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# The Schatten $p$-quasinorm on Euclidean Jordan algebras

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**Abstract:** In this article, we proved that a Schatten $p$-(quasi)norm for $0 < p < 1$, defined on Euclidean Jordan algebras, satisfied a relaxed triangle inequality with an optimal constant $2^{\frac{1}{p}-1}$; hence, it indeed induced a quasinorm. This confirmed the validity of a conjecture raised by Huang, Chen, and Hu.

**Keywords:** Schatten $p$-quasinorm; Euclidean Jordan algebra; spectral function; majorization
**Mathematics Subject Classification:** 15A42, 17C20

## 1. Introduction

A quasinorm $\|\cdot\|$ is a weaker notion of a norm in the sense that it is positive definite and absolutely homogeneous, but only satisfies a relaxed version of the triangle inequality,

$$\|x + y\| \le K( \|x\| + \|y\| )$$

for some constant $K \ge 1$. The most widely used quasinorm is a $p$-quasinorm $\|u\|_p = \left( \sum_{i=1}^{n} |u_i|^p \right)^{\frac{1}{p}}$ for $0 < p < 1$ and $u \in \mathcal{R}^n$. $p$-quasinorms have been successfully applied in low-rank minimization problems such as low-rank matrix recovery problems, recommendation system, and robust principal component analysis, to name a few. For more details, we refer to [10] and references therein.

Going beyond Euclidean spaces, one prevalent choice of a (quasi)norm for matrices is the so-called Schatten $p$-(quasi)norm, defined by the $p$-(quasi)norm of the singular values. It is well known that it is a norm when $1 \le p \le \infty$ and a quasinorm when $0 < p < 1$. In particular, for $A \in \mathcal{S}^n$, where $\mathcal{S}^n$ is the space of $n \times n$ real symmetric matrices, we have

$$|||A|||_p = \|\lambda(A)\|_p \, .$$

Here, $\lambda(A)$ denotes the vector in $\mathcal{R}^n$ whose entries are the eigenvalues of $A$, arranged in nonincreasing order.

Recently, Huang-Chen-Hu [7] introduced Schatten $p$-(quasi)norms on $\mathcal{R}^n$ and on the Jordan spin algebra $\mathcal{L}^n$, then investigated how these two (quasi)norms are related. Realizing that $\mathcal{S}^n$, $\mathcal{R}^n$, and $\mathcal{L}^n$ are all particular instances of (simple) Euclidean Jordan algebras, where each element $x$ in a Euclidean Jordan algebra $\mathcal{V}$ can be associated with its corresponding eigenvalue vector $\lambda(x)$, Huang-Chen-Hu [7, Conjecture 1] conjectured whether the following functional $x \in \mathcal{V} \mapsto \|\lambda(x)\|_p$ for $0 < p < 1$ is indeed a quasinorm for general Euclidean Jordan algebras. In this manuscript, we give an affirmative answer to their conjecture by proving the following theorem.

**Theorem 1.1.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra of rank $n$. For $0 < p < 1$, the functional*

$$x \mapsto |||x|||_p = \|\lambda(x)\|_p = \left[ \sum_{i=1}^{n} |\lambda_i(x)|^p \right]^{\frac{1}{p}}$$

*is a quasinorm on $\mathcal{V}$.*

The rest of the paper is organized as follows: Section 2 provides necessary definitions and properties of Euclidean Jordan algebras. We derive a proof of Theorem 1.1 in Section 3 and conclude the paper with some conjectures in Section 4.

## 2. Preliminary

Throughout the manuscript, $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathcal{R}^n_+$ is the nonnegative orthant in $\mathcal{R}^n$. When $n = 1$, we omit the superscript and simply write $\mathcal{R}$ and $\mathcal{R}_+$. For $u = (u_1, \ldots, u_n) \in \mathcal{R}^n$, we write $u \geq 0$ in $\mathcal{R}^n$ if $u_i \geq 0$ for all $i$ or, equivalently, $u \in \mathcal{R}^n_+$. We write $u^\downarrow = (u^\downarrow_1, \ldots, u^\downarrow_n) \in \mathcal{R}^n$ for the vector obtained by rearranging the entries of $u$ in nonincreasing order. For any function $f : \mathcal{R} \to \mathcal{R}$, we define $f(u) = (f(u_1), \ldots, f(u_n))$; in particular, $|u| = (\,|u_1|\,, \ldots, |u_n|\,)$.

