AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 4135–4160.
DOI:10.3934/math.2024203
Received: 12 November 2023
Revised: 11 December 2023
Accepted: 19 December 2023
Published: 15 January 2024

Research article

Multiplicity of nontrivial solutions for a class of fractional Kirchhoff equations

Liuyang Shao[1,*], Haibo Chen[2], Yicheng Pang[1] and Yingmin Wang[1]

[1] School of Mathematics and Statistics, GuiZhou University of Finance and Economics, Guiyang, Guizhou, 550025, China

[2] School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China

* **Correspondence:** Email: sliuyang316@163.com.

Abstract: In this article, we study a class of fractional Kirchhoff with a superlinear nonlinearity:

$$\begin{cases} M(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)(-\triangle)^{\alpha} u + \lambda V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^{\alpha}(\mathbb{R}^N), & N \geq 1, \end{cases} \qquad (1.1)$$

where $\lambda > 0$ is a parameter, a and b are positive numbers satisfying $M(t) = am(t) + b$, $m : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. $V : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous. f satisfies $\lim\limits_{|t|\to\infty} f(x,t)/|t|^{k-1} = Q(x)$ uniformly in $x \in \mathbb{R}^N$ for each $2 < k < 2^*_\alpha, (2^*_\alpha = \frac{2N}{N-2\alpha})$. We investigated the effects of functions m and Q on the solution. By applying the variational method, we obtain the existence of multiple solutions. Furthermore, it is worth mentioning that the ground state solution has also been obtained.

Keywords: fractional Kirchhoff; varied method; multiplicity; nontrivial solution
Mathematics Subject Classification: 35B38, 35D30, 35J10, 35J20, 35J62

1. Introduction and main results

In this article, we investigate the existence of nontrivial solutions for the following fractional Kirchhoff equations with steep potential well:

$$\begin{cases} M(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)(-\triangle)^{\alpha} u + \lambda V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^{\alpha}(\mathbb{R}^N), & N \geq 1, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $(-\triangle)^{\alpha}$ stands for the fractional Laplacian, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous. a, λ are positive parameters, b is a nonnegative parameter, $2^*_\alpha := \frac{2N}{N-2\alpha}$ is the critical Sobolev exponent.

Equation (1.1) is related to the stationary analogue of the equation

$$\left\{\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2 dx\right) u_{xx} = 0, \right. \tag{1.2}$$

where P_0, h, E, L are constants, and (1.2) was proposed by Kirchhoff [1] as an extension of D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In [2], Fiscella and Valdinoci first proposed a stationary fractional Kirchhoff variational model with homogeneous Dirichlet boundary conditions and critical nonlinearity:

$$\begin{cases} M(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)(-\triangle)^{\alpha} u = \lambda f(x,u) + |u|^{2_s^*-2} u & \text{in } \mathbb{R}^N, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.3}$$

where M is a continuous Kirchhoff function whose model case is given by $M(t) = a + bt$. They proved the existence of a solution for the truncated problem. They also obtained the sign of the weak solutions of problem (1.3). There are some interesting results about fractional Kirchhoff equations (see [3–17] and their references). On the other hand, some studies have focused on the existence and multiplicity of solutions for fractional Kirchhoff equations see [18–26]. In particular, in [22], they studied the following fractional Kirchhoff equation

$$\left(p + q(1-s)\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\triangle)^s u = g(u), \text{ in } \mathbb{R}^N,$$

Under the Berestycki-Lions type assumptions. Applying minimax arguments, they established a multiplicity result for the above equation, provided that q is sufficiently small.

In another study [23], the authors studied the existence of multiple solutions for the following fractional p-Kirchhoff equation

$$\begin{cases} M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)(-\triangle)_p^s u = \lambda |u|^{q-2} u + \frac{|u|^{r-2}u}{|x|^{\alpha}} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N/\Omega. \end{cases}$$

Applying fibering maps and Nehari manifold, they obtained that the existence of multiple solutions to the above equation for both Hardy-Sobolev subcritical and critical cases.

