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Research article

On the A_{α^-}-spectra of graphs and the relation between A_α- and A_{α^-}-spectra

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Abstract: Let G be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. For any real number $\alpha \in [0, 1]$, Nikiforov defined the A_α-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

The eigenvalues of the matrix $A_\alpha(G)$ form the A_α-spectrum of G. The A_α-spectral radius of G is the largest eigenvalue of $A_\alpha(G)$ denoted by $\rho_\alpha(G)$. In this paper, we propose the A_{α^-}-matrix of G as

$$A_{\alpha^-}(G) = \alpha D(G) + (\alpha - 1)A(G), \quad 0 \leq \alpha \leq 1.$$

Let the A_{α^-}-spectral radius of G be denoted by $\lambda_{\alpha^-}(G)$, and let $S^\alpha_\beta(G)$ and $S^{\alpha^-}_\beta(G)$ be the sum of the β^{th} powers of the A_α and A_{α^-} eigenvalues of G, respectively. We determine the A_{α^-}-spectra of some graphs and obtain some bounds of the A_{α^-}-spectral radius. Moreover, we establish a relationship between the A_α-spectral radius and A_{α^-}-spectral radius. Indeed, for $\alpha \in (\frac{1}{2}, 1)$, we show that $\lambda_{\alpha^-} \leq \rho_\alpha$, and we prove that if G is connected, then the equality holds if and only if G is bipartite. Employing this relation, we obtain some upper bounds of $\lambda_{\alpha^-}(G)$, and we prove that the A_{α^-}-spectrum and A_α-spectrum are equal if and only if G is a bipartite connected graph. Furthermore, we generalize the relation established by S. Akbari et al. in (2010) as follows: for $\alpha \in [\frac{1}{2}, 1)$, if $0 < \beta \leq 1$ or $2 \leq \beta \leq 3$, then $S^\alpha_\beta(G) \geq S^{\alpha^-}_\beta(G)$, and if $1 \leq \beta \leq 2$, then $S^\alpha_\beta(G) \leq S^{\alpha^-}_\beta(G)$, where the equality holds if and only if G is a bipartite graph such that $\beta \notin \{1, 2, 3\}$.

Keywords: Laplacian; singnless Laplacian; A_{α^-}-spectral radius; A_α-matrix; sum of powers of A_α-eigenvalues
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1. Introduction

Let

$$G = (V(G), E(G))$$

be a simple undirected graph on n vertices such that $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, for $v_i \in V(G)$, and $d_G(v_i) = d(v_i) = d_i$ denotes the degree of vertex v_i and the set of vertices adjacent to v_i, denoted by $N(v_i)$, refers to the neighborhood of v_i. The maximum and the minimum degree of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. $\lambda(M)$, $\lambda_k(M)$ and $\lambda_{min}(M)$ denote the largest eigenvalue, the k-th largest eigenvalue, and the smallest eigenvalue of a matrix M, respectively. The set of all eigenvalues for a matrix M together with their multiplicities is called the M-spectrum. A real symmetric matrix M is called positive semidefinite if $\lambda_{min}(M) \geq 0$. Denote by K_n, P_n, C_n and $K_{s,n-s}$ the complete graph, path, cycle, and complete bipartite graph with n vertices, respectively. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of the degrees of G, respectively. The signless Laplacian $Q(G)$ of G is defined as

$$Q(G) = D(G) + A(G).$$

The Laplacian $L(G)$ of G is defined as

$$L(G) = D(G) - A(G).$$

We denote the eigenvalues of $Q(G)$ and $L(G)$ by $\rho_k(G)$ and $\lambda_k(G)$, respectively, where $1 \leq k \leq n$ (we drop G when it is clear from the context). In particular, the largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G, denoted by $\rho(G)$, and the largest eigenvalue of $L(G)$ is called the Laplacian spectral radius of G, denoted by $\lambda(G)$.

In [1], Nikiforov introduced the A_α-matrix of G, which is the convex linear combination of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G), \quad 0 \leq \alpha \leq 1.$$

Clearly,

$$A_0(G) = A(G), \;\; A_1(G) = D(G) \;\text{ and }\; 2A_{\frac{1}{2}}(G) = Q(G).$$

Thus, the matrix $A_\alpha(G)$ is a generalization of the adjacency matrix and the signless Laplacian matrix. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-spectral radius of G, denoted by $\rho_\alpha(G)$. Nikiforov [1] had studied the matrix $A_\alpha(G)$, and he has investigated many properties on $A_\alpha(G)$, including bounds on the k-th largest (especially, the largest, the second largest, and the smallest) eigenvalue of $A_\alpha(G)$, the positive semidefiniteness of $A_\alpha(G)$, etc. From then on, the study of A_α-spectra and the A_α-spectral radius for the graph has attracted the attention of many researchers. In [2], Nikiforov et al. gave an upper bound of the spectral radius of $A_\alpha(T_\Delta)$, where T_Δ is the tree of maximal degree Δ. Also, he obtained several bounds on the spectral radius of A_α of general graphs. The graphs with

$$\lambda_k(A_\alpha(G)) = \alpha n - 1, \;\; 2 \leq k \leq n$$

had been characterized by Lin et al. in [3]. In [4], Guo and Zhou gave upper bounds for the A_α-spectral radius for some graphs under some conditions. Guo and Zhang [5]; obtained a sharp upper bound on the A_α-spectral radius for $\alpha \in [\frac{1}{2}, 1)$, and proved that for two connected graphs G_1 and G_2,

$\rho_\alpha(G_1) > \rho_\alpha(G_2)$ under certain conditions. For further related studies, one may see [6–8].

The research on $L(G)$ has shown that it is a remarkable matrix in many respects. Since; $L(G)$ is just the difference between $D(G)$ and $A(G)$, then to understand to what extent each of the summands $-A(G)$ and $D(G)$ determines the properties of $L(G)$, and motivated by the above works, we introduce the A_{α^-}-matrix defined by

$$A_{\alpha^-}(G) = \alpha D(G) + (\alpha - 1)A(G), \quad 0 \leq \alpha \leq 1. \tag{1.1}$$

Through our study, several facts indicate that the study of the $A_{\alpha^-}(G)$ family is of a unique importance. First of all, obviously

$$A(G) = -A_{0^-}(G), \quad D(G) = A_{1^-}(G) \quad \text{and} \quad L(G) = 2A_{\frac{1}{2}^-}(G).$$

Since $A_{\frac{1}{2}^-}(G)$ is essentially equivalent to $L(G)$, then the matrix $A_{\alpha^-}(G)$ can be regarded as a generalization of the Laplacian matrix $L(G)$. Since

$$A_1(G) = A_{1^-}(G) = D(G);$$

unless otherwise specified, we only consider the case of $0 \leq \alpha < 1$ in this paper.

