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Research article

Representations and cohomologies of modified λ-differential Hom-Lie algebras

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Abstract: In this paper, we introduce the concept and representations of modified λ-differential Hom-Lie algebras. We then develop the cohomology of modified λ-differential Hom-Lie algebras with coefficients in a suitable representation. As applications, abelian extensions and skeletal modified λ-differential Hom-Lie 2-algebras are characterized in terms of cohomology groups.

Keywords: Hom-Lie algebras; modified λ-differential operator; cohomology; extension; Hom-Lie 2-algebras
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1. Introduction

Hom-Lie algebraic structures first appeared in [1] as a generalization of Lie algebras, as part of describing the deformations of Witt and Virasoro algebras. Because some q-deformations of Witt and Virasoro algebras naturally have the structure of Hom-Lie algebras [2]. Since then, many people have paid special attention to this algebraic structure, for example, quadratic Hom-Lie algebras have been studied in [3]; the concept of Hom-Lie 2-algebras has been introduced in [4], which is a categorified Hom-Lie algebra; matched pair of Hom-Lie algebras has been defined in [5], which was equivalent to Hom-Lie bialgebra. Further research on Hom-Lie algebras could be found in [6–9] and references cited therein. All these provide a good starting point for further discussion and research.

Our second research object is a modified λ-differential operator. Semenov-Tian-Shansky [10] solved the solution of the modified classical Yang-Baxter equation, which was called the modified r-matrix in [11]. Inspired by the case of modified r-matrix, Peng et al. introduced the concepts of modified λ-differential operators and modified λ-differential Lie algebras in [12]. This modified algebraic structure has been extended to other algebraic structures, such as modified Rota-Baxter associative algebras of weight λ [13], modified Rota-Baxter Leibniz algebras of weight λ [14, 15], modified λ-differential Lie triple systems [16] and modified λ-differential 3-Lie algebras [17]. Motivated by the work of [1, 4, 12], it is natural and meaningful to study modified λ-differential Hom-Lie algebras and modified λ-differential Hom-Lie 2-algebras. This becomes our first motivation for writing the present paper.

The representation theory of an algebraic object reveals some hidden profound structures. An example is that the structure of complex semi-simple Lie algebra is revealed by its representation theory. Additionally, cohomology theories occupy a central position as they enable for example to control extension problems. In particular, the cohomology of Lie algebras has been defined by Chevalley and Eilenberg [18]. Further, based on Chevalley and Eilenberg's work, representation theory and cohomology of Hom-Lie algebras have been established by Sheng [6]. In [19], Tang et al. constructed the representations and cohomologies of Lie algebras with derivations and applied them to study the central extension of Lie algebras with derivations. Recently, representation theories and cohomology theories of various kinds of algebras have been developed, see [11, 13–17, 20, 21] and references cited therein. Motivated by the recent work of [6,16,17,19], in this paper, we will investigate representations, cohomologies and abelian extensions of modified λ-differential Hom-Lie algebras. This becomes another motivation for writing the present paper.

The paper is organized as follows. In Section 2, we introduce the concept of modified λ-differential Hom-Lie algebras, and give its representations. We show that $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ if and only if $(\mathfrak{L} \oplus \mathfrak{V}, \varphi \oplus \varphi_{\mathfrak{V}})$ is a modified λ-differential Hom-Lie algebra (Proposition 2.17). Inspired by the matched pair of Hom-Lie algebras [5], the matched pair of modified λ-differential Hom-Lie algebras is also studied. And then we propose the dual representation of the modified λ-differential Hom-Lie algebra (Proposition 2.19). In Section 3, we first introduce the cochain map Δ (Lemma 3.1), and then we establish a cohomology theory of modified λ-differential Hom-Lie algebras with coefficients in a representation from the cochain map Δ. In Section 4, we prove that any abelian extension of a modified λ-differential Hom-Lie algebra has a representation and a 2-cocycle. It is further proved that they are classified by the second cohomology group (Theorem 4.5). Finally, in Section 5, we introduce the concept of modified λ-differential Hom-Lie 2-algebras and show that skeletal modified λ-differential Hom-Lie 2-algebras are classified by 3-cocycles of modified λ-differential Hom-Lie algebras (Theorem 5.4).

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the vector spaces and (multi) linear maps are taken over $\mathbb{K}$.

2. Representations of modified λ-differential Hom-Lie algebras

In this section, we propose the concept of modified λ-differential Hom-Lie algebras. Then, we introduce the representation and dual representation of modified λ-differential Hom-Lie algebras and give some examples. First, let's recall some definitions and results about hom-Lie algebra and its representations from [1, 6].

Definition 2.1. [1] (i) A Hom-Lie algebra is a triple $(\mathfrak{L}, [-,-], \alpha)$ consisting of a vector space $\mathfrak{L}$, a skew-symmetric map $[-,-] : \mathfrak{L}\otimes\mathfrak{L} \to \mathfrak{L}$, and a linear transformation α on $\mathfrak{L}$ (called the twist) satisfying $\alpha([a, b]) = [\alpha(a), \alpha(b)]$ such that

$$[\alpha(a), [b, c]] + [\alpha(c), [a, b]] + [\alpha(b), [c, a]] = 0, \tag{2.1}$$

for any $a, b, c \in \mathfrak{L}$. We often denote a Hom-Lie algebra as above by $(\mathfrak{L}, \alpha)$. Furthermore, $(\mathfrak{L}, \alpha)$ is called a regular Hom-Lie algebra if α is an invertible linear transformation.
(ii) Let $\mathfrak{I}$ be a subspace of a Hom-Lie algebra $(\mathfrak{L}, \alpha)$. Then $\mathfrak{I}$ is called an ideal of $\mathfrak{L}$, if $[\mathfrak{I}, \mathfrak{L}] \subseteq \mathfrak{I}$ and $\alpha(\mathfrak{I}) \subseteq \mathfrak{I}$. Moreover, an ideal $\mathfrak{I}$ of $\mathfrak{L}$ is said to be an abelian ideal in $\mathfrak{L}$ if $[\mathfrak{I}, \mathfrak{I}] = 0$.
(iii) A homomorphism of Hom-Lie algebras $f : (\mathfrak{L}, \alpha) \to (\mathfrak{L}', \alpha')$ is a linear map $f : \mathfrak{L} \to \mathfrak{L}'$ such that

$$f \circ \alpha = \alpha' \circ f, \quad f[a, b] = [f(a), f(b)]'.$$

Example 2.2. Let $\mathfrak{L}$ be a 4-dimensional Lie algebra with a basis ε_1, ε_2, ε_3 and ε_4 defined by $[\varepsilon_1, \varepsilon_2] = -\varepsilon_3$, $[\varepsilon_1, \varepsilon_3] = \varepsilon_2$, $[\varepsilon_2, \varepsilon_4] = -\varepsilon_2$ and $[\varepsilon_3, \varepsilon_4] = \varepsilon_3$. The linear transformation α on $\mathfrak{L}$ is defined by $\alpha(\varepsilon_1) = -\varepsilon_1$, $\alpha(\varepsilon_2) = \varepsilon_2$, $\alpha(\varepsilon_3) = -\varepsilon_3$ and $\alpha(\varepsilon_4) = \varepsilon_4$. Then $(\mathfrak{L}, \alpha)$ is a Hom-Lie algebra.

Definition 2.3. [6] (i) A representation of a Hom-Lie algebra $(\mathfrak{L}, \alpha)$ on a Hom-vector space $(\mathfrak{V}, \beta)$ is a linear map $\rho : \mathfrak{L} \to \mathrm{End}(\mathfrak{V})$, such that

$$\rho(\alpha(a)) \circ \beta = \beta \circ \rho(a), \tag{2.2}$$

$$\rho([a, b]) \circ \beta = \rho(\alpha(a)) \circ \rho(b) - \rho(\alpha(b))) \circ \rho(a), \tag{2.3}$$

for all $a, b \in \mathfrak{L}$. We denote a representation by $(\mathfrak{V}, \beta; \rho)$.
(ii) If $(\mathfrak{V}, \beta; \rho)$ is a representation of $(\mathfrak{L}, \alpha)$, then the space $\mathfrak{L} \oplus \mathfrak{V}$ becomes a Hom-Lie algebra with the maps

$$\alpha \oplus \beta(a + u) = \alpha(a) + \beta(u), \tag{2.4}$$

$$[a + u, b + v]_{\ltimes} = [a, b] + \rho(a)v - \rho(b)u, \forall a, b \in \mathfrak{L}, u, v \in \mathfrak{V}. \tag{2.5}$$

We denote this Hom-Lie algebra by $\mathfrak{L} \ltimes \mathfrak{V}$.

