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Research article

Approximation by the modified λ-Bernstein-polynomial in terms of basis function

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Abstract: In this article by means of shifted knots properties, we introduce a new type of coupled Bernstein operators for Bézier basis functions. First, we construct the operators based on shifted knots properties of Bézier basis functions then investigate the Korovkin's theorem, establish a local approximation theorem, and provide a convergence theorem for Lipschitz continuous functions and Peetre's K-functional. In addition, we also obtain an asymptotic formula of the type Voronovskaja.

Keywords: Bernstein-polynomial; λ-Bernstein-polynomial; Bézier basis function; shifted knots; modulus of continuity; Lipschitz maximal functions; Peetre's K-functional
Mathematics Subject Classification: 33C45, 41A25, 41A36

1. Introduction

The famous Bernstein polynomial, denoted by $B_n(g)(\psi)$, is defined as:

$$B_s(g;\psi) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) b_{s,i}(\psi),$$

where $s \in \mathbb{N}$ (positive integers) and $b_{s,i}(ɥ)$ are the Bernstein polynomials of degree s at most defined by

$$b_{s,i}(ɥ) = \binom{s}{i} ɥ^i (1-ɥ)^{s-i} \, (i = 0, 1, \cdots, s; ɥ \in [0,1])$$

and

$$b_{s,i}(ɥ) = 0 (i < 0 \text{ or } i > s),$$

where $g \in C[0,1]$ is the function to be approximated and n is a positive integer. The Bernstein polynomial is a linear combination of powers of $ɥ$ and $(1-ɥ)$, with coefficients given by the function g evaluated at equidistant points between 0 and 1. The Bernstein polynomial provides a sequence of polynomial approximations to g, which converges uniformly to g on the interval $[0,1]$ as n approaches infinity. This means that the polynomial approximations become arbitrarily close to g for all values of $ɥ$ in the interval $[0,1]$.

It is very easy to verify the recursive relation for the Bernstein polynomials. The recursive relationship for Bernstein polynomials $b_{s,i}(ɥ)$ is very simple to prove such that

$$b_{s,i}(ɥ) = (1-ɥ) b_{s-1,i}(ɥ) + ɥ b_{s-1,i-1}(ɥ).$$

In 2010, Cai et al. defined the Bernstein-polynomials by the introduction of new Bézier bases with shape parameter $\lambda \in [-1,1]$, known as the λ-Bernstein operators as follows:

$$B_{s,\lambda}(g; ɥ) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) \tilde{b}_{s,i}(\lambda; ɥ) \tag{1.1}$$

where the new Bernstein basis function $\tilde{b}_{s,i}(\lambda; ɥ)$ in terms of the Bernstein polynomial $b_{s,i}(ɥ)$ is defined by Ye et al. [1] as follows:

$$\begin{aligned}
\tilde{b}_{s,0}(\lambda; ɥ) &= b_{s,0}(ɥ) - \frac{\lambda}{s+1} b_{s+1,1}(ɥ), \\
\tilde{b}_{s,i}(\lambda; ɥ) &= b_{s,i}(ɥ) + \lambda \Big(\frac{s-2i+1}{s^2-1} b_{s+1,i}(ɥ) \\
&\quad - \frac{s-2i-1}{s^2-1} b_{s+1,i+1}(ɥ) \Big), \text{ for } 1 \le i \le s-1, \\
\tilde{b}_{s,s}(\lambda; ɥ) &= b_{s,s}(ɥ) - \frac{\lambda}{s+1} b_{s+1,s}(ɥ).
\end{aligned}$$

In 2010, Gadjiev et al. introduced the recent Bernstein type Stancu polynomials by means of shifted knots [2] such as:

$$S_{s,\mu,\beta}(g; ɥ) = \left(\frac{s+\nu_2}{m}\right)^m \sum_{i=0}^{s} \binom{s}{i} \left(ɥ - \frac{\mu_2}{s+\nu_2}\right)^i \left(\frac{s+\mu_2}{m+\nu_2} - ɥ\right)^{s-i} g\left(\frac{i+\mu_1}{s+\nu_1}\right) \tag{1.2}$$

where $ɥ \in [\frac{\mu_2}{m+\nu_2}, \frac{s+\mu_2}{s+\nu_2}]$ and $\mu_i, \ \nu_i, \ i = 1, 2$ are positive real numbers provided $0 \le \mu_2 \le \mu_1 \le \nu_1 \le \nu_2$.

As a result of research conducted in the approximation process, Bernstein type operators have been obtained by researchers within the past few years, for example, a new family of Bernstein-Kantorovich operators [3], q-Bernstein shifted operators [4], the Stancu variant of Bernstein-Kantorovich operators [5], Genuine modified Bernstein-Durrmeyer operators [6], Bézier bases with

Schurer polynomials [7], generalized Bernstein-Schurer operators [8], the approximation of Bernstein type operators [9] and Bernstein operators based on Bézier bases [10], etc. For more details and recent published research we refer the reader to [11–23].

2. Operators and basic estimation

We take the Bernstein basis function $b_{s,i}^{\mu,\nu}$ by means of shifted knots (see [2]) as follows:

$$b_{s,i}^{\mu,\nu}(ɥ) = \binom{s}{i}\left(ɥ - \frac{\mu}{s+\nu}\right)^{i}\left(\frac{s+\mu}{s+\nu} - ɥ\right)^{s-i}. \tag{2.1}$$

We take the Bézier bases function $\tilde{b}_{s,i}^{\mu,\nu}$ by means of Bernstein basis function $b_{s,i}^{\mu,\nu}$ (see [1]) as follows:

$$\begin{aligned}
\tilde{b}_{s,0}^{\mu,\nu}(\lambda; ɥ) &= b_{s,0}^{\mu,\nu}(ɥ) - \frac{\lambda}{s+1} b_{s+1,1}^{\mu,\nu}(ɥ),\\
\tilde{b}_{s,j}^{\mu,\nu}(\lambda; y) &= b_{s,j}^{\mu,\nu}(ɥ) + \lambda\Big(\frac{s-2j+1}{s^2-1} b_{s+1,j}^{\mu,\nu}(ɥ)\\
&\qquad - \frac{s-2j-1}{s^2-1} b_{s+1,j+1}^{\mu,\nu}(ɥ)\Big), \text{ for } 1 \le j \le s-1,\\
\tilde{b}_{s,s}^{\mu,\nu}(\lambda; ɥ) &= b_{s,s}^{\mu,\nu}(ɥ) - \frac{\lambda}{s+1} b_{s+1,s}^{\mu,\nu}(ɥ).
\end{aligned}$$

Thus, for all $\frac{\mu}{s+\nu} \le ɥ \le \frac{s+\mu}{s+\nu}$ and the real number $0 \le \mu \le \nu$, we define the new λ-Bernstein shifted knots operators $B_{s,\lambda}^{\mu,\nu}$ in terms of Bézier bases function $\tilde{b}_{s,i}^{\mu,\nu}$ as follows:

$$B_{s,\lambda}^{\mu,\nu}(g; ɥ) = \left(\frac{s+\nu}{s}\right)^{s} \sum_{i=0}^{s} \tilde{b}_{s,i}^{\mu,\nu}(\lambda; ɥ) g\left(\frac{i}{s}\right) \tag{2.2}$$

where $C[0,1]$ is the set of all continuous functions defined on $[0,1]$ and $s \in \mathbb{N}$ (the set of positive integers). Clearly, for the choice $\mu = \nu = 0$ in the equality (2.2), our new operators $B_{s,\lambda}^{\mu,\nu}$ reduced to the operators of the equality (1.1) defined by Cai et al. [24].

This paper is structured generally as follows: We look at the moments and central moments of our new operators, (2.2). We investigate a Korovkin approximation theorem, prove a local approximation theorem, provide a convergence theorem for Lipschitz continuous functions and produce a Voronovskaja asymptotic formula.

