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Research article

On a class of Lyapunov's inequality involving λ-Hilfer Hadamard fractional derivative

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Abstract: In this paper, we presented and proved a general Lyapunov's inequality for a class of fractional boundary problems (FBPs) involving a new fractional derivative, named λ-Hilfer. We proved a criterion of existence which extended that of Lyapunov concerning the ordinary case. We used this criterion to solve the fractional differential equation (FDE) subject to the Dirichlet boundary conditions. In order to do so, we invoked some properties and essential results of λ-Hilfer fractional boundary value problem (HFBVP). This result also retrieved all previous Lyapunov-type inequalities for different types of boundary conditions as mixed. The order that we considered here only focused on $1 < r \leq 2$. General Hartman-Wintner-type inequalities were also investigated. We presented an example in order to provide an application of this result.

Keywords: Lyapunov's inequality; fractional integral; Green's function; Hilfer Hadamard integral
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1. Introduction

Over the past years, fractional differential theory has gained attention and importance in various fields. It has been developed extensively for different fields such as physics, mechanical and electrical theory, economics, engineering sciences, and other related fields. This is particularly relevant for fractional differential equations (FDEs) where Lyapunov's inequality represents a necessary condition of non-existence of non-trivial solutions to the second-order differential equation (SODE).

$$\begin{cases} m''(t) + B(t)m(t) = 0, \quad a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{1.1}$$

where $B \in C^0([a_1, a_2], \mathbb{R})$.

In [9], Lyapunov proved an important inequality to problem (1.1), represented by

$$(a_2 - a_1) \int_{a_1}^{a_2} |B(s)| ds > 4, \ a_1 < t < a_2. \tag{1.2}$$

From [10], several papers were derived. We refer the reader to the following references [1–8,12,15,16,17]. For a Riemann-Liouville derivative, Ferreira [3] showed that if $m \neq 0$ is a solution of the following fractional boundary problem

$$\begin{cases} ({}_{a_1}D^r m)(t) + B(t)m(t) = 0, \quad a_1 < t < a_2, \quad 1 < r \leq 2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{1.3}$$

then

$$(a_2 - a_1)^{r-1} \int_{a_1}^{a_2} |B(s)| \, ds > \tfrac{\Gamma(r)\, r^r}{(r-1)^{r-1}}. \tag{1.4}$$

For a Caputo fractional boundary problem (FBP), it was proved in [3] that a necessary condition of existence to the FBP

$$\begin{cases} ({}_a^C D^r m)(t) + B(t)m(t) = 0, \quad a_1 < t < a_2, \quad 1 < r \leq 2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{1.5}$$

is represented by the following integral inequality

$$(a_2 - a_1)^{r-1} \int_{a_1}^{a_2} |B(s)| \, ds > \Gamma(r)\, (4)^{r-1}. \tag{1.6}$$

For similar types of fractional differential equations with different boundary conditions and orders, one may consider the following references ([11,18–20]).

In Section 5, we treat a new class of the form

$$\begin{cases} ({}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+} m)(t) + \frac{f(B(t),m(t))}{a_2 - a_1} = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0. \end{cases}$$

As a particular case of our result, we present a Lyapunov's inequality (LPI) for the Hadamard (FBP)

$$\begin{cases} ({}^{\mathbb{H}}D^r m)(t) + B(t)u(t) = 0, \quad a_1 < t < a_2, \quad 1 < r \leq 2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{1.7}$$

where a_1 and a_2 are consecutive zeros of the solution m satisfying $1 < a_1 < a_2$. We investigate this Hadamard fractional problem (1.7) by showing that for $r \in (1, 2]$, the function B satisfies

$$\int_{a_1}^{a_2} B(t) \frac{dt}{t} \geq \frac{\Gamma(r) 4^{r-1}}{(\log a_2 - \log a_1)^{r-1}}. \tag{1.8}$$

The novelty that we deal here with, focuses λ-Hilfer type of differentiation, where λ is supposed to be a positive non-decreasing function. In fact, we present an LPI for the following Hilfer fractional boundary value problem (HFBVP)

$$\begin{cases} \left({}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+}\right) m(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{1.9}$$

where $1 < r,\beta \leq 2$, $r \leq \beta$, and a_1 and a_2 are constants.

The function $B \in C^0([a_1, a_2], \mathbb{R})$, and the operator ${}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+}$ denotes a λ-Hilfer derivative operator of order r and type β. In this note, we point out that the trivial solutions are of non-interest. We further investigate problem (1.9) by showing that it admits a solution for r, $\beta \in (1, 2]$, $r \leq \beta$, provided that B satisfies

$$\int_{a_1}^{a_2} \lambda'(s)|B(s)|ds \geq \frac{\Gamma(r)}{\left(\frac{\lambda(a_2)-\lambda(a_1)}{4}\right)^{r-1}}. \tag{1.10}$$

Below, we look for solutions of the following HFBVP

$$\begin{cases} \left({}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+}\right) m(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0. \end{cases}$$

So far, with a construction of an appropriate Green's function (GF) to the HFBVP given by Eq (1.9), none of the raised results in the literature recourse to HFBVP, where λ-Hilfer derivative is used instead of the ordinary derivative of FDE given by Eq (1.1), which is classical.

In a more general context, in Section 5, we prove a generalization to all previous results as well as [13, 14].

Thus, it is key to investigate the following HFBVP, where for different values of r and β, we obtain a large set of new solutions. The main step here is to prove that the corresponding non-trivial solution to Eq (1.9) exists provided that

$$\int_{a_1}^{a_2} \lambda'(s)|B(s)|ds \geq \frac{\Gamma(r)}{\left(\frac{\lambda(a_2)-\lambda(a_1)}{4}\right)^{r-1}}, \tag{1.11}$$

is satisfied, where $\lambda'(t) \neq 0$ and λ is assumed to be positive and non-decreasing.