Example 2.4. [6] For any non-negative integer s, any Hom-Lie algebra $(\mathfrak{L}, \alpha)$ is a representation over itself with

$$\mathrm{ad}^s : \mathfrak{L} \to \mathrm{End}(\mathfrak{L}), a \mapsto (b \mapsto [\alpha^s(a), b]), \forall a, b \in \mathfrak{L}.$$

It is called the α^s-adjoint representation over the Hom-Lie algebra.

Definition 2.5. (i) Let $\lambda \in \mathbb{K}$ and $(\mathfrak{L}, \alpha)$ be a Hom-Lie algebra. A modified λ-differential operator on $\mathfrak{L}$ is a linear map $\varphi : \mathfrak{L} \to \mathfrak{L}$, such that

$$\varphi \circ \alpha = \alpha \circ \varphi, \tag{2.6}$$

$$\varphi[a, b] = [\varphi(a), b] + [a, \varphi(b)] + \lambda[a, b], \ \forall \ a, b \in \mathfrak{L}. \tag{2.7}$$

(ii) A modified λ-differential Hom-Lie algebra is a triple $(\mathfrak{L}, \alpha, \varphi)$ consisting of a Hom-Lie algebra $(\mathfrak{L}, \alpha)$ and a modified λ-differential operator φ.
(iii) A homomorphism between two modified λ-differential Hom-Lie algebras $(\mathfrak{L}_1, \ \alpha_1, \varphi_1)$ and $(\mathfrak{L}_2, \alpha_2, \varphi_2)$ is a Hom-Lie algebra homomorphism $f : (\mathfrak{L}_1, \alpha_1) \to (\mathfrak{L}_2, \alpha_2)$ such that $f \circ \varphi_1 = \varphi_2 \circ f$. Furthermore, f is called an isomorphism from $\mathfrak{L}_1$ to $\mathfrak{L}_2$ if f is nondegenerate.

Remark 2.6. (i) A modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ with $\alpha = \mathrm{id}_{\mathfrak{L}}$ is nothing but a modified λ-differential Lie algebra. See [12] for more details about modified λ-differential Lie algebra. It follows that the results established in this paper can be naturally adapted to the context of modified λ-differential Lie algebra.
(ii) Let φ be a modified λ-differential operator on $\mathfrak{L}$. If $\lambda = 0$, then φ is a derivation on $\mathfrak{L}$. We denote the set of all derivations on $\mathfrak{L}$ by $\mathrm{Der}(\mathfrak{L})$. See [6] for derivations of Hom-Lie algebras.

Proposition 2.7. *A linear map $\varphi : \mathfrak{L} \to \mathfrak{L}$ is a modified λ-differential operator if and only if $\varphi + \lambda \mathrm{id}_{\mathfrak{L}}$ is a derivation on $\mathfrak{L}$.*

Example 2.8. An identity map $\mathrm{id}_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}$ is a modified (-1)-differential operator.

Example 2.9. Let $(\mathfrak{L}, \alpha, \varphi)$ be a modified λ-differential Hom-Lie algebra. Then, for $k \in \mathbb{K}$, $(\mathfrak{L}, \alpha, k\varphi)$ is a modified $(k\lambda)$-differential Hom-Lie algebra.

Example 2.10. Let $(\mathfrak{L}, \alpha)$ be the 4-dimensional Hom-Lie algebra given in Example 2.2. Then, the operator

$$\varphi = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & k \end{pmatrix}$$

is a modified $(-k)$-differential operator on $\mathfrak{L}$, for $k, k_1 \in \mathbb{K}$.

Definition 2.11. A representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ is a quadruple $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ such that the following conditions are satisfied:
(i) $(\mathfrak{V}, \beta; \rho)$ is a representation of the Hom-Lie algebra $(\mathfrak{L}, \alpha)$;
(ii) $\varphi_{\mathfrak{V}} : \mathfrak{V} \to \mathfrak{V}$ is a linear map satisfying the following equations

$$\varphi_{\mathfrak{V}} \circ \beta = \beta \circ \varphi_{\mathfrak{V}}, \tag{2.8}$$

$$\varphi_{\mathfrak{V}}(\rho(a)v) = \rho(\varphi(a))v + \rho(a)\varphi_{\mathfrak{V}}(v) + \lambda\rho(a)v, \tag{2.9}$$

for any $a \in \mathfrak{L}$ and $v \in \mathfrak{V}$.

Remark 2.12. Let $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ be a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$. If $\lambda = 0$, then $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ is a representation of the Hom-Lie algebra with a derivation. One can refer to [19] for more information about Lie algebra with a derivation. It follows that the results established in this paper can be naturally adapted to the context of Hom-Lie algebras with derivations.

Proposition 2.13. *$(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ if and only if $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}} + \lambda \mathrm{id}_{\mathfrak{V}})$ is a representation of the Hom-Lie algebra with a derivation $(\mathfrak{L}, \alpha, \varphi + \lambda \mathrm{id}_{\mathfrak{L}})$.*

Proof. Equation (2.9) is equivalent to

$$(\varphi_{\mathfrak{V}} + \lambda \mathrm{id}_{\mathfrak{V}})(\rho(a)v) = \rho((\varphi + \lambda \mathrm{id}_{\mathfrak{L}})(a))v + \rho(a)(\varphi_{\mathfrak{V}} + \lambda \mathrm{id}_{\mathfrak{V}})(v).$$

The proposition follows. □

Example 2.14. For any non-negative integer s, $(\mathfrak{L}, \alpha; \mathrm{ad}^s, \varphi)$ is an α^s-adjoint representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$.

When $\varphi_{\mathfrak{V}} = \mathrm{id}_{\mathfrak{V}}$ and $\varphi = -\lambda \mathrm{id}_{\mathfrak{L}}$ in Eq (2.9), we can easily get the following example.

Example 2.15. Let $(\mathfrak{V}, \beta; \rho)$ be a representation of the Hom-Lie algebra $(\mathfrak{L}, \alpha)$. Then, for $k \in \mathbb{K}$, $(\mathfrak{V}, \beta; \rho, \mathrm{id}_{\mathfrak{V}})$ is a representation of a modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, -\lambda \mathrm{id}_{\mathfrak{L}})$.

We can get the following example when both sides of Eq (2.9) are multiplied by scalar $k \in \mathbb{K}$ at the same time.

Example 2.16. Let $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$ be a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L},\alpha,\varphi)$. Then, for $k \in \mathbb{K}$, $(\mathfrak{V},\beta;\rho,k\varphi_{\mathfrak{V}})$ is a representation of a modified $(k\lambda)$-differential Hom-Lie algebra $(\mathfrak{L},\alpha,k\varphi)$.

In the next proposition, we discuss the case of direct sum $\mathfrak{L} \oplus \mathfrak{V}$, where $(\mathfrak{L},\alpha,\varphi)$ is the modified λ-differential Hom-Lie algebra and $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$ is its representation.

Proposition 2.17. *Let $(\mathfrak{L},\alpha)$ be a Hom-Lie algebra and $(\mathfrak{V},\beta;\rho)$ be a representation of it. Then $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L},\alpha,\varphi)$ if and only if $\mathfrak{L}\oplus\mathfrak{V}$ is a modified λ-differential Hom-Lie algebra with maps $\alpha\oplus\beta$, $[-,-]_{\ltimes}$ (see Eqs* (2.4) *and* (2.5)*) and $\varphi\oplus\varphi_{\mathfrak{V}}$ is defined as follows*

$$\varphi\oplus\varphi_{\mathfrak{V}}(a+u) :=\varphi(a)+\varphi_{\mathfrak{V}}(u),$$

for any $a+u \in \mathfrak{L}\oplus\mathfrak{V}$. We denote this semidirect product modified λ-differential Hom-Lie algebra by $(\mathfrak{L}\ltimes\mathfrak{V},\varphi\oplus\varphi_{\mathfrak{V}})$.

Proof. In view of [6], $(\mathfrak{V},\beta;\rho)$ is a representation of $(\mathfrak{L},\alpha)$ if and only if $(\mathfrak{L}\oplus\mathfrak{V},[-,-]_{\ltimes},\alpha\oplus\beta)$ is a Hom-Lie algebra.