Peng and Xia (see [24]) considered the existence, multiplicity and concentration of non-trivial solutions of the following concave-convex elliptic equations involving fractional Laplacian:

$$\begin{cases} (-\triangle)^{\alpha} u + V_{\lambda}(x) u = \alpha(x)|u|^{q-2} u + b(x)|u|^{p-2} u & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N. \end{cases}$$

They obtained multiplicity of solutions by applying Nehari manifold decomposition research.

Inspired by some of the previous results, and unlike other literature, we mainly discuss the influence of the number of solutions for functions m and f. We find that the number of differently solutions are obtained when the assumptions about m and f are different. Moreover, we also discuss the existence of ground state solution.

In the next step, we assume the potential function $V(x)$ as follows:
(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V \geq 0$ on $\mathbb{R}^N$;
(V_2) There exists $c > 0$ such that the set

$$\{V < c\} := \{x \in \mathbb{R}^N | V(x) < c\}$$

is nonempty and has finite measure;
(V_3) Let $\Omega = intV^{-1}(0)$ be a nonempty and smooth boundary with $\overline{\Omega} = V^{-1}(0)$.
$(V_1) - (V_3)$ are introduced by Bartsch and Wang, please refer to reference [27].
Finally, We can state the following our main results.

Theorem 1.1. *Assume that $(V_1) - (V_3)$ are satisfied, $N \geq 1$. Especially, under the following hypotheses $(F_1) - (F_3)$ on m and f for any $2 < k < 2^*_\alpha$;*
(F_1) there exists a constant $m_\infty > 0$ such that $\lim\limits_{t\to\infty} \frac{m(t)}{t^{\frac{k-2}{2}}} = m_\infty$ and $\int_\delta^\eta m(t)dt \geq \frac{2(\eta-\delta)}{k} m(\eta)$ for every $0 \leq \delta < \eta$;
(F_2) there exist $Q \in L^\infty(\mathbb{R}^N)$ and $0 \leq \mu < N - \frac{k(N-2\alpha)}{2}$ satisfying $Q \not\equiv 0$ on $\overline{\Omega}$ and $\liminf\limits_{|x|\to\infty} |x|^\mu Q(x) > 0$ such that $\lim\limits_{t\to\infty} \frac{f(x,t)}{t^{k-1}} = Q(x)$ uniformly for $x \in \mathbb{R}^N$;
(F_3) For any fixed real constant $x \in (0, \infty)$, $t \to \frac{f(x,t)}{t^{k-1}}$ is nondecreasing function. Then there exists $\widetilde{\gamma} > 0$ such that equation (1.1) admits at least one positive solution for all $\lambda > \widetilde{\gamma}$ and $a > 0$.
We now assume that the function m satisfies the following assumptions instead of condition (F_1):
(F_4) The function *$m(t)$ is nondecreasing for $t \in (0, +\infty)$;*
(F_5) *There are constants greater than zero m_0, σ* and *ϱ_0 such that $m(t) \geq m_0 t^\sigma$ for every $t \geq \varrho_0$.*
Then we have the following result.

Theorem 1.2. *Assume that $(V_1) - (V_3), (F_4)$ and (F_5) with $\sigma \geq \frac{2N}{N-2\alpha}$ are satisfied, $N \geq 3$. In addition, for any real number $2 < k < 2^*_\alpha$, let us suppose that the function f satisfies condition $(F_1), (F_2)$. Then there exist positive numbers $\widetilde{a}_*, \widetilde{\gamma}_* > 0$ such that for every $0 < a < \widetilde{a}_*$* and *$\lambda > \widetilde{\gamma}_*$, Equation (1.1) admits at least two positive solutions $u^1_{a,\lambda}$ and $u^2_{a,\lambda}$ satisfying $\mathcal{J}_{a,\lambda}(u^1_{a,\lambda}) < 0 < \mathcal{J}_{a,\lambda}(u^2_{a,\lambda})$. In particular, $u^1_{a,\lambda}$ is also a ground state solution of Eq (1.1).*
(F_6) there exist the function $Q(x)$ satisfying $Q(x) \not\equiv 0$ on $\overline{\Omega}$ and *$Q(x) \leq c^*|x|^{\frac{k(N-2\alpha)}{2}-N}$ for some $c^* > 0$ and for all $x \in \mathbb{R}^N$ such that $\lim\limits_{t\to\infty} \frac{f(x,t)}{t^{k-1}} = Q(x)$ uniformly in $x \in \mathbb{R}^N$.*