Lemma 2.1. *Let $g(t) = 1, t, t^2$ then for all $s \in \mathbb{N} \setminus \{1\}$, the operators $B_{s,\lambda}^{\mu,\nu}$ defined by* (2.2)*, have the following equalities:*

$$\begin{aligned}
B_{s,\lambda}^{\mu,\nu}(1; ɥ) &= 1,\\
B_{s,\lambda}^{\mu,\nu}(t; ɥ) &= \left(\frac{s+\nu}{s} - \frac{2\lambda}{s(s-1)}\right)\left(ɥ - \frac{\mu}{s+\nu}\right) + \frac{\lambda}{s(s-1)}\left(\frac{s+\nu}{s}\right)^{s}\left(ɥ - \frac{\mu}{s+\nu}\right)^{s+1}\\
&\quad - \frac{\lambda}{s(s-1)}\left(\frac{s+\nu}{s}\right)^{s}\left(\frac{s+\mu}{s+\nu} - ɥ\right)^{s+1} + \frac{\lambda}{s(s-1)}\left(\frac{s+\nu}{s}\right),
\end{aligned}$$

$$
\begin{aligned}
B^{\mu,\nu}_{s,\lambda}(t^2;ɥ) = \frac{1}{s}\Big[\Big(\frac{s+\nu}{s}\Big)^s + \frac{2\lambda}{s-1}\Big]\Big(ɥ - \frac{\mu}{s+\nu}\Big)\\
&+ \Big(\frac{s+\nu}{s}\Big)\Big[\frac{s-1}{s}\frac{s+\nu}{s} - \frac{4\lambda}{s^2}\Big]\Big(ɥ - \frac{\mu}{s+\nu}\Big)^2\\
&+ \lambda\Big(\frac{s+\nu}{s}\Big)^s\Big[\frac{(s+1)^2}{s^2(s-1)} + \frac{1}{s+1}\Big]\Big(ɥ - \frac{\mu}{s+\nu}\Big)^{s+1}\\
&+ \frac{\lambda}{s^2(s-1)}\Big(\frac{s+\nu}{s}\Big)^s\Big(\frac{s+\mu}{s+\nu} - ɥ\Big)^{s+1} - \frac{\lambda}{s^2(s-1)}\Big(\frac{s+\nu}{s}\Big).
\end{aligned}
$$

Proof. We proof the equalities as follows:

$$
\begin{aligned}
B^{\mu,\nu}_{s,\lambda}(1;ɥ) &= \Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} \tilde{b}^{\mu,\nu}_{s,i}(\lambda;ɥ)\\
&= \Big(\frac{s+\nu}{s}\Big)^s \Big\{ \sum_{i=0}^{s} b^{\mu,\nu}_{s,i}(ɥ) - \frac{\lambda}{s+1} b^{\mu,\nu}_{s+1,1}(ɥ) + \lambda\frac{s-2+1}{s^2-1} b^{\mu,\nu}_{s+1,1}(ɥ)\\
&\quad - \lambda\frac{s-2-1}{s^2-1} b^{\mu,\nu}_{s+1,2}(ɥ) + \lambda\frac{s-4+1}{s^2-1} b^{\mu,\nu}_{s+1,2}(ɥ)\\
&\quad - \lambda\frac{s-4-1}{s^2-1} b^{\mu,\nu}_{s+1,3}(ɥ) + \cdots + \lambda\frac{s-2(s-1)+1}{s^2-1} b^{\mu,\nu}_{s+1,s-1}(ɥ)\\
&\quad - \lambda\frac{s-2(s-1)-1}{s^2-1} b^{\mu,\nu}_{s+1,s}(ɥ) - \frac{\lambda}{s+1} b^{\mu,\nu}_{s+1,s}(ɥ)\Big\}\\
&= \Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} b^{\mu,\nu}_{s,i}(ɥ)\\
&= \Big(\frac{s+\nu}{s}\Big)^s \Big(ɥ - \frac{\mu}{s+\nu} + \frac{s+\mu}{s+\nu} - ɥ\Big)^s\\
&= 1,\\
B^{\mu,\nu}_{s,\lambda}(t;ɥ) &= \Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} \frac{i}{s}\tilde{b}^{\mu,\nu}_{s,i}(\lambda;ɥ)\\
&= \Big(\frac{s+\nu}{s}\Big)^s \Big[\sum_{i=0}^{s-1} \frac{i}{s}\Big\{ b^{\mu,\nu}_{s,i}(ɥ) + \lambda\Big(\frac{s-2i+1}{s^2-1} b^{\mu,\nu}_{s+1,i}(ɥ) - \frac{s-2i-1}{s^2-1} b^{\mu,\nu}_{s+1,i+1}(ɥ)\Big)\Big\}\\
&\quad + b^{\mu,\nu}_{s,s}(ɥ) - \frac{\lambda}{s+1} b^{\mu,\nu}_{s+1,s}(ɥ)\Big]\\
&= \Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} \frac{i}{s} b^{\mu,\nu}_{s,i}(ɥ) + \lambda\Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} \frac{i}{s}\frac{s-2i+1}{s^2-1} b^{\mu,\nu}_{s+1,i}(ɥ)\\
&\quad - \lambda\Big(\frac{s+\nu}{s}\Big)^s \sum_{i=1}^{s-1} \frac{i}{s}\frac{s-2i-1}{s^2-1} b^{\mu,\nu}_{s+1,i+1}(ɥ),
\end{aligned}
$$

where we can examine the as follows:

$$
\Big(\frac{s+\nu}{s}\Big)^s \sum_{i=0}^{s} \frac{i}{s} b^{\mu,\nu}_{s,i}(ɥ) = \Big(\frac{s+\nu}{s}\Big)^s \Big(ɥ - \frac{\mu}{s+\nu}\Big) \sum_{i=0}^{s-1} b^{\mu,\nu}_{s-1,i}(ɥ)
$$