Next, suppose $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$ is a representation of $(\mathfrak{L},\alpha,\varphi)$, then for any $a,b \in \mathfrak{L}, u,v \in \mathfrak{V}$, by Eqs (2.6)–(2.9), we have

$$\begin{aligned}
&\alpha\oplus\beta(\varphi\oplus\varphi_{\mathfrak{V}}(a+u)) = \alpha\oplus\beta(\varphi(a)+\varphi_{\mathfrak{V}}(u)) = \alpha(\varphi(a))+\beta(\varphi_{\mathfrak{V}}(u)) = \varphi(\alpha(a))+\varphi_{\mathfrak{V}}(\beta(u))\\
&= \varphi\oplus\varphi_{\mathfrak{V}}(\alpha\oplus\beta(a+u)),\\
&\varphi\oplus\varphi_{\mathfrak{V}}([a+u,b+v]_{\ltimes})\\
&= \varphi[a,b]+\varphi_{\mathfrak{V}}(\rho(a)(v))-\varphi_{\mathfrak{V}}(\rho(b)(u))\\
&= [\varphi(a),b]+[a,\varphi(b)]+\lambda[a,b]+\rho(\varphi(a))v+\rho(a)\varphi_{\mathfrak{V}}(v)+\lambda\rho(a)v\\
&\quad -\rho(\varphi(b))u-\rho(b)\varphi_{\mathfrak{V}}(u)-\lambda\rho(b)u\\
&= ([\varphi(a),b]+\rho(\varphi(a))v-\rho(b)\varphi_{\mathfrak{V}}(u))+([a,\varphi(b)]+\rho(a)\varphi_{\mathfrak{V}}(v)-\rho(\varphi(b))u)\\
&\quad +\lambda([a,b]+\rho(a)v-\rho(b)u)\\
&= [\varphi\oplus\varphi_{\mathfrak{V}}(a+u),b+v]_{\ltimes}+[a+u,\varphi\oplus\varphi_{\mathfrak{V}}(b+v)]_{\ltimes}+\lambda[a+u,b+v]_{\ltimes}.
\end{aligned}$$

So $(\mathfrak{L}\ltimes\mathfrak{V},\varphi\oplus\varphi_{\mathfrak{V}})$ is a modified λ-differential Hom-Lie algebra.

Conversely, assume that $(\mathfrak{L}\ltimes\mathfrak{V},\varphi\oplus\varphi_{\mathfrak{V}})$ is a modified λ-differential Hom-Lie algebra, then for any $a \in \mathfrak{L}$ and $u \in \mathfrak{V}$, we have

$$\begin{aligned}
&\alpha\oplus\beta(\varphi\oplus\varphi_{\mathfrak{V}}(a+u)) = \varphi\oplus\varphi_{\mathfrak{V}}(\alpha\oplus\beta(a+u)),\\
&\varphi\oplus\varphi_{\mathfrak{V}}[a+0,0+u]_{\ltimes} = [\varphi\oplus\varphi_{\mathfrak{V}}(a+0),0+u]_{\ltimes}+[a+0,\varphi\oplus\varphi_{\mathfrak{V}}(0+u)]_{\ltimes}+\lambda[a+0,0+u]_{\ltimes},
\end{aligned}$$

which imply that

$$\begin{aligned}
\varphi_{\mathfrak{V}}\circ\beta =&\beta\circ\varphi_{\mathfrak{V}},\\
\varphi_{\mathfrak{V}}(\rho(a)u) =&\rho(\varphi(a))u+\rho(a)\varphi_{\mathfrak{V}}(u)+\lambda\rho(a)u.
\end{aligned}$$

Therefore, $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$ is a representation of $(\mathfrak{L},\alpha,\varphi)$. □

In [5], Sheng and Bai introduced the concept of matched pair of Hom-Lie algebras, and proved that matched pair of Hom-Lie algebras are equivalent to Hom-Lie bialgebras. Motivated by Sheng and Bai's work, we propose the concept of a matched pair of modified λ-differential Hom-Lie algebra.

Proposition 2.18. *Let $(\mathfrak{L}_1, \alpha_1, \varphi_1)$ and $(\mathfrak{L}_2, \alpha_2, \varphi_2)$ be two modified λ-differential Hom-Lie algebras, such that $(\mathfrak{L}_1, \mathfrak{L}_2, \alpha_1, \alpha_2; \rho_1, \rho_2)$ is a matched pair of Hom-Lie algebras with linear maps $\rho_1 : \mathfrak{L}_1 \to \mathrm{End}(\mathfrak{L}_2)$ and $\rho_2 : \mathfrak{L}_2 \to \mathrm{End}(\mathfrak{L}_1)$, where the Hom-Lie algebra structure on $\mathfrak{L}_1 \bowtie \mathfrak{L}_2$ is given by*

$$\begin{aligned}
&[a + x, b + y]_{\bowtie} :=[a, b] + \rho_1(a)(y) - \rho_1(b)(x) + [x, y] + \rho_2(x)(b) - \rho_2(y)(a),\\
&\alpha_1 \oplus \alpha_2(a + x) :=\alpha_1(a) + \alpha_2(x),
\end{aligned}$$

for any $a, b \in \mathfrak{L}_1$ and $x, y \in \mathfrak{L}_2$. Moreover, if $(\mathfrak{L}_2, \alpha_2; \rho_1, \varphi_2)$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}_1, \alpha_1, \varphi_1)$ and $(\mathfrak{L}_1, \alpha_1; \rho_2, \varphi_1)$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}_2, \alpha_2, \varphi_2)$, define

$$\varphi_1 \oplus \varphi_2(a + x) :=\varphi_1(a) + \varphi_2(x),$$

then $(\mathfrak{L}_1 \bowtie \mathfrak{L}_2, \alpha_1 \oplus \alpha_2, \varphi_1 \oplus \varphi_2)$ is a modified λ-differential Hom-Lie algebra. We call $(\mathfrak{L}_1, \mathfrak{L}_2, \alpha_1, \alpha_2, \rho_1, \rho_2, \varphi_1, \varphi_2)$ a matched pair of modified λ-differential Hom-Lie algebra.

Proof. Using Eqs (2.6)–(2.9), we can verify it. □

More information on representation and matched pair of Hom-Lie algebras, one can refer to [5, 6].

In [3], Benayadi and Makhlouf developed the dual representation of a Hom-Lie algebra, it is further studied in [7] with reference to purely Hom-Lie bialgebras. Based on the stated references, at the end of this section, we propose the dual representation of the modified λ-differential Hom-Lie algebra as follow:

Proposition 2.19. *Let $(\mathfrak{L}, \alpha, \varphi)$ be a modified λ-differential regular Hom-Lie algebra, and $(\mathfrak{V}, \beta; \rho, \varphi_V)$ be a representation of it with β an invertible linear map. Define $\rho^\star : \mathfrak{L} \to \mathrm{End}(\mathfrak{V}^*)$ and $(\beta^{-1})^*, \varphi_{\mathfrak{V}}^* : \mathfrak{V}^* \to \mathfrak{V}^*$ respectively by*

$$\begin{aligned}
&\langle \rho^\star(a)\xi, u\rangle = -\langle \xi, \rho(\alpha^{-1}(a))\beta^{-2}(u)\rangle,\\
&\langle (\beta^{-1})^*(\xi), u\rangle = \langle \xi, \beta^{-1}(u)\rangle, \quad \langle \varphi_{\mathfrak{V}}^*(\xi), u\rangle = -\langle \xi, \varphi_{\mathfrak{V}}(u)\rangle, \quad \forall a \in \mathfrak{L}, u \in \mathfrak{V}, \xi \in \mathfrak{V}^*.
\end{aligned}$$

Then $(\mathfrak{V}^*, (\beta^{-1})^*; \rho^\star, \varphi_{\mathfrak{V}}^*)$ is a representation of $(\mathfrak{L}, \alpha, \varphi)$, which is called the dual representation of $(\mathfrak{L}, \alpha, \varphi)$.

Proof. It can be obtained directly from the result of [7] by using Proposition 2.17. □

We have the following result from Proposition 2.19 and Example 2.14.

Example 2.20. Let $(\mathfrak{L}, \alpha, \varphi)$ be a modified λ-differential regular Hom-Lie algebra. Define $\mathrm{ad}^\star : \mathfrak{L} \to \mathrm{End}(\mathfrak{L}^*)$ and $(\alpha^{-1})^*, \varphi^* : \mathfrak{L}^* \to \mathfrak{L}^*$ respectively by

$$\begin{aligned}
&\langle \mathrm{ad}_a^\star(\xi), b\rangle = -\langle \xi, [\alpha^{-1}(a), \alpha^{-2}(b)]\rangle,\\
&\langle (\alpha^{-1})^*(\xi), a\rangle = \langle \xi, \alpha^{-1}(a)\rangle, \quad \langle \varphi^*(\xi), a\rangle = -\langle \xi, \varphi(a)\rangle, \quad \forall a, b \in \mathfrak{L}, \xi \in \mathfrak{L}^*.
\end{aligned}$$

Then, $(\mathfrak{L}^*, (\alpha^{-1})^*; \mathrm{ad}^\star, \varphi^*)$ is a representation of the modified λ-differential regular Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ on $\mathfrak{L}^*$, which is called the coadjoint representation of $(\mathfrak{L}, \alpha, \varphi)$.

