) = \beta(D)$. Then, $des(T_B) = des(D) < \infty$ and $\alpha(T_B) = \beta(T_B) < \infty$ by the assumption $\alpha(A) + \alpha(D) = \beta(A) + \beta(D)$. Hence, $\alpha(T_B^*) = \beta(T_B^*) < \infty$ and $asc(T_B^*) = des(T_B) < \infty$ by [1, Proposition 3.1]. Therefore, we get $asc(T_B) = asc(T_B^{**}) = des(T_B^*) < \infty$ by Lemma 1.6. This means T_B is Browder. □

According to Theorems 2.1–2.3 and Remark 2.1, we obtain the theorem below, which is the main result of this section.

Theorem 2.4. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. Then, T_B is Browder for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if A is left Browder, D is right Browder, and $\alpha(A) + \alpha(D) = \beta(A) + \beta(D)$.*

Immediately, we get the following corollary, in which the set $\bigcap_{B\in C_D^+(\mathcal{K},\mathcal{H})} \sigma_b(T_B)$ is estimated:

Corollary 2.1. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. Then,*

$$\bigcap_{B\in C_D^+(\mathcal{K},\mathcal{H})} \sigma_b(T_B) = \sigma_{lb}(A) \cup \sigma_{rb}(D) \cup \{\lambda \in \mathbb{C} : \alpha(A - \lambda I) + \alpha(D - \lambda I)$$

$$\neq \beta(A - \lambda I) + \beta(D - \lambda I)\}.$$

Remark 2.2. *Theorem 2.4 and Corollary 2.1 extend the results of bounded case in [5, 18].*

Theorem 2.5. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. Then,*

$$\sigma_b(T_B) = \sigma_b(A) \cup \sigma_b(D) \tag{2.1}$$

for every $B \in C_D^+(\mathcal{K}, \mathcal{H})$, if and only if $\lambda \in \sigma_{asc}(D) \setminus (\sigma_{lb}(A) \cup \sigma_{rb}(D))$ implies $\alpha(A - \lambda I) + \alpha(D - \lambda I) \neq \beta(A - \lambda I) + \beta(D - \lambda I)$, where $\sigma_{asc}(D) = \{\lambda \in \mathbb{C} : asc(D - \lambda I) = \infty\}$.

Proof. Equation (2.1) holds for every $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if

$$\sigma_{asc}(D) \setminus (\sigma_{lb}(A) \cup \sigma_{rb}(D)) \subset \sigma_b(T_B)$$

by [2, Theorem 3.3]. That is,

$$\sigma_{asc}(D) \setminus (\sigma_{lb}(A) \cup \sigma_{rb}(D)) \subset \bigcap_{B\in C_D^+(\mathcal{K},\mathcal{H})} \sigma_b(T_B).$$

This induces

$$\sigma_{asc}(D) \setminus (\sigma_{lb}(A) \cup \sigma_{rb}(D)) \subset \{\lambda \in \mathbb{C} : \alpha(A - \lambda I) + \alpha(D - \lambda I) \neq \beta(A - \lambda I) + \beta(D - \lambda I)\}.$$

□

3. Spectra of operator matrix

Theorem 3.1. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ with dense domains. Then, T_B is right invertible for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if D is right invertible and $\alpha(D) \geq \beta(A)$.*

Proof. Necessity. If T_B is right invertible for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$, then D is right invertible and T_B has the following representation:

$$T_B = \begin{bmatrix} A_1 & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \\ 0 & 0 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(A) \\ \mathcal{N}(D) \\ \mathcal{N}(D)^\perp \cap \mathcal{D}(A) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{bmatrix}.$$

Obviously, $D_1 : \mathcal{N}(D)^\perp \cap \mathcal{D}(A) \to \mathcal{K}$ is a bijection. Set $P = \begin{bmatrix} I & 0 & -B_{12}D_1^{-1} \\ 0 & I & -B_{22}D_1^{-1} \\ 0 & 0 & I \end{bmatrix}$, then

$$PT_B = \begin{bmatrix} A_1 & B_{11} & 0 \\ 0 & B_{21} & 0 \\ 0 & 0 & D_1 \end{bmatrix} := \widehat{T_B}.$$

Thus, $\mathcal{R}(\widehat{T_B}) = \mathcal{H} \oplus \mathcal{K}$ since P is bijective, and hence $\alpha(D) \geq \beta(A)$.