### 2.1. Euclidean Jordan algebras

A triple $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ is called a *Euclidean Jordan algebra* if a finite dimensional inner product space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is equipped with the Jordan product $\circ$ satisfying commutativity, Jordan identity, and associativity with the inherited inner product. We refer [1] for detailed definitions and properties of Euclidean Jordan algebras. Throughout the manuscript, $\mathcal{V}$ denotes a Euclidean Jordan algebra of rank $n$ with the unit element $e$. The *symmetric cone* $\mathcal{V}_+$ of $\mathcal{V}$ is the cone of square elements, i.e., $\mathcal{V}_+ = \{x \circ x : x \in \mathcal{V}\}$. We write $x \geq y$ provided that $x - y \in \mathcal{V}_+$.

A Euclidean Jordan algebra is said to be *simple* if it cannot be written as the direct product of nonzero Euclidean Jordan algebras. A classification theorem [1, Corollary IV.1.5 and Theorem V.3.7] says that every nonzero Euclidean Jordan algebra is the direct product of simple Euclidean Jordan algebras, and each of simple Euclidean Jordan algebras is isomorphic to one of the five algebras given below.

- The algebras $\mathcal{S}^n$, $\mathcal{H}^n$, and $\mathcal{Q}^n$ of $n \times n$ real symmetric, complex Hermitian, and quaternion Hermitian matrices, respectively. In these three matrix algebras, the Jordan product and the inner product are defined, respectively, by $X \circ Y = \frac{1}{2}(XY + YX)$ and (the real part of) $\mathrm{tr}(XY)$.
- The algebra $\mathcal{O}^3$ of $3 \times 3$ octonion Hermitian matrices, with the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ and the inner product defined by the real part of $\mathrm{tr}(XY)$.

- The Jordan spin algebra $\mathcal{L}^n$ for $n \geq 3$. (Definition is given in Example 2.2 below.)

An element $c \in \mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as the sum of two nonzero idempotents. A set $\{e_1, \dots, e_n\}$ of primitive idempotents in $\mathcal{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ when } i \neq j \quad \text{and} \quad \sum_{i=1}^{n} e_i = e.$$

The (complete) spectral decomposition theorem [1, Theorem III.1.2] asserts that, for every $x \in \mathcal{V}$, there exist uniquely determined real numbers $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ and a Jordan frame $\{e_1, \dots, e_n\}$ such that

$$x = \lambda_1(x)e_1 + \cdots + \lambda_n(x)e_n.$$

These real numbers $\lambda_1(x), \dots, \lambda_n(x)$ are called the *eigenvalues* of $x$. We define the *eigenvalue map* $\lambda : \mathcal{V} \to \mathcal{R}^n$ by $\lambda(x) = (\lambda_1(x), \dots, \lambda_n(x))$; thus, $\lambda(x)$ is a vector in $\mathcal{R}^n$ consisting of the eigenvalues of $x$ written in nonincreasing order.

A linear map $\phi : \mathcal{V} \to \mathcal{V}$ is called an *algebra automorphism* if it is invertible and satisfies

$$\phi(x \circ y) = \phi(x) \circ \phi(y)$$

for all $x, y \in \mathcal{V}$. The set of all algebra automorphisms of $\mathcal{V}$ is denoted by $\mathrm{Aut}(\mathcal{V})$. It is known [1, Theorem IV.2.5] that algebra automorphisms preserve eigenvalues, that is, $\lambda(\phi(x)) = \lambda(x)$ for all $x \in \mathcal{V}$ and $\phi \in \mathrm{Aut}(\mathcal{V})$.

For $0 < p < \infty$, we define a functional $|||\cdot|||_p : \mathcal{V} \to \mathcal{R}$ by

$$x \mapsto |||x|||_p := \|\lambda(x)\|_p = \left[ \sum_{i=1}^{n} |\lambda_i(x)|^p \right]^{\frac{1}{p}} \tag{2.1}$$

and $|||x|||_\infty := \|\lambda(x)\|_\infty = \max_i |\lambda_i(x)|$. It is easy to see that the above is positive definite and absolute homogeneous. Moreover, when $1 \leq p \leq \infty$, it also satisfies the triangle inequality [12, Theorem 4.1; 8, Example 6], i.e.,

$$|||x + y|||_p \leq |||x|||_p + |||y|||_p.$$

This norm is referred to as the *Schatten p-norm* (also, the terms such as the *spectral p-norm* or *trace p-norm* are often used in the literature). We refer [3, 5, 8, 12] for resent works on the Schatten $p$-norm on Euclidean Jordan algebras.