In this paper, we study some properties on $A_{\alpha^-}(G)$ and obtain some bounds of the largest eigenvalue of $A_{\alpha^-}(G)$. For the relation between the Q-spectrum and L-spectrum of a connected graph G, it is well known that G is bipartite if and only if $Q(G)$ and $L(G)$ has the same spectrum ([9, Proposition 1.3.10]), and we extend this relation on the A_α- and A_{α^-}-spectrum. In [10], Akbari et al. proved that, for a graph G of order n and any real number β, if $0 < \beta \leq 1$ or $2 \leq \beta \leq 3$, then

$$\sum_{k=1}^{n} \rho_k^\beta \geq \sum_{k=1}^{n} \lambda_k^\beta;$$

and if $1 \leq \beta \leq 2$, then

$$\sum_{k=1}^{n} \rho_k^\beta \leq \sum_{k=1}^{n} \lambda_k^\beta.$$

In this work we present the relation between the sum of powers of A_α and A_{α^-} eigenvalues, which is a generalization of the relation between the sum of powers of $Q(G)$ and $L(G)$ eigenvalues in [10].

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas used later. In Section 3, we derive new basic properties of $A_{\alpha^-}(G)$. In Section 4, we present the A_{α^-}-spectra of the complete graphs and the complete split graphs. Section 5 gives some bounds of the largest eigenvalue of $A_{\alpha^-}(G)$. In Section 6, we deduce the relation between the A_α- and A_{α^-}-spectral radius. Finally, in Section 7, we prove the relation between the sum of powers of A_α and A_{α^-} eigenvalues.

2. Preliminaries

Although Weyl's inequalities have been known for almost a century (see, e.g., [11]), their equality case was first established by So in [12], and for convenience we state Weyl and So's complete theorem as follows:

Theorem 2.1. *Let A and B be Hermitian matrices of order n, and let $1 \leq i \leq n$ and $1 \leq j \leq n$. Then,*

$$\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A + B), \ \ if \ \ i + j \geq n + 1, \tag{2.1}$$

$$\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B), \ \ if \ \ i + j \leq n + 1. \tag{2.2}$$

In either of these inequalities, the equality holds if and only if there exists a nonzero n-vector that is an eigenvector to each of the three eigenvalues involved.

A simplified version of (2.1) and (2.2) gives

$$\lambda_k(A) + \lambda_{min}(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda(B). \tag{2.3}$$

We shall need the following lemmas for our new results.

Lemma 2.2. *[3] Let G be a graph of order n. If $e \in E(G)$ and $\alpha \geq \frac{1}{2}$, then,*

$$\rho_i(A_\alpha(G)) \geq \rho_i(A_\alpha(G - e)), \quad for \ \ 1 \leq i \leq n.$$

Lemma 2.3. *[13] Let B be a real symmetric nonnegative irreducible matrix and λ be the largest eigenvalue of B. $Z \in R^n$. If $Z^t BZ = \lambda$ and $\|Z\| = 1$, then $BZ = \lambda Z$.*

Lemma 2.4. *[5] Let G be a connected graph with $n \geq 4$ vertices and m edges. If $\alpha \in [\frac{1}{2}, 1)$, then,*

$$\rho_\alpha(G) \leq \max\left\{\alpha\Delta(G), (1-\alpha)(m - \frac{n-1}{2})\right\} + 2\alpha.$$

Equality holds if and only if $\alpha = \frac{n-1}{n+1}$ and $G = K_n$.

3. Basic properties of $A_{\alpha^-}(G)$

For a graph G of order n, suppose that λ is an eigenvalue of $A_{\alpha^-}(G)$ and x is an eigenvector of $A_{\alpha^-}(G)$ with respect to λ. We use $x(v)$ to denote the entry of x corresponding to the vertex $v \in V(G)$. It is clear that the system of eigenequations for the matrix $A_{\alpha^-}(G)$ is

$$\lambda x(v) = \alpha d_G(v)x(v) + (\alpha - 1) \sum_{u \in N(v)} x(u). \tag{3.1}$$

If G is a graph of order n with $A_{\alpha^-}(G) = A_{\alpha^-}$, and x is a real vector, the quadratic form $\langle A_{\alpha^-}x, x\rangle$ can be represented in several equivalent ways:

$$\langle A_{\alpha^-}x, x\rangle = \sum_{uv \in E(G)} (\alpha x(u)^2 + 2(\alpha - 1)x(u)x(v) + \alpha x(v)^2), \tag{3.2}$$

$$\langle A_{\alpha^-}x, x\rangle = (2\alpha - 1) \sum_{u \in V(G)} x(u)^2 d_G(u) + (1 - \alpha) \sum_{uv \in E(G)} (x(u) - x(v))^2, \tag{3.3}$$

$$\langle A_{\alpha^-}x, x\rangle = \alpha \sum_{u \in V(G)} x(u)^2 d_G(u) + 2(\alpha - 1) \sum_{uv \in E(G)} x(u)x(v). \tag{3.4}$$

Each of these representations can be useful in proofs.

Now, we give some of the spectral properties of the A_{α^-}-matrix. Let us call the largest eigenvalue of $A_{\alpha^-}(G)$ the A_{α^-}-spectral radius of G, and denote it as $\lambda_{\alpha^-}(G)$. Let us also denote the smallest eigenvalue of $A_{\alpha^-}(G)$ as $\mu_{\alpha^-}(G)$. Since $A_{\alpha^-}(G)$ is a real symmetric matrix, and by using Rayleigh's principle, the following result holds:

Proposition 3.1. *If $\alpha \in [0, 1)$ and G is a graph of order n, then*

$$\lambda_{\alpha^-}(G) = \max_{\|x\|_2=1} \langle A_{\alpha^-}(G)x, x\rangle \ \text{ and } \mu_{\alpha^-}(G) = \min_{\|x\|_2=1} \langle A_{\alpha^-}(G)x, x\rangle. \tag{3.5}$$

Moreover, if x is a unit n-vector, then,

$$\lambda_{\alpha^-}(G) = \langle A_{\alpha^-}(G)x, x\rangle$$

if and only if x is an eigenvector to $\lambda_{\alpha^-}(G)$, and

$$\mu_{\alpha^-}(G) = \langle A_{\alpha^-}(G)x, x\rangle$$

if and only if x is an eigenvector to $\mu_{\alpha^-}(G)$.