$$
\begin{aligned}
&= \left(\frac{s+\nu}{s}\right)\left(ч - \frac{\mu}{s+\nu}\right),\\
\lambda\left(\frac{s+\nu}{s}\right)^s \sum_{i=0}^{s} \frac{i}{s}\frac{s-2i+1}{s^2-1} b^{\mu,\nu}_{s+1,i}(ч) &= \lambda\left(\frac{s+\nu}{s}\right)^s \left[\frac{1}{s-1}\sum_{i=0}^{s}\frac{i}{s} b^{\mu,\nu}_{s+1,i}(ч) - \frac{2}{s^2-1}\sum_{i=0}^{s}\frac{i^2}{s} b^{\mu,\nu}_{s+1,i}(ч)\right]\\
&= \lambda\frac{s+1}{s(s-1)}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)\sum_{i=0}^{s-1} b^{\mu,\nu}_{s,i}(ч)\\
&\quad - \lambda\frac{2}{s-1}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)^2\sum_{i=0}^{s-2} b^{\mu,\nu}_{s-1,i}(ч)\\
&\quad - \lambda\frac{2}{s(s-1)}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)\sum_{i=0}^{s-1} b^{\mu,\nu}_{s,i}(ч)\\
&= \lambda\frac{s+1}{s(s-1)}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)\left[\left(\frac{s}{s+\nu}\right)^s - \left(ч - \frac{\mu}{s+\nu}\right)^s\right]\\
&\quad - \lambda\frac{2}{s-1}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)^2\left[\left(\frac{s}{s+\nu}\right)^{s-1} - \left(ч - \frac{\mu}{s+\nu}\right)^{s-1}\right]\\
&\quad - \lambda\frac{2}{s(s-1)}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)\left[\left(\frac{s}{s+\nu}\right)^s - \left(ч - \frac{\mu}{s+\nu}\right)^s\right]\\
&= \lambda\frac{1}{s}\left(ч - \frac{\mu}{s+\nu}\right) - \lambda\frac{1}{s}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)^{s+1}\\
&\quad - \lambda\frac{2}{s-1}\left(\frac{s+\nu}{s}\right)\left(ч - \frac{\mu}{s+\nu}\right)^2 + \lambda\frac{2}{s-1}\left(\frac{s+\nu}{s}\right)^s\left(ч - \frac{\mu}{s+\nu}\right)^{s+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&- \lambda\left(\frac{s+\nu}{s}\right)^s \sum_{i=1}^{s-1}\frac{i}{s}\frac{s-2i+1}{s^2-1} b^{\mu,\nu}_{s+1,i+1}(ч)\\
=& - \lambda\left(\frac{s+\nu}{s}\right)^s \frac{1}{s}\left(ч - \frac{\mu}{s+\nu}\right)\sum_{i=1}^{s-1} b^{\mu,\nu}_{s,i}(ч) + \lambda\left(\frac{s+\nu}{s}\right)^s\frac{1}{s(s+1)}\sum_{i=1}^{s-1} b^{\mu,\nu}_{s+1,i+1}(ч)\\
&+ \lambda\left(\frac{s+\nu}{s}\right)^s\frac{2}{s-1}\left(ч - \frac{\mu}{s+\nu}\right)^2\sum_{i=0}^{s-2} b^{\mu,\nu}_{s-1,i}(ч) - \lambda\left(\frac{s+\nu}{s}\right)^s\frac{2}{s(s-1)}\left(ч - \frac{\mu}{s+\nu}\right)\sum_{i=1}^{s-1} b^{\mu,\nu}_{s,i}(ч)\\
&+ \lambda\left(\frac{s+\nu}{s}\right)^s\frac{2}{s(s^2-1)}\sum_{i=1}^{s-1} b^{\mu,\nu}_{s+1,i+1}(ч)\\
=& - \lambda\left(\frac{s+\nu}{s}\right)^s\frac{1}{s}\left(ч - \frac{\mu}{s+\nu}\right)\left[\left(\frac{s}{s+\nu}\right)^s - \left(\frac{s+\mu}{s+\nu} - ч\right)^s - \left(ч - \frac{\mu}{s+\nu}\right)^s\right]\\
&+ \lambda\left(\frac{s+\nu}{s}\right)^s\frac{1}{s(s+1)}\left[\left(\frac{s}{s+\nu}\right)^{s+1} - \left(\frac{s+\mu}{s+\nu} - ч\right)^{s+1} - \left(ч - \frac{\mu}{s+\nu}\right)^{s+1}\right.\\
&\left. - (s+1)\left(ч - \frac{\mu}{s+\nu}\right)\left(\frac{s+\mu}{s+\nu} - ч\right)^s\right]\\
&+ \lambda\left(\frac{s+\nu}{s}\right)^s\frac{2}{s-1}\left(ч - \frac{\mu}{s+\nu}\right)^2\left[\left(\frac{s}{s+\nu}\right)^{s-1} - \left(ч - \frac{\mu}{s+\nu}\right)^{s-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda\left(\frac{s+\nu}{s}\right)^{s}\frac{2}{s(s-1)}\left(ɥ-\frac{\mu}{s+\nu}\right)\left[\left(\frac{s}{s+\nu}\right)^{s}-\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s}-\left(ɥ-\frac{\mu}{s+\nu}\right)^{s}\right]\\
&+\lambda\left(\frac{s+\nu}{s}\right)^{s}\frac{2}{s(s^2-1)}\left[\left(\frac{s}{s+\nu}\right)^{s+1}-\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s+1}-\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1}\right.\\
&\left.-(s+1)\left(ɥ-\frac{\mu}{s+\nu}\right)\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s}\right]\\
=&-\lambda\frac{s+1}{s(s-1)}\left(ɥ-\frac{\mu}{s+\nu}\right)+\lambda\left(\frac{s+\nu}{s}\right)\frac{2}{s-1}\left(ɥ-\frac{\mu}{s+\nu}\right)^{2}-\lambda\frac{1}{s-1}\left(\frac{s+\nu}{s}\right)^{s}\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1}\\
&-\lambda\frac{1}{s(s-1)}\left(\frac{s+\nu}{s}\right)^{s}\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s+1}+\lambda\frac{1}{s(s-1)}\left(\frac{s+\nu}{s}\right),
\end{aligned}
$$

which gives $B^{\mu,\nu}_{s,\lambda}(t;ɥ)$.

Similarly for $g(t)=t^2$, we find

$$
\begin{aligned}
B^{\mu,\nu}_{s,\lambda}(t^2;ɥ)&=\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=0}^{s}\frac{i^2}{s^2}\tilde{b}^{\mu,\nu}_{s,i}(\lambda;ɥ)\\
&=\left(\frac{s+\nu}{s}\right)^{s}\left[\sum_{i=0}^{s-1}\frac{i^2}{s^2}\left\{b^{\mu,\nu}_{s,i}(ɥ)+\lambda\left(\frac{s-2i+1}{s^2-1}b^{\mu,\nu}_{s+1,i}(ɥ)-\frac{s-2i-1}{s^2-1}b^{\mu,\nu}_{s+1,i+1}(ɥ)\right)\right\}\right.\\
&\left.\quad+b^{\mu,\nu}_{s,s}(ɥ)-\frac{\lambda}{s+1}b^{\mu,\nu}_{s+1,s}(ɥ)\right]\\
&=\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=0}^{s}\frac{i^2}{s^2}b^{\mu,\nu}_{s,i}(ɥ)+\lambda\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=0}^{s}\frac{i^2}{s^2}\frac{s-2i+1}{s^2-1}b^{\mu,\nu}_{s+1,i}(ɥ)\\
&\quad-\lambda\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=1}^{s-1}\frac{i^2}{s^2}\frac{s-2i-1}{s^2-1}b^{\mu,\nu}_{s+1,i+1}(ɥ).
\end{aligned}
$$

By simple calculations, we get

$$
\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=0}^{s}\frac{i^2}{s^2}b^{\mu,\nu}_{s,i}(ɥ)=\left(\frac{s+\nu}{s}\right)^{2}\frac{s-1}{s}\left(ɥ-\frac{\mu}{s+\nu}\right)^{2}+\left(\frac{s+\nu}{s}\right)^{s}\frac{1}{s}\left(y-\frac{\mu}{s+\nu}\right),
$$

$$
\begin{aligned}
\lambda\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=0}^{s}\frac{i^2}{s^2}\frac{s-2i+1}{s^2-1}b^{\mu,\nu}_{s+1,i}(ɥ)&=\lambda\frac{1}{s^2}\left(ɥ-\frac{\mu}{s+\nu}\right)+\lambda\left(\frac{s+\nu}{s}\right)\frac{s-5}{s(s-1)}\left(ɥ-\frac{\mu}{s+\nu}\right)^{2}\\
&-\lambda\left(\frac{s+\nu}{s}\right)^{2}\frac{2}{s}\left(ɥ-\frac{\mu}{s+\nu}\right)^{3}+\lambda\left(\frac{s+\nu}{s}\right)^{s}\frac{(s+1)^2}{s^2(s-1)}\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1},
\end{aligned}
$$

and

$$
\begin{aligned}
-\lambda\left(\frac{s+\nu}{s}\right)^{s}\sum_{i=1}^{s-1}\frac{i^2}{s^2}\frac{s-2i-1}{s^2-1}b^{\mu,\nu}_{s+1,i+1}(ɥ)&=\lambda\frac{s+1}{s^2(s-1)}\left(y-\frac{\mu}{s+\nu}\right)-\lambda\left(\frac{s+\nu}{s}\right)\frac{1}{s}\left(y-\frac{\mu}{s+\nu}\right)^{2}\\
&+\lambda\left(\frac{s+\nu}{s}\right)^{2}\frac{2}{s}\left(ɥ-\frac{\mu}{s+\nu}\right)^{3}+\lambda\left(\frac{s+\nu}{s}\right)^{s}\frac{1}{s+1}\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1}\\
&+\lambda\left(\frac{s+\nu}{s}\right)^{s}\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s+1}-\lambda\left(\frac{s+\nu}{s}\right)\frac{1}{s^2(s-1)}.
\end{aligned}
$$