Here, we list two examples of the Schatten $p$-(quasi)norm defined on $\mathcal{R}^n$.

**Example 2.1.** The $n$-dimensional Euclidean space $\mathcal{R}^n$ can be made into a (non-simple) Euclidean Jordan algebra by imposing the Jordan product as the component-wise product. In this algebra, the eigenvalues of $u \in \mathcal{R}^n$ are nothing but the entries of $u$. Hence, the functional (2.1) is reduced to the standard $p$-(quasi)norm.

**Example 2.2.** The Jordan spin algebra $\mathcal{L}^n$ for $n \geq 3$ is an $n$-dimensional Euclidean space equipped with the Jordan product defined as follows: For a vector $z \in \mathcal{R}^n$, we write $z = (z_1, \bar{z})$, where $z_1 \in \mathcal{R}$ and $\bar{z} \in \mathcal{R}^{n-1}$. For $x = (x_1, \bar{x})$ and $y = (y_1, \bar{y})$, the Jordan product is defined by

$$(x_1, \bar{x}) \circ (y_1, \bar{y}) = (x_1 y_1 + \langle \bar{x}, \bar{y} \rangle, y_1 \bar{x} + x_1 \bar{y}).$$

For $n \geq 3$, the Jordan spin algebra $\mathcal{L}^n$ has rank 2 and the eigenvalues of $x = (x_1, \bar{x})$ in $\mathcal{L}^n$ are $x_1 \pm \|\bar{x}\|$. Hence, in this algebra, the functional (2.1) simplifies to

$$|||x||| = \left(\left|x_1 + \|\bar{x}\|\right|^p + \left|x_1 - \|\bar{x}\|\right|^p\right)^{\frac{1}{p}}.$$

It has been observed in [7, Theorem 3.7] that the above functional satisfies

$$|||x + y|||_p \leq 2^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right)$$

for $0 < p < 1$ and $x, y \in \mathcal{L}^n$. Hence, it is a quasinorm.

### 2.2. *Majorization*

For two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathcal{R}^n$, we say that $u$ is *weakly majorized* by $v$ and write $u \prec_w v$, provided

$$\sum_{i=1}^{k} u_i^{\downarrow} \leq \sum_{i=1}^{k} v_i^{\downarrow}$$

for all $1 \leq k \leq n$. Additionally, if the inequality above becomes an equality for $k = n$, then we say that $u$ is *majorized* by $v$ and write $u \prec v$. An $n \times n$ real matrix $D$ is called *doubly stochastic* if all the entries are nonnegative and each row and column sums to 1.

We collect some properties of (weak) majorization, which will be used frequently in the sequel. Proofs can be found in [11, 15].

**Proposition 2.1.** *For vectors $u, v \in \mathcal{R}^n$, the following hold:*

*(a) $u \prec v$ if, and only if, $u = Dv$ for some $n \times n$ doubly stochastic matrix $D$.*
*(b) For any convex function $f : \mathcal{R} \to \mathcal{R}$, $u \prec v$ implies $f(u) \prec_w f(v)$.*
*(c) In particular, if $u \prec v$, then $|u| \prec_w |v|$.*

The concepts of (weak) majorization can be naturally generalized to Euclidean Jordan algebras via the eigenvalue map. For $x, y \in \mathcal{V}$, we say that $x$ is majorized (weakly majorized) by $y$ on $\mathcal{V}$ and write $x \prec y$ ($x \prec_w y$) if $\lambda(x) \prec \lambda(y)$ ($\lambda(x) \prec_w \lambda(y)$) on $\mathcal{R}^n$. Various inequalities in the setting of Euclidean Jordan algebras have been obtained/generalized with an aid of (weak) majorization; see [2, 6, 9, 12, 14].

## 3. Schatten $p$-quasinorm on Euclidean Jordan algebras

We start with the following elementary result on $p$-quasinorm on $\mathcal{R}^n$.