By using these relations, the following result is evident:

Proposition 3.2. *If $\alpha \in [0, 1)$ and G is a graph, then*

$$\lambda_{\alpha^-}(G) = \max\{\lambda_{\alpha^-}(H) : H \ is \ a \ component \ of \ G\},$$

$$\mu_{\alpha^-}(G) = \min\{\mu_{\alpha^-}(H) : H \ is \ a \ component \ of \ G\}.$$

It is clear that if G is a d-regular graph of order n, then

$$A_{\alpha^-}(G) = \alpha d I_n + (\alpha - 1)A(G),$$

and so there is a linear correspondence between the spectra of $A_{\alpha^-}(G)$ and of $A(G)$,

$$\lambda_k(A_{\alpha^-}(G)) = \alpha d + (\alpha - 1)\lambda_{n-k+1}(A(G)), \quad 1 \le k \le n. \tag{3.6}$$

In particular, if $k = n$, then

$$\mu_{\alpha^-}(G) = (2\alpha - 1)d$$

for any $\alpha \in [0, 1]$. As a consequence of Weyl's inequality (2.3), the following result is immediate:

Proposition 3.3. *If $\alpha \in [0, 1]$ and G is a graph with*

$$A(G) = A \ \text{ and } \ A_{\alpha^-}(G) = A_{\alpha^-},$$

then,

$$\alpha\delta + (\alpha - 1)\lambda_{n-k+1}(A) \le \lambda_k(A_{\alpha^-}) \le \alpha\Delta + (\alpha - 1)\lambda_{n-k+1}(A).$$

An important property of the Laplacian $L(G)$ is that it is positive semidefinite. This is certainly not true for $A_{\alpha^-}(G)$ if $\alpha < \frac{1}{2}$ and G is a regular graph, but if $\alpha \ge \frac{1}{2}$, then $A_{\alpha^-}(G)$ is like $L(G)$. We give this result in the following:

Proposition 3.4. *If $\alpha \ge \frac{1}{2}$, and G is a graph, then $A_{\alpha^-}(G)$ is positive semidefinite, and if $\alpha > \frac{1}{2}$ and G has no isolated vertices, then $A_{\alpha^-}(G)$ is positive definite. Moreover, if $\alpha < \frac{1}{2}$ and G is a regular graph, then $A_{\alpha^-}(G)$ is not positive semidefinite.*

Proof. Let x be a nonzero vector. If $\alpha \leq \frac{1}{2}$, then for any edge $uv \in E(G)$, we see that

$$\langle A_{\alpha^-}(G)x, x\rangle \geq (2\alpha - 1)(x(u)^2 + x(v)^2) + (1 - \alpha)(x(u) - x(v))^2 \geq 0. \tag{3.7}$$

Hence, $A_{\alpha^-}(G)$ is positive semidefinite. Now, assume that G has no isolated vertices. Choose a vertex u with $x(u) \neq 0$ and let $uv \in E(G)$. Then, (3.7) becomes a strict inequality, and so $A_{\alpha^-}(G)$ is positive definite. Finally, suppose that G is a $d-$regular graph, $\alpha < \frac{1}{2}$, and let A be its adjacency matrix, then $\lambda(A) = d$. Thus

$$\mu_{\alpha^-}(G) = (2\alpha - 1)d < 0.$$

□

By the above proposition we get the following lemma which gives a relation between the A_{α^-}-eigenvalues of G and the A_{α^-} eigenvalues of spanning subgraphs of G.

Lemma 3.5. *Let G be a graph of order n and let $\alpha \in [\frac{1}{2}, 1)$. If $G' = G - e$, where $e \in E(G)$, then,*

$$\lambda_i(G') \leq \lambda_i(G) \text{ for } 1 \leq i \leq n.$$

Proof. Let $e = uv$ such that $u, v \in V(G)$. It is easy to see that

$$A_{\alpha^-}(G) = A_{\alpha^-}(G') + M,$$

where M is the matrix of order n indexed by the vertices of G having $(u, v)^{th}$ and $(v, u)^{th}$ entries both equal to $\alpha - 1$, and the $(u, u)^{th}$ and $(v, v)^{th}$ entries both equal to α and all other entries equal to zero, hence M is an A_{α^-}-matrix of a graph containing only one edge. Since $\alpha \in [\frac{1}{2}, 1)$, then $A_{\alpha^-}(G)$, $A_{\alpha^-}(G')$ and M are positive semidefinite and Weyl's inequalities (2.3) imply that $\lambda_i(G') \leq \lambda_i(G)$. □

4. The A_{α^-}-spectra of some graphs

Equality (3.6) and the fact the eigenvalues of $A(K_n)$ are $\{n - 1, -1, \cdots, -1\}$ give the spectrum of $A_{\alpha^-}(K_n)$ as follows:

Proposition 4.1. *The eigenvalues of $A_{\alpha^-}(K_n)$ are*

$$\mu_{\alpha^-}(K_n) = (2\alpha - 1)(n - 1) \text{ and } \lambda_k(A_{\alpha^-}(K_n)) = \alpha(n - 2) + 1 \text{ for } 1 \leq k \leq n - 1.$$

If $S \subseteq V(G)$, then we use $G[S]$ to denote the subgraph of G induced by S. Recall that $G[S]$ is an independent set if no two vertices of S are adjacent and $G[S]$ is a clique if it is a complete subgraph of G. The graph $K_r \vee (n - r)K_1$ is called a complete split graph, denoted by $CS_{r,n-r}$. The work in the following proposition is motivated by the proof of [3, Proposition 2.4].

Proposition 4.2. *Let G be a graph with $A_{\alpha^-}(G)$, and $\alpha \in [0, 1)$. Let $S \subseteq V(G)$ and $|S| = k$. Suppose that $d_G(u) = d$ for each vertex $u \in S$, and $N(v)\backslash\{w\} = N(w)\backslash\{v\}$ for any two vertices $v, w \in S$. Then, we have the following statements:*

(i) If $G[S]$ is a clique, then $\alpha(d - 1) + 1$ is an eigenvalue of $A_{\alpha^-}(G)$ with multiplicity at least $k - 1$.

(ii) If $G[S]$ is an independent set, then αd is an eigenvalue of $A_{\alpha^-}(G)$ with multiplicity at least $k - 1$.