Thus, finally, we get $B^{\mu,\nu}_{s,\lambda}(t^2;ɥ)$. □

Lemma 2.2. *For the operators $B_{s,\lambda}^{\mu,\nu}$, we get the following central moments:*

$$\begin{aligned}
B_{s,\lambda}^{\mu,\nu}(t-ɥ;ɥ) &= \left[\left(\frac{s+\nu}{s}\right)-\frac{2}{s(s-1)}-1\right]ɥ \\
&+\frac{1}{s(s-1)}\left(\frac{s+\nu}{s}\right)^{s}\left[\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1}-\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s+1}\right] \\
&+\frac{1}{s(s-1)}\left(\frac{s+\nu}{s}\right)+\frac{2\mu}{s(s-1)(s+\nu)}-\frac{\mu}{s}, \\
B_{s,\lambda}^{\mu,\nu}\left((t-ɥ)^2;ɥ\right) &= \left[\frac{1}{s}\left\{\left(\frac{s+\nu}{s}\right)^{s}+\frac{2}{s-1}\right\}-2\left(\frac{s+\nu}{s}-\frac{2ɥ}{s(s-1)}\right)\right]\left(ɥ-\frac{\mu}{s+\nu}\right) \\
&+\left(\frac{s+\nu}{s}\right)^{s}\left[\left(\frac{(s+1)^2}{s^2(s-1)}+\frac{1}{s+1}+\frac{ɥ}{s(s-1)}\right)\right]\left(ɥ-\frac{\mu}{s+\nu}\right)^{s+1} \\
&+\left(\frac{s+\nu}{s}\right)\left[\frac{s-1}{s}\frac{s+\nu}{s}-\frac{4}{s^2}\right]\left(ɥ-\frac{\mu}{s+\nu}\right)^{2}+ɥ^2 \\
&+\left(\frac{s+\nu}{s}\right)^{s}\left[\left(\frac{1}{s^2(s-1)}-\frac{ɥ}{s(s-1)}\right)\right]\left(\frac{s+\mu}{s+\nu}-ɥ\right)^{s+1} \\
&+\left(\frac{s+\nu}{s}\right)\left[\left(-\frac{1}{s^2(s-1)}+\frac{ɥ}{s(s-1)}\right)\right].
\end{aligned}$$

3. Convergence of operators $B_{s,\lambda}^{\mu,\nu}$

We use the properties of the modulus of smoothness in this section of the paper so that we can obtain convergence of the sequence of operators of $B_{s,\lambda}^{\mu,\nu}$ defined by (2.2). We can determine the maximum oscillation of ϕ for any $\delta > 0$ by taking $\omega(f;\delta)$, which is the modulus of smoothness of the function ϕ of order one satisfying $\lim_{\delta\to 0+}\omega(\phi;\delta) = 0$, and

$$\omega(\phi;\delta) = \sup_{|ɥ_1-ɥ_2|\le\delta} |\,\phi(ɥ_1)-\phi(ɥ_2)\,|; \quad y_1, ɥ_2 \in [0,1], \tag{3.1}$$

$$|\,\phi(ɥ_1)-\phi(ɥ_2)\,| \le \left(1+\frac{|\,ɥ_1-ɥ_2\,|}{\delta}\right)\omega(\phi;\delta). \tag{3.2}$$

Theorem 3.1. *[25] Let $\{P\}_{s\ge 1}$ be any sequence of positive linear operators defined in $C[u,v] \to C[x_1,x_2]$ such that $[ɥ_1,ɥ_2] \subseteq [u,v]$ then*

(1) For all $\phi \in C[u,v]$ and $ɥ \in [x_1,x_2]$, it follows that:

$$\begin{aligned}
|P_s(\phi;ɥ)-\phi(ɥ)| &\le |\phi(ɥ)||P_s(1;ɥ)-1| \\
&\quad + \{P_s(1;ɥ)+\frac{1}{\delta}\sqrt{P_s((t-ɥ)^2;ɥ)}\sqrt{P_s(1;ɥ)}\}\omega(\phi;\delta),
\end{aligned}$$

(2) for all $\varphi' \in C[u,v]$ and $ɥ \in [x_1,x_2]$, it follows that:

$$\begin{aligned}
|P_s(\varphi;ɥ)-\varphi(ɥ)| &\le |\varphi(ɥ)||P_s(1;ɥ)-1| + |\varphi'(ɥ)||P_s(t-ɥ;ɥ)| \\
&\quad + P_s((t-ɥ)^2;ɥ)\{\sqrt{P_s(1;ɥ)}+\frac{1}{\delta}\sqrt{P_s((t-ɥ)^2;ɥ)}\}\omega(\varphi';\delta).
\end{aligned}$$

Theorem 3.2. *For any $\varphi \in C[0,1]$, the set of all continuous functions on $[0,1]$ and ɥ $\in [0,1]$, the operators $B^{\mu,\nu}_{s,\lambda}$ are defined by* (2.2) *satisfying:*

$$|B^{\mu,\nu}_{s,\lambda}(\varphi; ɥ) - \varphi(ɥ)| \;\leq\; 2\omega\left(\varphi;\ \sqrt{\delta^{\mu,\nu}_{s,\lambda}(ɥ)}\right),$$

where $\delta^{\mu,\nu}_{s,\lambda}(ɥ) = B^{\mu,\nu}_{s,\lambda}((t-ɥ)^2; ɥ)$.

Proof. By taking into account (1) from Theorem 3.1 and using Lemmas 2.1 and 2.2 we are able to prove the inequality

$$\begin{aligned}|B^{\mu,\nu}_{s,\lambda}(\varphi; ɥ) - \varphi(ɥ)| \leq |\varphi(ɥ)||B^{\mu,\nu}_{s,\lambda}(1; ɥ) - 1| + \Big\{ & B^{\mu,\nu}_{s,\lambda}(1; ɥ) \\ & + \frac{1}{\delta}\sqrt{B^{\mu,\nu}_{s,\lambda}((t-ɥ)^2; ɥ)}\sqrt{B^{\mu,\nu}_{s,\lambda}(1; ɥ)}\Big\}\omega(\varphi; \delta).\end{aligned}$$

We suppose $\delta = \sqrt{B^{\mu,\nu}_{s,\lambda}((t-ɥ)^2; ɥ)} = \sqrt{\delta^{\mu,\nu}_{s,\lambda}(ɥ)}$, which is our required result. □

Theorem 3.3. *Let ɥ $\in [0,1]$, then for any $\varphi' \in C[0,1]$ operators $B^{\mu,\nu}_{s,\lambda}$ are as follows:*

$$|B^{\mu,\nu}_{s,\lambda}(\varphi; ɥ) - \varphi(ɥ)| \;\leq\; |\varphi'(ɥ)|\zeta^{\mu,\nu}_{s,\lambda}(ɥ) + 2\,\delta^{\mu,\nu}_{s,\lambda}(ɥ)\,\omega\Big(\varphi';\ \sqrt{\delta^{\mu,\nu}_{s,\lambda}(ɥ)}\Big),$$

where $\zeta^{\mu,\nu}_{s,\lambda}(ɥ) = \max_{ɥ\in[0,1]}\left|B^{\mu,\nu}_{s,\lambda}((t-ɥ); ɥ)\right|$ and $\delta^{\mu,\nu}_{s,\lambda}(ɥ)$ are defined by Theorem 3.2.

Proof. If we consider (2) from Theorem 3.1 and Lemmas 2.1 and 2.2, then it is easy to get

$$\begin{aligned}|B^{\mu,\nu}_{s,\lambda}(\varphi; ɥ) - \varphi(ɥ)| &\leq |\varphi(ɥ)||B^{\mu,\nu}_{s,\lambda}(1; ɥ) - 1| + |\varphi'(ɥ)||B^{\mu,\nu}_{s,\lambda}(t-ɥ; ɥ)| \\ &\quad + B^{\mu,\nu}_{s,\lambda}((t-ɥ)^2; ɥ)\Big\{1 + \frac{\sqrt{B^{\mu,\nu}_{s,\lambda}((t-ɥ)^2; ɥ)}}{\delta}\Big\}\omega(\varphi'; \delta) \\ &\leq |\varphi'(ɥ)|\zeta^{\mu,\nu}_{s,\lambda}(ɥ) + 2\,\delta^{\mu,\nu}_{s,\lambda}(ɥ)\,\omega\Big(\varphi';\ \sqrt{\delta^{\mu,\nu}_{s,\lambda}(ɥ)}\Big),\end{aligned}$$

where we take $\zeta^{\mu,\nu}_{s,\lambda}(ɥ) = \max_{ɥ\in[0,1]}\left|B^{\mu,\nu}_{s,\lambda}((t-ɥ); ɥ)\right|$. □

The next step is to estimate some local direct approximations of our new operators $B^{\mu,\nu}_{s,\lambda}$ by using a Lipschitz-type maximal function, which we assume to be $Lip^{\vartheta}_{\mathcal{M}}$. Thus, for any $0 < \vartheta \leq 1$, the Lipschitz-type maximal function $Lip^{\vartheta}_{\mathcal{M}}$ is defined in the form of any positive real parameters β_1, β_2 (see [26] for more details) such that:

$$Lip^{\vartheta}_{\mathcal{M}} = \Big\{\Phi \in C_B[0,1] : |\Phi(t) - \Phi(ɥ)| \leq \mathcal{M}\frac{|t-ɥ|^{\vartheta}}{(\beta_1 ɥ^2 + \beta_2 ɥ + t)^{\frac{\vartheta}{2}}};\ ɥ,\ t \in [0,1]\Big\},$$

where $C_B[0,1]$ is the set of all continuous and bounded functions on $[0,1]$ and $\mathcal{M}$ is any positive constant.