Sufficiency. There are two cases to consider.

Case I: Let $\alpha(D) = \infty$, then we can define a Unitary operator $U : \mathcal{N}(D) \to \mathcal{H}$. Set

$$B = \begin{bmatrix} U & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D) \\ \mathcal{N}(D)^\perp \end{bmatrix} \to \mathcal{H}.$$

Clearly, T_B is right invertible.

Cases II: Assume that $\alpha(D) < \infty$, then we can define the right invertible operator $R : \mathcal{N}(D) \to \mathcal{R}(A)^\perp$. Set

$$B = \begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(D) \\ \mathcal{N}(D)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

Clearly, T_B is right invertible. □

From the above theorem and Theorems 5.2.1, 5.2.3 of [15], we can obtain the next results immediately.

Theorem 3.2. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ with dense domains. Then, T_B is invertible for some $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if A is left invertible, D is right invertible, and $\alpha(D) \geq \beta(A)$.*

Corollary 3.1. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ with dense domains. Then,*

$$\bigcap_{B \in C_D^+(\mathcal{K}, \mathcal{H})} \sigma(T_B) = \sigma_{ap}(A) \cup \sigma_\delta(D) \cup \{\lambda \in \mathbb{C} : \alpha(D - \lambda I) \neq \beta(A - \lambda I)\}.$$

Remark 3.1. *Theorem 3.2 and Corollary 3.1 are also valid for bounded operator matrix $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. These conclusions extend the results in [8, 9, 12, 13].*

From Corollary 3.1, we have the following theorem, which extends the results of [3, 7, 10, 12].

Theorem 3.3. *Let $A \in C(\mathcal{H})$ and $D \in C(\mathcal{K})$ be given operators with dense domains. Then,*

$$\sigma(T_B) = \sigma(A) \cup \sigma(D) \tag{3.1}$$

for every $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if

$$\lambda \in \sigma_{p,1}(D) \cap \sigma_{r,1}(A) \Rightarrow \alpha(D - \lambda I) \neq \beta(A - \lambda I)$$

where $\sigma_{p,1}(D) = \{\lambda \in \mathbb{C} : \mathcal{N}(D - \lambda I) \neq \{0\}, \mathcal{N}(D - \lambda I) = \mathcal{K}\}$ and $\sigma_{r,1}(A) = \{\lambda \in \mathbb{C} : \mathcal{N}(A - \lambda I) = \{0\}, \mathcal{R}(A - \lambda I) = \overline{\mathcal{R}(A - \lambda I)} \neq \mathcal{H}\}$.

Proof. Equation (3.1) holds for every $B \in C_D^+(\mathcal{K}, \mathcal{H})$ if and only if

$$\sigma_{p,1}(D) \cap \sigma_{r,1}(A) \subset \sigma(T_B)$$

by [7, Corollary 2]. That is,

$$\sigma_{p,1}(D) \cap \sigma_{r,1}(A) \subset \bigcap_{B \in C_D^+(\mathcal{K}, \mathcal{H})} \sigma(T_B).$$

This induces

$$\sigma_{p,1}(D) \cap \sigma_{r,1}(A) \subset \{\lambda \in \mathbb{C} : \alpha(D - \lambda I) \neq \beta(A - \lambda I)\}$$

by Corollary 3.1, $\sigma_{r,1}(A) \cap \sigma_{ap}(A) = \emptyset$, and $\sigma_{p,1}(D) \cap \sigma_\delta(D) = \emptyset$. □

Applying Theorems 2.5 and 3.3 for the Hamiltonian operator matrix, we obtain the next result.