Proof. Let $S = \{v_1, v_2, \cdots, v_k\}$. Clearly, $d_1 = \cdots = d_k = d$. Let $z_1, z_2, \cdots, z_{k-1}$ be vectors such that

$$\begin{cases} z_i(v_1) = 1, \\ z_i(v_{i+1}) = -1, \\ z_i(v) = 0, \quad \text{if } v \in V(G)\backslash\{v_1, v_{i+1}\}, \end{cases}$$

for $i = 1, \cdots, k-1$. Assume that $G[S]$ is a clique. It is easy to obtain that

$$A_{\alpha^-}(G)z_i = (\alpha(d_1 - 1) + 1, 0, \cdots, 0, \alpha(1 - d_i) - 1, 0, \cdots, 0)' = (\alpha(d - 1) + 1)z_i,$$

for $i = 1, \cdots, k-1$. Hence, $\alpha(d-1)+1$ is an eigenvalue of $A_{\alpha^-}(G)$ and $z_1, z_2, \cdots, z_{k-1}$ are eigenvectors of $A_{\alpha^-}(G)$ corresponding to $\alpha(d-1)+1$. In addition, since $z_1, z_2, \cdots, z_{k-1}$ are linearly independent, the multiplicity of $\alpha(d-1)+1$ is at least $k-1$. Now, supposing that $G[S]$ is an independent set, we have

$$A_{\alpha^-}(G)z_i = (\alpha d_1, 0, \cdots, 0, -\alpha d_i, 0, \cdots, 0)' = \alpha d z_i,$$

for $i = 1, \cdots, k-1$. Since $z_1, z_2, \cdots, z_{k-1}$ are linearly independent, it follows that αd is an eigenvalue of $A_{\alpha^-}(G)$ with multiplicity at least $k-1$. Thus, the proof is completed. □

Consider an $n \times n$ real symmetric matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1t} \\ S_{21} & S_{22} & \cdots & S_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ S_{t1} & S_{t2} & \cdots & S_{tt} \end{pmatrix},$$

whose rows and columns are partitioned according to a partitioning $P_1, P_2, \cdots, P_t$ of $\{1, 2, \cdots, n\}$. The quotient matrix B of the matrix S is the $t \times t$ matrix whose entries are the average row sums of the blocks S_{ij} of S. The partition is equitable if each block S_{ij} of S has constant row sum.

Lemma 4.3. *[14] Let B be an equitable quotient matrix of a symmetric real matrix S. If λ is an eigenvalue of B, then λ is also an eigenvalue of S.*

Now, we can determine all A_{α^-}-eigenvalues of $CS_{r,n-r}$ as follows:

Proposition 4.4. *The A_{α^-}-spectrum of $CS_{r,n-r}$ contains $\alpha(n-2)+1$ with multiplicity $r-1$, αr with multiplicity $n-r-1$, and the remaining two A_{α^-}-eigenvalues are*

$$\frac{\alpha(n + 2(r-1)) + 1 \pm \sqrt{(\alpha(n + 2(r-1)) + 1)^2 + 4r(1 - 2\alpha)(n - r + \alpha(r-1))}}{2}. \tag{4.1}$$

Proof. We can write $A_{\alpha^-}(CS_{r,n-r})$ as

$$A_{\alpha^-}(CS_{r,n-r}) = \begin{pmatrix} (1-\alpha)J_{r,r} + (\alpha n - 1)I_r & (1-\alpha)J_{r,n-r} \\ (1-\alpha)J_{n-r,r} & \alpha r I_{n-r,n-r} \end{pmatrix}.$$

Then, the quotient matrix of $A_{\alpha^-}(CS_{r,n-r})$ is equitable and it can be written in the form

$$B(A_{\alpha^-}(CS_{r,n-r})) = \begin{pmatrix} (n-r)\alpha + r - 1 & (1-\alpha)(n-r) \\ (1-\alpha)r & \alpha r \end{pmatrix}.$$

Thus, by Lemma 4.3, the eigenvalues of $B(A_{\alpha^-}(CS_{r,n-r}))$ are eigenvalues of $A_{\alpha^-}(CS_{r,n-r})$, and according to Proposition 4.2, we get the result. □

5. The largest eigenvalue $\lambda_{\alpha^-}(G)$

In this section we give a few general bounds on $\lambda_{\alpha^-}(G)$.

Proposition 5.1. *Let G be a graph, with*

$$\Delta(G) = \Delta, \ \ A(G) = A \ \ \textit{and} \ \ D(G) = D.$$

The following inequalities hold for $\lambda_{\alpha^-}(G)$:

$$\lambda_{\alpha^-}(G) \geq \alpha\Delta + (\alpha - 1)\lambda(A), \tag{5.1}$$

$$\lambda_{\alpha^-}(G) \geq (2\alpha - 1)\lambda(A). \tag{5.2}$$

Proof. Inequality (5.1) follows by Weyl's inequalities (2.3) because

$$\begin{aligned}\alpha\Delta + (\alpha - 1)\lambda(A) &= \alpha\lambda(D) + (\alpha - 1)\lambda(A)\\ &= \lambda(\alpha D) + \lambda_{min}((\alpha - 1)A)\\ &\leq \lambda(\alpha D + (\alpha - 1)A)\\ &= \lambda_{\alpha^-}(G).\end{aligned}$$

To prove the inequality (5.2), let H be a component of G such that $\lambda(A) = \lambda(A(H))$. Let x be a positive unit vector to $\lambda(A(H))$. For every edge $uv \in E(H)$, the AM-GM inequality implies that

$$\begin{aligned}2x(u)x(v) &= 2\alpha x(u)x(v) + 2(1 - \alpha)x(u)x(v)\\ &\leq \alpha\frac{(x(u) + x(v))^2}{2} + 2(1 - \alpha)x(u)x(v)\\ &= \frac{1}{2}(\alpha x(u)^2 + 2(\alpha - 1)x(u)x(v) + \alpha x(v)^2) + (3 - 2\alpha)x(u)x(v).\end{aligned}$$

Summing this inequality over all edges $uv \in E(H)$, and using (3.2), we get

$$\begin{aligned}\lambda(A) = \lambda(A(H)) &= \langle A(H)x, x\rangle\\ &\leq \frac{1}{2}\langle A_{\alpha^-}(H)x, x\rangle + \frac{1}{2}(3 - 2\alpha)\lambda(A);\end{aligned}$$

and then we get

$$2\lambda(A) - (3 - 2\alpha)\lambda(A) \leq \langle A_{\alpha^-}(H)x, x\rangle \leq \lambda_{\alpha^-}(G);$$

hence

$$(2\alpha - 1)\lambda(A) \leq \lambda_{\alpha^-}(G).$$

□

Having inequality (5.2) in hand, if $\alpha \geq \frac{1}{2}$, then every lower bound of $\lambda(A)$ gives a lower bound on $\lambda_{\alpha^-}(G)$, but if $\alpha < \frac{1}{2}$, then every upper bound of $\lambda(A)$ gives a lower bound on $\lambda_{\alpha^-}(G)$. We mention just two such bounds.