Theorem 3.4. *For all $\Phi \in Lip_{\mathcal{M}}^{\vartheta}$, operators $B_{s,\lambda}^{\mu,\nu}$ satisfy*

$$|B_{s,\lambda}^{\mu,\nu}(\Phi;\text{ɥ}) - \Phi(\text{ɥ})| \leq \mathcal{M}\Big(\frac{\delta_{s,\lambda}^{\mu,\nu}(\text{ɥ})}{(\beta_1\text{ɥ}^2+\beta_2\text{ɥ})}\Big)^{\frac{\vartheta}{2}},$$

where $\delta_{s,\lambda}^{\mu,\nu}(\text{ɥ})$ is given by Theorem 3.2.

Proof. We suppose that the function $\Phi \in Lip_{\mathcal{M}}^{\vartheta}$ is valid for all $0 < \vartheta \leq 1$. We would need to verify first that the results of Theorem 3.4 are valid for $\vartheta = 1$. Therefore, it is easy to get the result for any $\beta_1,\ \beta_2 \geq 0$ such that $(\beta_1\text{ɥ}^2+\beta_2\text{ɥ}+t)^{-1/2} \leq (\beta_1\text{ɥ}^2+\beta_2\text{ɥ})^{-1/2}$. Consider the Cauchy-Schwarz inequality, thus we have

$$\begin{aligned}|B_{s,\lambda}^{\mu,\nu}(\Phi;\text{ɥ}) - \Phi(\text{ɥ})| &\leq |B_{s,\lambda}^{\mu,\nu}(|\Phi(t)-\Phi(\text{ɥ})|;\text{ɥ})| + \Phi(\text{ɥ})\,|(1;\text{ɥ})-1| \\ &\leq B_{s,\lambda}^{\mu,\nu}\Big(\frac{|t-\text{ɥ}|}{(\beta_1\text{ɥ}^2+\beta_2\text{ɥ}+t)^{\frac{1}{2}}};\text{ɥ}\Big) \\ &\leq \mathcal{M}(\beta_1\text{ɥ}^2+\beta_2\text{ɥ})^{-1/2}B_{s,\lambda}^{\mu,\nu}(|t-\text{ɥ}|;\text{ɥ}) \\ &\leq \mathcal{M}\sqrt{\frac{B_{s,\lambda}^{\mu,\nu}((t-\text{ɥ})^2;\text{ɥ})}{\beta_1\text{ɥ}^2+\beta_2\text{ɥ}}}.\end{aligned}$$

As a result, we conclude that the statement is correct for $\vartheta = 1$. Next, we'll check to see if the statement is also true when $\vartheta \in (0,1)$. We apply the monotonicity property to the operators $B_{s,\lambda}^{\mu,\nu}$ and use the Hölder's inequality to obtain

$$\begin{aligned}\Big|B_{s,\lambda}^{\mu,\nu}(\Phi;\text{ɥ}) - \Phi(\text{ɥ})\Big| &\leq B_{s,\lambda}^{\mu,\nu}(\Big|\Phi(t)-\Phi(\text{ɥ})\Big|;\text{ɥ}) \\ &\leq \Big(B_{s,\lambda}^{\mu,\nu}(\Big|\Phi(t)-\Phi(\text{ɥ})\Big|^{\frac{2}{\vartheta}};\text{ɥ})\Big)^{\frac{\vartheta}{2}}\Big(B_{s,\lambda}^{\mu,\nu}(1;\text{ɥ})\Big)^{\frac{2-\vartheta}{2}} \\ &\leq \mathcal{M}\Big\{\frac{B_{s,\lambda}^{\mu,\nu}((t-\text{ɥ})^2;\text{ɥ})}{t+\beta_1\text{ɥ}^2+\beta_2 y}\Big\}^{\frac{\vartheta}{2}} \\ &\leq \mathcal{M}(\beta_1\text{ɥ}^2+\beta_2\text{ɥ})^{-\vartheta/2}\Big\{B_{s,\lambda}^{\mu,\nu}\Big((t-\text{ɥ})^2;\text{ɥ}\Big)\Big\}^{\frac{\vartheta}{2}} \\ &\leq \mathcal{M}(\beta_1\text{ɥ}^2+\beta_2\text{ɥ})^{-\vartheta/2}\Big(B_{s,\lambda}^{\mu,\nu}(t-\text{ɥ})^2;\text{ɥ})\Big)^{\frac{\vartheta}{2}} \\ &= \mathcal{M}\Big(\frac{\delta_{s,\lambda}^{\mu,\nu}(\text{ɥ})}{(\beta_1\text{ɥ}^2+\beta_2\text{ɥ})}\Big)^{\frac{\vartheta}{2}}.\end{aligned}$$

The statement is valid when $0 < \vartheta < 1$, thus we complete the proof. □

On the other hand, we employ the Lipschitz maximum function to establish another another local approximation property for the operators of $B_{s,\lambda}^{\mu,\nu}$. Assume $\Phi \in C_B[0,1]$ and $t, y \in [0,1]$ have the same class of all Lipschitz type maximal functions (see [27]).

$$\omega_{\vartheta}(\Phi;\text{ɥ}) = \sup_{t\neq\text{ɥ},\ t\in[0,1]} \frac{|\ \Phi(t)-\Phi(\text{ɥ})\ |}{|\ t-\text{ɥ}\ |^{\vartheta}}, \tag{3.3}$$

where $0 < \vartheta \leq 1$.

Theorem 3.5. *For all* $\Phi \in C_B[0,1]$ *and* $\upsilon \in [0,1]$*, operators* $B^{\mu,\nu}_{s,\lambda}$ *satisfy*

$$\left|B^{\mu,\nu}_{s,\lambda}(\Phi;\upsilon) - \Phi(\upsilon)\right| \leq \left(\delta^{\mu,\nu}_{s,\lambda}(\upsilon)\right)^{\frac{\vartheta}{2}} \omega_{\vartheta}(\Phi;\upsilon),$$

where $\omega^*_{\vartheta}(\Phi;\upsilon)$ *is defined by* (3.3) *and* $\delta^{\mu,\nu}_{s,\lambda}(\upsilon)$ *is obtained by Theorem 3.2.*

Proof. One can write by taking the the Hölder inequality,

$$\begin{aligned}\left|B^{\mu,\nu}_{s,\lambda}(\Phi;\upsilon) - \Phi(\upsilon)\right| &\leq B^{\mu,\nu}_{s,\lambda}\left(|\Phi(t) - \Phi(\upsilon)|;\upsilon\right)\\ &\leq \omega_{\vartheta}(\Phi;\upsilon) \mid B^{\mu,\nu}_{s,\lambda}\left(|t-\upsilon|^{\vartheta};\upsilon\right)\\ &\leq \omega_{\vartheta}(\Phi;\upsilon)\left(B^{\mu,\nu}_{s,\lambda}(1;\upsilon)\right)^{\frac{2-\vartheta}{2}}\left(B^{\mu,\nu}_{s,\lambda}(|t-\upsilon|^2;\upsilon)\right)^{\frac{\vartheta}{2}}\\ &= \omega_{\vartheta}(\Phi;\upsilon)\left(B^{\mu,\nu}_{s,\lambda}\left((t-\upsilon)^2;\upsilon\right)\right)^{\frac{\vartheta}{2}},\end{aligned}$$

where the set of all continuously bounded functions on $[0,1]$ was indicated by $C_B[0,1]$. The anticipated outcome now completes the proof. □

4. Some direct theorems of operators $B^{\mu,\nu}_{s,\lambda}$

For our new operators $B^{\mu,\nu}_{s,\lambda}$ defined by Eq (2.2) this section can provide some direct approximation findings in the space of Peetre's K-functional. Simply, for $\Phi \in C[0,1]$, we define the fundamental concept of Peetre's K-functional supposing $K_p(\Phi;\delta)$:

Thus for any $\delta > 0$, the Peetre's $K-$functional is defined by

$$K_p(\Phi;\delta) = \inf\left\{(\| \Phi - \varphi \|_{C[0,1]} + \delta \| \varphi'' \|_{C[0,1]}) : \varphi, \varphi', \varphi'' \in C[0,1]\right\}. \tag{4.1}$$

From [28], for an absolute positive constant C we have

$$\begin{aligned}&K_p(\Phi;\delta) \leq C\omega_{\delta}(\Phi;\sqrt{\delta}),\ \delta > 0,\\ &K_p(\Phi;\delta) \leq C\{\omega_{\delta}(\Phi;\sqrt{\delta}) + \min(1,\delta)\|\Phi\|_{C[0,1]}\},\end{aligned}$$

where $\omega_{\delta}(\Phi;\delta)$ is defined for the modulus of smoothness in order two and given as:

$$\omega_{\delta}(\Phi;\delta) = \sup_{0<\theta<\delta}\ \sup_{\upsilon\in[0,1])} |\Phi(\upsilon + 2\theta) - 2\Phi(\upsilon+\theta) + \Phi(\upsilon)|. \tag{4.2}$$

Theorem 4.1. *For an arbitrary* $\Psi \in C[0,1]$*, let's define the auxiliary operators* $A^{\mu,\nu}_{s,\lambda}$ *such that*

$$A^{\mu,\nu}_{s,\lambda}(\Psi;\upsilon) = B^{\mu,\nu}_{s,\lambda}(\Psi;\upsilon) + \Psi(\upsilon) - \Psi\Big(B^{\mu,\nu}_{s,\lambda}(\Phi;\upsilon)\Big), \tag{4.3}$$

then, for every $\Phi \in C[0,1]$ *we get that*

$$\left|B^{\mu,\nu}_{s,\lambda}(\Phi;\upsilon) - \Phi(\upsilon)\right| \leq C\omega_{\delta}\left(\Phi; \frac{\sqrt{\delta^{\mu,\nu}_{s,\lambda}(\upsilon) + \left(\tau^{\mu,\nu}_{s,\lambda}(\upsilon)\right)^2}}{2}\right) + \omega_{\vartheta}\Big(\Phi;\tau^{\mu,\nu}_{s,\lambda}(\upsilon)\Big),$$

where $\tau^{\mu,\nu}_{s,\lambda}(\upsilon) = B^{\mu,\nu}_{s,\lambda}((t-\upsilon);\upsilon)$ *and* $\delta^{\mu,\nu}_{s,\lambda}(\upsilon)$ *is defined by Theorem 3.2.*

Proof. When $i = 0, 1$ and $\Psi_i = t^i$ are taken into consideration, it is simple to prove that $A^{\mu,\nu}_{s,\lambda}(\Psi_0; \text{ч}) = 1$ and

$$A^{\mu,\nu}_{s,\lambda}(\Psi_1; \text{ч}) = B^{\mu,\nu}_{s,\lambda}(\Psi_1; \text{ч}) + \text{ч} - B^{\mu,\nu}_{s,\lambda}(\Psi_1; \text{ч}) = \text{ч}.$$

We can deduce the equality from the Taylor series expression

$$\Lambda(t) = \Lambda(\text{ч}) + (t - \text{ч})\Lambda'(\text{ч}) + \int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta, \ \Lambda \in C^2[0, 1]. \tag{4.4}$$

Apply $A^{\mu,\nu}_{s,\lambda}$, and then

$$\begin{aligned} A^{\mu,\nu}_{s,\lambda}(\Lambda; \text{ч}) - \Lambda(\text{ч}) &= \Lambda'(\text{ч})A^{\mu,\nu}_{s,\lambda}(t - \text{ч}; \text{ч}) + A^{\mu,\nu}_{s,\lambda}\Big(\int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta; \text{ч}\Big) \\ &= A^{\mu,\nu}_{s,\lambda}\Big(\int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta; \text{ч}\Big) \\ &= B^{\mu,\nu}_{s,\lambda}\Big(\int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta; \text{ч}\Big) + \int_{\text{ч}}^{\text{ч}} (\text{ч} - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta; \text{ч} \\ &\quad - \int_{\text{ч}}^{B^{\mu,\nu}_{s,\lambda}(t;\text{ч})} \Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч}) - \vartheta\Big)\Lambda''(\vartheta)\mathrm{d}\vartheta, \\ | A^{\mu,\nu}_{s,\lambda}(\Lambda; \text{ч}) - \Lambda(\text{ч}) | &\le \left|B^{\mu,\nu}_{s,\lambda}\Big(\int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta; \text{ч}\Big)\right| \\ &\quad + \left| \int_{\text{ч}}^{B^{\mu,\nu}_{s,\lambda}(t;\text{ч})} \Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч}) - \vartheta\Big)\Lambda''(\vartheta)\mathrm{d}\vartheta\right|. \end{aligned}$$

We know the inequality

$$\left| \int_{\text{ч}}^{t} (t - \vartheta)\Lambda''(\vartheta)\mathrm{d}\vartheta\right| \le (t - \text{ч})^2 \parallel \Lambda'' \parallel$$

and

$$\left| \int_{\text{ч}}^{B^{\mu,\nu}_{s,\lambda}(t;\text{ч})} \Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч}) - \vartheta\Big)\Lambda''(\vartheta)\mathrm{d}\vartheta\right| \le \Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч}) - \text{ч}\Big)^2 \parallel \Lambda'' \parallel .$$

Thus we get

$$| A^{\mu,\nu}_{s,\lambda}(\Lambda; \text{ч}) - \Lambda(\text{ч}) |\le \Big\{ B^{\mu,\nu}_{s,\lambda}\Big((t - \text{ч})^2; \text{ч}\Big) + \Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч}) - \text{ч}\Big)^2 \Big\} \parallel \Lambda'' \parallel .$$

On the other hand we deduce that

$$\parallel B^{\mu,\nu}_{s,\lambda}(\Psi; \text{ч}) \parallel\le\parallel \Psi \parallel,$$

and

$$| A^{\mu,\nu}_{s,\lambda}(\Psi; \text{ч}) |\le| B^{\mu,\nu}_{s,\lambda}(\Psi; \text{ч}) | + | \Psi(\text{ч}) | + \left|\Psi\Big\{B^{\mu,\nu}_{s,\lambda}(\Psi; \text{ч})\Big\}\right| \le 3 \parallel \Psi \parallel . \tag{4.5}$$

By accounting for (4.4) and (4.5) we arrive at

$$\begin{aligned} \left|B^{\mu,\nu}_{s,\lambda}(\Phi; \text{ч}) - \Phi(\text{ч})\right| &\le \left|A^{\mu,\nu}_{s,\lambda}(\Phi - \Lambda; \text{ч}) - (\Phi - \Lambda)(\text{ч})\right| \\ &\quad + \left|A^{\mu,\nu}_{s,\lambda}(\Lambda; \text{ч}) - \Lambda(\text{ч})\right| + \left|\Phi(\text{ч}) - \Phi\Big(B^{\mu,\nu}_{s,\lambda}(t; \text{ч})\Big)\right| \end{aligned}$$

$$\leq 4\,\|\Phi - \Lambda\| + \omega_{\vartheta}\Big(\Phi; B_{s,\lambda}^{\mu,\nu}\left((t-ɥ);ɥ\right)\Big)$$
$$+ \left\{B_{s,\lambda}^{\mu,\nu}\left((t-ɥ)^2;ɥ\right) + \|\ \Lambda''\ \| \left(B_{s,\lambda}^{\mu,\nu}(t-ɥ;ɥ)\right)^2\right\}.$$