**Lemma 3.1.** *For $u = (u_1, \ldots, u_n) \in \mathcal{R}_+^n$ and $0 < p < 1$, we have*

$$\|u\|_1 \leq \|u\|_p \leq n^{\frac{1}{p}-1} \|u\|_1 .$$

**Proposition 3.1.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra of rank n, then*

$$|||x + y|||_p \leq n^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right)$$

*for $0 < p < 1$ and $x, y \in \mathcal{V}$.*

*Proof.* Note that, for any $x, y \in \mathcal{V}$, we have $\lambda(x+y) \prec \lambda(x) + \lambda(y)$ by [4, Theorem 4.5], which implies $|\lambda(x+y)| \prec_w |\lambda(x) + \lambda(y)|$. It follows that

$$\sum_{i=1}^{n} |\lambda_i(x+y)| \leq \sum_{i=1}^{n} |\lambda_i(x) + \lambda_i(y)| \leq \sum_{i=1}^{n} |\lambda_i(x)| + |\lambda_i(y)| \,. \tag{3.1}$$

Therefore, for any $0 < p < 1$, we have

$$\begin{aligned} |||x+y|||_p &= \left[ \sum_{i=1}^{n} |\lambda_i(x+y)|^p \right]^{\frac{1}{p}} \\ &\leq n^{\frac{1}{p}-1} \sum_{i=1}^{n} |\lambda_i(x+y)| \\ &\leq n^{\frac{1}{p}-1} \sum_{i=1}^{n} |\lambda_i(x)| + |\lambda_i(y)| \\ &\leq n^{\frac{1}{p}-1} \left( \left[ \sum_{i=1}^{n} |\lambda_i(x)|^p \right]^{\frac{1}{p}} + \left[ \sum_{i=1}^{n} |\lambda_i(y)|^p \right]^{\frac{1}{p}} \right) \\ &= n^{\frac{1}{p}-1} \left( |||x|||_p + |||y|||_p \right), \end{aligned}$$

where the first and the last inequalities are due to Lemma 3.1 and the second inequality is by (3.1). This completes the proof.

Hence, the above proposition alone already gives an answer to the conjecture raised by Huang-Chen-Hu [7] in the affirmative. The subsequent part of this section is devoted to proving that the optimal quasinorm constant $K$ is indeed $2^{\frac{1}{p}-1}$ for any Euclidean Jordan algebras of rank $n \geq 2$.

The first lemma extends Thompson's matrix triangle inequality to Euclidean Jordan algebras. The second lemma is a generalization of well-known majorization relation on $\mathcal{R}^n$ to Euclidean Jordan algebras.

**Lemma 3.2.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra, then for $a, b \in \mathcal{V}$, there exist $\phi, \psi \in \mathrm{Aut}(\mathcal{V})$ such that*

$$|a+b| \leq \phi(|a|) + \psi(|b|). \tag{3.2}$$

*Proof.* It was originally proved by Tao-Kong-Luo-Kiu in [12, Theorem 3.1] for simple Euclidean Jordan algebras by case-by-case analysis. Later, the first author extended the result to general Euclidean Jordan algebra in [13, Theorem 3.2] and gave a direct proof.

**Lemma 3.3.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra of rank n. For $x, y \in \mathcal{V}_+$, we have*

$$\begin{bmatrix} \lambda(x) \\ \lambda(y) \end{bmatrix} \prec \begin{bmatrix} \lambda(x+y) \\ 0 \end{bmatrix}. \tag{3.3}$$

*Proof.* The lemma has been shown in [12, Theorem 5.1] when $\mathcal{V}$ is simple. For non-simple $\mathcal{V}$, for simplicity we assume $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are simple algebras of rank $n_1$ and $n_2$, respectively (hence, $n = n_1 + n_2$). Recall that for any $z = (z_1, z_2) \in \mathcal{V}_1 \times \mathcal{V}_2$, the eigenvalues of $z$

consist of the eigenvalues of $z_1$ and $z_2$ in some order. Thus, for given $x = (x_1, x_2)$, $y = (y_1, y_2)$, where $x_i, y_i \in \mathcal{V}_i$ for $i = 1, 2$, there exist permutation matrices $\sigma_1, \sigma_2, \sigma_3 \in \Sigma_n$ such that

$$\lambda\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \sigma_1 \begin{bmatrix} \lambda(x_1) \\ \lambda(x_2) \end{bmatrix}, \quad \lambda\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \sigma_2 \begin{bmatrix} \lambda(y_1) \\ \lambda(y_2) \end{bmatrix}, \quad \lambda\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \sigma_3 \begin{bmatrix} \lambda(x_1 + y_1) \\ \lambda(x_2 + y_2) \end{bmatrix}.$$