Using the coadjoint representation $(\mathfrak{L}^*, (\alpha^{-1})^*; \mathrm{ad}^\star, \varphi^*)$, we can get a semidirect product modified λ-differential Hom-Lie algebra structure on $\mathfrak{L} \oplus \mathfrak{L}^*$.

Example 2.21. Let $(\mathfrak{L}, \alpha, \varphi)$ be a modified λ-differential Hom-Lie algebra. Then there is a natural modified λ-differential Hom-Lie algebra $(\mathfrak{L} \oplus \mathfrak{L}^*, [-,-]_*, \alpha \oplus \alpha^*, \varphi \oplus \varphi^*)$, where

$$\begin{aligned}
[a+\xi, b+\eta]_* :=& [a,b] + \mathrm{ad}_a^\star(\eta) - \mathrm{ad}_b^\star(\xi),\\
\varphi \oplus \varphi^*(a+\xi) :=& \varphi(a) + \varphi^*(\xi),\\
\alpha \oplus (\alpha^{-1})^*(a+\xi) :=& \alpha(a) + (\alpha^{-1})^*(\xi),
\end{aligned}$$

for all $a, b \in \mathfrak{L}$ and $\xi, \eta \in \mathfrak{L}^*$.

3. Cohomology of modified λ-differential Hom-Lie algebras

In this section, we develop the cohomology of a modified λ-differential Hom-Lie algebra with coefficients in its representation.

In [18], Chevalley and Eilenberg introduced the cohomology theory of Lie algebras. Furthermore, based on Chevalley and Eilenberg's work, Sheng developed the cohomology theory of Hom-Lie algebras in [6], which can be described as follows.

Let $(\mathfrak{V}, \beta; \rho)$ be a representation of the Hom-Lie algebra $(\mathfrak{L}, \alpha)$. Denote the n-cochains of $\mathfrak{L}$ with coefficients in representation $\mathfrak{V}$ by

$$C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) := \{f \in \mathrm{Hom}(\mathfrak{L}^{\otimes n}, \mathfrak{V}) \mid \beta(f(a_1, \cdots, a_n)) = f(\alpha(a_1), \cdots, \alpha(a_n))\}.$$

The coboundary operator $\delta : C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \to C^{n+1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$, for $a_1, \cdots, a_{n+1} \in \mathfrak{L}$ and $f \in C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$, as

$$\begin{aligned}
\delta f(a_1, \cdots, a_{n+1}) =& \sum_{i=1}^{n+1} (-1)^{i+1} \rho(\alpha^{n-1}(a_i)) f(a_1, \cdots, \widehat{a_i}, \cdots, a_{n+1})\\
&+ \sum_{i<j} (-1)^{i+j} f([a_i, a_j], \alpha(a_1), \cdots, \widehat{\alpha(a_i)}, \cdots, \widehat{\alpha(a_j)}, \cdots, \alpha(a_{n+1})).
\end{aligned} \tag{3.1}$$

Then, it was proved that $\delta^2 = 0$. Please see [6] for more details. Let us denote by $\mathcal{H}^*_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$, the cohomology group associated to the cochain complex $(C^*_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}), \delta)$.

For any $n \geq 1$, we define a linear map $\Delta : C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \to C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$ by

$$\Delta f(a_1, \cdots, a_n) = \sum_{i=1}^{n} f(a_1, \cdots, \varphi(a_i), \cdots, a_n) + (n-1)\lambda f(a_1, \cdots, a_n) - \varphi_{\mathfrak{V}}(f(a_1, \cdots, a_n)). \tag{3.2}$$

Lemma 3.1. *The map Δ is a cochain map, i.e., $\Delta \circ \delta = \delta \circ \Delta$. In other words, the following diagram is commutative:*

$$\begin{tikzcd}
C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \arrow[r, "\delta"] \arrow[d, "\Delta"] & C^{n+1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \arrow[d, "\Delta"] \\
C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \arrow[r, "\delta"] & C^{n+1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}).
\end{tikzcd}$$

Proof. For any $f \in C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$, we have

$$\begin{aligned}\delta(\Delta f)(a_1, \cdots, a_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}\rho(\alpha^{n-1}(a_i))(\Delta f)(a_1, \cdots, \widehat{a_i}, \cdots, a_{n+1})\\ &+ \sum_{i<j}(-1)^{i+j}(\Delta f)([a_i, a_j], \alpha(a_1), \cdots, \widehat{\alpha(a_i)}, \cdots, \widehat{\alpha(a_j)}, \cdots, \alpha(a_{n+1}))\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\Delta(\delta f)(a_1, \cdots, a_{n+1}) = &\sum_{i=1}^{n+1}(\delta f)(a_1, \cdots, \varphi(a_i), \cdots, a_{n+1}) + n\lambda(\delta f)(a_1, \cdots, a_{n+1})\\ &- \varphi_{\mathfrak{V}}((\delta f)(a_1, \cdots, a_{n+1})).\end{aligned} \tag{3.4}$$

Using Eqs (2.6)–(2.9) and further expanding Eqs (3.3) and (3.4), we can get the conclusion that $\Delta \circ \delta = \delta \circ \Delta$. □

Let $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ be a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, [-,-], \alpha, \varphi)$. We define the set of modified λ-differential Hom-Lie algebra 1-cochains to be $C^1_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) = C^1_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V})$. For $n \geq 2$, we define the set of modified λ-differential Hom-Lie algebra n-cochains by

$$C^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) := C^n_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \oplus C^{n-1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}).$$

Define a linear map $\partial : C^1_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) \to C^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$ by

$$\partial(f) = (\delta f, -\Delta f), \forall f \in C^1_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}).$$

Then, for $n \geq 2$, we define the linear map $\partial : C^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) \to C^{n+1}_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$ by

$$\partial(f, g) = (\delta f, \delta g + (-1)^n \Delta f), \forall (f, g) \in C^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}).$$

In view of Lemma 3.1, we have the following theorem.

Theorem 3.2. *The map ∂ is a coboundary operator, i.e., $\partial \circ \partial = 0$.*

Let $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ be a representation of a modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$, then we have an cochain complex $(C^*_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}), \partial)$. Denote the set of n-cocycles by $\mathcal{Z}^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$ and the set of n-coboundaries by $\mathcal{B}^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$. The n-th cohomology group of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ with coefficients in the representation $(\mathfrak{V}, \beta; \rho, \varphi_{\mathfrak{V}})$ by

$$\mathcal{H}^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) = \frac{\mathcal{Z}^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})}{\mathcal{B}^n_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})}.$$

Obviously, there is a short exact sequence of cochain complexes:

$$0 \to C^{*-1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \longrightarrow C^*_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) \longrightarrow C^*_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \to 0.$$

So as to induces a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{H}^p_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) \to \mathcal{H}^p_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \to \mathcal{H}^{p+1}_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V}) \to \mathcal{H}^{p+1}_{\mathrm{HLie}}(\mathfrak{L}, \mathfrak{V}) \to \cdots.$$

4. Abelian extensions of modified λ-differential Hom-Lie algebras

Motivated by the extensions of Hom-Lie algebras [8, 9], in this section, we prove that any abelian extension of a modified λ-differential Hom-Lie algebra has a representation and a 2-cocycle. It is further proved that they are classified by the second cohomology.