Corollary 5.2. *Let G be a graph such that $\alpha \geq \frac{1}{2}$. If G is of order n and has m edges, then*

$$\lambda_{\alpha^-}(G) \geq (2\alpha - 1)\frac{2m}{n}.$$

Corollary 5.3. *Let G be a connected graph such that $\alpha < \frac{1}{2}$. If G is of order n and has m edges, then*

$$\lambda_{\alpha^-}(G) \geq (2\alpha - 1)\sqrt{2m - n + 1}.$$

Proposition 5.4. *Let G be a graph of order n. If $\alpha \in (\frac{1}{2}, 1]$, then,*

$$\lambda_{\alpha^-}(G) \leq \alpha(n-2) + 1.$$

Moreover,

$$\lambda_{\alpha^-}(G) = \alpha(n-2) + 1$$

with multiplicity $k - 1$ if G has k vertices of degree $n - 1$.

Proof. Applying Lemma 3.5 leads to

$$\lambda_{\alpha^-}(G) \leq \lambda_{\alpha^-}(K_n) = \alpha(n-2) + 1.$$

If G has k vertices of degree $n-1$, then it follows from Proposition 4.2 that $\alpha(n-2)+1$ is an eigenvalue of $A_{\alpha^-}(G)$ with multiplicity at least $k - 1$, and since

$$\lambda_{\alpha^-}(G) \leq \alpha(n-2) + 1,$$

we get

$$\lambda_{\alpha^-}(G) = \alpha(n-2) + 1$$

with multiplicity $k - 1$. □

6. The relationship between A_{α^-}- and A_α-spectral radius

Let G be a connected graph. Merris [15] pointed out $\lambda(L(G)) \leq \rho(Q(G))$, and the equality holds if G is a bipartite graph. In the next result, we generalize this result to the A_{α^-}- and A_α-spectral radius of a connected graph.

Theorem 6.1. *Let G be a graph of order n, $\alpha \in (0, 1)$, $\lambda_{\alpha^-}(G) = \lambda_{\alpha^-}$ and $\rho_\alpha(G) = \rho_\alpha$. We have $\lambda_{\alpha^-} \leq \rho_\alpha$. Moreover, if G is connected, then the equality holds if and only if G is bipartite.*

Proof. Let

$$V(G) = \{v_1, v_2, \cdots, v_n\}, \quad x = (x_1, x_2, \cdots, x_n)^t \in R^n$$

be an arbitrary vector such that $\|x\| = 1$. Let

$$y = (y_1, y_2, \cdots, y_n)^t \in R^n$$

be a unit eigenvector of $A_{\alpha^-}(G)$ belonging to λ_{α^-} and

$$y' = (y'_1, y'_2, \cdots, y'_n)^t \in R^n$$

be a unit eigenvector of $A_\alpha(G)$ belonging to ρ_α. Let

$$|y| = (|y_1|, |y_2|, \cdots, |y_n|)^t.$$

First, we prove that $\lambda_{\alpha^-} \leq \rho_\alpha$.

$$\begin{aligned}\lambda_{\alpha^-} &= \max x^t A_{\alpha^-} x\\ &= \max x^t(\alpha D + (\alpha-1)A)x\\ &= \max\left(\alpha\sum_{i=1}^{n} x_i^2 d_i + 2(\alpha-1)\sum_{v_iv_j\in E(G)} x_ix_j\right)\\ &= y^t A_{\alpha^-} y\\ &= \alpha\sum_{i=1}^{n} y_i^2 d_i + 2(\alpha-1)\sum_{v_iv_j\in E(G)} y_iy_j\end{aligned}$$

and

$$\begin{aligned}\rho_\alpha &= \max x^t A_\alpha x\\ &= \max x^t\,(\alpha D + (1-\alpha)A)\,x\\ &= \max\left(\alpha\sum_{i=1}^{n} x_i^2 d_i + 2(1-\alpha)\sum_{v_iv_j\in E(G)} x_ix_j\right)\\ &= y'^t A_\alpha y'\\ &= \alpha\sum_{i=1}^{n} y_i'^2 d_i + 2(1-\alpha)\sum_{v_iv_j\in E(G)} y_i'y_j'.\end{aligned}$$

Thus,

$$\begin{aligned}\lambda_{\alpha^-} &= \alpha\sum_{i=1}^{n} y_i^2 d_i + 2(\alpha-1)\sum_{v_iv_j\in E(G)} y_iy_j\\ &\leq \alpha\sum_{i=1}^{n} y_i^2 d_i + 2(1-\alpha)\sum_{v_iv_j\in E(G)} |y_iy_j|\\ &= |y^t|A_\alpha|y|\\ &\leq \max x^t A_\alpha x\\ &= \rho_\alpha. \end{aligned} \tag{6.1}$$

Now, if G is bipartite, then the matrix A_{α^-} and the matrix A_α are similar by a diagonal matrix D' with diagonal entries ±1, that is, $A_\alpha = D'A_{\alpha^-}D'^{-1}$. Therefore, A_{α^-} and A_α have the same spectrum, and thus we get $\lambda_{\alpha^-} = \rho_\alpha$. Finally, when G is connected and $\lambda_{\alpha^-} = \rho_\alpha$, all inequalities (6.1) must be equalities. By Lemma 2.3 and the equality

$$|y^t|A_\alpha|y| = \rho_\alpha;$$

we know that $|y|$ is an eigenvector of A_α belonging to ρ_α. So, $|y| = \pm y'$. Using the Perron-Frobenius' theorem for $A_\alpha(G)$, we have $y' > 0$, $|y| = y'$, and $|y_i| > 0$, $i = 1, 2, \cdots, n$.

Since

$$\alpha\sum_{i=1}^{n} y_i^2 d_i + 2(\alpha-1)\sum_{v_iv_j\in E(G)} y_iy_j = \alpha\sum_{i=1}^{n} y_i^2 d_i + 2(1-\alpha)\sum_{v_iv_j\in E(G)} |y_iy_j|;$$

we get

$$-\sum_{v_iv_j\in E(G)} y_iy_j = \sum_{v_iv_j\in E(G)} |y_iy_j|,$$

hence, $|y_iy_j| = -y_iy_j$ when $v_iv_j \in E(G)$. Therefore, $y_iy_j < 0$ if $v_iv_j \in E(G)$.