Since $\mathcal{V}_1, \mathcal{V}_2$ are simple, we have

$$\begin{bmatrix} \lambda(x_i) \\ \lambda(y_i) \end{bmatrix} \prec \begin{bmatrix} \lambda(x_i + y_i) \\ 0 \end{bmatrix}$$

for $i = 1, 2$. Thus, for each $i$, there exists a $2n_i \times 2n_i$ doubly stochastic matrix $D_i$ such that

$$\begin{bmatrix} \lambda(x_i) \\ \lambda(y_i) \end{bmatrix} = D_i \begin{bmatrix} \lambda(x_i + y_i) \\ 0 \end{bmatrix}.$$

This can be written as a single matrix equation

$$\begin{bmatrix} \lambda(x_1) \\ \lambda(y_1) \\ \lambda(x_2) \\ \lambda(y_2) \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} \lambda(x_1 + y_1) \\ 0 \\ \lambda(x_2 + y_2) \\ 0 \end{bmatrix}.$$

Defining a block diagonal matrix $D$ whose diagonal blocks are $D_1, D_2$, one can easily verify that $D$ is a $2n \times 2n$ doubly stochastic matrix. Now, choose permutation matrices $\Sigma_1, \Sigma_2 \in \Sigma_{2n}$ so that

$$\Sigma_1 \begin{bmatrix} \lambda(x_1) \\ \lambda(x_2) \\ \lambda(y_1) \\ \lambda(y_2) \end{bmatrix} = \begin{bmatrix} \lambda(x_1) \\ \lambda(y_1) \\ \lambda(x_2) \\ \lambda(y_2) \end{bmatrix} \quad \text{and} \quad \Sigma_2 \begin{bmatrix} \lambda(x_1 + y_1) \\ \lambda(x_2 + y_2) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda(x_1 + y_1) \\ 0 \\ \lambda(x_2 + y_2) \\ 0 \end{bmatrix}.$$

Finally, define a $2n \times 2n$ matrix $A$ by

$$A := \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \Sigma_1^{-1} D \Sigma_2 \begin{bmatrix} \sigma_3^{-1} & 0 \\ 0 & I_n \end{bmatrix}.$$

As the product of doubly stochastic matrices is doubly stochastic, $A$ is doubly stochastic and satisfies

$$\begin{bmatrix} \lambda(x) \\ \lambda(y) \end{bmatrix} = A \begin{bmatrix} \lambda(x + y) \\ 0 \end{bmatrix},$$

implying the desired majorization.

**Theorem 3.4.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra of rank $n$. Let $a, b \in \mathcal{V}$, and $f : \mathcal{R}_+ \to \mathcal{R}_+$ be an increasing concave function such that $f(0) \geq 0$, then*

$$\sum_{i=1}^{n} f(\lambda_i(|a + b|)) \leq \sum_{i=1}^{n} f(\lambda_i(|a|)) + \sum_{i=1}^{n} f(\lambda_i(|b|)).$$

*Proof.* We first remark that the result was proved in [12, Theorem 5.5] for simple Euclidean Jordan algebras, and the proof given here is in the same vein as the proof of the original result. From Lemma 3.2, there exist $\phi, \psi \in \mathrm{Aut}(\mathcal{V})$ such that $|a+b| \leq \phi(|a|) + \psi(|b|)$. It follows that $\lambda_i(|a+b|) \leq \lambda_i(\phi(|a|) + \psi(|b|))$ for all $i$. Since $f$ is increasing on $\mathcal{R}_+$, we consequently get

$$f(\lambda_i(|a+b|)) \leq f(\lambda_i(\phi(|a|) + \psi(|b|))) \tag{3.4}$$

for all $i = 1, \dots, n$. Now, from Lemma 3.3 applied to $\phi(|a|), \psi(|b|) \in \mathcal{V}_+$, we have

$$\begin{bmatrix} \lambda(|a|) \\ \lambda(|b|) \end{bmatrix} = \begin{bmatrix} \lambda(\phi(|a|)) \\ \lambda(\phi(|b|)) \end{bmatrix} \prec \begin{bmatrix} \lambda(\phi(|a|) + \psi(|b|)) \\ 0 \end{bmatrix},$$

where the equality follows from the fact that the eigenvalues are invariant under algebra automorphisms. Since $-f$ is convex on $\mathcal{R}_+$, the above implies

$$\begin{bmatrix} -f(\lambda(|a|)) \\ -f(\lambda(|b|)) \end{bmatrix} \prec_w \begin{bmatrix} -f(\lambda(\phi(|a|) + \psi(|b|))) \\ -f(0) \end{bmatrix}. \tag{3.5}$$