Definition 4.1. (i) Let $(\mathfrak{L}, \alpha, \varphi)$ be a modified λ-differential Hom-Lie algebra and $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ an abelian modified λ-differential Hom-Lie algebra with the trivial product. An abelian extension of $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ is a short exact sequence of modified λ-differential Hom-Lie algebras

$$0 \longrightarrow (\mathfrak{V}, \beta, \varphi_{\mathfrak{V}}) \xrightarrow{i} (\hat{\mathfrak{L}}, \hat{\alpha}, \hat{\varphi}) \xrightarrow{p} (\mathfrak{L}, \alpha, \varphi) \longrightarrow 0$$

such that $\hat{\alpha}|_{\mathfrak{V}} = \beta$ and $[\mathfrak{V}, \mathfrak{V}]_{\hat{\mathfrak{L}}} = 0$, i.e., $\mathfrak{V}$ is an abelian ideal of $\hat{\mathfrak{L}}$.
(ii) A section of an abelian extension $(\hat{\mathfrak{L}}, \hat{\alpha}, \hat{\varphi})$ of $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ is a linear map $s : \mathfrak{L} \to \hat{\mathfrak{L}}$ such that $s \circ \alpha = \hat{\alpha} \circ s$ and $p \circ s = \mathrm{id}_{\mathfrak{L}}$.
(iii) Let $(\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1)$ and $(\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2)$ be two abelian extensions of $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$. They are said to be equivalent if there is an isomorphism of modified λ-differential Hom-Lie algebras $f : (\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1) \to (\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2)$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & (\mathfrak{V}, \beta, \varphi_{\mathfrak{V}}) \arrow[r, "i_1"] \arrow[d, equal] & (\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1) \arrow[r, "p_1"] \arrow[d, "f"'] & (\mathfrak{L}, \alpha, \varphi) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathfrak{V}, \beta, \varphi_{\mathfrak{V}}) \arrow[r, "i_2"] & (\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2) \arrow[r, "p_2"] & (\mathfrak{L}, \alpha, \varphi) \arrow[r] & 0.
\end{tikzcd} \tag{4.1}$$

Now for an abelian extension $(\hat{\mathfrak{L}}, \hat{\alpha}, \hat{\varphi})$ of $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ with a section $s : \mathfrak{L} \to \hat{\mathfrak{L}}$, we define a linear map $\varrho : \mathfrak{L} \to \mathrm{End}(\mathfrak{V})$ by

$$\varrho(a)u := [s(a), u]_{\hat{\mathfrak{L}}}, \quad \forall a \in \mathfrak{L}, u \in \mathfrak{V}.$$

Proposition 4.2. *With the above notations, $(\mathfrak{V}, \beta; \varrho, \varphi_{\mathfrak{V}})$ is a representation of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ and does not depend on the choice of the section s. Moreover, equivalent abelian extensions give the same representation.*

Proof. First, for any other section $s' : \mathfrak{L} \to \hat{\mathfrak{L}}$, $a \in \mathfrak{L}$, we have

$$p(s(a) - s'(a)) = p(s(a)) - p(s'(a)) = a - a = 0.$$

Thus, there is an $u \in \mathfrak{V}$, so that $s'(a) = s(a) + u$. Note that $\mathfrak{V}$ is an abelian ideal of $\hat{\mathfrak{L}}$, this yields that

$$[s'(a), v]_{\hat{\mathfrak{L}}} = [s(a) + u, v]_{\hat{\mathfrak{L}}} = [s(a), v]_{\hat{\mathfrak{L}}}.$$

This means that ϱ does not depend on the choice of the section s.

Secondly, for any $a, b \in \mathfrak{L}$ and $u \in \mathfrak{V}$, by $\mathfrak{V}$ is an abelian ideal of $\hat{\mathfrak{L}}$ and $[s(a), s(b)]_{\hat{\mathfrak{L}}} - s([a, b]) \in \mathfrak{V}$, we have

$$\begin{aligned}
\varrho(\alpha(a))\beta(u) &= [s(\alpha(a)), \beta(u)]_{\hat{\mathfrak{L}}} = [\beta(s(a)), \beta(u)]_{\hat{\mathfrak{L}}} = \beta([s(a), u]_{\hat{\mathfrak{L}}}) \\
&= \beta(\varrho(a)u),
\end{aligned}$$

$$
\begin{aligned}
&\varrho(\alpha(a))\varrho(b)u-\varrho(\alpha(b)))\varrho(a)u\\
=&[s(\alpha(a)),[s(b),u]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}-[s(\alpha(b)),[s(a),u]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}\\
=&-[\beta(u),[s(a),s(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}\\
=&[[s(a),s(b)]_{\hat{\mathfrak{L}}},\beta(u)]_{\hat{\mathfrak{L}}}\\
=&[s([a,b]),\beta(u)]_{\hat{\mathfrak{L}}}\\
=&\varrho([a,b])\beta(u).
\end{aligned}
$$

This shows that $(\mathfrak{V},\beta;\varrho)$ is a representation of a Hom-Lie algebra $(\mathfrak{L},\alpha)$.

On the other hand, by $\hat{\varphi}(s(a))-s(\varphi(a))\in\mathfrak{V}$, we have

$$
\begin{aligned}
\varphi_{\mathfrak{V}}(\varrho(a)u)=&\varphi_{\mathfrak{V}}([s(a),u]_{\hat{\mathfrak{L}}})\\
=&[\varphi_{\mathfrak{V}}(s(a)),u]_{\hat{\mathfrak{L}}}+[s(a),\varphi_{\mathfrak{V}}(u)]_{\hat{\mathfrak{L}}}+\lambda[s(a),u]_{\hat{\mathfrak{L}}}\\
=&\varrho(\varphi(a))u+\varrho(a)\varphi_{\mathfrak{V}}(u)+\lambda\varrho(a)u.
\end{aligned}
$$

Hence, $(\mathfrak{V},\beta;\varrho,\varphi_{\mathfrak{V}})$ is a representation of $(\mathfrak{L},\alpha,\varphi)$.

Finally, suppose that $(\hat{\mathfrak{L}}_1,\hat{\alpha}_1,\hat{\varphi}_1)$ and $(\hat{\mathfrak{L}}_2,\hat{\alpha}_2,\hat{\varphi}_2)$ are equivalent abelian extensions of $(\mathfrak{L},\alpha,\varphi)$ by $(\mathfrak{V},\beta,\varphi_{\mathfrak{V}})$ with the associated isomorphism $f:(\hat{\mathfrak{L}}_1,\hat{\alpha}_1,\hat{\varphi}_1)\to(\hat{\mathfrak{L}}_2,\hat{\alpha}_2,\hat{\varphi}_2)$ such that the diagram in (4.1) is commutative. We choose sections s_1 and s_2 of p_1 and p_2 respectively, we have $p_2fs_1(a)=p_1s_1(a)=a=p_2s_2(a)$, then $fs_1(a)-s_2(a)\in\mathrm{Ker}(p_2)\cong\mathfrak{V}$. Moreover,

$$
[s_1(a),u]_{\hat{\mathfrak{L}}_1}=[fs_1(a),u]_{\hat{\mathfrak{L}}_2}=[s_2(a),u]_{\hat{\mathfrak{L}}_2}.
$$

Therefore, equivalent abelian extensions give the same ϱ. □

Now for an abelian extension $(\hat{\mathfrak{L}},\hat{\alpha},\hat{\varphi})$ of $(\mathfrak{L},\alpha,\varphi)$ by $(\mathfrak{V},\beta,\varphi_{\mathfrak{V}})$ with a section $s:\mathfrak{L}\to\hat{\mathfrak{L}}$. Define the following maps $\varpi:\mathfrak{L}\times\mathfrak{L}\to\mathfrak{V}$ and $\tau:\mathfrak{L}\to\mathfrak{V}$ respectively by

$$
\begin{aligned}
\varpi(a,b)&=[s(a),s(b)]_{\hat{\mathfrak{L}}}-s([a,b]),\\
\tau(a)&=\hat{\varphi}(s(a))-s(\varphi(a)),\quad\forall a,b\in\mathfrak{L}.
\end{aligned}
$$

We transfer the modified λ-differential Hom-Lie algebra structure on $\hat{\mathfrak{L}}$ to $\mathfrak{L}\oplus\mathfrak{V}$ by endowing $\mathfrak{L}\oplus\mathfrak{V}$ with a multiplication $[-,-]_{\varpi}$ a linear map $\alpha\oplus\beta$ and a modified λ-differential operator φ_τ defined by

$$
\begin{aligned}
[a+u,b+v]_{\varpi}&=[a,b]+\varrho(a)v-\varrho(b)u+\varpi(a,b), &(4.2)\\
\alpha\oplus\beta(a+u)&=\alpha(a)+\beta(u),\\
\varphi_\tau(a+u)&=\varphi(a)+\tau(a)+\varphi_{\mathfrak{V}}(u),\ \forall a,b\in\mathfrak{L},\ u,v\in\mathfrak{V}. &(4.3)
\end{aligned}
$$

Proposition 4.3. *The 4-tuple $(\mathfrak{L}\oplus\mathfrak{V},[-,-]_{\varpi},\alpha\oplus\beta,\varphi_\tau)$ is a modified λ-differential Hom-Lie algebra if and only if (ϖ,τ) is a 2-cocycle of the modified λ-differential Hom-Lie algebra $(\mathfrak{L},\alpha,\varphi)$ with the coefficient in $(\mathfrak{V},\beta;\varrho,\varphi_{\mathfrak{V}})$. In this case,*

$$
0\longrightarrow(\mathfrak{V},\beta,\varphi_{\mathfrak{V}})\xrightarrow{\ i\ }(\mathfrak{L}\oplus\mathfrak{V},\alpha\oplus\beta,\varphi_\tau)\xrightarrow{\ p\ }(\mathfrak{L},\alpha,\varphi)\longrightarrow 0
$$

is an abelian extension.