Theorem 1.3. *Assume that $(V_1) - (V_3)$ are satisfied, $N \geq 3$. Furthermore, for any real number $2 < k < 2^*_\alpha$, and let us suppose that conditions $(F_1), (F_6)$ and (F_3) are satisfied. Then for each $0 < a < \frac{1}{m_\infty \widetilde{\mu}_0^k}$ there exists $\overline{\gamma} > 0$ such that Equation (1.1) admits at least one positive solution for all $\lambda > \overline{\gamma}$.*

Theorem 1.4. *Assume that $(V_1) - (V_3), (F_3)$ are satisfied, $N \geq 3$. Furthermore, for every $2 < k < 2^*_\alpha$, and let us suppose that conditions (F_4) with $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) and (F_3) hold. Then there exists constants $\overline{a}_*, \overline{\gamma}_* > 0$ such that for every $0 < a < \overline{a}_*$ and $\lambda > \overline{\gamma}_*$, Equation (1.1) admits at least two nontrivial solutions $u^1_{a,\lambda} > 0$ and $u^2_{a,\lambda} > 0$ satisfying $\mathcal{J}_{a,\lambda}(u^1_{a,\lambda}) < 0 < \mathcal{J}_{a,\lambda}(u^2_{a,\lambda})$. Especially, $u^1_{a,\lambda}$ is also a ground state solution of Eq (1.1).*

Remark 1.5. For each $k \in [1, 2^*_\alpha - 1)$, given the definition

$$\lambda_0^k := \inf_{u \in E_0} \frac{\left(\int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} dxdy\right)^{\frac{k+1}{2}}}{\int_\Omega Q(x)|u|^{k+1} dx} > 0, \tag{1.4}$$

where $\Gamma = \mathbb{R}^{2N}|(\Omega^c \times \Omega^c), \Omega^c = \mathbb{R}^N|\Omega$, $Q(x)$ is bounded on $\overline{\Omega}$ with $Q^+ \neq 0$,

$$E_0 = \{\varphi \in L^2(\Omega)| \int_\Gamma \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{N+2\alpha}} dxdy < \infty, \varphi(x) = 0, \ if \ x \in \Omega\}$$

Which is achieved by $\phi_k \in E_0$ with $\int_\Omega Q(x)|u|^{k+1}dx = 1$ and $\phi_k > 0$ a.e. in Ω, by Fatou's Lemma and the compactness embedding theorem from E_0 into $L^{k+1}(\Omega)$ [28]. Under conditions (F_6) and (1.4), it is easing seen that for any $1 < k < 2^*_\alpha$, the minimum problem

$$\widetilde{\mu}_0^k := \inf_{u \in E_0} \frac{(\int_\Gamma |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{\frac{k}{2}}}{\int_\Omega |x|^{(\frac{k(N-2\alpha)}{2} - N)} |u|^k dx} > \frac{\overline{\nu}_1^{(k)}}{c^*} > 0. \tag{1.5}$$

Notation. We will use the following notation: C represents various positive constants; $\to$ ($\rightharpoonup$) means various strong (weak) convergence; $o(1)$ means $o(1) \to 0$ as $n \to \infty$; and $B_\rho(0)$ represents a ball centered at the origin with radius $\rho > 0$.

The remaining content of this article as follows: Section 2 introduces some preliminary results and some results will be used. Section 3 proving the main results.

2. Variational setting and preliminaries

We now collect some preliminary results for the fractional Laplacian. A complete introduction of fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ can be found in [29].