Let

$$U = \{v_i : y_i > 0\} \ \text{ and } \ W = \{v_j : y_j < 0\}.$$

For each edge $e = v_iv_j$, we have $y_iy_j < 0$, one of the vertices of edge e is in U, and the other is in W. So, G is a bipartite graph. □

Remark 6.2. *If G is not bipartite and is not connected, and λ_{α^-} is the largest eigenvalue of A_{α^-} for a bipartite connected component of G, then the equality in Theorem 6.1 still holds.*

Example 6.3. *Take*

$$G = C_3 \cup C_4.$$

Then, G is not bipartite and not connected. Now, $A_{\frac{1}{4}^-}(G)$ has a spectrum $2^{[1]},\ 1.25^{[2]},\ 0.5^{[2]},\ -1^{[2]}$ (where $\lambda^{[i]}$ means the eigenvalue λ is repeated i times in the spectrum). On the other hand, $A_{\frac{1}{4}}(G)$ has a spectrum $2^{[2]},\ 0.5^{[2]},\ -0.25^{[2]},\ -1^{[1]}$. Then, we have

$$\lambda_{\frac{1}{4}^-}(G) = \rho_{\frac{1}{4}}(G) = 2.$$

Note that 2 is the largest eigenvalue of $A_{\frac{1}{4}^-}(C_4)$ as well.

Now, we introduce the relationship between the A_{α^-}- and A_α-spectra of bipartite graphs, which is a generalization [9, Proposition 1.3.10], and it follows from the proof of the above theorem.

Corollary 6.4. *Let G be a connected graph. Then, G is bipartite if and only if the A_{α^-}-spectrum and A_α-spectrum are equal.*

Remark 6.5. *In fact, if G is bipartite and is not connected, the A_{α^-}-spectrum still equals the A_α-spectrum.*

Example 6.6. *Take $G = P_3 \cup C_4$. Then G is bipartite and not connected. We have that $A_{\frac{3}{4}^-}(G)$ has a spectrum $2^{[1]},\ 1.64039^{[1]},\ 1.5^{[2]},\ 1^{[1]},\ 0.75^{[1]},\ 0.609612^{[1]}$. In contrast, $A_{\frac{3}{4}}(G)$ has the same spectrum as $A_{\frac{3}{4}^-}(G)$, although G is not connected.*

According to Corollary 6.4 and [1, Propositions 38 and 39], we get the following two results:

Corollary 6.7. *The A_{α^-}-spectrum and the A_α-spectrum of the complete bipartite graph $K_{a,b}$ are equal, that is, if $a \geq b \geq 1$ and $\alpha \in (0, 1)$, the eigenvalues of $A_{\alpha^-}(K_{a,b})$ are*

$$\lambda_{\alpha^-}(K_{a,b}) = \frac{1}{2}\left(\alpha(a+b) + \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)}\right),$$

$$\mu_{\alpha^-}(K_{a,b}) = \frac{1}{2}\left(\alpha(a+b) - \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)}\right),$$

$$\lambda_k(A_{\alpha^-}(K_{a,b})) = \alpha a \ \ for \ \ 1 < k \leq b,$$

$$\lambda_k(A_{\alpha^-}(K_{a,b})) = \alpha b \ \ for \ \ b < k < a+b.$$

Corollary 6.8. *The A_{α^-}-spectrum and the A_α-spectrum of the star $K_{1,n-1}$ are equal, that is, the eigenvalues of $A_{\alpha^-}(K_{1,n-1})$ are*

$$\lambda_{\alpha^-}(K_{1,n-1}) = \frac{1}{2}\left(\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\right),$$

$$\mu_{\alpha^-}(K_{1,n-1}) = \frac{1}{2}\left(\alpha n - \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\right),$$

$$\lambda_k(A_{\alpha^-}(K_{1,n-1})) = \alpha \ \ for \ \ 1 < k < n.$$

Indeed, many practical results of Theorem 6.1 can be deduced, and here are some of them.

Proposition 6.9. *Let G be a connected graph with $n \geq 4$ vertices and m edges. If $\alpha \in [\frac{1}{2}, 1)$, then*

$$\lambda_{\alpha^-}(G) < \max\left\{\alpha\Delta(G), (1-\alpha)(m - \frac{n-1}{2})\right\} + 2\alpha. \tag{6.2}$$

Proof. Let

$$\lambda_{\alpha^-}(G) = \lambda_{\alpha^-} \ \text{ and } \ \rho_\alpha(G) = \rho_\alpha.$$

Then, Theorem 6.1 and Lemma 2.4 lead to

$$\lambda_{\alpha^-} \leq \max\left\{\alpha\Delta(G), (1-\alpha)(m - \frac{n-1}{2})\right\} + 2\alpha. \tag{6.3}$$

Suppose that the equality in (6.3) holds, thus $\rho_\alpha = \lambda_{\alpha^-}$, and so G is bipartite and

$$\rho_\alpha = \max\left\{\alpha\Delta(G), (1-\alpha)(m - \frac{n-1}{2})\right\} + 2\alpha.$$

Therefore, by Lemma 2.4, $G = K_n$, and thus G is bipartite if and only if $n = 2$, but $n \geq 4$, and hence the inequality is strict. □

Theorem 6.1 and [2, Theorem 2] lead directly to the next result.

Proposition 6.10. *If T is a tree of order n and $\alpha \in [0, 1]$, then*

$$\lambda_{\alpha^-}(T) \leq \frac{n\alpha + \sqrt{n^2\alpha^2 + 4(n-1)(1-2\alpha)}}{2}.$$

The equality holds if and only if T is the star $K_{1,n-1}$.

By Theorem 6.1 and [1, Proposition 20], we get the next result.

Proposition 6.11. *If G is a graph with no isolated vertices, then*

$$\lambda_{\alpha^-}(G) \leq \max_{u \in V(G)}\left\{\alpha d(u) + \frac{1-\alpha}{d(u)} \sum_{uv \in E(G)} d(v)\right\}.$$

If $\alpha \in (\frac{1}{2}, 1)$ and G is connected, the equality holds if and only if G is a regular bipartite graph.

In this part we give two explicit expressions for the sums and the sum of squares of the eigenvalues of A_{α^-} and A_α, considering [1, Propositions 34 and 35].

Proposition 6.12. *If G is a graph of order n and has m edges, then*

$$\sum_{i=1}^{n} \lambda_i(A_{\alpha^-}(G)) = \sum_{i=1}^{n} \lambda_i(A_\alpha(G)) = trA_{\alpha^-}(G) = trA_\alpha(G);$$

where

$$\sum_{i=1}^{n} \lambda_i(A_{\alpha^-}(G)) = \alpha \sum_{u\in V(G)} d(u) = 2\alpha m.$$

A similar formula for the sum of the squares of the A_{α^-} and A_α-eigenvalues is given as follows:

Proposition 6.13. *If G is a graph of order n and has m edges, then*

$$\sum_{i=1}^{n} \lambda_i^2(A_{\alpha^-}(G)) = \sum_{i=1}^{n} \lambda_i^2(A_\alpha(G)) = trA^2_{\alpha^-}(G) = trA^2_\alpha(G);$$

where

$$\sum_{i=1}^{n} \lambda_i^2(A_{\alpha^-}(G)) = \alpha^2 \sum_{u\in V(G)} d^2(u) + 2(1-\alpha)^2 m.$$