Proof. The 4-tuple $(\mathfrak{L}\oplus\mathfrak{V}, [-,-]_{\varpi}, \alpha\oplus\beta, \varphi_\tau)$ is a modified λ-differential Hom-Lie algebra if and only if

$$[a+u, b+v]_{\varpi} = -[b+v, a+u]_{\varpi}, \tag{4.4}$$
$$\alpha\oplus\beta([a+u, b+v]_{\varpi}) = [\alpha(a)+\beta(u), \alpha(b)+\beta(v)]_{\varpi}, \tag{4.5}$$
$$[\alpha(a)+\beta(u), [b+v, c+w]_{\varpi}]_{\varpi} + [\alpha(c)+\beta(w), [a+u, b+v]_{\varpi}]_{\varpi} + [\alpha(b)+\beta(v), [c+w, a+u]_{\varpi}]_{\varpi} = 0, \tag{4.6}$$
$$\varphi_\tau(\alpha(a)+\beta(u)) = \alpha\oplus\beta(\varphi_\tau(a+u)), \tag{4.7}$$
$$\varphi_\tau[a+u, b+v]_{\varpi} = [\varphi_\tau(a+u), b+v]_{\varpi} + [a+u, \varphi_\tau(b+v)]_{\varpi} + \lambda[a+u, b+v]_{\varpi}, \tag{4.8}$$

for any $a, b, c \in \mathfrak{L}$ and $u, v, w \in \mathfrak{V}$. Further, Eqs (4.4)–(4.8) are equivalent to the following equations:

$$\varpi(a, b) = -\varpi(b, a), \tag{4.9}$$
$$\beta(\varpi(a, b)) = \varpi(\alpha(a), \alpha(b)), \tag{4.10}$$
$$\varrho(\alpha(a))\varpi(b, c) - \varrho(\alpha(b))\varpi(a, c) + \varrho(\alpha(c))\varpi(a, b) - \varpi([a, b], \alpha(c)) + \varpi([a, c], \alpha(b)) - \varpi([b, c], \alpha(a)) = 0, \tag{4.11}$$
$$\tau(\alpha(a)) = \beta(\tau(a)), \tag{4.12}$$
$$\tau[a, b] + \varphi_{\mathfrak{V}}\varpi(a, b) = \varrho(a)(\tau(b)) - \varrho(b)(\tau(a)) + \varpi(\varphi(a), b) + \varpi(a, \varphi(b)) + \lambda\varpi(a, b). \tag{4.13}$$

By Eqs (4.10) and (4.12), we get $(\varpi, \tau) \in C^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$. Using Eqs (4.11) and (4.13), we have $\delta\varpi = 0$ and $\delta\tau + \Delta\varpi = 0$, respectively. Therefore, $\partial(\varpi, \tau) = (\delta\varpi, \delta\tau + \Delta\varpi) = 0$, that is, (ϖ, τ) is a 2-cocycle.

Conversely, if (ϖ, τ) is a 2-cocycle of the modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ with the coefficient in $(\mathfrak{V}, \beta; \varrho, \varphi_{\mathfrak{V}})$, then we have $\partial(\varpi, \tau) = (\delta\varpi, \delta\tau + \Delta\varpi) = 0$, in which Eqs (4.10)–(4.13) hold. So $(\mathfrak{L}\oplus\mathfrak{V}, [-,-]_{\varpi}, \alpha\oplus\beta, \varphi_\tau)$ is a modified λ-differential Hom-Lie algebra. □

Proposition 4.4. *Let $(\hat{\mathfrak{L}}, \hat{\alpha}, \hat{\varphi})$ be an abelian extension of a modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ and $s : \mathfrak{L} \to \hat{\mathfrak{L}}$ a section. If the pair (ϖ, τ) is a 2-cocycle of $(\mathfrak{L}, \alpha, \varphi)$ with the coefficient in $(\mathfrak{V}, \beta; \varrho, \varphi_{\mathfrak{V}})$ constructed using the section s, then its cohomology class does not depend on the choice of s.*

Proof. Let $s_1, s_2 : \mathfrak{L} \to \hat{\mathfrak{L}}$ be two distinct sections, then we can get two corresponding 2-cocycles (ϖ_1, τ_1) and (ϖ_2, τ_2) respectively. We define a linear map $\iota : \mathfrak{L} \to \mathfrak{V}$ by $\iota(a) = s_1(a) - s_2(a)$. Then

$$\begin{aligned}
&\varpi_1(a, b)\\
&= [s_1(a), s_1(b)]_{\hat{\mathfrak{L}}_1} - s_1([a, b])\\
&= [s_2(a) + \iota(a), s_2(b) + \iota(b)]_{\hat{\mathfrak{L}}_1} - (s_2([a, b]) + \iota([a, b]))\\
&= [s_2(a), s_2(b)]_{\hat{\mathfrak{L}}_2} + \varrho(a)\iota(b) - \varrho(b)\iota(a) - s_2([a, b]) - \iota([a, b])\\
&= ([s_2(a), s_2(b)]_{\hat{\mathfrak{L}}_2} - s_2([a, b])) + \varrho(a)\iota(b) - \varrho(b)\iota(a) - \iota([a, b])\\
&= \varpi_2(a, b) + \delta\iota(a, b)
\end{aligned}$$

and

$$\tau_1(x) = \hat{\varphi}(s_1(a)) - s_1(\varphi(a))$$

$$
\begin{aligned}
&= \hat{\varphi}(s_2(a) + \iota(a)) - (s_2(\varphi(a)) + \iota(\varphi(a)))\\
&= (\hat{\varphi}(s_2(a)) - s_2(\varphi(a))) + \hat{\varphi}(\iota(a)) - \iota(\varphi(a))\\
&= \tau_2(a) + \varphi_{\mathfrak{V}}(\iota(a)) - \iota(\varphi(a))\\
&= \tau_2(x) - \Delta\iota(a).
\end{aligned}
$$

Hence, $(\varpi_1, \tau_1) - (\varpi_2, \tau_2) = (\delta\iota, -\Delta\iota) = \partial(\iota) \in \mathcal{B}^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$, that is (ϖ_1, τ_1) and (ϖ_2, τ_2) are in the same cohomological class in $\mathcal{H}^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$. □

Next we are ready to classify abelian extensions of a modified λ-differential Hom-Lie algebra.

Theorem 4.5. *Abelian extensions of a modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ are classified by the second cohomology group $\mathcal{H}^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$ of $(\mathfrak{L}, \alpha, \varphi)$ with coefficients in the representation $(\mathfrak{V}, \beta; \varrho, \varphi_{\mathfrak{V}})$.*

Proof. Assume that$(\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1)$ and $(\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2)$ are equivalent abelian extensions of $(\mathfrak{L}, \alpha, \varphi)$ by $(\mathfrak{V}, \beta, \varphi_{\mathfrak{V}})$ with the associated isomorphism $f : (\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1) \to (\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2)$ such that the diagram in (4.1) is commutative. Let s_1 be a section of $(\hat{\mathfrak{L}}_1, \hat{\alpha}_1, \hat{\varphi}_1)$. As $p_2 \circ f = p_1$, we get

$$
p_2 \circ (f \circ s_1) = p_1 \circ s_1 = \mathrm{id}_{\mathfrak{L}}.
$$

That is, $f \circ s_1$ is a section of $(\hat{\mathfrak{L}}_2, \hat{\alpha}_2, \hat{\varphi}_2)$. Denote $s_2 := f \circ s_1$. Since f is an isomorphism of modified λ-differential Hom-Lie algebras such that $f|_{\mathfrak{V}} = \mathrm{id}_{\mathfrak{V}}$, we have

$$
\begin{aligned}
\varpi_2(a, b) &= [s_2(a), s_2(b)]_{\hat{\mathfrak{L}}_2} - s_2([a, b])\\
&= [f(s_1(a)), f(s_1(b))]_{\hat{\mathfrak{L}}_2} - f(\sigma_1([a, b]))\\
&= f([s_1(a), s_1(b)]_{\hat{\mathfrak{L}}_1} - s_1([a, b]))\\
&= f(\varpi_1(a, b))\\
&= \varpi_1(a, b)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_2(a) &= \hat{\varphi}_2(s_2(a)) - s_2(\varphi(a)) = \hat{\varphi}_2(f(s_1(a))) - f(s_1(\varphi(a)))\\
&= f(\hat{\varphi}_1(s_1(a)) - s_1(\varphi(x)))\\
&= f(\tau_1(a))\\
&= \tau_1(a).
\end{aligned}
$$

So, all equivalent abelian extensions give rise to the same element in $\mathcal{H}^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$.