In the sequel, in order to achieve the aim that we look for, we need to use some fractional tools and lemmas. Therefore, new concepts of λ-Hilfer fractional derivatives and integrals of problem (1.9) are considered. To accomplish our results, we adopt the following notations.

Let us denote by $[a_1, a_2]$ the finite interval on $\mathbb{R}$, $0 < a_1 < a_2$, and by $SCF([a_1, a_2])$, $SCF^n([a_1, a_2])$ and $ASCF^n([a_1, a_2])$ respectively, the space of continuous functions, n times absolutely continuous, and n times continuous and continuously differentiable functions. Also, we define

$$\|m\|_{SCF([a_1,a_2])} = \max_{t\in[a,b]} m(t), \tag{1.12}$$

and

$$ASCF^n([a_1, a_2]) = \{f : (a_1, a_2] \to \mathbb{R}; f^{(n-1)} \in [a_1, a_2]\}. \tag{1.13}$$

2. Definitions and auxiliary fractional tools

Definition 2.1. *If $\xi \in SCF([a_1, a_2])$ and $r > 0$, then*

$$I^r_{a_1^+}\xi(t) = \frac{1}{\Gamma(r)}\int_{a_1}^{t}\frac{\xi(s)}{(t-s)^{1-r}}ds,$$

is defined in the Riemann-Liouville (RL)-sense.

Definition 2.2. *Let $r \geq 0$, and $n = [r] + 1$. If $\xi \in ASCF^n([a_1, a_2])\mathbb{R}) \cap L^1[a_1, a_2]$, then*

$$cD^r_{a_1^+}\xi(t) = \frac{1}{\Gamma(n-r)}\int_{a_1}^{t}\frac{\xi^{(n)}(s)}{(t-s)^{r-n+1}}\,ds,$$

exists almost everywhere on $[a_1, a_2]$ ($[r]$ is the entire part of r) and defined in the Caputo sense [8].

Lemma 2.1. *Let $r, \beta > 0$ and $n = [r] + 1$. Then*

$$cD^r_{0^+}\,t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-r)}\,t^{\beta-r-1},$$

and for $\beta > n$, $cD^r_{o^+}\,t^k = 0$, $k = 0, .., n-1$ hold.

Lemma 2.2. *For $r > 0$, and $\xi \in C(0, 1)$* [8]*,*

$$cD^r_{a_1^+}\xi(t) = 0,$$

has a solution

$$g(t) = c_1 + c_2t + c_3t^2 + ... + c_nt^{n-1},$$

where, $c_i \in \mathbb{R}$, $i = 0, ..., n$, and $n = [r] + 1$, (r non-integer).

For a more general context, we define a generalized Riemann-Liouville fractional integral and derivatives as follows:

Definition 2.3. *Let $r > 0$ and n be a smallest integer greater than or equal to r. Then,* [8]

$$\begin{aligned} D^r_{a_1,\xi}f(t) &= \left(\frac{1}{\xi'(t)}\right)\left(\frac{d}{dt}\right)^n I^{n-r}_{a_1,\xi}f \\ &= \frac{1}{\Gamma(n-r)}\left(\frac{1}{\xi'(t)}\right)\left(\frac{d}{dt}\right)^n\int_{a}^{t}\frac{\xi'(s)}{(\xi(t)-\xi(s))^{r-n+1}}\,f(t)ds, a.e\ t \in [a_1, a_2], \end{aligned} \tag{2.1}$$

provided that

$$\left(\frac{1}{\xi'(t)}\right)\left(\frac{d}{dt}\right)I^{n-r}_{a_1,\xi}f,$$

exists. Let $f \in L^1((a_1, a_2), \mathbb{R})$. Then, the fractional integral of order $r > 0$ of f with respect to the function ξ is defined by

$$I^r_{a_1+,\xi}f(t) = \frac{1}{\Gamma(r)}\int_{a_1}^{t}\frac{\xi'(s)}{(\xi(t)-\xi(s))^{r-1}}\,f(t)ds,\ \ a.e\ t \in [a_1, a_2].$$

Now, we consider the new definition λ-Hilfer derivative. This novelty will be twofold. First, in the RL-sense, and second, in the Caputo-sense. In light of this, we will provide properties and lemmas involving this new concept of derivatives and theorems that are useful for the result of this paper and its application.

Theorem 2.1. *Let* $1 < r \leq \beta \leq 2$, $B \in C([a_1, a_2])$ *and* λ' *is a positive function. Then, the solution of* $m \neq 0$

$$\begin{cases} \left({}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+}\right) m(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{2.2}$$

is given by

$$m(t) = \frac{1}{\Gamma(r)} \int_{a_1}^{t} GF(t,s)m(s)B(s)ds \ + \ \frac{1}{\Gamma(r)} \int_{t}^{a_2} GF(t,s)m(s)B(s)ds,$$

where $GF(t, s)$ *is defined by*

$$\Gamma(r)GF(t,s) \ = \begin{cases} (\lambda(t) - \lambda(a_1))^{l-1}\, \lambda'(a_2) \left(\frac{\lambda(a_2)-\lambda(s)}{\lambda(a_2)-\lambda(a_1)}\right)^{r-1} \ - \lambda'(t)\,(\lambda(t) - \lambda(s))^{r-1}, \\ a_1 \leq s \leq t \\ (\lambda(t) - \lambda(a_1))^{l-1}\, \lambda'(a_2) \left(\frac{\lambda(a_2)-\lambda(s)}{\lambda(a_2)-\lambda(a_1)}\right)^{r-1} \\ t \leq s \leq a_2. \end{cases} \tag{2.3}$$