For any $\alpha \in (0, 1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2\alpha}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

It is widely known that $\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} dxdy = c_\alpha^{-1} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx$ where $c_\alpha = \frac{1}{2} \left(\int_{\mathbb{R}^N} \frac{1-cos\xi}{|\xi|^{N+2\alpha}} d\xi \right)^{-1}$ we endow the space $H^\alpha(\mathbb{R}^N)$ with the norm

$$E = \left\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty \right\} \quad E_\lambda = \left\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) u^2 dx < +\infty \right\}$$

It is Hilbert space equipped by the following norm

$$\langle u, u \rangle = \|u\|^2_{H^\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|(-\triangle)^{\frac{\alpha}{2}} u|^2 + V(x)u^2) dx, \ u \in H^\alpha(\mathbb{R}^N)$$

and

$$\langle u, u \rangle_\lambda = \|u\|^2_{H^\alpha(\mathbb{R}^N),\lambda} = \int_{\mathbb{R}^N} (|(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x)u^2) dx, \ u \in H^\alpha(\mathbb{R}^N).$$

$H^\alpha(\mathbb{R}^N)$ is also the completion of $C_0^\infty(\mathbb{R}^N)$ with $\|\cdot\|_{H^\alpha(\mathbb{R}^N)}$ and it is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [1, 2^*_\alpha]$. The homogeneous space $D^{\alpha,2}(\mathbb{R}^N)$ is

$$D^{\alpha,2}(\mathbb{R}^N) := \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2\alpha}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

and it is also the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$||u||_{D^{\alpha,2}} = \left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx\right)^{\frac{1}{2}}.$$

The energy functional is defined $\mathcal{J}^{\mu}_{\lambda,a}$ on E_λ as follows.

$$\mathcal{J}_{\lambda,a}(u) = \frac{a}{2}\widehat{m}\left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx\right) + \frac{1}{2}\left(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^N} \lambda V(x)u^2dx\right) - \int_{\mathbb{R}^N} F(x,u)dx \tag{2.1}$$

for all $u \in H^\alpha(\mathbb{R}^N)$.

Furthermore, it is easy to prove that we can obtain $\mathcal{J}_{\lambda,a} \in C^1(H^\alpha(\mathbb{R}^N),\mathbb{R})$, and if ϕ is a solution of Eq (1.1), we can get,

$$\langle \mathcal{J}'_{\lambda,a}(u),\phi\rangle = \left[am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx) + b\right]\int_{\mathbb{R}^N} (-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}\phi dx + \int_{\mathbb{R}^N} \lambda V(x)u\phi dx - \int_{\mathbb{R}^N} f(x,u)\phi dx \tag{2.2}$$

for all $\phi \in H^\alpha(\mathbb{R}^N)$.
The following inequalities will be applied to some related theorems.
For any $\lambda > 0$. By condition (V_1) and fractional Gagliardo-Nirenberg inequality (see [30]), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} u^2dx &= \int_{\{V\geq c\}} u^2dx + \int_{\{V<c\}} u^2dx \\ &\leq \frac{1}{c}\int_{\{V\geq c\}} V(x)u^2dx + \left(|\{V<c\}|\int_{\mathbb{R}^N} u^4dx\right)^{\frac{1}{2}} \\ &\leq \frac{1}{c}\int_{\mathbb{R}^N} V(x)u^2dx + \beta^2|\{V<c\}|^{\frac{1}{2}}||u||^{\frac{N}{2}}_{D^{\alpha,2}}||u||^{\frac{4-N}{4}}_{L^2} \\ &\leq \frac{1}{c}\int_{\mathbb{R}^N} V(x)u^2dx + \frac{N\beta^{\frac{8}{N}}}{4}|\{V<c\}|^{\frac{2}{N}}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx + (1-\frac{N}{4})\int_{\mathbb{R}^N} u^2dx\end{aligned}$$

which conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} u^2dx &\leq \frac{4}{Nc}\int_{\mathbb{R}^N} V(x)u^2dx + \beta^{\frac{8}{N}}|\{V<c\}|^{\frac{2}{N}}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx \\ &\leq (1+\beta^{\frac{8}{N}}|\{V<c\}|^{\frac{2}{N}})||u||^2_{H^\alpha(\mathbb{R}^N),\lambda}\end{aligned}$$

