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Research article

Three new soft separation axioms in soft topological spaces

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Abstract: Soft ω-almost-regularity, soft ω-semi-regularity, and soft ω-$T_{2\frac{1}{2}}$ as three novel soft separation axioms are introduced. It is demonstrated that soft ω-almost-regularity is strictly between "soft regularity" and "soft almost-regularity"; soft ω-$T_{2\frac{1}{2}}$ is strictly between "soft $T_{2\frac{1}{2}}$" and "soft T_2", and soft ω-semi-regularity is a weaker form of both "soft semi-regularity" and "soft ω-regularity". Several sufficient conditions for the equivalence between these new three notions and some of their relevant ones are given. Many characterizations of soft ω-almost-regularity are obtained, and a decomposition theorem of soft regularity by means of "soft ω-semi-regularity" and "soft ω-almost-regularity" is obtained. Furthermore, it is shown that soft ω-almost-regularity is heritable for specific kinds of soft subspaces. It is also proved that the soft product of two soft ω-almost regular soft topological spaces is soft ω-almost regular. In addition, the connections between our three new conceptions and their topological counterpart topological spaces are discussed.

Keywords: soft almost regularity; soft ω-openness; soft $R\omega$-openness; soft regularity; soft ω-regularity; soft semi-regularity
Mathematics Subject Classification: 54A40, 54D05

1. Introduction and preliminaries

The need for theories that cope with uncertainty emerges from daily experiences with complicated challenges requiring ambiguous facts. Molodstov's [1] soft set is a contemporary mathematical approach to coping with these difficulties. Soft collection logic is founded on the parameterization principle, which argues that complex things must be seen from several perspectives, with each aspect providing only a partial and approximate representation of the full item. Molodstov [1] was a pioneer in the application of soft sets in a variety of domains, emphasizing their advantages over probability theory and fuzzy set theory, which deal with ambiguity or uncertainty.

Following that, Maji et al. [2] began researching soft set operations such as soft unions and soft intersections. To overcome the shortcomings of these operations, Ali et al. [3] created and showed new operations such as limited union, intersection, and complement of a soft set. Babitha and Sunil [4] investigated numerous aspects of linkages and functions in a soft setting. Qin and Hong [5] developed novel kinds of soft equal relations and showed some algebraic properties of them. Their pioneering work paved the way for subsequent papers (for more detail, see [6, 7] and the references listed therein). Soft set theory has lately been a popular method among academics for dealing with uncertainty in a wide range of fields, including information theory [8], computer sciences [9], engineering [10], and medical sciences [11].

Soft topology was introduced by Shabir and Naz in [12]. Since then, many soft topological notions, including soft separation axioms [13–16], soft covering axioms [17–22], soft connectedness [23–26], and different weak and strong types of soft continuity, have been developed and investigated in recent years. The equivalence between the enriched and extended soft topologies was discussed in [27].

Separation axioms provide a way to study certain properties of compact and Lindelof spaces, as well as a way to categorize spaces and mappings into distinct families. As a result, topological scholars who presented various kinds of soft separation axioms became interested in soft separation axioms. Generally speaking, they can be separated into two classes: Soft points and ordinary points, based on the subjects being studied. While the authors in [14–16, 28] examined soft separation axioms using ordinary points, the authors in [13, 29–33] and others have applied the concept of soft points. In the present work, we introduce soft ω-almost-regularity, soft ω-semi-regularity, and soft ω-$T_{2\frac{1}{2}}$ as three novel soft separation axioms.

This article is organized as follows:

In Section 1, after the introduction, we provide a few definitions that are relevant to this paper.

In Section 2, we define soft ω-almost-regularity as a new soft separation axiom that lies between soft regularity and soft almost-regularity. We introduce many characterizations of this type of soft separation axiom. Also, we provide several sufficient conditions establishing the equivalence between this newly introduced axiom and its relevant counterparts. Moreover, we establish that soft ω-almost-regularity is heritable for specific types of soft subspaces. Furthermore, we show that soft ω-almost-regularity is a productive soft property. In addition, we investigated the links between this class of soft topological spaces and its analogs in general topology.

In Section 3, we define soft ω-semi-regularity and soft ω-$T_{2\frac{1}{2}}$ as two new soft separation axioms. We show that soft ω-semi-regularity is a weaker form of both soft semi-regularity and soft ω-regularity, and soft ω-$T_{2\frac{1}{2}}$ lies strictly between soft $T_{2\frac{1}{2}}$ and soft T_2. Also, we provide several sufficient conditions establishing the equivalence between these newly introduced axioms and their relevant counterparts. Moreover, a decomposition theorem for soft regularity through the interplay of soft ω-semi-regularity and soft ω-almost-regularity is obtained. In addition, we investigated the links between these classes of soft topological spaces and their analogs in general topology.

This paper follows the notions and terminologies as appear in [34–36]. Topological spaces and soft topological spaces, respectively, shall be abbreviated as TS and STS.

The following definitions will be used in the remainder of the paper:

Definition 1.1. A TS (H,β) is called

(a) [37] almost-regular (A-R, for simplicity) if for every $z \in H$ and every $N \in SC(H,\beta)$ such that

$z \in H - N$, we find $U, V \in \beta$ such that $z \in U$, $N \subseteq V$, and $U \cap V = \emptyset$;

(b) [38] semi-regular (S-R, for simplicity) if $RO(H,\beta)$ forms a base for β;

(c) [39] ω-almost-regular (ω-A-R, for simplicity) if for every $z \in H$ and every $N \in S\omega C(H,\beta)$ such that $z \in H - N$, we find $U, V \in \beta$ such that $z \in U$, $N \subseteq V$, and $U \cap V = \emptyset$;

(d) [39] ω-semi-regular (ω-S-R, for simplicity) if $R\omega O(H,\beta)$ forms a base for β.

Definition 1.2. A STS (H, φ, Σ) is called

(a) [13] soft T_2 if for every two soft points $a_s, b_t \in SP(H,\Sigma)$, we find $K, W \in \varphi$ such that $a_s \widetilde{\in} K$, $b_y \widetilde{\in} W$, and $K \widetilde{\cap} W = 0_\Sigma$;

(b) [13] soft regular if for every $a_z \in SP(H,\Sigma)$ and every $K \in \varphi$ such that $a_z \widetilde{\in} K$, we find $G \in \varphi$ such that $a_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$;

(c) [32] soft $T_{2\frac{1}{2}}$ if for every two soft points $a_s, b_t \in SP(H,\Sigma)$, we find $K, W \in \varphi$ such that $a_s \widetilde{\in} K$, $b_y \widetilde{\in} W$, and $Cl_\varphi(K) \widetilde{\cap} Cl_\varphi(W) = 0_\Sigma$;

(d) [31] soft almost-regular (soft A-R, for simplicity) if for every $r_z \in SP(H,\Sigma)$ and every $G \in SC(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} 1_\Sigma - G$, we find $S, T \in \varphi$ such that $r_z \widetilde{\in} S$, $G \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_\Sigma$.

(d) [33] soft ω-regular for every $a_z \in SP(H,\Sigma)$ and every $K \in \varphi$ such that $a_z \widetilde{\in} K$, we find $G \in \varphi$ such that $a_z \widetilde{\in} G \widetilde{\subseteq} Cl_{\varphi_\omega}(G) \widetilde{\subseteq} K$.

(e) [22] fully if $G(r) \neq \emptyset$ for every $G \in \varphi - \{0_\Sigma\}$ and $r \in \Sigma$.

2. Soft ω-almost-regular spaces

In this section, we define soft ω-almost-regularity as a new soft separation axiom that lies between soft regularity and soft almost-regularity. We introduce many characterizations of this type of soft separation axiom. Also, we provide several sufficient conditions establishing the equivalence between this newly introduced axiom and its relevant counterparts. Moreover, we establish that soft ω-almost-regularity is heritable for specific types of soft subspaces. Furthermore, we show that soft ω-almost-regularity is a productive soft property. In addition, we investigated the links between this class of soft topological spaces and its analogs in general topology.

Definition 2.1. An STS (H, φ, Σ) is called soft ω-almost-regular (soft ω-A-R, for simplicity) if for every $r_z \in SP(H,\Sigma)$ and every $G \in S\omega C(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} 1_\Sigma - G$, we find $S, T \in \varphi$ such that $r_z \widetilde{\in} S$, $G \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_\Sigma$.

Several characterizations of soft ω-almost-regularity are listed in the following theorem.

Theorem 2.2. The following are equivalent for any STS (H, φ, Σ):

(1) (H, φ, Σ) is soft ω-A-R.

(2) For every $r_z \in SP(H,\Sigma)$ and every $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$, we find $L \in \varphi$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} K$.

(3) For every $r_z \in SP(H,\Sigma)$ and every $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$, we find $L \in SO(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} K$.

(4) For every $r_z \in SP(H,\Sigma)$ and every $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$, we find $L \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} K$.

(5) For every $r_z \in SP(H,\Sigma)$ and every $K \in \varphi$ such that $r_z \widetilde{\in} K$, there is $L \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(K)\right)$.

(6) For every $r_z \in SP(H,\Sigma)$ and every $K \in \varphi$ such that $r_z \widetilde{\in} K$, there is $L \in \varphi$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(K)\right)$.

(7) For every $r_z \in SP(H,\Sigma)$ and every $G \in S\omega C(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} 1_\Sigma - G$, there are $S, T \in \varphi$ such that $r_z \widetilde{\in} S$, $G \widetilde{\subseteq} T$, and $Cl_\varphi(S) \widetilde{\cap} Cl_\varphi(T) = 0_\Sigma$.

(8) For every $G \in S\omega C(H,\varphi,\Sigma)$, $G = \widetilde{\cap}\left\{Cl_\varphi(K) : K \in \varphi \text{ and } G \widetilde{\subseteq} K\right\}$.

(9) For every $G \in S\omega C(H,\varphi,\Sigma)$, $G = \widetilde{\cap}\left\{Y : Y \in \varphi^c \text{ and } G \widetilde{\subseteq} Int_\varphi(Y)\right\}$.

(10) For every $L \in SS(H,\Sigma)$ and every $M \in S\omega O(H,\varphi,\Sigma)$ such that $L \widetilde{\cap} M \neq 0_\Sigma$, there is $K \in \varphi$ such that $L \widetilde{\cap} K \neq 0_\Sigma$ and $Cl_\varphi(K) \widetilde{\subseteq} M$.

(11) For every $L \in SS(H,\Sigma) - \{0_\Sigma\}$ and every $M \in S\omega C(H,\varphi,\Sigma)$ such that $L \widetilde{\cap} M = 0_\Sigma$, there are $S, T \in \varphi$ such that $L \widetilde{\cap} S \neq 0_\Sigma$ and $M \widetilde{\subseteq} T$.

Proof. (1) $\longrightarrow$ (2): Let $r_z \in SP(H,\Sigma)$ and $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$. Then, $r_z \widetilde{\notin} 1_\Sigma - K \in S\omega C(H,\varphi,\Sigma)$ and by (a) there exist $L, T \in \varphi$ such that $r_z \widetilde{\in} L$, $1_\Sigma - K \widetilde{\subseteq} T$, and $L \widetilde{\cap} T = 0_\Sigma$. Thus, $r_z \widetilde{\in} L \widetilde{\subseteq} 1_\Sigma - T \widetilde{\subseteq} K$ with $1_\Sigma - T \in \varphi^c$, and so $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} 1_\Sigma - T \widetilde{\subseteq} K$. This ends the proof.