Proof. By [8], the solution of FDE given by Eq (2.2) follows as

$$\begin{aligned} m(t) &= c_1(\lambda(t) - \lambda(a_1))^{l-1} + c_2(\lambda(t) - \lambda(a_1))^{l-2} \\ &- \frac{1}{\Gamma(r)} \int_{a_1}^{t} \lambda'(s)(\lambda(t) - \lambda(s))^{r-1}\, m(s)B(s)ds. \end{aligned}$$

Using the boundary conditions in (2.2), we found

$$c_1 \ = 0, \ \text{ and } \quad c_2 = \frac{1}{\Gamma(r)} \int_{a_1}^{a_2} \left(\frac{(\lambda(t) - \lambda(s))^{r-1}}{(\lambda(a_2) - \lambda(a_1))^{l-1}} \right) \lambda'(a_2)\, B(s)m(s)ds.$$

Therefore,

$$\begin{aligned} m(t) = \left(\frac{\lambda(t) - \lambda(a_1}{\lambda(a_2) - \lambda(a_1)} \right)^{l-1} \frac{1}{\Gamma(r)} \int_{a_1}^{a_2} (\lambda(a_2) - \lambda(s))^{r-1}\ \lambda'(s)m(s)B(s)ds \\ - \frac{1}{\Gamma(r)} \int_{a_1}^{t} (\lambda(t) - \lambda(s))^{r-1}\, \lambda'(s)m(s)B(s)ds. \end{aligned}$$

3. Main result and its consequences

The contribution that we investigate below is focused on the use of a λ-Hilfer fractional integral, which is more general than the Riemann-Liouville and Hadamard integrals. The employed and necessary tools in establishing the result and its consequences are the construction of GF and the use of its maximum value on the considered interval. It is noteworthy that the novelty presented in getting the maximum value is the use of maximum principles for functions ordinary differential equations. This power tool is a simpler way for finding either the existence or non-existence of solutions of differential equations. This includes ordinary and fractional.

We are motivated by the research of Chidouh and Torres [13] and Kirane and Berikbol [14]. Let us recall that in [14], the authors investigated the following LPI for the non-linear RL-FDE

$$\begin{cases} \left(D^{r}_{a_1+}\right) m(t) + B(t)m(t) = 0, a_1 < t < b_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.1}$$

where, $B \in L^1([a_1, a_2], \mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is a concave and a non decreasing function. They showed that if $m \neq 0$ satisfying (3.1), then the following inequality

$$\int_{a_1}^{a_2} |B(s)| ds > \frac{\Gamma(r) 4^{r-1} \eta}{f(\eta)}, \tag{3.2}$$

is satisfied, where η stands for the maximum of $|m|$ over $[a_1, a_2]$.

The authors in [14] showed that if $m \neq 0$ satisfying

$$\begin{cases} \left(D^{r,l}_{a+}\right) m(t) + B(t)m(t) = 0, \ a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.3}$$

where $1 < r \leq l \leq 2$, then the following Lyapunov's inequality is satisfied

$$\Gamma(r) \, \|m\| > \frac{(l + r - 2)^{(l+r-2)}}{(r-1)^{r-1}} \, \frac{(a_2 - a_1)^{(l-r)}}{((l-1)a_2 - (r-1)a_1)^{l-r}} \, f(\|m\|) \int_{a_1}^{a_2} |B(s)| ds.$$

The result that we establish in this paper generalizes all the results presented above, and the fractional differential equation that we consider is more general than the one in the literature. We deal with a λ-Hilfer derivative operator and the corresponding function GF to the HFBVP. Precisely, we state the properties of GF which we will recourse to later in the sequel.

Theorem 3.1. *Assume that $m \neq 0$ satisfying*

$$\begin{cases} \left({}^{\mathbb{H}}D^{r,\beta;\lambda}_{a_1+}\right) m(t) + B(t)m(t) = 0, \ a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.4}$$

where $1 < r \leq l \leq 2$, $B \in C^0([a_1, a_2])$, and λ' is positive. Then, the GF defined in Eq (2.3) is positive and realizes the following inequality for all $(t, s) \in [a_1, a_2] \times [a_1, a_2]$

$$GF(t, s) \leq GF(s, s),$$

where $GF(s, s)$ is defined by

$$GF(s, s) \ = \left(\frac{\lambda(s) - \lambda(a)}{\lambda(a_2) - \lambda(a_1)} \right)^{l-1} - (\lambda(a_2) - \lambda(s))^{r-1}, \ \ s \in (a_1, a_2).$$

Furthermore, $GF'(s, s) = 0$ is achieved at

$$s = \frac{(l-1)\lambda(a_1) + (r-1)\lambda(a_2)}{l + r - 2}.$$

Then, the largest value of GF is given by

$$\max_{s \in [a_1, a_2]} G(s, s) = GF(\frac{(l-1)\lambda(a_1) + (r-1)\lambda(a_2)}{l + r - 2}, \frac{(l-1)\lambda(a_1) + (r-1)\lambda(a_2)}{l + r - 2}).$$

Proof. We first consider

$$G(t,s) = (\lambda(t) - \lambda(a_1))^{r-1}\left(\frac{(\lambda(a_2) - \lambda(s))}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1}.$$

Due to the fact that $\lambda' > 0$, $GF > 0$ too. However, for $s \le t$, GF is given by

$$GF(t,s) := (\lambda(t) - \lambda(a_1))^{r-1}\left(\frac{\lambda)(a_2) - \lambda(s)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1} - (\lambda(t) - \lambda(s))^{r-1}.$$