Theorem 3.4. *Let $H = \begin{bmatrix} A & B \\ 0 & -A^* \end{bmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(A^*) \subset \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ be a Hamiltonian operator matrix, where A is a densely defined closed operator and B is a self adjoint operator. Then, the following assertions hold:*

(1) *For every $C \in C_{A^*}^+(\mathcal{K}, \mathcal{H})$,*

$$\sigma_b(H) = -\sigma_b(A)^* \cup \sigma_b(A) \tag{3.2}$$

if and only if

$$\lambda \in \sigma_{asc}(A^*) \setminus (-\sigma_{lb}(A) \cup \sigma_{rb}(A^*)) \Rightarrow \alpha(A - \lambda I) + \alpha(A + \overline{\lambda} I) \neq \beta(A - \lambda I) + \beta(A + \overline{\lambda} I).$$

(2) *For every $C \in C_{A^*}^+(\mathcal{K}, \mathcal{H})$,*

$$\sigma(H) = -\sigma(A)^* \cup \sigma(A) \tag{3.3}$$

if and only if

$$\lambda \in (-\sigma_{r,1}(A)^* \cap \sigma_{r,1}(A)) \setminus \{\lambda \in \mathbb{C} : Re\lambda = 0\} \Rightarrow \beta(A + \overline{\lambda} I) \neq \beta(A - \lambda I).$$

In particular, if $\sigma_{r,1}(A)$ does not include symmetric points about the imaginary axis, then (3.3) *holds.*

Example 3.1. *Consider the plate bending equation in the domain* $\{(x, y) : 0 < x < 1, 0 < y < 1\}$

$$D(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^2\omega = 0$$

with boundary conditions

$$\omega(x, 0) = \omega(x, 1) = 0,$$

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = 0, y = 0, 1.$$

Set

$$\theta = \frac{\partial\omega}{\partial x}, q = D(\frac{\partial^3\omega}{\partial x^3} + \frac{\partial^3\omega}{\partial y^3}), m = -D(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2})$$

then the equation can be written as the Hamiltonian system (see [19])

$$\frac{\partial}{\partial x}\begin{pmatrix}\omega\\ \theta\\ q\\ m\end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -\frac{1}{D}\\ 0 & 0 & 0 & \frac{\partial^2}{\partial y^2}\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{pmatrix}\omega\\ \theta\\ q\\ m\end{pmatrix}$$

and the corresponding Hamiltonian operator matrix is given by

$$H = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{d^2}{dy^2} & 0 & 0 & -\frac{1}{D}\\ 0 & 0 & 0 & \frac{d^2}{dy^2}\\ 0 & 0 & -1 & 0\end{bmatrix} = \begin{bmatrix} A & B\\ 0 & -A^*\end{bmatrix}$$

with domain $\mathcal{D}(A) \oplus \mathcal{D}(A^*) \subset \mathcal{H} \oplus \mathcal{H}$*, where* $\mathcal{H} = L_2(0, 1) \oplus L_2(0, 1)$*,* $\mathcal{A} = AC[0, 1]$*, and*

$$A = \begin{bmatrix} 0 & 1\\ -\frac{d^2}{dy^2} & 0\end{bmatrix}, B = \begin{bmatrix} 0 & 0\\ 0 & -\frac{1}{D}\end{bmatrix},$$

$$\mathcal{D}(A) = \left\{\begin{pmatrix}\omega\\ \theta\end{pmatrix} \in \mathcal{H} : \omega(0) = \omega(1) = 0, \omega^{'} \in \mathcal{A}, \omega^{''} \in \mathcal{H}\right\}.$$

By a simple calculation, we get $\sigma_{r,1}(A) = \emptyset$ *and* $\sigma(A) = \{k\pi : k = \pm 1, \pm 2, ...\}$*. By Theorem 3.4, we get*

$$\sigma(H) = -\sigma(A^*) \cup \sigma(A) = \{k\pi : k = \pm 1, \pm 2, ...\}.$$

On the other hand, we can easily calculate that

$$\sigma(H) = \{k\pi : k = \pm 1, \pm 2, ...\} = -\sigma(A^*) \cup \sigma(A).$$

4. Conclusions

Certain spectral and Browder spectral properties of closed operator matrix $T_B = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ in a Hilbert space are considered in this paper. Specifically, some sufficient and necessary conditions are described under which T_B is Browder (resp., invertible) for some closable operator B with $\mathcal{D}(B) \supset \mathcal{D}(D)$. Additionally, a sufficient and necessary condition is obtained under which the Browder spectrum (resp., spectrum) of T_B coincides with the union of the Browder spectrum (resp., spectrum) of its diagonal entries. These results also hold for bounded operator matrix. As an application, the corresponding properties of the Hamiltonian operator matrix from elasticity theory are given.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We are grateful to the referees for their valuable comments on this paper.