(2) $\longrightarrow$ (3): Let $r_z \in SP(H,\Sigma)$ and $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$. By (2) we find $M \in \varphi$ such that $r_z \widetilde{\in} M \widetilde{\subseteq} Cl_\varphi(M) \widetilde{\subseteq} K$. Set $L = Int_\varphi\left(Cl_\varphi(M)\right)$. Then, $L \in SO(H,\varphi,\Sigma)$. Since $L \widetilde{\subseteq} Cl_\varphi(M) \widetilde{\subseteq} K$, $Cl_\varphi(L) \widetilde{\subseteq} Cl_\varphi(M) \widetilde{\subseteq} K$. This completes the proof.

(3) $\longrightarrow$ (4): Let $r_z \in SP(H,\Sigma)$ and $K \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} K$. By (3) we find $L \in SO(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} K$. Since $L \in SO(H,\varphi,\Sigma)$, and by Theorem 3 of [36], we have $RO(H,\varphi,\Sigma) \widetilde{\subseteq} R\omega O(H,\varphi,\Sigma)$, $L \in S\omega O(H,\varphi,\Sigma)$. This completes the proof.

(4) $\longrightarrow$ (5): Let $r_z \in SP(H,\Sigma)$ and $K \in \varphi$ such that $r_z \widetilde{\in} K$. Since by Theorem 9 of [36] $Int_\varphi\left(Cl_{\varphi_\omega}(K)\right) \in S\omega O(H,\varphi,\Sigma)$, by (4) there is $L \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(K)\right)$. This completes the proof.

(5) $\longrightarrow$ (6): Let $r_z \in SP(H,\Sigma)$ and $K \in \varphi$ such that $r_z \widetilde{\in} K$. Then, by (5) we find $L \in S\omega O(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(K)\right)$. Since by Theorem 3 of [36] we have $R\omega O(H,\varphi,\Sigma) \widetilde{\subseteq} \varphi$, then $L \in \varphi$. This completes the proof.

(6) $\longrightarrow$ (7): Let $r_z \in SP(H,\Sigma)$ and $G \in S\omega C(H,\varphi,\Sigma)$ such that $r_z \widetilde{\in} 1_\Sigma - G$. Since by Theorem 3 of [36] $R\omega O(H,\varphi,\Sigma) \widetilde{\subseteq} \varphi$, we have $r_z \widetilde{\in} 1_\Sigma - G \in \varphi$. So, by (6) we find $N \in \varphi$ such that $r_z \widetilde{\in} N \widetilde{\subseteq} Cl_\varphi(N) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(1_\Sigma - G)\right) = 1_\Sigma - G$. Again, by (6) we find $S \in \varphi$ such that $r_z \widetilde{\in} S \widetilde{\subseteq} Cl_\varphi(S) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(N)\right) \widetilde{\subseteq} Cl_\varphi(N) \widetilde{\subseteq} 1_\Sigma - G$. Let $T = 1_\Sigma - Cl_\varphi(N)$. Then, $S, T \in \varphi$ and $r_z \widetilde{\in} S$. Since $Cl_\varphi(N) \widetilde{\subseteq} 1_\Sigma - G$, then $G \widetilde{\subseteq} 1_\Sigma - Cl_\varphi(N) = T$.

Claim. $Cl_\varphi(S) \widetilde{\cap} Cl_\varphi(T) = 0_\Sigma$.

Proof of Claim. Suppose to the contrary that there is $a_x \widetilde{\in} Cl_\varphi(S) \widetilde{\cap} Cl_\varphi(T)$. Since $a_x \widetilde{\in} Cl_\varphi(T)$ and $a_x \widetilde{\in} Cl_\varphi(S) \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(N)\right) \in \varphi$, then $Int_\varphi\left(Cl_{\varphi_\omega}(N)\right) \widetilde{\cap} T \neq 0_\Sigma$. Since $Int_\varphi\left(Cl_{\varphi_\omega}(N)\right) \widetilde{\subseteq} Cl_\varphi(N)$, then $Cl_\varphi(N) \widetilde{\cap} T = Cl_\varphi(N) \widetilde{\cap} \left(1_\Sigma - Cl_\varphi(N)\right) \neq 0_\Sigma$, a contradiction.

This completes the proof.

(7) $\longrightarrow$ (8): Let $G \in S\omega C(H,\varphi,\Sigma)$. Then, for each $r_z \widetilde{\in} 1_\Sigma - G$, there exist $S_{r_z}, T_{r_z} \in \varphi$ such that $r_z \widetilde{\in} S_{r_z}$, $G \widetilde{\subseteq} T_{r_z}$, and $Cl_\varphi\left(S_{r_z}\right) \widetilde{\cap} Cl_\varphi\left(T_{r_z}\right) = 0_\Sigma$. Thus, $G \widetilde{\subseteq} T_{r_z}$ and $r_z \widetilde{\notin} Cl_\varphi\left(T_{r_z}\right)$.

Claim. $G = \widetilde{\cap}\left\{Cl_\varphi\left(T_{r_z}\right) : r_z \widetilde{\in} 1_\Sigma - G\right\}$.

Proof of Claim. For every $r_z \widetilde{\in} 1_\Sigma - G$, we have $G \widetilde{\subseteq} T_{r_z} \widetilde{\subseteq} Cl_\varphi\left(T_{r_z}\right)$, and so $G \widetilde{\subseteq} \widetilde{\cap}\left\{Cl_\varphi\left(T_{r_z}\right) : r_z \widetilde{\in} 1_\Sigma - G\right\}$. To show that $\widetilde{\cap}\left\{Cl_\varphi\left(T_{r_z}\right) : r_z \widetilde{\in} 1_\Sigma - G\right\} \widetilde{\subseteq} G$, let $r_z \widetilde{\in} 1_\Sigma - G$. Then, $r_z \widetilde{\notin} Cl_\varphi\left(T_{r_z}\right)$, and thus

$r_z \widetilde{\notin} \widetilde{\cap} \left\{ Cl_\varphi \left(T_{r_z} \right) : r_z \widetilde{\in} 1_\Sigma - G \right\}$.

By the above claim, we conclude that $G \widetilde{\subseteq} \widetilde{\cap} \left\{ Cl_\varphi(T) : T \in \varphi \text{ with } G \widetilde{\subseteq} T \right\} \widetilde{\subseteq} \widetilde{\cap} \left\{ Cl_\varphi \left(T_{r_z} \right) : r_z \widetilde{\in} 1_\Sigma - G \right\} = G$. This completes the proof.

(8) $\longrightarrow$ (9): Obvious.

(9) $\longrightarrow$ (10): Let $L \in SS(H, \Sigma)$ and $M \in S\omega O(H, \varphi, \Sigma)$ such that $L \widetilde{\cap} M \neq 0_\Sigma$. Pick $a_x \widetilde{\in} L \widetilde{\cap} M$. Since $M \in S\omega O(H, \varphi, \Sigma)$, $1_\Sigma - M \in S\omega C(H, \varphi, \Sigma)$, and by (9) $1_\Sigma - M = \widetilde{\cap} \left\{ Y : Y \in \varphi^c \text{ with } 1_\Sigma - M \widetilde{\subseteq} Int_\varphi(Y) \right\}$. Since $a_x \widetilde{\in} M$, then $a_x \widetilde{\notin} \widetilde{\cap} \left\{ Y : Y \in \varphi^c \text{ with } 1_\Sigma - M \widetilde{\subseteq} Int_\varphi(Y) \right\}$, and thus we find $Y \in \varphi^c$ such that $1_\Sigma - M \widetilde{\subseteq} Int_\varphi(Y)$ and $a_x \widetilde{\notin} Y$. Let $S = 1_\Sigma - Y$. Then, $S \in \varphi$, $S \widetilde{\subseteq} 1_\Sigma - Int_\varphi(Y) \widetilde{\subseteq} M$, and $a_x \widetilde{\in} S \widetilde{\cap} L$. Since $1_\Sigma - Int_\varphi(Y) \in \varphi^c$ and $S \widetilde{\subseteq} 1_\Sigma - Int_\varphi(Y) \widetilde{\subseteq} M$, then $Cl_\varphi(S) \widetilde{\subseteq} M$. This completes the proof.

(10) $\longrightarrow$ (11): Let $L \in SS(H, \Sigma) - \{0_\Sigma\}$ and $M \in S\omega C(H, \varphi, \Sigma)$ such that $L \widetilde{\cap} M = 0_\Sigma$. Then, $1_\Sigma - M \in S\omega O(H, \varphi, \Sigma)$ such that $L \widetilde{\cap} (1_\Sigma - M) = L \neq 0_\Sigma$. Thus, by (10) we find $S \in \varphi$ such that $L \widetilde{\cap} S \neq 0_\Sigma$ and $Cl_\varphi(S) \widetilde{\subseteq} 1_\Sigma - M$. Let $T = 1_\Sigma - Cl_\varphi(S)$. Then, $T \in \varphi$, $M \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = S \widetilde{\cap} \left(1_\Sigma - Cl_\varphi(S) \right) = 0_\Sigma$.

(11) $\longrightarrow$ (1): $r_z \in SP(H, \Sigma)$ and every $G \in S\omega C(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} 1_\Sigma - G$. Then, $r_z \widetilde{\cap} G = 0_\Sigma$, and by (11) there exist $S, T \in \varphi$ such that $r_z \widetilde{\cap} S \neq 0_\Sigma$, $G \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_\Sigma$. Since $r_z \widetilde{\cap} S \neq 0_\Sigma$, then $r_z \widetilde{\in} S$. This ends the proof.

In Theorems 2.3, 2.4, 2.7, and Corollary 2.8, we discuss the connections between soft almost-regularity and its analog in traditional topological spaces. Also, in Theorems 2.5, 2.6, 2.9, and Corollary 2.10, we discuss the connections between soft ω-almost-regularity and its analog in traditional topological spaces.

Theorem 2.3. If (H, φ, Σ) is full and soft A-R, then (H, φ_r) is A-R for all $r \in \Sigma$.

Proof. Let (H, φ, Σ) be full and soft A-R. Let $r \in \Sigma$. Let $z \in H$ and let $W \in \varphi_r$ such that $z \in W$. Choose $K \in \varphi$ such that $K(r) = W$. Since $r_z \widetilde{\in} K \in \varphi$, by Theorem 3.4 (iv) of [31], we find $L \in \varphi$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi \left(Cl_\varphi(K) \right)$. By Proposition 7 of [12], $Cl_{\varphi_r}(L(r)) \subseteq \left(Cl_\varphi(L) \right)(r)$. Also, by Theorem 12 (c) of [36], $\left(Int_\varphi \left(Cl_\varphi(K) \right) \right)(r) = Int_\varphi \left(Cl_\varphi(K(r)) \right)$. Therefore, we have

$$\begin{aligned} z &\in L(r) \\ &\subseteq Cl_{\varphi_r}(L(r)) \\ &\subseteq \left(Cl_\varphi(L) \right)(r) \\ &\subseteq \left(Int_\varphi \left(Cl_\varphi(K) \right) \right)(r) \\ &= Int_\varphi \left(Cl_\varphi(K(r)) \right) \\ &= Int_{\varphi_r} \left(Cl_{\varphi_r}(W) \right). \end{aligned}$$

Hence, by Theorem 2.2 (d) of [37], it follows that (H, φ_r) is A-R.