For $\lambda \geq \frac{4}{Nc}(1+\beta^{\frac{8}{N}}|\{V<c\}|^{\frac{2}{N}})^{-1}$.
where $S_\alpha := \inf\limits_{u\in D^{\alpha,2},u\neq 0} \frac{\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}}u|^2dx}{(\int_{\mathbb{R}^N} |u|^{2^*_\alpha}dx)^{\frac{2}{2^*_\alpha}}}$, which is introduced in [31](see Theorem 1.1).

$$\int_{\mathbb{R}^N} |u|^rdx \leq \left(\int_{\mathbb{R}^N} |u|^2dx\right)^{\frac{2^*_\alpha-r}{2^*_\alpha-2}}\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha}dx\right)^{\frac{r-2}{2^*_\alpha-2}}$$

$$= \left(\int_{\{V \geq c\}} |u|^2 dx + \int_{\{V < c\}} |u|^2 dx \right)^{\frac{2_\alpha^* - r}{2_\alpha^* - 2}} \left(S_\alpha^{-2_\alpha^*} \left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{2_\alpha^*}{2}} \right)^{\frac{r-2}{2_\alpha^*-2}}$$

$$\leq \left(\frac{1}{\lambda c} \int_{\mathbb{R}^N} \lambda V(x) u^2 dx + |\{V < c\}|^{\frac{2_\alpha^*-2}{2_\alpha^*}} S_\alpha^{-2} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{r-2}{2_\alpha^*-2}}$$

$$\cdot \left[S_\alpha^{-2_\alpha^*} \left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x) u^2 dx \right)^{\frac{2_\alpha^*}{2}} \right]^{\frac{r-2}{2_\alpha^*-2}}$$

$$\leq \left[max \left\{ \frac{1}{\lambda c}, S_\alpha^{-2} |\{V < c\}|^{\frac{2_\alpha^*-2}{2_\alpha^*}} \right\} \int_{\mathbb{R}^N} (|(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x) u^2) dx \right]^{\frac{2_\alpha^*-r}{2_\alpha^*-2}} S_\alpha^{\frac{-2_\alpha^*(r-2)}{2_\alpha^*-2}}$$

$$\cdot \left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x) u^2 dx \right)^{\frac{2_\alpha^*(r-2)}{2_\alpha^*-2}}$$

$$\leq |\{V < c\}|^{\frac{2_\alpha^*-r}{2_\alpha^*}} S_\alpha^{-r} \|u\|^r_{H^\alpha(\mathbb{R}^N),\lambda} \tag{2.4}$$

when $\lambda \geq \frac{S_\alpha^2}{c} |\{V < c\}|^{\frac{2-2_\alpha^*}{2_\alpha^*}}$. Since the imbedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ $(2 \leq q < \infty)$ is continuous for $N = 1, 2$. Similar to (2.4) we have $\int_{\mathbb{R}^N} |u|^r dx \leq S_\alpha^{-r} (1 + \beta^{\frac{8}{N}} |\{V < c\}|^{\frac{2}{N}})^{\frac{r}{2}} \|u\|^r_{H^\alpha(\mathbb{R}^N),\lambda}$ we set

$$\gamma_N := \begin{cases} \frac{4}{Nc(1+\beta^{\frac{8}{N}} |\{V<c\}|^{\frac{2}{N}})} & \text{if } N = 1, 2, \\ \frac{S_\alpha^2}{c} |\{V < c\}|^{-\frac{2}{N}}, & \text{if } N \geq 3; \end{cases} \tag{2.5}$$

and

$$\tau_{r,N} := \begin{cases} S_\alpha^{-r} (1 + \beta^{\frac{8}{N}} |\{V < c\}|^{\frac{2}{N}})^{\frac{r}{2}} & \text{if } N = 1, 2, \\ |\{V < c\}|^{\frac{2_\alpha^*-r}{2_\alpha^*}} S_\alpha^{-r}, & \text{if } N \geq 3; \end{cases} \tag{2.6}$$

3. Proofs of Theorem 1.1–1.4

To complete the proof of Theorem 1.1-1.3, we need the following result.