In order to show that this function is positive, we handle the expression $(\lambda(t) - \lambda(s))^{r-1}$ as follows

$$(\lambda(t) - \lambda(s))^{r-1} = \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a_1)}\right)^{r-1}\left(\lambda(a_2) - \lambda(a_1) + \frac{\lambda(s) - \lambda(a_1)}{\lambda(t) - \lambda(a_1)}\right)^{r-1}.$$

Now, due to the following fact

$$\begin{aligned}
\left(\lambda(a_1) + \frac{(\lambda(s) - \lambda(a_1))(\lambda(a_2) - \lambda(a_1))}{\lambda(t) - \lambda(a_1)}\right) &\ge \lambda(s) \\
\Leftrightarrow \frac{\lambda(a_1)(\lambda(t) - \lambda(a_1)) + (\lambda(s) - \lambda(a_1))(\lambda(a_2) - \lambda(a_1))}{\lambda(t) - \lambda(a_1)} & \\
&\ge \lambda(s) \\
\Leftrightarrow \lambda(a_1)(\lambda(t) - \lambda(a_1)) + (\lambda(s) - \lambda(a_1))(\lambda(a_2) - \lambda(a_1)) & \\
&\ge 0 \\
\Leftrightarrow \lambda(s) \ge \lambda(a) & \\
\Leftrightarrow s \ge a, &
\end{aligned}$$

since the function λ is supposed to be an increasing function, we then conclude that the Green's function GF is positive for $t \le s$. Indeed, let us write

$$\begin{aligned}
\left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1} - (\lambda(t) - \lambda(s))^{r-1} &= \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1}((\lambda(a_2) - \lambda(s))^{r-1} - \\
&\quad \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a)}\right)^{(r-1)-(l-1)}(\lambda(a_2) - \lambda(a_1))^{r-1}) \\
&= \left(\frac{\lambda(t) - \lambda(a)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1} \\
&\quad (1 - \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a_1)}\right)^{(r-1)-(l-1)}\left(\frac{\lambda(a_2) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right)^{(r-1)}) \\
&= \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-1} \\
&\quad (1 - \left(\frac{\lambda(t) - \lambda(a)}{\lambda(a_2) - \lambda(a_1)}\right)^{(r-l)}\left(\frac{\lambda(a_2) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right)^{(r-1)}).
\end{aligned}$$

Now since $t \le a_2$, $s \ge a_1$, $n - 1 < r < n$, and $l = r + \beta(n - r)$, in light of

$$\left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right)\left(\frac{\lambda(a_2) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right) \le 1,$$

the previous inequality is with positive right side. Thus, GF is positive for all $t, s \in [a_1, a_2]$ in view of λ', which is positive.

The next aim is to show that

$$GF(t, s) \leq GF(s, s).$$

For this matter, below is the differentiation of GF defined in Eq (2.3) with respect to t for fixed $s \in (a_1, a_2)$, where $t \leq s$, we get

$$\begin{aligned} \Gamma(r)(GF)_t(t, s) &= (l-1)(\lambda(t) - \lambda(a_1))^{l-2}(\lambda(a_2) - \lambda(s))^{r-1} \\ &- (r-1)(\lambda(t) - \lambda(a_1))^{l-2}. \end{aligned} \tag{3.5}$$

Similarly, we re-write the expression $(\lambda(t) - \lambda(s))^{r-2}$ as

$$\begin{aligned} (\lambda(t) - \lambda(s))^{l-2} &= \lambda(t) - \lambda(a_1) + \lambda(a_1) - \lambda(s))^{l-2} \\ &= \left(\frac{\lambda(t) - \lambda(s)}{\lambda(a_2) - \lambda(a_1)}\right)^{l-2} \left(\lambda(a_2) - \left(\lambda(a_1) + \frac{\lambda(s) - \lambda(a_1))}{\lambda(t) - \lambda(a_1))}\right)\right)^{r-2}. \end{aligned} \tag{3.6}$$

Now, inserting (3.6) into (3.5), one may observe that $(GF)_t$ is negative for $t \leq s$ and therefore G, is a decreasing function on this interval.

For the case $s \leq t$, with a routine calculation one may see that the function $(GF)_t$ is a positive one and consequently the function GF is an increasing function.

This enables us to complete the proof of Theorem 3.1. Let us define h for fixed s in (a_1, a_2) by

$$h(s) := GF(s, s) = \frac{(\lambda(s) - \lambda(a_1))^{l-1}(\lambda(a_2) - \lambda(s))^{r-1}}{(\lambda(a_2) - \lambda(a_1))^{r-1}}.$$

Notice that for $s = a_1$ or $s = a_2$, $h(s) = 0$.

Let us differentiate h with respect to s. After reduction and simplification of some terms, we obtain

$$\begin{aligned} h'(s) &= \frac{1}{(\lambda(a_2) - \lambda(a_1))^{l-1}}(l-1)(\lambda(s) - \lambda(a))^{l-2}(\lambda(a_2) - \lambda(s))^{r-1} \\ &- (r-1)(\lambda(a_2) - \lambda(s))^{r-2}(\lambda(s) - \lambda(a_1))^{l-2}. \end{aligned} \tag{3.7}$$

A corresponding unique solution of the derivative of h is therefore achieved by

$$s = s^* = \frac{(l-1)\lambda(a_1) + (r-1)\lambda(a_2)}{l + r - 2},$$

which leads us to conclude that the function h on $(a_1, s*)$ is increasing and on $(s*, a_2)$ it is decreasing. Thus, we deduce that

$$\begin{aligned} \max_{s \in [a_1, a_2]} h(s) = h(s*) &= h\left(\frac{(l-1)\lambda(a_1) + (r-1)\lambda(a_2)}{l + r - 2}\right) \\ &= \left(\frac{(l-1)\lambda(a_1)}{l + r - 2}\right)^{l-1} \left(\frac{(r-1)\lambda(a_2)}{l + r - 2}\right)^{r-1}. \end{aligned} \tag{3.8}$$