Conversely, given two cohomologous 2-cocycles (ϖ_1, τ_1) and (ϖ_2, τ_2) in $\mathcal{H}^2_{\mathrm{MDHLie}}(\mathfrak{L}, \mathfrak{V})$, we can construct two abelian extensions $(\mathfrak{L} \oplus \mathfrak{V}, [-, -]_{\varpi_1}, \alpha \oplus \beta, \varphi_{\tau_1})$ and $(\mathfrak{L} \oplus \mathfrak{V}, [-, -]_{\varpi_2}, \alpha \oplus \beta, \varphi_{\tau_2})$ via Proposition 4.3. Then there is a linear map $\gamma : \mathfrak{L} \to \mathfrak{V}$ such that

$$
(\varpi_1, \tau_1) - (\varpi_2, \tau_2) = \partial(\gamma) = (\delta\gamma, -\Delta\gamma).
$$

Define a linear map $f_\gamma : \mathfrak{L} \oplus \mathfrak{V} \to \mathfrak{L} \oplus \mathfrak{V}$ by $f_\gamma(a + u) := a + \gamma(a) + u,\ a \in \mathfrak{L}, u \in \mathfrak{V}$. Then we have f_γ is an isomorphism of these two abelian extensions. □

5. Skeletal modified λ-differential Hom-Lie 2-algebras

In this section, we introduce the notion of modified λ-differential Hom-Lie 2-algebras and show that modified λ-differential Hom-Lie 2-algebras are classified by 3-cocycles of modified λ-differential Hom-Lie algebras.

We first recall the definition of Hom-Lie 2-algebra [4], which is a categorization of a Hom-Lie algebra.

A Hom-Lie 2-algebra is a 7-tuple $(\mathfrak{L}_0, \mathfrak{L}_1, h, l_2, l_3, \alpha_0, \alpha_1)$, where $h : \mathfrak{L}_1 \to \mathfrak{L}_0$ is a linear map, $l_2 : \mathfrak{L}_i \times \mathfrak{L}_j \to \mathfrak{L}_{i+j}$ are bilinear maps, $\alpha_0 \in \mathrm{End}(\mathfrak{L}_0)$ and $\alpha_1 \in \mathrm{End}(\mathfrak{L}_1)$ satisfying $\alpha_0 \circ h = h \circ \alpha_1$, $l_3 : \mathfrak{L}_0 \times \mathfrak{L}_0 \times \mathfrak{L}_0 \to \mathfrak{L}_1$ is a skew-symmetric trilinear map satisfying $l_3 \circ (\alpha_0 \times \alpha_0 \times \alpha_0) = \alpha_1 \circ l_3$, such that for any $a, b, c, x \in \mathfrak{L}_0$ and $m, n \in \mathfrak{L}_1$, the following equations are satisfied:

$$l_2(a, b) = -l_2(b, a), \tag{5.1}$$

$$l_2(a, m) = -l_2(m, a), \tag{5.2}$$

$$l_2(m, n) = 0, \tag{5.3}$$

$$hl_2(a, m) = l_2(a, hm), \tag{5.4}$$

$$l_2(hm, n) = l_2(m, hn), \tag{5.5}$$

$$\alpha_0(l_2(a, b)) = l_2(\alpha_0(a), \alpha_0(b)), \tag{5.6}$$

$$\alpha_1(l_2(a, m)) = l_2(\alpha_0(a), \alpha_1(m)), \tag{5.7}$$

$$hl_3(a, b, c) = l_2(\alpha_0(a), l_2(b, c)) + l_2(\alpha_0(b), l_2(c, a)) + l_2(\alpha_0(c), l_2(a, b)), \tag{5.8}$$

$$hl_3(a, b, hm) = l_2(\alpha_0(a), l_2(b, m)) + l_2(\alpha_0(b), l_2(m, a)) + l_2(\alpha_1(m), l_2(a, b)), \tag{5.9}$$

$$\begin{aligned} &l_3(l_2(a, b), \alpha_0(c), \alpha_0(x)) + l_2(l_3(a, b, x), \alpha_0^2(c)) + l_3(\alpha_0(a), l_2(b, x), \alpha_0(c)) + l_3(l_2(a, x), \alpha_0(b), \alpha_0(c)) \\ &= l_2(l_3(a, b, c), \alpha_0^2(x)) + l_3(l_2(a, c), \alpha_0(b), \alpha_0(x)) + l_3(\alpha_0(a), l_2(b, c), \alpha_0(x)) \\ &\quad + l_2(\alpha_0^2(a), l_3(b, c, x)) + l_2(l_3(a, c, x), \alpha_0^2(b)) + l_3(\alpha_0(a), l_2(c, x), \alpha_0(b)). \end{aligned} \tag{5.10}$$

Motivated by [4, 19–21], we propose the definition of a modified λ-differential Hom-Lie 2-algebra.

Definition 5.1. A modified λ-differential Hom-Lie 2-algebra consists of a Hom-Lie 2-algebra $\mathcal{L} = (\mathfrak{L}_0, \mathfrak{L}_1, h, l_2, l_3, \alpha_0, \alpha_1)$ and a modified λ-differential 2-operator $\phi = (\phi_0, \phi_1, \phi_2)$ of $\mathcal{L}$, where $\phi_0 : \mathfrak{L}_0 \to \mathfrak{L}_0$, $\phi_1 : \mathfrak{L}_1 \to \mathfrak{L}_1$ are linear maps and $\phi_2 : \mathfrak{L}_0 \times \mathfrak{L}_0 \to \mathfrak{L}_1$ is a skew-symmetric bilinear map, and for any $a, b, c \in \mathfrak{L}_0, m \in \mathfrak{L}_1$, they satisfy the following equations:

$$\phi_0 \circ h = h \circ \phi_1, \tag{5.11}$$

$$\phi_0 \circ \alpha_0 = \alpha_0 \circ \phi_0, \tag{5.12}$$

$$\phi_1 \circ \alpha_1 = \alpha_1 \circ \phi_1, \tag{5.13}$$

$$\phi_2 \circ (\alpha_0 \times \alpha_0) = \alpha_1 \circ \phi_2, \tag{5.14}$$

$$h\phi_2(a, b) + \phi_0(l_2(a, b)) = l_2(\phi_0(a), b) + l_2(a, \phi_0(b)) + \lambda l_2(a, b), \tag{5.15}$$

$$\phi_2(a, hm) + \phi_1(l_2(a, m)) = l_2(\phi_0(a), m) + l_2(a, \phi_1(m)) + \lambda l_2(a, m), \tag{5.16}$$

$$\begin{aligned} &l_3(\phi_0(a), b, c) + l_3(a, \phi_0(b), c) + l_3(a, b, \phi_0(c)) + 2\lambda l_3(a, b, c) - \phi_1(l_3(a, b, c)) \\ &= -l_2(\phi_2(a, b), \alpha_0(c)) + l_2(\phi_2(a, c), \alpha_0(b)) + l_2(\alpha_0(a), \phi_2(b, c)) - \phi_2(l_2(a, b), \alpha_0(c)) \\ &\quad + \phi_2(l_2(a, c), \alpha_0(b)) + \phi_2(\alpha_0(a), l_2(b, c)). \end{aligned} \tag{5.17}$$

We denote a modified λ-differential Hom-Lie 2-algebra by $(\mathcal{L}, \phi)$.

A modified λ-differential Hom-Lie 2-algebra is said to be skeletal (resp. strict) if $h = 0$ (resp. $l_3 = 0, \phi_2 = 0$).

First we have the following trivial example of strict modified λ-differential Hom-Lie 2-algebra.

Example 5.2. For any modified λ-differential Hom-Lie algebra $(\mathfrak{L}, \alpha, \varphi)$, $(\mathfrak{L}_0 = \mathfrak{L}_1 = \mathfrak{L}, h = 0, l_2 = [-,-], \alpha_0 = \alpha_1 = \alpha, \phi_0 = \phi_1 = \varphi)$ is a strict modified λ-differential Hom-Lie 2-algebra.