Taking the infimum over all $\Lambda \in C^2[0,1]$ and applying Peetre's K-functional properties, we get

$$\left|B_{s,\lambda}^{\mu,\nu}(\Phi;ɥ) - \Phi(ɥ)\right| \leq 4K_p\Bigg(\Phi; \frac{\delta_{s,\lambda}^{\mu,\nu}(ɥ) + \left(B_{s,\lambda}^{\mu,\nu}\left((t-ɥ);ɥ\right)\right)^2}{4} + \omega_{\vartheta}\Big(\Phi; B_{s,\lambda}^{\mu,\nu}\left((t-ɥ);ɥ\right)\Big)$$
$$\leq C\omega_{\delta}\Bigg(\Phi; \frac{\sqrt{\delta_{s,\lambda}^{\mu,\nu}(ɥ) + \left(B_{s,\lambda}^{\mu,\nu}\left((t-ɥ);ɥ\right)\right)^2}}{2}\Bigg) + \omega_{\vartheta}\Big(\Phi; B_{s,\lambda}^{\mu,\nu}\left(t-ɥ;ɥ\right)\Big).$$

As a result, we have our desired proof. □

5. Approximation to operators $B_{s,\lambda}^{\mu,\nu}$ in weighted spaces

In this section, we explore the approximation in weighted space, which is the well-known Korovkin's type theorems, for our new operators $B_{s,\lambda}^{\mu,\nu}$. Remember that for each $\varphi \in C[0,1]$, the equipped normed function on $\varphi(ɥ)$ is given by $\|\ \varphi\ \|_{C[0,1]} = \sup_{ɥ\in[0,1]} |\varphi(ɥ)|$ for the real valued continuous function $\varphi(ɥ)$.

Theorem 5.1. *[29, 30] Any positive linear operator sequences K_s that act on $[a,b]$ such that $\lim\limits_{s\to\infty} K_S\left(t^i;ɥ\right) = ɥ^i$, are uniformly on $[a,b]$ for all $i = 0,1,2$. Then for every $\varphi \in C[a,b]$, the operators $\lim\limits_{s\to\infty} K_s(\varphi) = \varphi$ uniformly converge for any compact subset of $[a,b]$.*

Theorem 5.2. *For every $\varphi \in C[0,1]$ and $y \in C[0,1]$, the sequence of positive operators $B_{s,\lambda}^{\mu,\nu}$ uniformly convergence on each compact subset of $[0,1]$ such that*

$$B_{s,\lambda}^{\mu,\nu}(\varphi;ɥ) \Rightarrow \varphi(ɥ),$$

where $\Rightarrow$ stands for uniformly.

Proof. In order to demonstrate the convergence of our new operators sufficiently so that we may utilize the condition of uniformity for operators $B_{s,\lambda}^{\mu,\nu}$ provided by Korovkin's theorem,

$$\lim_{s\to\infty} B_{s,\lambda}^{\mu,\nu}\left(t^i;ɥ\right) = ɥ^i, \quad i = 0,1,2, \quad s\to\infty\}.$$

If $s \to \infty$ we deduce that $B_{s,\lambda}^{\mu,\nu}(1;ɥ) = 1$ and

$$\lim_{s\to\infty} B_{s,\lambda}^{\mu,\nu}(t;ɥ) = ɥ, \quad \lim_{s\to\infty} B_{s,\lambda}^{\mu,\nu}(t^2;ɥ) = ɥ^2.$$

This is enough to get $B_{s,\lambda}^{\mu,\nu}(\varphi;ɥ) \Rightarrow \varphi(ɥ)$. □

Theorem 5.3. *[31, 32] For the operator $\{P_s\}_{s\geq1}$, which acts $C[0,1] \to C[0,1]$ satisfying $\lim_{s\to\infty} \|P_m(t^i) - ɥ^i\|_{C[0,1]} = 0,\ i = 0,1,2$ then $f \in C[0,1],\ s \in \mathbb{N}$ it follows that*

$$\lim_{s\to\infty} \|P_s(f) - f\|_{C[0,1]} = 0.$$

Theorem 5.4. *Assume* $B_{s,\lambda}^{\mu,\nu}$ *acts from* $C[0,1]$ *to* $C[0,1]$ *and has the property* $\lim_{s\to\infty}\|B_{s,\lambda}^{\mu,\nu}(t^i)-y^i\|_{C[0,1]}=0$*. Then, for all* $\varphi\in C[0,1]$*, we get the equality*

$$\lim_{s\to\infty}\|\, B_{s,\lambda}^{\mu,\nu}(\varphi)-\varphi\,\|_{C[0,1]}=0.$$

Proof. When we consider Theorem 5.3 and Korovkin's Theorem, it is simple to demonstrate that

$$\lim_{s\to\infty}\|\, B_{s,\lambda}^{\mu,\nu}(t^i)-y^i\,\|_{C[0,1]}=0,\quad i=0,1,2.$$

For $i=0$, we can easily deduce that from the Lemma 2.1, $\|\, B_{s,\lambda}^{\mu,\nu}(t^0)-t^0\,\|_{C[0,1]}=\sup_{\text{ɥ}\in[0,1]}\left|B_{s,\lambda}^{\mu,\nu}(1;\text{ɥ})-1\right|=0.$ For $i=1$, it is easy to obtain

$$\begin{aligned}\|\, B_{s,\lambda}^{\mu,\nu}(t)-\text{ɥ}\,\|_{C[0,1]} &= \sup_{\text{ɥ}\in[0,1]}\left|B_{s,\lambda}^{\mu,\nu}(t;\text{ɥ})-\text{ɥ}\right|\\ &= \sup_{\text{ɥ}\in[0,1]}\tau_{s,\lambda}^{\mu,\nu}(\text{ɥ}),\end{aligned}$$

since $s\to\infty$, then we deduce that $\|\, B_{s,\lambda}^{\mu,\nu}(t)-\text{ɥ}\,\|_{C[0,1]}\to 0$. Similarly if $i=2$, we have

$$\|\, B_{s,\lambda}^{\mu,\nu}(t^2)-\text{ɥ}^2\,\|_{C[0,1]} = \sup_{\text{ɥ}\in[0,1])}\left|B_{s,\lambda}^{\mu,\nu}(t^2;\text{ɥ})-\text{ɥ}^2\right|,$$

which gives $\|\, B_{s,\lambda}^{\mu,\nu}(t^2)-\text{ɥ}^2\,\|_{C[0,1]}\to 0$ whenever $s\to\infty$. These observations help us to acquire desired results. □

6. Voronovskaja type theorems

We begin the quantitative Voronovskaja-type approximation theorem for our new operators $B_{s,\lambda}^{\mu,\nu}$, which is primarily driven by [8, 33]. The definition of the modulus of smoothness that was covered in the preceding section is used for this purpose. This smoothness modulus is described by:

$$\omega_\chi(\varphi,\delta):=\sup_{0<|\rho|\le\delta}\left\{\left|f\left(\text{ɥ}+\frac{\rho\chi(\text{ɥ})}{2}\right)-\varphi\left(\text{ɥ}-\frac{\rho\chi(\text{ɥ})}{2}\right)\right|,\text{ɥ}\pm\frac{\rho\chi(\text{ɥ})}{2}\in[0,1]\right\}.$$

Here $\varphi\in C[0,]$ and $\chi(\text{ɥ})=(\text{ɥ}-\text{ɥ}^2)^{1/2}$, and the related Peetre's K-functional is known as

$$K_\chi(\varphi,\delta)=\inf_{g\in\omega_\chi[0,1]}\{\|\varphi-g\|+\delta\|\chi g'\|:g'\in C[0,1],\delta>0\},$$

where $\omega_\chi[0,]=\{g:\ g'\in C^*[0,1],\ \ \|\,\chi g'\,\|<\infty\}$ and $C^*[0,1]$ as for the set of absolutely continuous functions on intervals $[a,b]\subset[0,1]$. There exists a positive constant M such that

$$K_\chi(f,\delta)\le M\ \omega_\chi(f,\delta).$$

Theorem 6.1. *For all* $\varphi,\varphi',\varphi''\in C[0,1]$*, it follows that*

$$\left|B_{s,\lambda}^{\mu,\nu}(\varphi;\text{ɥ})-\varphi(\text{ɥ})-\tau_{s,\lambda}^{\mu,\nu}(\text{ɥ})\varphi'(\text{ɥ})-\frac{\delta_{s,\lambda}^{\mu,\nu}(\text{ɥ})+1}{2}\varphi''(\text{ɥ})\right|\le\frac{C}{s}\chi^2(\text{ɥ})\omega_\chi\left(\varphi'',\frac{1}{\sqrt{s}}\right),$$

where $y\in[0,1]$*,* $C>0$ *is a constant,* $\tau_{s,\lambda}^{\mu,\nu}(\text{ɥ})=B_{s,\lambda}^{\mu,\nu}(t-\text{ɥ};\text{ɥ})$ *and* $\delta_{s,\lambda}^{\mu,\nu}=B_{s,\lambda}^{\mu,\nu}\left((t-\text{ɥ})^2;\text{ɥ}\right)$ *are defined by Lemma 2.2.*