Now, before presenting an alternative proof which allows us to achieve the maximum value of the Green's function GF, we will explain its relevance. In dealing with this kind of Lyapunov-type inequality, one may see that once the Green function constructed, achieving its maximum value is not always an easy issue to overcome. This is due to the fractional operator in question. It may lead to a more complex analysis of the boundary value problem (cf. the difference between Riemann-Liouville and Caputo in [2, 3] related to this operator). The novelty here is to use the maximum principle to overcome this difficulty, which is valid for any fractional operator. It is worthwhile to mention that the method used to deal with Lyapunov's inequality (LPI) is based on an approach analyzing the components of the Green's function into two arguments (t and s). However, the analysis of GF might be very complex. In this case, we may need to utilize another tool to overcome the difficulty. The tool that we apply here is the maximum principle, which has, in general, a big impact in a class of ordinary differential equations and partial differential equations. In our context, we use the maximum principles of functions as follows.

The two functions ξ_1 and ξ_2 defined in an interval $[a_1, a_2]$ such that

$$0 < |\xi_1|(t, s) < |\xi_2|(t, s), \quad \text{where } t, s \in [a_1, a_2].$$

Then,

$$0 < \max_{t,s\in[a_1,a_2]} |\xi_1|(t, s) < \max_{t,s\in[a_1,a_2]} |\xi_2|(t, s), \quad \text{where } t, s \in [a_1, a_2].$$

To apply this principle to our boundary value problem (3.4), we split the GF defined in (2.3) into two functions (ξ_1 and ξ_2) as follows

$$\xi_1(t, s) := \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right)^{l-1} (\lambda(t) - \lambda(s))^{r-1},$$

and

$$\xi_2(t, s) := \left(\frac{\lambda(t) - \lambda(a_1)}{\lambda(a_2) - \lambda(s)}\right)^{l-1},$$

It is easy to see that $0 < \xi_1(t, s) \leq \xi_2(t, s)$ and therefore

$$0 < \max_{t,s\in[a_1,a_2]} |\xi_1|(t, s) < \max_{t,s\in[a_1,a_2]} |\xi_2|(t, s) := GF(s, s).$$

The function $GF(s, s)$ is a function of s denoted by $h(s)$. With the same computation as before, the function h attains its maximum at

$$s = s* = \frac{(l - 1)\lambda(a_1) + (r - 1)\lambda(a_2)}{l + r - 2},$$

and therefore the maximum of GF is

$$\max_{t,s\in[a_1,a_2]} GF(t, s) = \left(\frac{(l - 1)\lambda(a_1)}{l + r - 2}\right)^{l-1} \left(\frac{(r - 1)\lambda(a_2)}{l + r - 2}\right)^{r-1}.$$

Consequently, we get different corollaries covering different previous results.

Corollary 3.1. Let $m \neq 0$ satisfying

$$\begin{cases} ({}^H D^{r,\beta;\lambda}_{a_1+} m)(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.9}$$

where the function $B \in C^0([a_1, a_2])$, and $1 < r \leq \beta \leq 2$. Then,

$$\int_{a_1}^{a_2} \left(\frac{\lambda(s) - \lambda(a_1}{\lambda(a_2) - \lambda(a_1)} \right)^{l-1} (\lambda(a_2) - \lambda(a_1))^{r-1} \lambda'(s)|B(s)|ds \geq \Gamma(r), \tag{3.10}$$

is satisfied.

Remark 3.1. It is worth mentioning that for $r = 2$, we get an analogous inequality which may be viewed as a Hartman-Winter inequality for problem (3.4), We have

$$(\lambda(a_2) - \lambda(a_1))^{-1} \int_{a_1}^{a_2} (\lambda(s) - \lambda(a_1))(\lambda(a_2) - \lambda(s))\, \lambda'(s)|B(s)|ds \geq \Gamma(2) = 1. \tag{3.11}$$

We shall point out that this is due to the fact that λ is an increasing function. Now, we recall that the arithmetic-geometric-harmonic inequality follows as

$$(\lambda(s) - \lambda(a_1))(\lambda(a_2) - \lambda(s)) \leq \left(\frac{(\lambda(a_2) - \lambda(a_1))}{2} \right)^2.$$

Thus,

$$\left(\frac{\lambda(a_2) - \lambda(a_1)}{4} \right) \int_{a_1}^{a_2} \lambda'(s)|B(s)|ds \geq 1.$$

When $\lambda(x) = x$, we get

$$(s - a_1)(a_2 - s) \geq \frac{(a_2 - a_1)^2}{4}.$$

Therefore, we derive the following classical LPI

$$\frac{(a_2 - a_1)}{4} \int_a^b |B(s)|ds \geq 1.$$

Proof. We apply Theorem 3.1 and we obtain directly the desired result.