Proof. Let

$$A_{\alpha^-} = A_{\alpha^-}(G), \;\; A = A(G) \;\text{ and }\; D = D(G).$$

Calculating the square $A^2_{\alpha^-}$ and taking its trace, we find that

$$\begin{aligned} trA^2_{\alpha^-} &= tr(\alpha^2 D^2 + (1-\alpha)^2 A^2 + \alpha(\alpha-1)DA + \alpha(\alpha-1)AD) \\ &= \alpha^2 trD^2 + (1-\alpha)^2 trA^2 - \alpha(1-\alpha)trDA + \alpha(\alpha-1)trAD \\ &= \alpha^2 \sum_{u\in V(G)} d^2(u) + 2(1-\alpha)^2 m. \end{aligned}$$

□

7. Sum of powers of A_{α^-} and A_α eigenvalues

Pirzada et al. [16] introduced the sum of the β^{th} powers of the A_α eigenvalues of G as

$$S^\alpha_\beta(G) = \sum_{i=1}^{n} \rho_i^\beta.$$

Now, we have the notation

$$S^{\alpha^-}_\beta(G) = \sum_{i=1}^{n} \lambda_i^\beta$$

for the sum of the β^{th} powers of the A_{α^-} eigenvalues of G. The following theorem is a generalization [10, Theorem 2].

Theorem 7.1. *Let G be a graph of order $n > 1$, $\alpha \in [\frac{1}{2}, 1)$, and let β be a real number.*

(i) If $0 < \beta \leq 1$ or $2 \leq \beta \leq 3$, then
$$S_\beta^\alpha(G) \geq S_\beta^{\alpha^-}(G).$$

(ii) If $1 \leq \beta \leq 2$, then
$$S_\beta^\alpha(G) \leq S_\beta^{\alpha^-}(G).$$

For $\beta \in (0,1) \cup (2,3)$, the equality holds in (i) if and only if G is a bipartite graph. Moreover, for $\beta \in (1,2)$, the equality holds in (ii) if and only if G is a bipartite graph.

Proof. We recall that, for any real number r, the binomial series
$$\sum_{k=0}^{\infty} \binom{r}{k} x^k$$
converges to $(1+x)^r$ if $|x| < 1$. This also remains true for $x = -1$ if $r > 0$ (see, e.g., [17, p. 419]). By Lemma 3.5, we find that,
$$\lambda_{\alpha^-}(G) \leq \lambda_{\alpha^-}(K_n) = \alpha(n-2)+1 \ \text{ and } \ \rho_\alpha(G) \leq \rho_\alpha(K_n) = n-1.$$
Since A_α is positive semidefinite when $\alpha \in [\frac{1}{2}, 1)$, then if $\rho_i > 0$, we have
$$|\frac{\rho_i}{n} - 1| \leq |\frac{n-1}{n} - 1| = |\frac{-1}{n}| < 1$$
and if $\rho_i = 0$, we get
$$\frac{\rho_i}{n} - 1 = -1.$$
Therefore,
$$\begin{aligned}
\frac{S_\beta^\alpha(G)}{n^\beta} &= (\frac{\rho_1}{n})^\beta + \cdots + (\frac{\rho_n}{n})^\beta \\
&= \sum_{k=0}^{\infty} \binom{\beta}{k} (\frac{\rho_1}{n} - 1)^k + \cdots + \sum_{k=0}^{\infty} \binom{\beta}{k} (\frac{\rho_n}{n} - 1)^k \\
&= \sum_{k=0}^{\infty} \binom{\beta}{k} tr\left(\frac{1}{n}(\alpha D + (1-\alpha)A) - I\right)^k.
\end{aligned}$$
Also, since A_{α^-} is positive semidefinite when $\alpha \in [\frac{1}{2}, 1)$, then if $\lambda_i > 0$,
$$\left|\frac{\lambda_i}{n} - 1\right| \leq \left|\frac{\alpha(n-2)+1}{n} - 1\right| = \left|(\alpha - 1) + \frac{1-2\alpha}{n}\right| < 1$$
and if $\lambda_i = 0$, we have
$$\frac{\lambda_i}{n} - 1 = -1.$$
Thus, in a similar manner as above, we obtain that
$$\frac{S_\beta^{\alpha^-}(G)}{n^\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} tr\left(\frac{1}{n}(\alpha D + (\alpha-1)A) - I\right)^k.$$

We claim that

$$\text{if } k \text{ is even, } tr(\alpha D + (1-\alpha)A - nI)^k \leq tr(\alpha D + (\alpha - 1)A - nI)^k;$$

$$\text{if } k \text{ is odd, } tr(\alpha D + (1-\alpha)A - nI)^k \geq tr(\alpha D + (\alpha - 1)A - nI)^k.$$

When $((\alpha D - nI) + (1-\alpha)A)^k$ and $((\alpha D - nI) + (\alpha - 1)A)^k$ are expanded in terms of the powers of $\alpha D - nI$ and $(1-\alpha)A$, respectively, the terms appearing in both expansions, regardless of their signs, are the same. To prove this claim, we identify the sign of each term in both expansions. Consider the terms in the expansion of $((\alpha D - nI) + (1-\alpha)A)^k$, where there are exactly j factors equal to $\alpha D - nI$, for some $j = 0, 1, \cdots, k$. The sign of the trace for each of these terms is $(-1)^j$ or 0 because all entries of $\alpha D - nI$ and $(1-\alpha)A$ are non-positive and non-negative, respectively. On the other hand, in each term in the expansion of $((\alpha D - nI) + (\alpha - 1)A)^k$ all factors are matrices with non-positive entries, hence the sign of the trace of each term is $(-1)^k$ or 0. Therefore, the claim has been proven.

Now, note that if $0 < \beta < 1$ or $2 < \beta < 3$, then the sign of $\binom{\beta}{k}$ is $(-1)^{k-1}$, except that $\binom{\beta}{2} > 0$, for $2 < \beta < 3$. According to this, for $0 < \beta < 1$ and every k,

$$\binom{\beta}{k} tr(\alpha D + (1-\alpha)A - nI)^k \geq \binom{\beta}{k} tr(\alpha D + (\alpha - 1)A - nI)^k.$$

This inequality remains true for $2 \leq \beta \leq 3$ as

$$tr(\alpha D + (1-\alpha)A - nI)^2 = tr(\alpha D + (\alpha - 1)A - nI)^2,$$

since $trA_\alpha^2 = trA_{\alpha^-}^2$. Thus, Part ($i$) is proved. For $1 < \beta < 2$, the sign of $\binom{\beta}{k}$ is $(-1)^{k-1}$, except that $\binom{\beta}{1} > 0$. Since $trA_\alpha = trA_{\alpha^-}$, we have

$$tr(\alpha D + (1-\alpha)A - nI) = tr(\alpha D + (\alpha - 1)A - nI),$$

and so part (ii) is similarly proved.