Theorem 2.4. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, C(\mathcal{L}), \Sigma)$ is soft A-R iff $(D, \mathcal{L})$ is A-R.

Proof. Necessity. Let $(D, C(\mathcal{L}), \Sigma)$ be soft A-R. Pick $r \in \Sigma$. Since it is clear that $(D, C(\mathcal{L}), \Sigma)$ is full, then by Theorem 2.3, $(D, (C(\mathcal{L}))_r) = (D, \mathcal{L})$ is A-R.

Sufficiency. Let $(D, \mathcal{L})$ be A-R. Let $r_z \in SP(D, \Sigma)$ and let $C_U \in C(\mathcal{L})$ such that $r_z \widetilde{\in} C_U$. Then, we have $z \in U \in \mathcal{L}$. So, by Theorem 2.2 (d) of [37], we find $V \in \mathcal{L}$ such that $z \in V \subseteq Cl_{\mathcal{L}}(V) \subseteq Int_{\mathcal{L}}(Cl_{\mathcal{L}}(U))$. Thus, we have $C_V \in C(\mathcal{L})$ and $r_z \widetilde{\in} C_V \widetilde{\subseteq} Cl_{C(\mathcal{L})}(C_V) = C_{Cl_{\mathcal{L}}(V)} \widetilde{\subseteq} C_{Int_{\mathcal{L}}(Cl_{\mathcal{L}}(U))} = Int_{C(\mathcal{L})}(Cl_{C(\mathcal{L})}(C_U))$. Therefore, by Theorem 3.4 (iv) of [39], $(D, C(\mathcal{L}), \Sigma)$ is soft A-R.

Theorem 2.5. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, C(\mathcal{L}), \Sigma)$ is soft ω-A-R iff $(D, \mathcal{L})$ is ω-A-R.

Proof. *Necessity.* Let $(D, C(\mathcal{L}), \Sigma)$ be soft ω-A-R. Let $z \in D$ and $U \in \mathcal{L}$ such that $z \in U$. Pick $r \in \Sigma$. Then, we have $r_z \widetilde{\in} C_U \in C(\mathcal{L})$. Since $(D, C(\mathcal{L}), \Sigma)$ is soft ω-A-R, by Theorem 2.2 (5) we find $V \in \mathcal{L}$ such that $r_z \widetilde{\in} C_V \widetilde{\subseteq} Cl_{C(\mathcal{L})}(C_V) = C_{Cl_{\mathcal{L}}(V)} \widetilde{\subseteq} Int_{C(\mathcal{L})}\left(Cl_{(C(\mathcal{L}))_\omega}(C_V)\right) = C_{Int_{\mathcal{L}}\left(Cl_{\mathcal{L}_\omega}(U)\right)}$. Therefore, $z \in V \subseteq Cl_{\mathcal{L}}(V) \subseteq Int_{\mathcal{L}}(Cl_{\mathcal{L}_\omega}(U))$. This shows that $(D, \mathcal{L})$ is ω-A-R.

Sufficiency. Let $(D, \mathcal{L})$ be ω-A-R. Let $r_z \in SP(D, \Sigma)$ and let $C_U \in C(\mathcal{L})$ such that $r_z \widetilde{\in} C_U$. Then, we have $z \in U \in \mathcal{L}$. So, by Theorem 2.1 (e) of [39], we find $V \in \mathcal{L}$ such that $z \in V \subseteq Cl_{\mathcal{L}}(V) \subseteq Int_{\mathcal{L}}(Cl_{\mathcal{L}_\omega}(U))$. Thus, we have $C_V \in C(\mathcal{L})$ and $r_z \widetilde{\in} C_V \widetilde{\subseteq} Cl_{C(\mathcal{L})}(C_V) = C_{Cl_{\mathcal{L}}(V)} \widetilde{\subseteq} C_{Int_{\mathcal{L}}\left(Cl_{\mathcal{L}_\omega}(U)\right)} = Int_{C(\mathcal{L})}\left(Cl_{(C(\mathcal{L}))_\omega}(C_V)\right)$. Therefore, $(D, C(\mathcal{L}), \Sigma)$ is soft ω-A-R.

Theorem 2.6. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, C(\mathcal{L}), \Sigma)$ is soft regular iff $(D, \mathcal{L})$ is regular.

Proof. *Necessity.* Let $(D, C(\mathcal{L}), \Sigma)$ be soft regular. Let $z \in D$ and $U \in \mathcal{L}$ such that $z \in U$. Pick $r \in \Sigma$. Then, we have $r_z \widetilde{\in} C_U \in C(\mathcal{L})$. Since $(D, C(\mathcal{L}), \Sigma)$ is soft regular, we find $V \in \mathcal{L}$ such that $r_z \widetilde{\in} C_V \widetilde{\subseteq} Cl_{C(\mathcal{L})}(C_V) = C_{Cl_{\mathcal{L}}(V)} \widetilde{\subseteq} C_U$. Therefore, $z \in V \subseteq Cl_{\mathcal{L}}(V) \subseteq U$. This shows that $(D, \mathcal{L})$ is regular.

Sufficiency. Let $(D, \mathcal{L})$ be regular. Let $r_z \in SP(D, \Sigma)$ and let $C_U \in C(\mathcal{L})$ such that $r_z \widetilde{\in} C_U$. Then, we have $z \in U \in \mathcal{L}$. So, we find $V \in \mathcal{L}$ such that $z \in V \subseteq Cl_{\mathcal{L}}(V) \subseteq U$. Thus, we have $C_V \in C(\mathcal{L})$ and $r_z \widetilde{\in} C_V \widetilde{\subseteq} C_{Cl_{\mathcal{L}}(V)} = Cl_{C(\mathcal{L})}(C_V) \widetilde{\subseteq} C_U$. Therefore, $(D, C(\mathcal{L}), \Sigma)$ is soft regular.

Theorem 2.7. Let $\{(H, \mathcal{L}_r) : r \in \Sigma\}$ be a collection of TSs. Then, $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft A-R iff $(H, \mathcal{L}_r)$ is A-R for every $r \in \Sigma$.

Proof. *Necessity*. Let $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ be soft A-R and let $r \in \Sigma$. Let $z \in H$ and let $U \in \mathcal{L}_r$ such that $z \in U$. Then, $r_z \widetilde{\in} r_U \in \oplus_{r\in\Sigma}\mathcal{L}_r$. So, by Theorem 3.4 (iv) of [31], we find $L \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L) \widetilde{\subseteq} Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U))$. Thus, we have $z \in L(r) \in \mathcal{L}_r$ and $z \in L(r) \subseteq (Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L))(r) \subseteq (Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U)))(r)$.

In contrast, by Lemma 4.9 of [40], $(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L))(r) = Cl_{\mathcal{L}_r}(L(r))$ and $(Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U)))(r) = Int_{\mathcal{L}_r}((Cl_{\mathcal{L}_r}(r_U))(r)) = Int_{\mathcal{L}_r}(Cl_{\mathcal{L}_r}(U))$. Thus, by Theorem 3.4 (iv) of [31], $(H, \mathcal{L}_r)$ is A-R.

Sufficiency. Let $(H, \mathcal{L}_r)$ be A-R for every $r \in \Sigma$. Let $r_z \in SP(H, \Sigma)$ and let $K \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} K$. By Theorem 3.5 of [34], we find $U \in \mathcal{L}_r$ such that $r_z \widetilde{\in} r_U \widetilde{\subseteq} K$. Then, we have $z \in U \in \mathcal{L}_r$. So, by Theorem 2.1 (d) of [39], we find $V \in \mathcal{L}_r$ such that $z \in V \subseteq Cl_{\mathcal{L}_r}(V) \subseteq Int_{\mathcal{L}_r}(Cl_{\mathcal{L}_r}(U))$. Thus, we have $r_V \in \oplus_{r\in\Sigma}\mathcal{L}_r$ and

$$\begin{aligned} r_z &\widetilde{\in} r_V \\ &\widetilde{\subseteq} r_{Cl_{\mathcal{L}_r}(V)} \\ &= Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_V) \\ &\widetilde{\subseteq} r_{Int_{\mathcal{L}_r}\left(Cl_{\mathcal{L}_r}(U)\right)} \\ &= Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U)) \\ &\widetilde{\subseteq} Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(K)). \end{aligned}$$

Corollary 2.8. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, \tau(\mathcal{L}), \Sigma)$ is soft A-R iff $(D, \mathcal{L})$ is A-R.

Proof. For each $r \in \Sigma$, set $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$, and by Theorem 2.7 we get the result.

Theorem 2.9. Let $\{(H, \mathcal{L}_r) : r \in \Sigma\}$ be a collection of TSs. Then, $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft ω-A-R iff $(H, \mathcal{L}_r)$ is ω-A-R for every $r \in \Sigma$.

Proof. *Necessity*. Let $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ be soft ω-A-R and let $r \in \Sigma$. Let $z \in H$ and let $U \in \mathcal{L}_r$ such that $z \in U$. Then, $r_z \widetilde{\in} r_U \in \oplus_{r\in\Sigma}\mathcal{L}_r$. So, by Theorem 2.2 (e), we find $L \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L) \widetilde{\subseteq} Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}\left(Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(r_U)\right)$. Thus, we have $z \in L(r) \in \mathcal{L}_r$ and $z \in L(r) \subseteq (Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L))(r) \subseteq (Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U)))(r)$.

In contrast, by Lemma 4.9 of [40] and Theorem 8 of [35], $(Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(L))(r) = Cl_{\mathcal{L}_r}(L(r))$ and $Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}\left(Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(r_U)\right)(r) = Int_{\mathcal{L}_r}\left(\left(Cl_{(\mathcal{L}_r)_\omega}(r_U)\right)(r)\right) = Int_{\mathcal{L}_r}\left(Cl_{(\mathcal{L}_r)_\omega}(U)\right)$. Thus, by Theorem 2.1 (e) of [39] $(H, \mathcal{L}_r)$ is ω-A-R.