Proposition 5.3. *Let $(\mathcal{L}, \phi)$ be a modified λ-differential Hom-Lie 2-algebra.*
(i) If $(\mathcal{L}, \phi)$ is skeletal or strict, then $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0, \phi_0)$ is a modified λ-differential Hom-Lie algebra, where $[a,b]_{\mathfrak{L}_0} = l_2(a,b)$ for any $a, b \in \mathfrak{L}_0$.
(ii) If $(\mathcal{L}, \phi)$ is strict, then $(\mathfrak{L}_1, [-,-]_{\mathfrak{L}_1}, \alpha_1, \phi_1)$ is a modified λ-differential Hom-Lie algebra, where $[m,n]_{\mathfrak{L}_1} = l_2(hm, n) = l_2(m, hn)$ for any $m, n \in \mathfrak{L}_1$.
(iii) If $(\mathcal{L}, \phi)$ is skeletal or strict, then $(\mathfrak{L}_1, \alpha_1; \rho, \phi_1)$ is a representation of $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0, \phi_0)$ with linear map $\rho : \mathfrak{L}_0 \to \mathrm{End}(\mathfrak{L}_1)$ by $\rho(a)m = l_2(a,m)$ for any $a \in \mathfrak{L}_0, m \in \mathfrak{L}_1$.

Proof. (i) From Eqs (5.1), (5.6), (5.8), (5.12) and (5.15), it can be concluded that $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0, \phi_0)$ is a modified λ-differential Hom-Lie algebra.
(ii) By Eqs (5.2), (5.7), (5.9), (5.13) and (5.16), we can easily check that $(\mathfrak{L}_1, [-,-]_{\mathfrak{L}_1}, \alpha_1, \phi_1)$ is a modified λ-differential Hom-Lie algebra.
(iii) By Eqs (5.7) and (5.9), we get that $(\mathfrak{L}_1, \alpha_1; \rho)$ is a representation of Hom-Lie algebra $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0)$. Further, by Eqs (5.13) and (5.16), we have $(\mathfrak{L}_1, \alpha_1; \rho, \phi_1)$ is a representation of $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0, \phi_0)$. □

Theorem 5.4. *There is a one-to-one correspondence between skeletal modified λ-differential Hom-Lie 2-algebras and 3-cocycles of modified λ-differential Hom-Lie algebras.*

Proof. Let $(\mathcal{L}, \phi)$ be a skeletal modified λ-differential Hom-Lie 2-algebra. By Proposition 5.3, we can consider the cohomology of modified λ-differential Hom-Lie algebra $(\mathfrak{L}_0, [-,-]_{\mathfrak{L}_0}, \alpha_0, \phi_0)$ with coefficients in the representation $(\mathfrak{L}_1, \alpha_1; \rho, \phi_1)$. For any $a, b, c, x \in \mathfrak{L}_0$, combining Eqs (3.1) and (5.10), we have

$$
\begin{aligned}
&\delta l_3(a,b,c,x)\\
=&\rho(\alpha_0^2(a))l_3(b,c,x) - \rho(\alpha_0^2(b))l_3(a,c,x) + \rho(\alpha_0^2(c))l_3(a,b,x) - \rho(\alpha_0^2(x))l_3(a,b,c)\\
&- l_3([a,b]_{\mathfrak{L}_0}, \alpha_0(c), \alpha_0(x)) + l_3([a,c]_{\mathfrak{L}_0}, \alpha_0(b), \alpha_0(x)) - l_3([a,x]_{\mathfrak{L}_0}, \alpha_0(b), \alpha_0(c))\\
&+ l_3([b,x]_{\mathfrak{L}_0}, \alpha_0(a), \alpha_0(c)) - l_3([b,c]_{\mathfrak{L}_0}, \alpha_0(a), \alpha_0(x)) - l_3([c,x]_{\mathfrak{L}_0}, \alpha_0(a), \alpha_0(b))\\
=&l_2(\alpha_0^2(a), l_3(b,c,x)) - l_2(\alpha_0^2(b), l_3(a,c,x)) + l_2(\alpha_0^2(c), l_3(a,b,x)) - l_2(\alpha_0^2(x), l_3(a,b,c))\\
&- l_3(l_2(a,b), \alpha_0(c), \alpha_0(x)) + l_3(l_2(a,c), \alpha_0(b), \alpha_0(x)) - l_3(l_2(a,x), \alpha_0(b), \alpha_0(c))\\
&+ l_3(, l_2(b,x), \alpha_0(a)\alpha_0(c)) - l_3(l_2(b,c), \alpha_0(a), \alpha_0(x)) - l_3(l_2(c,x), \alpha_0(a), \alpha_0(b))\\
=&0.
\end{aligned}
$$

By Eqs (3.2) and (5.17), there holds that

$$
\begin{aligned}
&(\delta\phi_2 - \Delta l_3)(a,b,c) = \delta\phi_2(a,b,c) - \Delta l_3(a,b,c)\\
=&\rho(\alpha_0(a))\phi_2(b,c) - \rho(\alpha_0(b))\phi_2(a,c) + \rho(\alpha_0(c))\phi_2(a,b) - \phi_2([a,b]_{\mathfrak{L}_0}, \alpha_0(c))
\end{aligned}
$$

$$
\begin{aligned}
&+\phi_2([a,c]_{\mathfrak{L}_0},\alpha_0(b))-\phi_2([b,c]_{\mathfrak{L}_0},\alpha_0(a))-l_3(\phi_0(a),b,c)-l_3(a,\phi_0(b),c)-l_3(a,b,\phi_0(c))\\
&-2\lambda l_3(a,b,c)+\phi_1 l_3(a,b,c)\\
=&l_2(\alpha_0(a),\phi_2(b,c))-l_2(\alpha_0(b),\phi_2(a,c))+l_2(\alpha_0(c),\phi_2(a,b))-\phi_2(l_2(a,b),\alpha_0(c))\\
&+\phi_2(l_2(a,c),\alpha_0(b))-\phi_2(l_2(b,c),\alpha_0(a))-l_3(\phi_0(a),b,c)-l_3(a,\phi_0(b),c)-l_3(a,b,\phi_0(c))\\
&-2\lambda l_3(a,b,c)+\phi_1 l_3(a,b,c)\\
=&0.
\end{aligned}
$$

So, $\partial(l_3,\phi_2)=(\delta l_3,\delta\phi_2-\Delta l_3)=0$, that is $(l_3,\phi_2)\in C^3_{\mathrm{MDHLie}}(\mathfrak{L}_0,\mathfrak{L}_1)$ is a 3-cocycle of modified λ-differential Hom-Lie algebra $(\mathfrak{L}_0,[-,-]_{\mathfrak{L}_0},\alpha_0,\phi_0)$ with coefficients in the representation $(\mathfrak{L}_1,\alpha_1;\rho,\phi_1)$.

Conversely, suppose that $(l_3,\phi_2)\in C^3_{\mathrm{MDHLie}}(\mathfrak{L},\mathfrak{V})$ is a 3-cocycle of modified λ-differential Hom-Lie algebra $(\mathfrak{L},[-,-],\alpha,\varphi)$ with coefficients in the representation $(\mathfrak{V},\beta;\rho,\varphi_{\mathfrak{V}})$. Then $(\mathcal{L},\phi)$ is a skeletal modified λ-differential Hom-Lie 2-algebra, where $\mathcal{L}=(\mathfrak{L}_0=\mathfrak{L},\mathfrak{L}_1=\mathfrak{V},h=0,l_2,l_3,\alpha_0=\alpha,\alpha_1=\beta)$ and $\phi=(\phi_0=\varphi,\phi_1=\varphi_{\mathfrak{V}},\phi_2)$ with $l_2(a,b)=[a,b],l_2(a,m)=\rho(a)m$ for any $a,b\in\mathfrak{L}_0,m\in\mathfrak{L}_1$. □

6. Conclusions

In the current research, we introduce the concept of modified λ-differential Hom-Lie algebras, and give its representation and dual representation. Additionally, the matched pair of modified λ-differential Hom-Lie algebras is also defined. Subsequently, we construct the cohomology of modified λ-differential Hom-Lie algebras with coefficients in a representation by introduce a cochain map Δ. Finally, we prove that any abelian extension of a modified λ-differential Hom-Lie algebra can obtain a representation and a 2-cocycle. Furthermore, it is obtained that any abelian extension is classified by the second cohomology group. In addition, any skeletal modified λ-differential Hom-Lie 2-algebra can be classified by the third cohomology group.

It is worth noting that modified λ-differential Lie algebras and Hom-Lie algebras with derivations are special cases of modified λ-differential Hom-Lie algebras (see Remarks 2.6 and 2.12), so the conclusions in the present paper can also be adapted to modified λ-differential Lie algebras and Hom-Lie algebras with derivations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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