Now, we examine the equality case. Since A_α and A_{α^-} are similar if G is bipartite, it follows that the equality holds in both (i) and (ii). Since for any positive integer i, trA^i equals the total number of closed walks of length i in G, then if G is not bipartite, there exists an odd integer r such that $trA^r > 0$ (see [18, Lemma 2.5]). Hence,

$$tr(\alpha D + (1-\alpha)A - nI)^r > tr(\alpha D + (\alpha - 1)A - nI)^r;$$

and so the inequalities in both (i) and (ii) are strict. □

We know that the A_α-spectra and A_{α^-}-spectra of K_n are $\{(n-1)^{[1]}, (\alpha n - 1)^{[n-1]}\}$ and $\{((2\alpha - 1)(n-1))^{[1]}, (\alpha(n-2)+1)^{[n-1]}\}$, respectively. Therefore,

$$S_\beta^\alpha(K_n) - S_\beta^{\alpha^-}(K_n) = (n-1)^\beta + (n-1)(\alpha n - 1)^\beta - ((2\alpha - 1)(n-1))^\beta - (n-1)(\alpha(n-2)+1)^\beta.$$

By Theorem 7.1 and based on our numerical experiments, we propose the following conjecture:

Conjecture 7.2. *For every $\alpha \in (\frac{1}{2}, 1)$ and each integer $n \geq 3$, we have*

$$S_\beta^\alpha(K_n) - S_\beta^{\alpha^-}(K_n) \geq 0$$

for any $\beta \in [0,1] \cup [2,\infty)$ and

$$S_\beta^\alpha(K_n) - S_\beta^{\alpha^-}(K_n) \leq 0$$

for any $\beta \in (-\infty, 0) \cup (1,2)$.

8. Conclusions

In this paper, we have introduced the A_{α^-}-matrix of a graph G, which is a generalization of the Laplacian matrix $L(G)$, and we have studied the basic properties of A_{α^-}; and derived some bounds for its spectral radius. Furthermore, we have determined the A_{α^-}-spectra for the complete graph and the complete split graph. Building upon previous results, we have extended findings related to the spectral radius of $L(G)$ and $Q(G)$ matrices to the A_α- and A_{α^-}-spectral radius. Specifically, in Theorem 6.1, we have generalized Merris' [15] observation that $\lambda(L(G)) \leq \rho(Q(G))$ with equality holding for bipartite graphs. Additionally, we have extended the known relation that G is bipartite if and only if $Q(G)$ and $L(G)$ share the same spectrum, as demonstrated in Corollary 6.4. Finally, in Theorem 7.1; we have generalized a relation established by S. Akbari et al. in [10]; which relates the sum of powers of the eigenvalues of $Q(G)$ and $L(G)$. In conclusion, these findings have implications for various applications involving graph analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. V. Nikiforov, Merging the A- and Q-spectral theories, *Appl. Anal. Discrete Math.*, **11** (2017), 81–107. https://doi.org/10.2298/AADM1701081N
2. V. Nikiforov, G. Pastén, O. Rojo, R. L. Soto, On the A_α-spectra of trees, *Linear Algebra Appl.*, **520** (2017), 286–305. https://doi.org/10.1016/j.laa.2017.01.029
3. H. Lin, J. Xue, J. Shu, On the A_α-spectra of graphs, *Linear Algebra Appl.*, **556** (2018), 210–219. https://doi.org/10.1016/j.laa.2018.07.003
4. H. Guo, B. Zhou, On the α-spectral radius of graphs, *Appl. Anal. Discrete Math.*, **14** (2020), 431–458. https://doi.org/10.2298/AADM180210022G
5. S. Guo, R. Zhang, Ordering graphs by their largest (least) A_α-eigenvalues, *Linear Multilinear Algebra*, **70** (2022), 7049–7056. https://doi.org/10.1080/03081087.2021.1981811
6. M. Basunia, I. Mahato, M. R. Kannan, On the A_α-spectra of some join graphs, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 4269–4297. https://doi.org/10.1007/s40840-021-01166-z
7. H. Lin, X. Huang, J. Xue, A note on the A_α-spectral radius of graphs, *Linear Algebra Appl.*, **557** (2018), 430–437. https://doi.org/10.1016/j.laa.2018.08.008

8. J. Wang, J. Wang, X. Liu, F. Belardo, Graphs whose A_α-spectral radius does not exceed 2, *Discuss. Math. Graph Theory*, **40** (2020), 677–690. https://doi.org/10.7151/dmgt.2288

9. A. E. Brouwer, W. H. Haemers, *Spectra of graphs*, Springer Science & Business Media, 2012. https://doi.org/10.1007/978-1-4614-1939-6

10. S. Akbari, E. Ghorbani, J. H. Koolen, M. R. Oboudi, On sum of powers of the Laplacian and signless Laplacian eigenvalues of graphs, *Electron. J. Comb.*, **2017** (2010), R115. https://doi.org/10.37236/387

11. R. A. Horn, C. R. Johnson, *Matrix analysis*, 2 Eds., Cambridge University Press, 2012. https://doi.org/10.1017/CBO9781139020411

12. W. So, Commutativity and spectra of Hermitian matrices, *Linear Algebra Appl.*, **212-213** (1994), 121–129. https://doi.org/10.1016/0024-3795(94)90399-9

13. J. Shu, Y. Hong, K. Ren, A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph, *Linear Algebra Appl.*, **347** (2002), 123–129. https://doi.org/10.1016/S0024-3795(01)00548-1

14. L. You, M. Yan, W. So, W. Xi, On the spectrum of an equitable quotient matrix and its application, *Linear Algebra Appl.*, **577** (2019), 21–40. https://doi.org/10.1016/j.laa.2019.04.013

15. R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra Appl.*, **197-198** (1994), 143–176. https://doi.org/10.1016/0024-3795(94)90486-3

16. S. Pirzada, B. A. Rather, R. Ul Shaban, T. A. Chishti, On the sum of the powers of A_α eigenvalues of graphs and A_α-energy like invariant, *Bol. Soc. Paran. Mat.*, **40** (2022), 1–12. https://doi.org/10.5269/bspm.52469

17. R. G. Bartle, *The elements of real analysis*, 2 Eds., John Wiley & Sons, Inc., 1976.

18. N. Biggs, *Algebraic graph theory*, 2 Eds., Cambridge University Press, 1993. https://doi.org/10.1017/CBO9780511608704