Sufficiency. Let $(H, \mathcal{L}_r)$ be ω-A-R for every $r \in \Sigma$. Let $r_z \in SP(H, \Sigma)$ and let $K \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} K$. By Theorem 3.5 of [34] we find $U \in \mathcal{L}_r$ such that $r_z \widetilde{\in} r_U \widetilde{\subseteq} K$. Then, we have $z \in U \in \mathcal{L}_r$. So, by Theorem 2.1 (e) of [39] we find $V \in \mathcal{L}_r$ such that $z \in V \subseteq Cl_{\mathcal{L}_r}(V) \subseteq Int_{\mathcal{L}_r}\left(Cl_{(\mathcal{L}_r)_\omega}(U)\right)$. Thus, we have $r_V \in \oplus_{r\in\Sigma}\mathcal{L}_r$ and

$$\begin{aligned} r_z &\widetilde{\in} r_V \\ &\widetilde{\subseteq} r_{Cl_{\mathcal{L}_r}(V)} \\ &= Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_V) \\ &\widetilde{\subseteq} r_{Int_{\mathcal{L}_r}\left(Cl_{(\mathcal{L}_r)_\omega}(U)\right)} \\ &= Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}\left(Cl_{(\oplus_{r\in\Sigma}(\mathcal{L}_r)_\omega)}(r_U)\right) \\ &\widetilde{\subseteq} Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}\left(Cl_{\oplus_{r\in\Sigma}(\mathcal{L}_r)_\omega}(K)\right) \\ &= Int_{\oplus_{r\in\Sigma}\mathcal{L}_r}\left(Cl\left(_{\oplus_{r\in\Sigma}\mathcal{L}_r}\right)_\omega(K)\right). \end{aligned}$$

Corollary 2.10. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, \tau(\mathcal{L}), \Sigma)$ is soft ω-A-R iff $(D, \mathcal{L})$ is ω-A-R.

Proof. For each $r \in \Sigma$, set $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$ and by Theorem 2.9 we get the result.

Theorem 2.11. Soft regular STSs are soft ω-A-R.

Proof. Let (H, φ, Σ) be soft regular. Let $r_z \in SP(H, \Sigma)$ and $K \in S\omega O(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} K$. Since by Theorem 3 of [36] we have $R\omega O(H, \varphi, \Sigma) \subseteq \varphi$, then $K \in \varphi$. Since (H, φ, Σ) is soft regular, then we find $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Thus, by Theorem 2.2 (2) (H, φ, Σ) is soft ω-A-R.

Theorem 2.12. Soft ω-A-R STSs are soft A-R.

Proof. Let (H, φ, Σ) be soft ω-A-R. Let $r_z \in SP(H, \Sigma)$ and $K \in SO(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} K$. Since by Theorem 3 of [36] we have $RO(H, \varphi, \Sigma) \subseteq R\omega O(H, \varphi, \Sigma)$, then $K \in S\omega O(H, \varphi, \Sigma)$. Since (H, φ, Σ) is soft ω-A-R, then by Theorem 2.2 (b) there is $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Thus, by Theorem 2.2 (b) of [31], (H, φ, Σ) is soft A-R.

Theorem 2.13. Soft L-C soft ω-A-R STSs are soft regular.

Proof. Let (H, φ, Σ) be soft L-C and soft ω-A-R. Let $r_z \in SP(H, \Sigma)$ and $K \in \varphi$ such that $r_z \widetilde{\in} K$. Since (H, φ, Σ) is soft L-C, then by Theorem 5 of [36] $K \in S\omega O(H, \varphi, \Sigma)$. Since (H, φ, Σ) is soft ω-A-R, then by Theorem 2.2 (2) there is $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Therefore, (H, φ, Σ) is soft regular.

Theorem 2.14. Soft anti-L-C soft A-R STSs are soft ω-A-R.

Proof. Let (H, φ, Σ) be soft anti-L-C and soft A-R. Let $r_z \in SP(H, \Sigma)$ and $K \in S\omega O(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} K$. Since (H, φ, Σ) is anti-L-C, then by Theorem 6 of [36] $K \in SO(H, \varphi, \Sigma)$. Since (H, φ, Σ) is soft A-R, then by Theorem 3.4 (ii) of [31] there is $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Therefore, by Theorem 2.2 (b), (H, φ, Σ) is soft ω-A-R.

Theorem 2.15. For any STS (H, φ, Σ), $(H, \varphi_\omega, \Sigma)$ is soft A-R iff (H, φ, Σ) is soft ω-A-R.

Proof. Necessity. Let $(H, \varphi_\omega, \Sigma)$ be soft A-R. Let $r_z \in SP(H, \Sigma)$ and $K \in S\omega O(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} K$. By Theorem 7 of [36] $K \in SO(H, \varphi_\omega, \Sigma)$. Since $(H, \varphi_\omega, \Sigma)$ is soft A-R, then by Theorem 3.4 (ii) of [31] there is $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Therefore, by Theorem 2.2 (b) (H, φ, Σ) is soft ω-A-R.

Sufficiency. Let (H, φ, Σ) be soft ω-A-R. Let $r_z \in SP(H, \Sigma)$ and $K \in SO(H, \varphi_\omega, \Sigma)$ such that $r_z \widetilde{\in} K$. By Theorem 7 of [36] $K \in S\omega O(H, \varphi, \Sigma)$. Since (H, φ, Σ) is soft ω-A-R, then by Theorem 2.2 (2) there is $G \in \varphi$ such that $r_z \widetilde{\in} G \widetilde{\subseteq} Cl_\varphi(G) \widetilde{\subseteq} K$. Therefore, by Theorem 3.4 (ii) of [31] $(H, \varphi_\omega, \Sigma)$ is soft A-R.

The previously mentioned theorems lead to the following implications, yet Examples 2.16 and 2.17 that follow demonstrate that the opposite of these implications is false.

$$\text{Soft regular} \longrightarrow \text{Soft } \omega\text{-A-R} \longrightarrow \text{Soft A-R.}$$

The following two examples show that any of the conditions soft L-C and soft anti-L-C in Theorems 2.13 and 2.14 cannot be dropped:

Example 2.16. Consider $(\mathbb{R}, C(\Theta), \mathbb{Z})$, where Θ is the cofinite topology on $\mathbb{R}$. Since $(\mathbb{R}, \Theta)$ is not regular, by Theorem 2.6 $(\mathbb{R}, C(\Theta), \mathbb{Z})$ is not soft regular. In contrast, since $(\mathbb{R}, C(\Theta), \mathbb{Z})$ is anti-L-C, then by Theorem 6 of [36] $R\omega O(\mathbb{R}, C(\Theta), \mathbb{Z}) = RO(\mathbb{R}, C(\Theta), \mathbb{Z}) = \{0_{\mathbb{Z}}, 1_{\mathbb{Z}}\}$, and thus $(\mathbb{R}, C(\Theta), \mathbb{Z})$ is soft ω-A-R.

Example 2.17. Consider $(\mathbb{N}, C(\Theta), \{a, b\})$, where Θ is the cofinite topology on $\mathbb{N}$. Since $(\mathbb{N}, \Theta)$ is not regular, by Theorem 2.6 $(\mathbb{N}, C(\Theta), \{a, b\})$ is not soft regular. Since $(\mathbb{N}, C(\Theta), \{a, b\})$ is soft L-C, then by Theorem 2.13 $(\mathbb{N}, C(\Theta), \{a, b\})$is not soft ω-A-R. In contrast, since $RO(\mathbb{N}, C(\Theta), \{a, b\}) = \{0_{\{a,b\}}, 1_{\{a,b\}}\}$, then $(\mathbb{N}, C(\Theta), \{a, b\})$ is soft A-R.

The following lemma will be used in the next main result:

Lemma 2.18. Let (H, φ, Σ) be an STS. If C_Y is a soft dense subset of $(H, \varphi_\omega, \Sigma)$, then for any soft subset $H \in SS(Y, \Sigma)$ $Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H)) = Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y$.

Proof. Suppose that C_Y is a soft dense subset of $(H, \varphi_\omega, \Sigma)$ and let $H \in SS(Y, \Sigma)$. To see that $Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H)) \widetilde{\subseteq} Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y$, let $a_x \widetilde{\in} Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H))$. Since $Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H)) \in \varphi_Y$, then there is $M \in \varphi$ such that $Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H)) = M \widetilde{\cap} C_Y$. Thus, we have $a_x \widetilde{\in} M \widetilde{\cap} C_Y \widetilde{\subseteq} Cl_{(\varphi_\omega)_Y}(H) = \left(Cl_{\varphi_\omega}(H)\right) \widetilde{\cap} C_Y$.

Claim. $M \widetilde{\subseteq} Cl_{\varphi_\omega}(H)$.

Proof of Claim. Suppose to the contrary that $M \widetilde{\cap} \left(1_\Sigma - Cl_{\varphi_\omega}(H)\right) \neq 0_\Sigma$. Since $1_\Sigma - Cl_{\varphi_\omega}(H) \in \varphi_\omega$ and $M \in \varphi \subseteq \varphi_\omega$, then $M \widetilde{\cap} \left(1_\Sigma - Cl_{\varphi_\omega}(H)\right) \in \varphi_\omega$. Since C_Y is soft dense in $(H, \varphi_\omega, \Sigma)$, then $M \widetilde{\cap} \left(1_\Sigma - Cl_{\varphi_\omega}(H)\right) \widetilde{\cap} C_Y \neq 0_\Sigma$. Choose $b_y \widetilde{\in} M \widetilde{\cap} \left(1_\Sigma - Cl_{\varphi_\omega}(H)\right) \widetilde{\cap} C_Y$. Thus, we have $b_y \widetilde{\in} 1_\Sigma - Cl_{\varphi_\omega}(H)$ and $b_y \widetilde{\in} M \widetilde{\cap} C_Y \widetilde{\subseteq} \left(Cl_{\varphi_\omega}(H)\right) \widetilde{\cap} C_Y \widetilde{\subseteq} Cl_{\varphi_\omega}(H)$, a contradiction.

Therefore, by the above Claim, we must have $a_x \widetilde{\in} M \widetilde{\subseteq} Cl_{\varphi_\omega}(H)$, and hence $a_x \widetilde{\in} Int_\varphi(Cl_{\varphi_\omega}(H)$. Hence, $a_x \widetilde{\in} Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y$.

To see that $Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y \widetilde{\subseteq} Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H))$, let $a_x \widetilde{\in} Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y$. Since $a_x \widetilde{\in} Int_\varphi(Cl_{\varphi_\omega}(H)) \in \varphi$, then there is $M \in \varphi$ such that $a_x \widetilde{\in} M \widetilde{\subseteq} Cl_{\varphi_\omega}(H)$ and so $a_x \widetilde{\in} M \widetilde{\cap} C_Y \widetilde{\subseteq} Cl_{\varphi_\omega}(H) \widetilde{\cap} C_Y = Cl_{(\varphi_\omega)_Y}(H)$. Since $M \widetilde{\cap} C_Y \in \varphi_\omega$, then $a_x \widetilde{\in} Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H))$.

Theorems 2.19 and 2.21 establish that soft ω-almost-regularity is heritable for specific types of soft subspaces.

Theorem 2.19. If (H, φ, Σ) is a soft ω-A-R STS and C_Y is a soft dense subspace of $(H, \varphi_\omega, \Sigma)$, then (Y, φ_Y, Σ) is soft ω-A-R.