Corollary 3.2. Let $m \neq 0$ satisfying

$$\begin{cases} ({}^H D_{a_1+}^{r,\beta;\lambda} m)(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.12}$$

where the function $B \in C^0([a_1, a_2])$, and $1 < r \leq \beta \leq 2$. Then,

$$\left(\frac{\lambda(a_2) - \lambda(a_1)}{4} \right)^{r-1} \int_{a_1}^{a_2} \lambda'(s)|B(s)|ds \geq \Gamma(r). \tag{3.13}$$

Proof. One may use Theorem 3.1 to conclude that

$$\begin{aligned} \int_{a_1}^{a_2} \lambda'(s)B(s)\, ds \quad &\geq \quad \Gamma(r) \frac{1}{GF\left(\frac{(l-1)\lambda(a_1)+(r-1)\lambda(a_2)}{l+r-2}, \frac{(l-1)\lambda(a_1)+(r-1)\lambda(a_2)}{l+r-2} \right)} \\ &> \quad \Gamma(r) \frac{1}{\left(\left(\frac{(r-1)}{l+r-2} \right)^{l-1} \left(\frac{(l-1)}{l+r-2} \right)^{r-1} \right) (\lambda(a_2) - \lambda(a_1))^{r-1}}. \end{aligned} \tag{3.14}$$

Equivalently, we have

$$\left(\left(\frac{(r-1)}{l+r-2}\right)^{l-1}\left(\frac{(l-1)}{l+r-2}\right)^{r-1}\right)(\lambda(a_2)-\lambda(a_1))^{r-1}\int_{a_1}^{a_2}\lambda'(s)B(s)\,ds\geq\Gamma(r),$$

which achieves the proof of this corollary.

By setting $\lambda(x)=\log x$, we get the HFBVP (1.7) and (1.8), and based on the following observation:

$$\log a_2-\log a_1\leq\frac{a_2}{a_1}\qquad\text{for}\quad a_2>a_1>1,$$

the integral inequality (3.13) in Corollary 3.2 takes the following form

$$\left(\frac{a_2-a_1}{\left(\frac{(r-1)}{l+r-2}\right)^{l-1}\left(\frac{(l-1)}{l+r-2}\right)^{r-1}}\right)\int_{a_1}^{a_2}B(s)\,ds\geq\Gamma(r),$$

which conducts us to the following remark.

Remark 3.2. The criteria condition of existence of HFBVP (1.8) implies the existence condition of RL problem (1.2), $(s>a_1>1)$.

The next corollary represents the classical ordinary differential equation of order two where the necessary integral condition of existence is formulated and proved.

Corollary 3.3. Let $m\neq 0$ satisfying

$$\begin{cases}({}^{H}D^{r,l;\log}_{a_1+}m)(t)+B(t)m(t)=0, a_1<t<a_2,\\ m(a_1)=m(a_2)=0,\end{cases}\tag{3.15}$$

where the function $B\in C^0([a_1,a_2])$, and $1<r\leq l\leq 2$. Then,

(i)

$$(\log a_2-\log a_1)^{1-l}\int_{a_1}^{a_2}(\log s-\log a_1)^{r-1}|B(s)|\frac{ds}{s}\geq\frac{\Gamma(r)}{(\log a_2-\log s)^{l-1}}.\tag{3.16}$$

In addition, if $\lambda(x)=x$, then

(ii)

$$\int_{a_1}^{a_2}|B(s)|\frac{ds}{s}\geq\frac{\Gamma(r)}{(a_2-a_1)^r}\text{ and }\quad\int_{a_1}^{a_2}|B(s)|\frac{ds}{s}\geq\frac{\Gamma(r)}{(\frac{a_2}{a_1})^r},\tag{3.17}$$

for $a_2>a_1>1$.

Proof. We set $r=2$, and therefore $l=2$ and $\Gamma(2)=1$. Now, in light of (3.13), the desired inequality (3.16) is achieved.

Thus, if $\lambda(x)=\log x$, for $x>1$, we find

$$(\frac{a_2}{a_1})\int_{a_1}^{a_2}|B(s)|ds\geq 4,$$

from which the desired integral inequalities (3.17) and (3.16) are achieved.

Below, we move to the principal focus of proving a criterion of existence of non-trivial solutions involving the Hadamard $\log x$-Hilfer fractional derivative of order r and type l.

Corollary 3.4. Let $m \neq 0$ satisfying

$$\begin{cases} ({}^{H}D^{r,l;\log}_{a_1+}m)(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.18}$$

where the function $B \in C^0([a_1, a_2])$, and $1 < r \leq l \leq 2$. Then,

$$(i) \qquad (\log a_2 - \log a_1)^{1-l} \int_{a_1}^{a_2} (\log s - \log a_1)^{r-1} |B(s)| \frac{ds}{s} \geq \frac{\Gamma(r)}{(\log a_2 - \log s)^{l-1}}, \tag{3.19}$$

$$(ii) \qquad \int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq \frac{\Gamma(r)}{(a_2 - a_1)^r} \text{ and } \int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq \frac{\Gamma(r)}{(\frac{a_2}{a_1})^r}, \tag{3.20}$$

for $a_2 > a_1 > 1$.

For $r = 2$, we have a new bound for the LPI in the HFDE of second order which is:

$$\int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq (a_1 - a_2)^2, \text{and} \quad \int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq (\frac{a_2}{a_1})^2,$$

for $a_2 > a_1 > 1$.

Proof. (i) It is a direct application of (3.10).
(ii) It seems obvious to the reader that the function $f(x) := \log x - x$ is a decreasing one for $x > 1$, and therefore the requested integral inequality (3.20) is immediate.

On the other hand, by considering (3.20) and using the same observation as before, we get

$$(\frac{a_2}{a_1})^{r-1} \int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq \Gamma(r) 4^{r-1}.$$

Indeed, for $a_1 \leq s \leq a_2$ and $\log a_2 - \log a_1 \leq \frac{a_2}{a_1}$ for $a_2 > a_1 > 1$, the integral inequality (3.19) leads us to the following remark.

Remark 3.3. Under the same conditions as Corollary 3.4, we have

$$(\frac{a_2}{a_1})^{r-1} \int_{a_1}^{a_2} B(s) \frac{ds}{s} \geq \frac{\Gamma(r) 4^{r-1}}{a_1}.$$

Furthermore, if $m \neq 0$ satisfying

$$\begin{cases} ({}^{\mathbb{H}}D^{r,l;\lambda}_{a_1+}m)(t) + B(t)m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{3.21}$$

where the function $B \in C^0([a_1, a_2])$ and $1 < r \leq l \leq 2$. Then,

$$\int_{a_1}^{a_2} B(s)\, ds \geq \frac{4}{(\frac{a_2}{a_1})^{r-1}}, \tag{3.22}$$

since $a_1, a_2 > 1$.