Combining (3.4) and (3.5) together with the condition $f(0) \geq 0$, we see that

$$\begin{aligned} \sum_{i=1}^{n} f(\lambda_i(|a+b|)) &\leq \sum_{i=1}^{n} f\Big(\lambda_i(\phi(|a|) + \psi(|b|))\Big) \\ &\leq \sum_{i=1}^{n} f\Big(\lambda_i(\phi(|a|) + \psi(|b|))\Big) + nf(0) \\ &\leq \sum_{i=1}^{n} f(\lambda_i(|a|)) + f(\lambda_i(|b|)). \end{aligned}$$

This completes the proof.

As a consequence of Theorem 3.4, by taking $f(t) = t^p$ for $0 < p < 1$, we see that the functional $x \mapsto |||x|||_p^p$ satisfies the triangle inequality:

$$|||x+y|||_p^p \leq |||x|||_p^p + |||y|||_p^p. \tag{3.6}$$

We point out that the inequality (3.6) naturally produces a metric on $\mathcal{V}$ defined by $d(x,y) = |||x-y|||_p^p$. This has been observed in [12, Theorem 5.6] for simple Euclidean Jordan algebras. We now come to our main result.

**Theorem 3.5.** *Let $\mathcal{V}$ be any Euclidean Jordan algebra of rank n. For $x, y \in \mathcal{V}$ and $0 < p < 1$, we have*

$$|||x+y|||_p \leq 2^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right).$$

*Moreover, if $n \geq 2$, the constant $2^{\frac{1}{p}-1}$ is optimal in the sense that it cannot be replaced by any smaller constant.*

*Proof.* From (3.6), it is easy to see that

$$|||x+y|||_p \leq \left(|||x|||_p^p + |||y|||_p^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}\left(\frac{|||x|||_p^p}{2} + \frac{|||y|||_p^p}{2}\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}\left(|||x|||_p + |||y|||_p\right),$$

where the last inequality follows from the convexity of the map $t \mapsto t^{\frac{1}{p}}$ for $0 < p < 1$. This proves the desired inequality. Additionally, let $\{e_1, \ldots, e_n\}$ be any Jordan frame of $\mathcal{V}$. Putting $x = e_1$ and $y = e_2$ gives the equality; hence, the constant is optimal.

As mentioned earlier, the functional (2.1) possesses positive definiteness and absolute homogeneity. These together with Theorem 3.5 verify that Theorem 1.1 holds, thereby affirming the conjecture by Huang-Chen-Hu. Consequently, it is now appropriate to call the functional (2.1) the Schatten $p$-quasinorm on $\mathcal{V}$ when $0 < p < 1$.

## 4. Concluding remarks

In this paper, we studied the functional (2.1) in Euclidean Jordan algebras and showed that it is a quasinorm when $0 < p < 1$ with the optimal constant $2^{\frac{1}{p}-1}$.

We conclude the manuscript with the following conjectures. Note that Conjecture 4.1 reduces to Theorem 3.4 if we take $k = n$. Conjecture 4.2 generalizes Thompson's matrix triangle inequality (3.2) and implies Conjecture 4.1.

**Conjecture 4.1.** *Let $a, b \in \mathcal{V}$, and $f : \mathcal{R}_+ \to \mathcal{R}_+$ be an increasing concave function with $f(0) \geq 0$, then the following weak majorization relation holds:*

$$\sum_{i=1}^{k} f(\lambda_i(|a+b|)) \leq \sum_{i=1}^{k} f(\lambda_i(|a|)) + \sum_{i=1}^{k} f(\lambda_i(|b|))$$

*for $1 \leq k \leq n$.*

**Conjecture 4.2.** *Let $a, b \in \mathcal{V}$, and $f : \mathcal{R}_+ \to \mathcal{R}_+$ be an increasing concave function with $f(0) \geq 0$, then there exist $\phi, \psi \in \mathrm{Aut}(\mathcal{V})$ such that*

$$f(|a+b|) \leq \phi(f(|a|)) + \psi(f(|b|)).$$

We remark that the above conjectures hold in matrix algebra [10].

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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