Proof. Let $a_x \in SP(Y, \Sigma)$ and let $H \in S\omega O(Y, \varphi_Y, \Sigma)$ such that $a_x \widetilde{\in} H$. Since $H \in S\omega O(Y, \varphi_Y, \Sigma)$, then $Int_{\varphi_Y}(Cl_{(\varphi_Y)_\omega}(H)) = H$. Since by Theorem 15 of [35] $(\varphi_\omega)_Y = (\varphi_Y)_\omega$, then $Int_{\varphi_Y}(Cl_{(\varphi_\omega)_Y}(H)) = H$. So, by Lemma 2.18 $H = Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y$. Thus, we have $a_x \widetilde{\in} Int_\varphi(Cl_{\varphi_\omega}(H)) \in S\omega O(H, \varphi, \Sigma)$. Since (H, φ, Σ) is soft ω-A-R, then by Theorem 2.2 (2) there is $L \in \varphi$ such that $a_x \widetilde{\in} L \widetilde{\subseteq} Cl_\varphi(L) \widetilde{\subseteq} Int_\varphi(Cl_{\varphi_\omega}(H))$. Therefore, we have $a_x \widetilde{\in} L \widetilde{\cap} C_Y \in \varphi_Y$ and $Cl_{\varphi_Y}(L \widetilde{\cap} C_Y) = Cl_\varphi(L \widetilde{\cap} C_Y) \widetilde{\cap} C_Y \widetilde{\subseteq} Int_\varphi(Cl_{\varphi_\omega}(H)) \widetilde{\cap} C_Y = H$. This shows that (Y, φ_Y, Σ) is soft ω-A-R.

The following lemma will be used in the next main result:

Lemma 2.20. Let (H,φ,Σ) be an STS and let $C_Y \in S\omega O(H,\varphi,\Sigma) - \{0_\Sigma\}$, then $R\omega O(Y,\varphi_Y,\Sigma) \subseteq R\omega O(H,\varphi,\Sigma)$.

Proof. Let $C_Y \in S\omega O(H,\varphi,\Sigma) - \{0_\Sigma\}$ and let $H \in S\omega O(Y,\varphi_Y,\Sigma)$. Then, $H = Int_{\varphi_Y}(Cl_{(\varphi_Y)_\omega}(H))$. Since by Theorem 15 of [35], $(\varphi_\omega)_Y = (\varphi_Y)_\omega$, then $Cl_{(\varphi_Y)_\omega}(H) = Cl_{(\varphi_\omega)_Y}(H) = Cl_{\varphi_\omega}(H)\widetilde{\cap}C_Y$. Since by Theorem 3 of [36] $R\omega O(H,\varphi,\Sigma) \subseteq \varphi$, then $C_Y \in \varphi$ and so $Int_{\varphi_Y}(Cl_{(\varphi_Y)_\omega}(H)) = Int_\varphi((Cl_{(\varphi_Y)_\omega}(H)))$. Thus, $H = Int_\varphi(Cl_{\varphi_\omega}(H)\widetilde{\cap}C_Y) = Int_\varphi(Cl_{\varphi_\omega}(H))\widetilde{\cap}Int_\varphi(C_Y) = Int_\varphi(Cl_{\varphi_\omega}(H))\widetilde{\cap}C_Y$. Since $H\widetilde{\subseteq}C_Y$, then $Int_\varphi(Cl_{\varphi_\omega}(H))\widetilde{\subseteq}Int_\varphi(Cl_{\varphi_\omega}(C_Y)) = C_Y$ and thus, $Int_\varphi(Cl_{\varphi_\omega}(H))\widetilde{\cap}C_Y = Int_\varphi(Cl_{\varphi_\omega}(H))$. Therefore, $H = Int_\varphi(Cl_{\varphi_\omega}(H))$. Hence, $H \in S\omega O(H,\varphi,\Sigma)$.

Theorem 2.21. If (H,φ,Σ) is a soft ω-A-R STS and $C_Y \in S\omega O(H,\varphi,\Sigma) - \{0_\Sigma\}$, then (Y,φ_Y,Σ) is soft ω-A-R.

Proof. Let $a_x \in SP(Y,R)$ and let $H \in S\omega O(Y,\varphi_Y,\Sigma)$ such that $a_x\widetilde{\in}H$. By Lemma 2.20, $H \in S\omega O(H,\varphi,\Sigma)$. Since (H,φ,Σ) is soft ω-A-R, then by Theorem 2.2 (2) there is $L \in \varphi$ such that $a_x\widetilde{\in}L\widetilde{\subseteq}Cl_\varphi(L)\widetilde{\subseteq}H$. Therefore, we have $a_x\widetilde{\in}L\widetilde{\cap}C_Y \in \varphi_Y$ and $Cl_{\varphi_Y}(L) = Cl_\varphi(L)\widetilde{\cap}C_Y\widetilde{\subseteq}H$. Hence, (Y,φ_Y,Σ) is soft ω-A-R.

The following lemma will be used in Theorems 2.23 and 3.33:

Lemma 2.22. For any two STSs (Z,δ,Σ) and (W,ρ,Ψ), $(\delta\times\rho)_{\delta_\omega} \subseteq \delta_{\delta_\omega}\times\rho_{\delta_\omega}$.

Proof. Let $T \in (\delta\times\rho)_{\delta_\omega}$ and $(e,f)_{(z,w)}\widetilde{\in}T$. Then, by Theorem 20 of [36] we find $S \in S\omega O(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ such that $(e,f)_{(z,w)}\widetilde{\in}S = Int_{\delta\times\rho}\left(Cl_{(\delta\times\rho)_\omega}(S)\right)\widetilde{\subseteq}T$. Choose $L \in \delta$ and $M \in \rho$ such that $(e,f)_{(z,w)}\widetilde{\in}L\times M\widetilde{\subseteq}S\widetilde{\subseteq}T$. By Proposition 3 (b) of [33] we have $Cl_{\delta_\omega}(L)\times Cl_{\rho_\omega}(M)\widetilde{\subseteq}Cl_{(\delta\times\rho)_\omega}(L\times M)$, and so

$$\begin{aligned} L\times M &\widetilde{\subseteq} Int_\delta\left(Cl_{\delta_\omega}(L)\right)\times Int_\rho\left(Cl_{\rho_\omega}(M)\right)\\ &\widetilde{\subseteq} Int_{\delta\times\rho}\left(Cl_{\delta_\omega}(L)\times Cl_{\rho_\omega}(M)\right)\\ &\widetilde{\subseteq} Int_{\delta\times\rho}\left(Cl_{(\delta\times\rho)_\omega}(L\times M)\right)\\ &\widetilde{\subseteq} T. \end{aligned}$$

By Theorem 9 and Corollary 7 of [36], $Int_\delta\left(Cl_{\delta_\omega}(L)\right) \in \delta_{\delta_\omega}$ and $Int_\rho\left(Cl_{\rho_\omega}(M)\right) \in \rho_{\delta_\omega}$. It follows that $T \in \delta_{\delta_\omega}\times\rho_{\delta_\omega}$.

The following result shows that soft ω-almost-regularity is a productive soft property:

Theorem 2.23. The soft product of two soft ω-A-R STSs is soft ω-A-R.

Proof. Let (Z,δ,Σ) and (W,ρ,Ψ) be two soft ω-A-R STSs. Let $(e,f)_{(z,w)}\widetilde{\in}SP(Z\times W,\Sigma\times\Psi)$ and let $K \in S\omega O(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ such that $(e,f)_{(z,w)}\widetilde{\in}K$. Then, by Corollary 7 of [36] $G \in (\delta\times\rho)_{\delta_\omega}$. So, by Lemma 2.22 $K \in \delta_{\delta_\omega}\times\rho_{\delta_\omega}$. Thus, there are $L \in \delta_{\delta_\omega}$ and $M \in \rho_{\delta_\omega}$ such that $(e,f)_{(z,w)}\widetilde{\in}L\times M\widetilde{\subseteq}K$. By Corollary 7 of [36] we find $S \in S\omega O(Z,\delta,\Sigma)$ and $T \in S\omega O(W,\rho,\Psi)$ such that $(e,f)_{(z,w)}\widetilde{\in}S\times T\widetilde{\subseteq}L\times M\widetilde{\subseteq}G$. So, by Theorem 2.2 (2) there are $M \in \delta$ and $N \in \rho$ such that $e_z\widetilde{\in}M\widetilde{\subseteq}Cl_\delta(M)\widetilde{\subseteq}S$ and $f_w\widetilde{\in}N\widetilde{\subseteq}Cl_\rho(N)\widetilde{\subseteq}T$. Therefore, we have $M\times N \in \delta\times\rho$ and $(e,f)_{(z,w)}\widetilde{\in}M\times N\widetilde{\subseteq}Cl_{\delta\times\rho}(M\times N) = Cl_\delta(M)\times Cl_\rho(N)\widetilde{\subseteq}S\times T\widetilde{\subseteq}L\times M\widetilde{\subseteq}K$. Again, by Theorem 2.2 (2) $(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ is soft ω-A-R.

The following result shows that soft almost-regularity is a productive soft property:

Theorem 2.24. Let (Z,δ,Σ) and (W,ρ,Ψ) be two STSs. Then $(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ is soft A-R iff (Z,δ,Σ) and (W,ρ,Ψ) are both soft A-R.

Proof. Necessity. Let $(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ be soft A-R. To see that (Z,δ,Σ) is soft A-R, let $e_z \in SP(Z,\Sigma)$ and $G \in SO(Z,\delta,\Sigma)$ such that $e_z\widetilde{\in}G$. Choose $f_w\widetilde{\in}SP(W,F)$. Then, $(e,f)_{(z,w)}\widetilde{\in} \in G\times 1_\Psi \in SO(Z\times W,\delta\times\rho,\Sigma\times\Psi)$. Thus, by Theorem 3.4 (ii) of [31] we find $H \in \delta\times\rho$ such that $(e,f)_{(z,w)}\widetilde{\in}H\widetilde{\subseteq}Cl_{\delta\times\rho}(H)\widetilde{\subseteq}G\times 1_\Psi$. Choose $M \in \delta$ and $N \in \rho$ such that $(e,f)_{(z,w)}\widetilde{\in}M\times N\widetilde{\subseteq}H$. Thus,

$$\begin{array}{rcl} (e,f)_{(z,w)} & \widetilde{\in} & M\times N \\ & \widetilde{\subseteq} & Cl_{\delta}(M)\times Cl_{\rho}(N) \\ & = & Cl_{\delta\times\rho}(M\times N) \\ & \widetilde{\subseteq} & Cl_{\delta\times\rho}(H) \\ & \widetilde{\subseteq} & G\times 1_{\Psi}. \end{array}$$

Therefore, we have $e_z\widetilde{\in}M\widetilde{\subseteq}Cl_{\delta}(M)\widetilde{\subseteq}G$. Hence, by Theorem 3.4 (ii) of [31] (Z,δ,Σ) is soft A-R. Similarly, we can show that (W,ρ,Ψ) is soft A-R.