Proof. For any $\varphi \in C[0,1]$ we consider the Taylor series expansion as follows:

$$\varphi(t) - \varphi(\text{ч}) - \varphi'(\text{ч})(t - \text{ч}) = \int_{\text{ч}}^{t} \varphi''(\theta)(t-\theta)\mathrm{d}\theta,$$

then it is easy to get

$$\varphi(t) - \varphi(\text{ч}) - (t-\text{ч})\varphi'(\text{ч}) - \frac{\varphi''(\text{ч})}{2}((t-\text{ч})^2+1) \leq \int_{\text{ч}}^{t} (t-\theta)[\varphi''(\theta) - \varphi''(\text{ч})]\mathrm{d}\theta. \tag{6.1}$$

Therefore, (6.1) give us,

$$\begin{aligned}&\left|B_{s,\lambda}^{\mu,\nu}(\varphi;\text{ч}) - \varphi(\text{ч}) - B_{s,\lambda}^{\mu,\nu}(t-\text{ч};\text{ч})\varphi'(\text{ч}) - \frac{\varphi''(\text{ч})}{2}(B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^2;\text{ч}) + B_{s,\lambda}^{\mu,\nu}(1;\text{ч}))\right| \\ &\leq B_{s,\lambda}^{\mu,\nu}\Big(\Big|\int_{\text{ч}}^{t} |\varphi''(\theta)|t-\theta| - \varphi''(\text{ч})|\,\mathrm{d}\theta\Big|;\text{ч}\Big).\end{aligned} \tag{6.2}$$

From the right hand side of equality (6.2) we can estimate:

$$\left|\int_{\text{ч}}^{t} |t-\theta|\,|\varphi''(\theta) - \varphi''(\text{ч})|\,\mathrm{d}\theta\right| \leq 2 \parallel \varphi'' - g \parallel (t-\text{ч})^2 + 2 \parallel \chi g' \parallel \chi^{-1}(\text{ч})|t-\text{ч}|^3, \tag{6.3}$$

where $\varphi \in \omega_\chi[0,1]$. There exists constant $C > 0$ such that

$$B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^2;\text{ч}) \leq \frac{C}{2s}\chi^2(\text{ч}) \qquad \text{and} \qquad B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^4;\text{ч}) \leq \frac{C}{2s^2}\chi^4(\text{ч}). \tag{6.4}$$

Using the Cauchy-Schwarz inequality, we can conclude that

$$\begin{aligned}&\left|B_{s,\lambda}^{\mu,\nu}(\varphi;\text{ч}) - \varphi(\text{ч}) - \varphi'(\text{ч})B_{s,\lambda}^{\mu,\nu}(t-\text{ч};\text{ч}) - \frac{\varphi''(\text{ч})}{2}(B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^2;\text{ч}) + B_{s,\lambda}^{\mu,\nu}(1;\text{ч}))\right| \\ &\leq 2 \parallel \varphi'' - g \parallel B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^2;\text{ч}) + 2 \parallel \chi(\text{ч})g' \parallel \chi^{-1}(\text{ч})B_{s,\lambda}^{\mu,\nu}(|t-\text{ч}|^3;\text{ч}) \\ &\leq \frac{C}{s}\chi^2(\text{ч}) \parallel \varphi'' - g \parallel +2 \parallel \chi(\text{ч})g' \parallel \chi^{-1}(\text{ч})\{B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^2;\text{ч})\}^{1/2}\{B_{s,\lambda}^{\mu,\nu}((t-\text{ч})^4;\text{ч})\}^{1/2} \\ &\leq \frac{C}{s}\chi^2(\text{ч})\Big\{ \parallel \varphi'' - g \parallel + s^{-1/2} \parallel \chi(\text{ч})g' \parallel \Big\}.\end{aligned}$$

Taking the infimum over all $g \in \omega_\chi[0,1]$, we deduce that

$$\left|B_{s,\lambda}^{\mu,\nu}(\varphi;\text{ч}) - \varphi(\text{ч}) - \tau_{s,\lambda}^{\mu,\nu}(\text{ч})\varphi'(\text{ч}) - \frac{\delta_{s,\lambda}^{\mu,\nu}(\text{ч})+1}{2}\varphi''(\text{ч})\right| \leq \frac{C}{s}\chi^2(\text{ч})\omega_\chi\Big(\varphi'', \frac{1}{\sqrt{s}}\Big),$$

which completes the proof. □

Theorem 6.2. *For all* $\psi \in C_B[0,1]$ *which is the set of all continuous and bounded functions on* $[0,1]$*, we have*

$$\lim_{s\to\infty} s\left[B_{s,\lambda}^{\mu,\nu}(\psi;\text{ч}) - \psi(\text{ч}) - \tau_{s,\lambda}^{\mu,\nu}(\text{ч})\psi'(\text{ч}) - \frac{\delta_{s,\lambda}^{\mu,\nu}(\text{ч})}{2}\psi''(\text{ч})\right] = 0.$$

Proof. Let any $\psi \in C_B[0,1]$, then from Taylor's series expansion, we can write

$$\psi(t) = \psi(ɥ) + (t - ɥ)\psi'(ɥ) + \frac{1}{2}(t - ɥ)^2\psi''(ɥ) + (t - ɥ)^2\, Q_{ɥ}(t), \tag{6.5}$$

where $Q_{ɥ}(t) \in C[0,1]$ and is defined for the Peano form of the remainder, moreover, $Q_{ɥ}(t) \to 0$ as $t \to ɥ$. Applying the operators $B^{\mu,\nu}_{s,\lambda}(\cdot; ɥ)$ to the equality (6.5), it is easy to see

$$B^{\mu,\nu}_{s,\lambda}(\psi; ɥ) - \psi(ɥ) = \psi'(ɥ)B^{\mu,\nu}_{s,\lambda}(t - ɥ; ɥ) + \frac{\psi''(ɥ)}{2}B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2; ɥ) + B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2 Q_{ɥ}(t); ɥ).$$

From the Cauchy-Schwarz inequality, we get

$$B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2 Q_{ɥ}(t); ɥ) \leq \sqrt{B^{\mu,\nu}_{s,\lambda}(Q^2_{ɥ}(t); ɥ)}\sqrt{B^{\mu,\nu}_{s,\lambda}((t - ɥ)^4; ɥ)}. \tag{6.6}$$

We clearly observe here $\lim_{s\to\infty} B^{\mu,\nu}_{s,\lambda}(Q^2_{ɥ}(t); ɥ) = 0$ and therefore

$$\lim_{s\to\infty} s\{B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2 Q_{ɥ}(t); ɥ)\} = 0.$$

Thus, we have

$$\begin{aligned}\lim_{s\to\infty} s\{B^{\mu,\nu}_{s,\lambda}(\psi; ɥ) - \psi(ɥ)\} = \lim_{s\to\infty} s\{&B^{\mu,\nu}_{s,\lambda}(t - ɥ; ɥ)\psi'(ɥ) + \frac{\psi''(ɥ)}{2}B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2; ɥ)\\ &+ B^{\mu,\nu}_{s,\lambda}((t - ɥ)^2 Q_{ɥ}(t); ɥ)\}.\end{aligned}$$

□

7. Conclusions

In the present article, we conclude that our new operators (2.2) are the shifted knots variant of the Bézier basis of the λ-Bernstein operators defined by equality (1.1). For the choice $\mu = \nu = 0$ in the equality (2.2), then our new operators $B^{\mu,\nu}_{s,\lambda}$ reduced to the operators by the equality (1.1) defined by Cai et al. [24]. Consequently, we can say that the classical Bernstein-operators and λ-Bernstein operators with Bézier basis are special cases of our operators (2.2). These facts lead us to the conclusion that our new operators are more powerful than earlier varieties of operators.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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