4. Example

In order to illustrate the LPI which corresponds to the given HFBVP problems (3.3) and (3.4), we present the following application. The key in proving this result focuses on the important inequality based on Mittag-Leffler function.

The complex function

$$E_{r,l}(z) := \Sigma_{k=0}^{\infty} \frac{z^k}{rk + l}, \ r, l > 0, \ z \in C,$$

is analytic in the whole complex plane C.

Now by considering Sturm-Liouville eigenvalue problem

$$\begin{cases} ({}^{\mathbb{H}}D_{a+}^{r,l;\lambda} m)(t) + \tilde{\lambda} m(t) = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{4.1}$$

where $1 < r \leq l \leq 2$, and by setting $a_1 = 1.5, a_2 = 2$, we obtain

$$\begin{cases} ({}^{\mathbb{H}}D_{1.5+}^{r,l;\lambda} m)(t) + \tilde{\lambda} m(t) = 0, 1.5 < t < 2, \\ m(1.5) = m(2) = 0. \end{cases} \tag{4.2}$$

In the next theorem, the set of solutions of problem (4.1) is not empty if

$$|\tilde{\lambda}| \geq \frac{4}{(\lambda(2) - \lambda(1.5))^{r-1}} \Gamma(r), \tag{4.3}$$

is satisfied.

Theorem 4.1. *If $\tilde{\lambda}$ is an eigenvalue of Problem* (4.1) *satisfying*

$$|\tilde{\lambda}| \geq \frac{4}{(\lambda(2) - \lambda(1.5))^{r-1}} \Gamma(r), \tag{4.4}$$

then the solution exists.

Proof. We would like to show integral inequality (3.20). For that, we apply Theorem 3.1. In other words, we assume that $\tilde{\lambda}$ fulfills problem (4.1), there exists only one non-trivial solution depending on λ, where

$$\int_{a_1}^{a_2} |B(s)| \frac{ds}{s} \geq \frac{4}{(\lambda(a_2) - \lambda(a_1))^{r-1}}, \tag{4.5}$$

is valid. Or equivalently, we have

$$|\tilde{\lambda}| \geq \frac{4}{(\lambda(2) - \lambda(1.5))^{r-1}} \Gamma(r), \tag{4.6}$$

which results in the proof being completed.

Then, we got a family of fractional eigenvalue boundary problems associated to each eigenvalue $\tilde{\lambda}$ and therefore to each λ. As an example, for Hadamard fractional eigenvalue problem, we take $\lambda(x) = \log x$, and (4.2)

$$|\tilde{\lambda}_{\log}| \geq \frac{4}{(\log(2) - \log(1.5))^{r-1}} \Gamma(r),$$

and by setting $\lambda(x) = x$, we get

$$|\tilde{\lambda}_x| \geq \frac{4}{(2-1.5)^{r-1}}\Gamma(r),$$

which is equivalent to

$$|\lambda_x| \geq 2^{r+1}\Gamma(r),$$

and so on.

5. Lyapunov's inequality for a general non-linear boundary value problems (BVP)

We now have a general result compared to the one stated in [14]. The tools of this investigation are the concavity of the function ξ (defined under below) and a construction of an appropriate Green's function GF. Let $m \neq 0$ satisfying the following HFBVP

$$\begin{cases} ({}^{\mathbb{H}}D^{r,l;\lambda}_{a_1+}m)(t) + \frac{f(B(t),m(t))}{a_2-a_1} = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \tag{5.1}$$

where $1 < r \leq l \leq 2$, and the function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy (H_1) $f(B(t), m(t)) \leq p(B(t))\xi(m(t))$.

In turn, p and ξ satisfy (H_2) $p : \mathbb{R} \to \mathbb{R}$ continuous and positive and $\xi : \mathbb{R} \to \mathbb{R}$ is continuous, concave, and non- decreasing. The two functions, B and m, defined on $[a_1, a_2]$ into $\mathbb{R}^+$ and $\mathbb{R}$, respectively, are continuous.

The two assumed conditions, (H_1) and (H_2), enable us to establish the generalization of the previous results investigated recently in [13, 14].

Theorem 5.1. *Assume that H_1 and H_2 are satisfied, and let $m \neq 0$ satisfying the non-linear (HFBVP)* (5.1)*. Then, if*

$$\frac{g(||m||)}{||m||} \left(\left(\frac{(r-1)\lambda(a_1)}{l+r-2}\right)^{l-1}\left(\frac{(l-1)\lambda(a_2)}{l+r-2}\right)^{r-1}\right) \lambda'(a_2)(\lambda(a_2)-\lambda(a_1))^{r-1}\int_{a_1}^{a_2}|p(B(s))| \geq \Gamma(r), \tag{5.2}$$

is satisfied, then (5.1) *admits a non-null solution.*

In order to do so, we recall the Jensen's inequality which is needed to prove Theorem 5.1 and involves a class of concave and non-decreasing functions. However, it is worth mentioning that problem (5.1) is still open for other classes of functions.

It seems interesting to know which kind of functions may realize Lyapunov's inequality other than the one cited here.