Sufficiency. Let (Z,δ,Σ) and (W,ρ,Ψ) be soft A-R. Let $(e,f)_{(z,w)}\widetilde{\in}SP(Z\times W,\Sigma\times\Psi)$ and let $K\in SO(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ such that $(e,f)_{(z,w)}\widetilde{\in}K$. Choose $M\in\delta$ and $N\in\rho$ such that $(e,f)_{(z,w)}\widetilde{\in}M\times N\widetilde{\subseteq}K$. Since $e_z\widetilde{\in}M$ and $f_w\widetilde{\in}N$, by Theorem 3.4 (iv) of [31] there are $S\in SO(Z,\delta,\Sigma)$ and $T\in SO(W,\rho,\Psi)$ such that $e_z\widetilde{\in}S\widetilde{\subseteq}Cl_{\delta}(S)\widetilde{\subseteq}Int_{\delta}(Cl_{\delta}(M))$ and $f_w\widetilde{\in}T\widetilde{\subseteq}Cl_{\rho}(T)\widetilde{\subseteq}Int_{\rho}\left(Cl_{\rho}(N)\right)$. Thus, we have $S\times T\in SO(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ and

$$\begin{array}{rcl} (e,f)_{(z,w)} & \widetilde{\in} & S\times T \\ & \widetilde{\subseteq} & Cl_{\delta}(S)\times Cl_{\rho}(T) \\ & = & Cl_{\delta\times\rho}(S\times T) \\ & \widetilde{\subseteq} & Int_{\delta}(Cl_{\delta}(M))\times Int_{\rho}\left(Cl_{\rho}(N)\right) \\ & = & Int_{\delta\times\rho}\left(Cl_{\delta\times\rho}(M\times N)\right) \\ & \widetilde{\subseteq} & Int_{\varphi\times\rho}\left(Cl_{\delta\times\rho}(M\times N)\right) \\ & = & K. \end{array}$$

Therefore, by Theorem 3.4 (iv) of [31] $(Z\times W,\delta\times\rho,\Sigma\times\Psi)$ is soft A-R.

3. Soft ω-semi-regular and soft ω-$T_{2\frac{1}{2}}$ spaces

In this section, we define soft ω-semi-regularity and soft ω-$T_{2\frac{1}{2}}$ as two new soft separation axioms. We show that soft ω-semi-regularity is a weaker form of both soft semi-regularity and soft ω-regularity, and soft ω-$T_{2\frac{1}{2}}$ lies strictly between soft $T_{2\frac{1}{2}}$ and soft T_2. Also, we provide several sufficient conditions establishing the equivalence between these newly introduced axioms and their relevant counterparts. Moreover, a decomposition theorem for soft regularity through the interplay of soft ω-semi-regularity and soft ω-almost-regularity is obtained. In addition, we investigated the links between these classes of soft topological spaces and their analogs in general topology.

Definition 3.1. An STS (H,φ,Σ) is called soft ω-semi-regular (soft ω-S-R, for simplicity) if $R\omega O(H,\varphi,\Sigma)$ forms a soft base for φ.

Two characterizations of soft ω-semi-regularity are listed in the following theorem.

Theorem 3.2. For any STS (H,φ,Σ), T.F.A.E:

(a) (H,φ,Σ) is soft ω-S-R.

(b) For every $H\in\varphi-\{0_{\Sigma}\}$ and every $r_z\widetilde{\in}H$, we find $K\in\varphi$ such that $r_z\widetilde{\in}K\widetilde{\subseteq}Int_{\varphi}\left(Cl_{\varphi_{\omega}}(K)\right)\widetilde{\subseteq}H$.

(c) $\varphi_{\delta_{\omega}}=\varphi$.

Proof. (a) $\longrightarrow$ (b): Let $H\in\varphi-\{0_{\Sigma}\}$ and let $r_z\widetilde{\in}H$. By (a) we find $K\in S\omega O(H,\varphi,\Sigma)$ such that $r_z\widetilde{\in}K=Int_{\varphi}\left(Cl_{\varphi_{\omega}}(K)\right)\widetilde{\subseteq}H$.

(b) ⟶ (c): By Theorem 21 of [36] we have $\varphi_{\delta_\omega} \widetilde{\subseteq} \varphi$. To show that $\varphi \subseteq \varphi_{\delta_\omega}$, let $H \in \varphi - \{0_\Sigma\}$, then for every $r_z \widetilde{\in} H$ we find $K_{r_z} \in \varphi$ such that $r_z \widetilde{\in} K_{r_z} \widetilde{\subseteq} Int_\varphi\left(Cl_{\varphi_\omega}(K_{r_z})\right) \widetilde{\subseteq} H$. Let $K = \widetilde{\cup}_{r_z \widetilde{\in} H} Int_\varphi\left(Cl_{\varphi_\omega}(K_{r_z})\right)$. Since for every $r_z \widetilde{\in} H$ $Int_\varphi\left(Cl_{\varphi_\omega}(K_{r_z})\right) \in S\omega O(H, \varphi, \Sigma) \subseteq \varphi_{\delta_\omega}$, then $K \in \varphi_{\delta_\omega}$.

(c) ⟶ (a): Since $R\omega O(H, \varphi, \Sigma)$ is a soft base for φ_{δ_ω}, and by (c) $\varphi_{\delta_\omega} = \varphi$, then $R\omega O(H, \varphi, \Sigma)$ is a soft base for φ. Therefore, (H, φ, Σ) is soft ω-S-R.

Corollary 3.3. Every soft ω-regular STS is soft ω-S-R.

Proof. The proof follows from Theorem 25 of [36] and Theorem 3.2.

Theorem 3.4. Every soft S-R STS is soft ω-S-R.

Proof. Let (H, φ, Σ) be soft S-R. Then $\varphi_\delta = \varphi$. So, by Theorem 21 of [36] $\varphi = \varphi_\delta \subseteq \varphi_{\delta_\omega} \subseteq \varphi$, and thus $\varphi_{\delta_\omega} = \varphi$. Therefore, by Theorem 3.2, (H, φ, Σ) is soft ω-S-R.

Theorem 3.5. Every soft ω-S-R soft anti-L-C STS is soft S-R.

Proof. Let (H, φ, Σ) be soft ω-S-R soft anti-L-C. Since (H, φ, Σ) is soft ω-S-R, then $R\omega O(H, \varphi, \Sigma)$ is a soft base for φ. Since (H, φ, Σ) is soft anti-L-C, then by Theorem 6 of [36], $RO(H, \varphi, \Sigma) = R\omega O(H, \varphi, \Sigma)$. So, $RO(H, \varphi, \Sigma)$ is a soft base for φ. Hence, (H, φ, Σ) is soft S-R.

The following implications come from the previous theorems; nevertheless, Examples 3.15 and 3.16 show that the converses of these implications are not true.

soft S-R ⟶ soft ω-S-R
↑
soft ω-regular.

Theorem 3.6. Soft L-C STSs are soft ω-S-R.

Proof. Let (D, φ, Σ) be soft L-C. Then, by Theorem 5 of [36] $R\omega O(D, \varphi, \Sigma) = \varphi$. So, $R\omega O(D, \varphi, \Sigma)$ is a soft base for φ. Hence, (D, φ, Σ) is soft ω-S-R.

Theorem 3.7. Let (D, φ, Σ) be an STS. If $(D, \varphi_\omega, \Sigma)$ is soft ω-S-R, then $(D, \varphi_\omega, \Sigma)$ is soft S-R.

Proof. Let $(D, \varphi_\omega, \Sigma)$ be soft ω-S-R. Then, $R\omega O(D, \varphi_\omega, \Sigma)$ is a soft base for φ_ω. Since by Theorem 7 of [36] $R\omega O(D, \varphi_\omega, \Sigma) = RO(D, \varphi_\omega, \Sigma)$, then $RO(D, \varphi_\omega, \Sigma)$ is a soft base for φ_ω. Hence, $(D, \varphi_\omega, \Sigma)$ is soft S-R.

In Theorems 3.8, 3.9, 3.11, and Corollary 3.12, we discuss the connections between soft semi-regularity and its analog in traditional topological spaces. Also, in Theorems 3.10, 3.13, and Corollary 3.14, we discuss the connections between soft ω-semi-regularity and its analog in traditional topological spaces.

Theorem 3.8. If (H, φ, Σ) is full and soft S-R, then (H, φ_r) is S-R for all $r \in \Sigma$.

Proof. Let (H, φ, Σ) be full and soft S-R. Let $r \in \Sigma$. Let $z \in H$ and let $W \in \varphi_r$ such that $z \in W$. Choose $K \in \varphi$ such that $K(r) = W$. Since (H, φ, Σ) is soft S-R and $r_z \widetilde{\in} K \in \varphi$, we find $L \in SO(H, \varphi, \Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} K$ and so $z \in L(r) \subseteq K(r) = W$. In contrast, by Theorem 13 of [36], $L(r) \in SO(H, \varphi_r)$. This shows that (H, φ_r) is S-R for all $r \in \Sigma$.

Theorem 3.9. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, C(\mathcal{L}), \Sigma)$ is soft S-R iff $(D, \mathcal{L})$ is S-R.

Proof. Necessity. Let $(D, C(\mathcal{L}), \Sigma)$ be soft S-R. Pick $r \in \Sigma$. Since it is clear that $(D, C(\mathcal{L}), \Sigma)$ is full, then by Theorem 3.8 $(D, (C(\mathcal{L}))_r) = (D, \mathcal{L})$ is S-R.

Sufficiency. Let $(D, \mathcal{L})$ be S-R. Let $r_z \in SP(D, \Sigma)$ and let $C_U \in C(\mathcal{L})$ such that $r_z \widetilde{\in} C_U$. Then, we have $z \in U \in \mathcal{L}$. So, we find $V \in SO(D, \mathcal{L})$ such that $z \in Int_{\mathcal{L}}(Cl_{\mathcal{L}}(V)) \subseteq U$. Thus, we have $C_V \in C(\mathcal{L})$ and $r_z \widetilde{\in} C_{Int_{\mathcal{L}}(Cl_{\mathcal{L}}(V))} = Int_{C(\mathcal{L})}(Cl_{C(\mathcal{L})}(C_V)) \widetilde{\subseteq} C_U$. This shows that $(D, C(\mathcal{L}), \Sigma)$ is soft S-R.

Theorem 3.10. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, C(\mathcal{L}), \Sigma)$ is soft ω-S-R iff $(D, \mathcal{L})$ is ω-S-R.

Proof. Necessity. Let $(D, C(\mathcal{L}), \Sigma)$ be soft ω-S-R. Let $z \in D$ and $U \in \mathcal{L}$ such that $z \in U$. Pick $r \in \Sigma$. Then, we have $r_z \widetilde{\in} C_U \in C(\mathcal{L})$. Since $(D, C(\mathcal{L}), \Sigma)$ is soft ω-S-R, we find $C_V \in S\omega O(D, C(\mathcal{L}), \Sigma)$ such that $r_z \widetilde{\in} C_V \widetilde{\subseteq} C_U$. Therefore, we have $z \in V \in S\omega O(D, \mathcal{L})$ and $V \subseteq U$. This shows that $(D, \mathcal{L})$ is ω-S-R.

Sufficiency. Let $(D, \mathcal{L})$ be ω-S-R. Let $r_z \in SP(H, \Sigma)$ and let $C_U \in C(\mathcal{L})$ such that $r_z \widetilde{\in} C_U$. Then, we have $z \in U \in \mathcal{L}$. So, we find $V \in S\omega O(D, \mathcal{L})$ such that $z \in V \subseteq U$. Thus, we have $R\omega O(D, C(\mathcal{L}), \Sigma)$ and $r_z \widetilde{\in} C_V \widetilde{\subseteq} C_U$. This shows that $(D, C(\mathcal{L}), \Sigma)$ is soft ω-S-R.