This work is supported by the Basic Scientific Research Funds of Subordinate Universities of Inner Mongolia (Grant No. ZSLJ202213).

Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. T. Alvarez, F. Fakhfakh, M. Mnif, Characterization of closed densely defined semi-Browder linear operators, *Complex Anal. Oper. Th.*, **7** (2013), 1775–1786. http://dx.doi.org/10.1007/s11785-012-0238-6
2. Q. M. Bai, J. J. Huang, A. Chen, Essential, Weyl and Browder spectra of unbounded upper triangular operator matrices, *Linear Multilinear A.*, **64** (2016), 1583–1594. http://dx.doi.org/10.1080/03081087.2015.1111290
3. M. Barraa, M. Boumazgour, A note on the spectrum of an upper triangular operator matrix, *P. Am. Math. Soc.*, **131** (2003), 3083–3088. Available from: `https://www.ams.org/journals/proc/2003-131-10/S0002-9939-03-06862-X/`.
4. A. Ben-Israel, T. N. E. Greville, *Generalized inverse: Theory and applications*, 2 Eds., New York: Springer, 2003. http://dx.doi.org/10.1016/B978-0-12-775850-3.50017-0
5. X. H. Cao, Browder spectra for upper triangular operator matrices, *J. Math. Anal. Appl.*, **342** (2008), 477–484. https://doi.org/10.1016/j.jmaa.2007.11.059
6. S. R. Caradus, Operators with finite ascent and descent, *Pac. J. Math.*, **18** (1996), 437–449. http://dx.doi.org/10.2140/pjm.1966.18.437

7. A. Chen, Q. M. Bai, D. Y. Wu, Spectra of 2×2 unbounded operator matrices, *Sci. Sin. Math.*, **46** (2016), 157–168. Available from: `https://www.cqvip.com/qk/93904x/201602/668172618.html`.

8. D. S. Djordjevic, Perturbations of spectra of operator matrices, *J. Operat. Theor.*, **48** (2002), 467–486. Available from: `https://www.jstor.org/stable/24715580`.

9. H. K. Du, J. Pan, Perturbation of spectrum of 2×2 operator matrices, *P. Am. Math. Soc.*, **121** (1994), 761–766. http://dx.doi.org/10.1080/03081087.2015.1111290

10. H. Elbjaoui, E. H. Zerouali, Local spectral theory for 2×2 operator matrices, *Int. J. Math. Soc.*, **42** (2003), 2667–2672. Available from: `https://www.ams.org/journals/proc/1994-121-03/S0002-9939-1994-1185266-2/`.

11. I. Gohberg, S. Goldberg, M. Kaashoek, *Classes of linear operators*, Basel: Birkhäyser Verlag, **1** (1990).

12. J. K. Han, H. Y. Lee, W. Y. Lee, Invertible completions of 2×2 upper triangular operator matrices, *P. Am. Math. Soc.*, **128** (2000), 119–123. Available from: `https://www.ams.org/journals/proc/2000-128-01/S0002-9939-99-04965-5/`.

13. I. S. Hwang, W. Y. Lee, The boundedness below of 2×2 upper triangular operator matrices, *Integr. Equ. Oper. Th.*, **39** (2001), 267–276. Available from: `https://link.springer.com/article/10.1007/BF01332656`.

14. M. A. Kaashoek, Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor, *Math. Ann.*, **172** (1967), 105–115. Available from: `https://link.springer.com/article/10.1007/BF01350090`.

15. Y. R. Qi, *The quadratic numerical range and conpletion problems of unbounded operator matrices*, Hohhot: School of Mathematical Sciences of Inner Mongolia University, 2014.

16. A. E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, *Math. Ann.*, **163** (1996), 18–49. http://dx.doi.org/10.1007/bf02052483

17. A. E. Taylor, D. C. Lay, *Introduction to functional analysis*, 2 Eds., New York: Wiley, 1980.

18. S. F. Zhang, H. J. Zhong, L. Zhang, Perturbation of Browder spectrum of upper-triangular operator matrices, *Linear Multilinear A.*, **64** (2016), 502–511. http://dx.doi.org/10.1080/03081087.2015.1050349

19. W. X. Zhong, Method of separation of variables and Hamiltonian system, *Comput. Struct. Mech. Appl.*, **8** (1991), 229–240. http://dx.doi.org/10.1002/num.1690090107