Jensen's inequality

Lemma 5.1. *Let f be an integrable function defined on $[a_1, a_2]$, and let λ be a continuous and convex function defined at least on the set $[M_1, M_2]$ where M_1 is the infimum of f and M_2 is the supremum of f. Then,*

$$\lambda\left(\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f\right) \leq \left(\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\lambda(f)\right).$$

Proof. We consider a Banach space E defined by $E := (C[a, b])$ with the Chebyshev norm $\|m\| := \max_{t\in[a_1,a_2]} |m(t)|$. We have seen that the non-trivial solution m can be expressed in terms of GF as

$$m(t) = \int_{a_1}^{a_2} GF(t, s) f(p(B(t), \xi(m))\, ds,$$

where we replace $B(t)m(t)$ by $f(p(B(t), \xi(m(t)))$. Consequently, we obtain

$$\begin{aligned} \Gamma(r)\, \|m(t)\| &\leq \int_{a_1}^{a_2} \lambda'(s)\, GF(t, s) |f(B(t), \xi(m))|\, ds \qquad (5.3)\\ &\leq \int_{a_1}^{a_2} \max_{t\in[a_1,a_2]} \lambda'(s)\, p(B(t))\xi(m(s))\, ds \\ &\leq \left(\left(\frac{(r-1)}{l+r-2}\right)^{l-1} \left(\frac{(l-1)}{l+r-2}\right)^{r-1}\right) (\lambda(a_2) - \lambda(a_1))^{r-1} \\ &\quad \int_{a_1}^{a_2} \lambda'(s)\, \frac{p(B(s))\xi(m(s))}{a_2 - a_1}\, ds. \end{aligned}$$

In view of Lemma 5.1, and since ξ is concave and non-decreasing, we get

$$\begin{aligned} \Gamma(r)\, \|m(t)\| &\leq \left(\left(\frac{(r-1)}{l+r-2}\right)^{l-1} \left(\frac{(l-1)}{l+r-2}\right)^{r-1}\right) (\lambda(a_2) - \lambda(a_1))^{r-1} \\ &\quad |p(B(s))|\, \xi\left(\int_{a_1}^{a_2} \lambda'(s)\, \frac{p(B(s))\xi(m(s))}{|p(B(s))|(a_2 - a_1)}\, ds\right) \\ &\leq \left(\left(\frac{(r-1)}{l+r-2}\right)^{l-1} \left(\frac{(l-1)}{l+r-2}\right)^{r-1}\right) (\lambda(a_2) - \lambda(a))^{r-1} \\ &\quad \xi(\|m\|) \int_{a_1}^{a_2} \lambda'(s)\, \frac{|p(B(s)))|}{a_2 - a_1}\, ds, \qquad (5.4) \end{aligned}$$

from which the desired inequality is achieved.

If we set $\lambda(t) = t$, and $\xi(m) = m$, we obtain the linear case. Additionally, $p(B(t)) = B(t)$, then we retrieve the result obtained for Chidouh and Torres [13]. Also, if we consider the function ξ non-linear in its argument m and we set $p(B(t)) = B(t)$, we get the result found in Kirane and Berikbol [14].

6. On a wide class of Lyapunov-type inequalities

In this section, we present a large class of inequalities involving fractional derivatives by selecting some appropriate values of a_1, a_2, and r and l. Notice that $l := r + \beta(2 - r)$. We start varying parameters β and l, and therefore β tends toward 1, therefore $l = 2$, and by setting $p(B(s)) = B(s)(a_2 - a_1)$, we obtain the following generalized Lyapunov's inequality

$$\Gamma(r)\, \|m\| \leq \left(\frac{r-1}{r}\right) = (\frac{1}{r})^{(r-1)}\, (\lambda(b) - \lambda(a))^{r-1} \xi(\|m\|) \int_{a_1}^{a_2} \lambda'(s)\, \frac{|p(B(s)))|}{a_2 - a_1}\, ds \qquad (6.1)$$

$$\begin{cases} ({}^{\mathbb{H}}D^{r,1;\lambda}_{a_1+} m)(t) + \frac{f(B(t),m(t))}{a_2-a_1} = 0, a_1 < t < a_2, \\ m(a_1) = m(a_2) = 0, \end{cases} \qquad (6.2)$$

where $1 < r \leq l \leq 2$, and the function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy (H_1), and p and ξ to satisfy (H_2). The fractional derivative associated to this HFBVP is the λ-Caputo fractional derivative.

For the second part, by taking different eventual values of r, $p(B(s))$, and λ, we obtain several Lyapunov's inequalities. Here, we restrict ourselves to the following particular case.

Let us take $\lambda(x) = \log x$, $r = 2$, $p(B(s)) = B(s)$, and $[a_1, a_2] = [1, e]$, then the inequality given by Eq (6.1) takes the form

$$\|m\| \leq \frac{1}{4}\xi(\|m\|) \int_1^e \frac{|B(s)|}{e-1} \frac{ds}{s}.$$

7. Conclusions

The criterion of existence of non-trivial solutions for different classes of HFBVP is very interesting and considered by many researchers. The investigated questions related to this matter required an appropriate construction of the Green function. Although there have been great efforts by many researchers in determining Lyapunov's inequalities, the construction is not always a simple way to achieve this. In this article, we constructed a new one based on λ-Hilfer and its properties. The investigated tools led us to achieve a large class of non-linear boundary value problems involving λ-Hilfer. In a similar way, one may consider the same fractional differential equation subject to different boundary conditions. An implication to ponder is to consider λ-Hilfer for an FDE with a Ψ-Hilfer for a class of fractional derivative boundary conditions that may matter in different disciplines such as oscillation theory, physical problems, and related ones.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' contributions

Lakhdar Ragoub: Conceptualization, Methodology, Validation, Formal analysis, Investigation; J.F. Gómez-Aguilar: Formal analysis, Investigation, Methodology; Eduardo Pérez-Careta: Methodology, Validation, Formal analysis, Investigation; Dumitru Baleanu: Formal analysis, Investigation, Methodology. All authors read and approved the final manuscript.

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