Theorem 3.11. Let $\{(H, \mathcal{L}_r) : r \in \Sigma\}$ be a collection of TSs. Then $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft S-R iff $(H, \mathcal{L}_r)$ is S-R for every $r \in \Sigma$.

Proof. Necessity. Let $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ be soft S-R and let $r \in \Sigma$. Let $z \in H$ and let $U \in \mathcal{L}_r$ such that $z \in U$. Then, $r_z \widetilde{\in} r_U \in \oplus_{r\in\Sigma}\mathcal{L}_r$. So, we find $L \in SO(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} r_U$ and thus, $z \in L(r) \subseteq (r_U)(r) = U$. In contrast, by Theorem 14 of [36] $L(r) \in SO(H, \mathcal{L}_r)$. This shows that $(H, \mathcal{L}_r)$ is S-R.

Sufficiency. Let $(H, \mathcal{L}_r)$ be S-R for every $r \in \Sigma$. Let $r_z \in SP(H, \Sigma)$ and let $K \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} K$. Then, we have $z \in K(r) \in \mathcal{L}_r$ and so we find $V \in SO(H, \mathcal{L}_r)$ such that $z \in V \subseteq U$. Now, we have $r_z \widetilde{\in} r_V \widetilde{\subseteq} r_U \widetilde{\subseteq} K$, and by Theorem 14 of [36] $r_V \in SO(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$. This shows that $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft S-R.

Corollary 3.12. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, \tau(\mathcal{L}), \Sigma)$ is soft S-R iff $(D, \mathcal{L})$ is S-R.

Proof. For each $r \in \Sigma$, set $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$ and by Theorem 3.11 we get the result.

Theorem 3.13. Let $\{(H, \mathcal{L}_r) : r \in \Sigma\}$ be a collection of TSs. Then, $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft ω-S-R iff $(H, \mathcal{L}_r)$ is ω-S-R for every $r \in \Sigma$.

Proof. Necessity. Let $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ be soft ω-S-R and let $r \in \Sigma$. Let $z \in H$ and let $U \in \mathcal{L}_r$ such that $z \in U$. Then, $r_z \widetilde{\in} r_U \in \oplus_{r\in\Sigma}\mathcal{L}_r$. So, we find $L \in S\omega O(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ such that $r_z \widetilde{\in} L \widetilde{\subseteq} r_U$, and thus $z \in L(r) \subseteq (r_U)(r) = U$. In contrast, by Theorem 15 of [36] $L(r) \in S\omega O(H, \mathcal{L}_r)$. This shows that $(H, \mathcal{L}_r)$ is ω-S-R.

Sufficiency. Let $(H, \mathcal{L}_r)$ be ω-S-R for every $r \in \Sigma$. Let $r_z \in SP(H, \Sigma)$ and let $K \in \oplus_{r\in\Sigma}\mathcal{L}_r$ such that $r_z \widetilde{\in} K$. Then, we have $z \in K(r) \in \mathcal{L}_r$ and so we find $V \in S\omega O(H, \mathcal{L}_r)$ such that $z \in V \subseteq U$. Now, we have $r_z \widetilde{\in} r_V \widetilde{\subseteq} r_U \widetilde{\subseteq} K$, and by Theorem 15 of [36] $r_V \in S\omega O(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$. This shows that $(H, \oplus_{r\in\Sigma}\mathcal{L}_r, \Sigma)$ is soft ω-S-R.

Corollary 3.14. Let $(D, \mathcal{L})$ be a TS. Then, for any set Σ, $(D, \tau(\mathcal{L}), \Sigma)$ is soft ω-S-R iff $(D, \mathcal{L})$ is ω-S-R.

Proof. For each $r \in \Sigma$, set $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$ and by Theorem 3.13 we get the result.

The following two examples show, respectively, that each of Theorem 3.4 and Corollary 3.3 does not have to be true in all cases:

Example 3.15. Consider (H, φ, Σ) in Example 2.17. $RO(H, \varphi, \Sigma) = \{0_\Sigma, 1_\Sigma\}$ is not a soft base for φ and thus (H, φ, Σ) is not soft S-R. In contrast, by Theorem 3.6 (H, φ, Σ) is soft ω-S-R.

Example 3.16. Let $(D, \mathcal{L})$ be as in Example 3.9 of [39]. It is proved in [39] that $(D, \mathcal{L})$ is ω-S-R but not ω-regular. Therefore, by Corollaries 3.14 and 19 of [33], $(D, \tau(\mathcal{L}), \Sigma)$ is soft ω-S-R but not soft ω-regular.

The following main result introduces a decomposition of soft regularity in terms of soft ω-semi-regularity and soft ω-almost-regularity:

Theorem 3.17. An STS (H, φ, Σ) is soft regular iff it is soft ω-S-R and soft ω-A-R.

Proof. Necessity. Let (H, φ, Σ) be soft regular. Then, by Theorem 15 of [33] and Corollary 3.3 (H, φ, Σ) is soft ω-S-R. In contrast, by Theorem 2.11 (H, φ, Σ) is soft ω-A-R.

Sufficiency. Let (H, φ, Σ) be soft ω-S-R and soft ω-A-R. Let $H \in \varphi - \{0_\Sigma\}$ and let $r_z \widetilde{\in} H$. Since

(H,φ,Σ) is soft ω-S-R, then there is $G \in S\omega O(H,\varphi,\Sigma)$ such that $r_z\widetilde{\in}G\widetilde{\subseteq}H$. Since (H,φ,Σ) is soft ω-A-R, then by Theorem 2.2 (2) there is $T \in \varphi$ such that $r_z\widetilde{\in}T\widetilde{\subseteq}Cl_{\varphi}(T)\widetilde{\subseteq}G\widetilde{\subseteq}H$. Hence, (H,φ,Σ) is soft regular.

Definition 3.18. An STS (H,φ,Σ) is called soft ω-$T_{2\frac{1}{2}}$ if for every $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$, we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$.

In Theorems 3.19, 3.21, and Corollary 3.12, we discuss the connections between soft $T_{2\frac{1}{2}}$ spaces and their analogs in traditional topological spaces. Also, in Theorems 3.20, 3.23, and Corollary 3.24, we discuss the connections between soft ω-$T_{2\frac{1}{2}}$ spaces and their analogs in traditional topological spaces.

Theorem 3.19. If (H,φ,Σ) is soft $T_{2\frac{1}{2}}$, then (H,φ_r) is $T_{2\frac{1}{2}}$ for every $r \in \Sigma$.

Proof. Suppose that (H,φ,Σ) is soft $T_{2\frac{1}{2}}$ and let $r \in \Sigma$. Let $x,y \in Z$ such that $x \neq y$. Then, $r_x, r_y \in SP(S,D)$ such that $r_x \neq r_y$. Since (H,φ,Σ) is soft $T_{2\frac{1}{2}}$, we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $r_y\widetilde{\in}G$, and $Cl_{\varphi}(K)\widetilde{\cap}Cl_{\varphi}(G) = 0_\Sigma$. Thus, we have $x \in K(r) \in \varphi_r$, $y \in G(r) \in \varphi_r$, and by Proposition 7 of [12] $Cl_{\varphi_r}(K(r)) \cap Cl_{\varphi_r}(G(r)) \subseteq \left(Cl_{\varphi}(K)\right)(r) \cap \left(Cl_{\varphi}(G)\right)(r) = \left(Cl_{\varphi}(K)\widetilde{\cap}Cl_{\varphi}(G)\right)(r) = \emptyset$. This shows that (H,φ_r) is $T_{2\frac{1}{2}}$.

Theorem 3.20. If (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$, then (H,φ_r) is ω-$T_{2\frac{1}{2}}$ for every $r \in \Sigma$.

Proof. Suppose that (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$ and let $r \in \Sigma$. Let $x,y \in Z$ such that $x \neq y$. Then, $r_x, r_y \in SP(S,D)$ such that $r_x \neq r_y$. Since (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$, we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $r_y\widetilde{\in}G$, and $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$. Thus, we have $x \in K(r) \in \varphi_r$, $y \in G(r) \in \varphi_r$, and by Proposition 7 of [12] $Cl_{(\varphi_\omega)_r}(K(r)) \cap Cl_{(\varphi_\omega)_r}(G(r)) \subseteq \left(Cl_{\varphi_\omega}(K)\right)(r) \cap \left(Cl_{\varphi_\omega}(G)\right)(r) = \left(Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G)\right)(r) = \emptyset$. But by Theorem 7 of [35], $(\varphi_\omega)_r = (\varphi_r)_\omega$. This shows that (H,φ_r) is ω-$T_{2\frac{1}{2}}$.

Theorem 3.21. Let $\{(H,\mathcal{L}_R) : r \in \Sigma\}$ be a collection of TSs. Then, $(H,\oplus_{r\in\Sigma}\mathcal{L}_r,\Sigma)$ is soft $T_{2\frac{1}{2}}$ iff $(H,\mathcal{L}_r)$ is $T_{2\frac{1}{2}}$ for every $r \in \Sigma$.

Proof. Necessity. Suppose that $(H,\oplus_{r\in\Sigma}\mathcal{L}_r,\Sigma)$ is soft $T_{2\frac{1}{2}}$ and let $r \in \Sigma$. Then, by Theorem 3.19 $(H,(\oplus_{r\in\Sigma}\mathcal{L}_r)_r)$ is $T_{2\frac{1}{2}}$. On the other hand, by Theorem 3.7 of [34] $(\oplus_{r\in\Sigma}\mathcal{L}_r)_r = \mathcal{L}_r$.

Sufficiency. Suppose that $(H,\mathcal{L}_r)$ is $T_{2\frac{1}{2}}$ for every $r \in \Sigma$. Let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$.

Case 1. $r \neq s$. Then, $r_x\widetilde{\in}r_Z \in \oplus_{r\in\Sigma}\mathcal{L}_r$, $s_y\widetilde{\in}s_Z \in \oplus_{r\in\Sigma}\mathcal{L}_r$, and $Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_Z)\widetilde{\cap}Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(s_Z) = 0_\Sigma$.

Case 2. $r = s$. Then, $x \neq y$. Since $(H,\mathcal{L}_r)$ is $T_{2\frac{1}{2}}$, we find $U,V \in \mathcal{L}_r$ such that $x \in U$, $y \in V$, and $Cl_{\mathcal{L}_r}(U) \cap Cl_{\mathcal{L}_r}(V) = \emptyset$. Then, we have $r_x\widetilde{\in}r_U \in \oplus_{r\in\Sigma}\mathcal{L}_r$, $s_y\widetilde{\in}s_V \in \oplus_{r\in\Sigma}\mathcal{L}_r$ and $Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(r_U) \cap Cl_{\oplus_{r\in\Sigma}\mathcal{L}_r}(s_V) = 0_\Sigma$.

Corollary 3.22. Let $(D,\mathcal{L})$ be a TS. Then, for any set Σ, $(D,\tau(\mathcal{L}),\Sigma)$ is soft $T_{2\frac{1}{2}}$ iff $(D,\mathcal{L})$ is $T_{2\frac{1}{2}}$.

Proof. For each $r \in \Sigma$, put $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$. We get the result as a consequence of Theorem 3.21.

Theorem 3.23. Let $\{(H,\mathcal{L}_r) : r \in \Sigma\}$ be a collection of TSs. Then, $(H,\oplus_{r\in\Sigma}\mathcal{L}_r,\Sigma)$ is soft ω-$T_{2\frac{1}{2}}$ iff $(H,\mathcal{L}_r)$ is ω-$T_{2\frac{1}{2}}$ for every $r \in \Sigma$.

Proof. Necessity. Suppose that $(H,\oplus_{r\in\Sigma}\mathcal{L}_r,\Sigma)$ is soft ω-$T_{2\frac{1}{2}}$ and let $r \in \Sigma$. Then, by Theorem 3.20 $(H,(\oplus_{r\in\Sigma}\mathcal{L}_r)_r)$ is ω-$T_{2\frac{1}{2}}$. In contrast, by Theorem 3.7 of [35], $(\oplus_{r\in\Sigma}\mathcal{L}_r)_r = \mathcal{L}_r$.

Sufficiency. Suppose that $(H,\mathcal{L}_r)$ is ω-$T_{2\frac{1}{2}}$ for every $r \in \Sigma$. Let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$.

Case 1. $r \neq s$. Then, $r_x\widetilde{\in}r_Z \in \oplus_{r\in\Sigma}\mathcal{L}_r$, $s_y\widetilde{\in}s_Z \in \oplus_{r\in\Sigma}\mathcal{L}_r$, and $Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(r_Z)\widetilde{\cap}Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(s_Z) = 0_\Sigma$.

Case 2. $r = s$. Then, $x \neq y$. Since $(D,\mathcal{L})$ is ω-$T_{2\frac{1}{2}}$, we find $A,B \in \mathcal{L}_r$ such that $x \in A$, $y \in B$, and $A \cap B = \emptyset$. Then, we have $r_x\widetilde{\in}r_A \in \oplus_{r\in\Sigma}\mathcal{L}_r$, $s_y\widetilde{\in}s_B \in \oplus_{r\in\Sigma}\mathcal{L}_r$ and $Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(r_A)\widetilde{\cap}Cl_{(\oplus_{r\in\Sigma}\mathcal{L}_r)_\omega}(s_B) = 0_\Sigma$.

Corollary 3.24. Let $(D,\mathcal{L})$ be a TS. Then, for any set Σ, $(D,\tau(\mathcal{L}),\Sigma)$ is soft ω-$T_{2\frac{1}{2}}$ iff $(D,\mathcal{L})$ is ω-$T_{2\frac{1}{2}}$.

Proof. For each $r \in \Sigma$, put $\mathcal{L}_r = \mathcal{L}$. Then, $\tau(\mathcal{L}) = \oplus_{r\in\Sigma}\mathcal{L}_r$. The result follows from Theorem 3.23.

Theorem 3.25. If (H,φ,Σ) is soft $T_{2\frac{1}{2}}$, then (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$.
Proof. Let (H,φ,Σ) be soft $T_{2\frac{1}{2}}$ and let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$. Then, we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $Cl_{\varphi}(K)\widetilde{\cap}Cl_{\varphi}(G) = 0_\Sigma$. Since $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G)\widetilde{\subseteq}Cl_{\varphi}(K)\widetilde{\cap}Cl_{\varphi}(G) = 0_\Sigma$, then $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$. This shows that (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$.
Theorem 3.26. If (H,φ,Σ) is soft anti-L-C and soft ω-$T_{2\frac{1}{2}}$, then (H,φ,Σ) is soft $T_{2\frac{1}{2}}$.
Proof. Let (H,φ,Σ) be soft anti-L-C and soft ω-$T_{2\frac{1}{2}}$. Let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$. Since (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$, then we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$. Since (H,φ,Σ) is anti-L-C, then by Theorem 14 of [35] $Cl_{\varphi}(K)\widetilde{\cap}Cl_{\varphi}(G) = Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$. Hence, (H,φ,Σ) is soft $T_{2\frac{1}{2}}$.
Theorem 3.27. Every soft ω-$T_{2\frac{1}{2}}$ STS is soft T_2.
Proof. Let (H,φ,Σ) be ω-$T_{2\frac{1}{2}}$ and let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$. Then, we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$. Since $K\widetilde{\cap}G\widetilde{\subseteq}Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = 0_\Sigma$, then $K\widetilde{\cap}G = 0_\Sigma$. Hence, (H,φ,Σ) is soft T_2.
Theorem 3.28. If (H,φ,Σ) is soft L-C and soft T_2, then (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$.
Proof. Let (H,φ,Σ) be soft L-C and soft T_2. Let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$. Since (H,φ,Σ) is soft T_2, then we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $K\widetilde{\cap}G = 0_\Sigma$. Since (H,φ,Σ) is soft L-C, then by Corollary 5 of [35] $Cl_{\varphi_\omega}(K)\widetilde{\cap}Cl_{\varphi_\omega}(G) = K\widetilde{\cap}G = 0_\Sigma$. Hence, (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$.

The following example demonstrates that Theorem 3.25's converse does not have to be true in general:
Example 3.29. Let $(D,\mathcal{L})$ be the TS in Example 75 of [41]. Then $(D,\mathcal{L})$ is T_2 but not $T_{2\frac{1}{2}}$. Since $(D,\tau(\mathcal{L}),\mathbb{N})$ is soft L-C, by Corollary 5 of [35] it is soft T_2. Thus, by Theorem 3.28 $(D,\tau(\mathcal{L}),\mathbb{N})$ is soft ω-$T_{2\frac{1}{2}}$. On the other hand, by Corollary 2.22 $(D,\tau(\mathcal{L}),\mathbb{N})$ is not soft $T_{2\frac{1}{2}}$.

The following example demonstrates why Theorem 3.27 does not have to be true in general:
Example 3.30. Let $(D,\mathcal{L})$ be the TS in Example 81 of [41]. It is known that $(D,\mathcal{L})$ is T_2 but not $T_{2\frac{1}{2}}$. Then, by Corollary 7 of [33] and Corollary 2.22 $(D,\tau(\mathcal{L}),[0,1])$ is soft T_2 but not soft $T_{2\frac{1}{2}}$. Since $(D,\tau(\mathcal{L}),[0,1])$ is soft anti-L-C, then by Theorem 3.26 $(D,\tau(\mathcal{L}),[0,1])$ is not ω-$T_{2\frac{1}{2}}$.
Theorem 3.31. Every soft ω-regular T_2 STS is soft ω-$T_{2\frac{1}{2}}$.
Proof. Let (H,φ,Σ) be soft ω-regular and soft T_2. Let $r_x, s_y \in SP(H,\Sigma)$ such that $r_x \neq s_y$. Since (H,φ,Σ) is soft T_2, then we find $K,G \in \varphi$ such that $r_x\widetilde{\in}K$, $s_y\widetilde{\in}G$, and $K\widetilde{\cap}G = 0_\Sigma$. Since (H,φ,Σ) is soft ω-regular, then we find $L,M \in \varphi$ such that $r_x\widetilde{\in}L\widetilde{\subseteq}Cl_{\varphi_\omega}(L)\widetilde{\subseteq}K$ and $s_y\widetilde{\in}M\widetilde{\subseteq}Cl_{\varphi_\omega}(M)\widetilde{\subseteq}G$. Therefore, we have $r_x\widetilde{\in}L$, $s_y\widetilde{\in}M$, and $Cl_{\varphi_\omega}(L)\widetilde{\cap}Cl_{\varphi_\omega}(M)\widetilde{\subseteq}K\widetilde{\cap}G = 0_\Sigma$. This proves that (H,φ,Σ) is soft ω-$T_{2\frac{1}{2}}$.
Question 3.32. Is it true that every soft ω-$T_{2\frac{1}{2}}$ STS is soft ω-regular?
Theorem 3.33. If (Z,β,Σ) and (W,ρ,Ψ) are two soft ω-S-R STSs such that the soft product $(Z\times W,\beta\times\rho,\Sigma\times\Psi)$ is soft ω-S-R, then both of (Z,β,Σ) and (W,ρ,Ψ) are soft ω-S-R.
Proof. Since $(Z\times W,\beta\times\rho,\Sigma\times\Psi)$ is soft ω-S-R, then by Theorem 3.2 $(\beta\times\rho)_{\delta_\omega} = \beta\times\rho$. So, by Lemma 2.22, $\beta\times\rho\widetilde{\subseteq}\beta_{\delta_\omega}\times\rho_{\delta_\omega}$. Hence, $\beta = \beta_{\delta_\omega}$ and $\rho = \rho_{\delta_\omega}$. Therefore, again by Theorem 3.2 (Z,β,Σ) and (W,ρ,Ψ) are soft ω-S-R.

4. Conclusions

Soft separation axioms are a collection of requirements for categorizing a system of STSs based on certain soft topological features. These axioms are often expressed in terms of classes of soft sets.

In this work, “soft ω-almost-regular”, “soft ω-semi-regular”, and “soft ω-$T_{2\frac{1}{2}}$” are defined as three new notions of soft separation axioms (Definitions 2.1, 3.1, 3.18). Several characterizations of soft ω-almost-regularity (Theorems 2.2) and soft ω-semi-regularity (Theorem 3.2) are given. It is proved that soft ω-almost-regularity lies strictly between regularity and almost-regularity (Theorems 2.11, 2.12 and Examples 2.16, 2.17); soft ω-semi-regularity is a weaker form of both soft semi-regularity and soft ω-regularity (Corollary 3.3, Theorem 3.4 and Examples 3.15, 3.16); soft ω-$T_{2\frac{1}{2}}$ lies strictly between soft $T_{2\frac{1}{2}}$ and soft T_2 (Theorems 3.25, 3.27 and Examples 3.29, 3.30). Several sufficient conditions for the equivalence between these new three notions and some of their relevant ones are given (Theorems 2.13, 2.14, 3.5, 3.6, 3.26, 3.28). A decomposition theorem of soft regularity by means of soft ω-semi-regularity and soft ω-almost-regularity is given (Theorem 3.17). It is shown that soft ω-almost-regularity is heritable for specific kinds of soft subspaces (Theorems 2.19, 2.21). Soft product theorems regarding soft almost regular spaces (Theorem 2.23), soft ω-almost regular spaces (Theorem 2.24), and soft ω-semi-regular spaces (Theorem 3.33). Finally, the article delves into the connections between the newly proposed as well as some known soft axioms and their counterparts in traditional topological spaces, facilitating a bridging of concepts between the soft and classical realms (Theorems 2.3–2.7, 3.8–3.11, 3.13, 3.19–3.21, 3.23, and Corollaries 2.8, 2.10, 3.12, 3.14, 3.22, 3.24).

In the next work, we intend to: 1) Define and investigate soft ω-almost-normality; 2) investigate the behavior of these new soft separation ideas under various kinds of soft mappings; and 3) find an application for our new two conceptions in the “decision-making problem”, “information systems”, or “expert systems”.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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