Lemma 3.1. *If assumptions* $(V_1) - (V_3)$ *and* $(F_1), (F_2)$ *are satisfied. Then for each* $\lambda \geq \gamma_N$ *there eixists* $\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > 0$ *and a constant* $\varrho_0 > 0$ *such that*

$$\inf \mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \ with \ \|u\|_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > \varrho_0$$

Proof. It follows from assumption $(F_1) - (F_2)$ that we have

$$f(x,t) \leq Q(x) t^{k-1} \ \ for \ all \ t \geq 0 \tag{3.1}$$

and

$$F(x,t) \leq \frac{1}{k} Q(x) t^k \ \ for \ all \ t \geq 0. \tag{3.2}$$

Thus, using (2.6) and (3.2), we have, for $u \in E$ and $\lambda \geq \gamma_N$

$$\int_{\mathbb{R}^N} F(x,u) dx \leq \frac{|Q|_\infty \tau_{k,N}}{k} \|u\|^k_{H^\alpha(\mathbb{R}^N),\lambda}.$$

We can infer that

$$\mathcal{J}_{\lambda,a}(u) = \frac{a}{2}\widehat{m}\left(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx\right) + \left(\frac{b}{2}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\lambda V(x)u^2dx\right) - \int_{\mathbb{R}^N}F(x,u)dx$$
$$\geq min\left\{\frac{1}{2},\frac{b}{2}\right\}||u||^2_{H^\alpha(\mathbb{R}^N),\lambda} - \frac{|Q|_\infty\tau_{k,N}}{k}||u||^k_{H^\alpha(\mathbb{R}^N),\lambda}.$$

Hence, there exists $||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > 0$ small enough, we can get that there exists a constant $\varrho_0 > 0$ such that

$$\inf \mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \; with \; ||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > \varrho_0$$

for all $\lambda \geq \gamma_N$, where $2 < k < 2^*_\alpha$. which is what we wanted to prove. □

By the following Lemma 3.2 and Lemma 3.3, we are able to proof that the functional $\mathcal{J}^\mu_{\lambda,a}$ satisfies the mountain pass geometry.

Lemma 3.2. *If assumptions* $(V_1)-(V_3)$, (F_6) and (F_3) *are satisfied. Then there exist* $||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > 0$ *and a constant* $\varrho_0 > 0$ *such that*

$$\inf\left\{\mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \; with \; ||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0\right\} > \varrho_0$$

for each $a > 0$ *and* $\lambda \geq \gamma_N$.

Proof. By assumption (F_6) and (F_3), we have

$$f(x,t) \leq c^*|x|^{(\frac{k(N-2\alpha)}{2}-N)}t^{k-1} \quad for \; all \; t \geq 0 \tag{3.3}$$

and

$$F(x,t) \leq \frac{c^*}{k}|x|^{(\frac{k(N-2\alpha)}{2}-N)}t^k \quad for \; all \; t \geq 0. \tag{3.4}$$

Then, using (1.4) and (3.4), for each $u \in E$ and $\lambda \geq \gamma_N$, we obtain

$$\int_{\mathbb{R}^N}F(x,u)dx \leq \frac{c^*}{k}\int_{\mathbb{R}^N}|x|^{(\frac{k(N-2\alpha)}{2}-N)}|u|^kdx \leq \frac{c^*}{k\overline{v}_1^{(k)}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{\frac{k}{2}}$$
$$\leq \frac{c^*}{k\overline{v}_1^{(k)}}||u||^k_{H^\alpha(\mathbb{R}^N),\lambda} \tag{3.5}$$

So we can infer that

$$\mathcal{J}_{\lambda,a}(u) \geq \frac{1}{2}\left(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^N}\lambda V(x)u^2dx\right) - \frac{c^*}{k\overline{v}_1^{(k)}}||u||^k_{H^\alpha(\mathbb{R}^N),\lambda}$$
$$\geq min\{\frac{b}{2},\frac{1}{2}\}||u||^2_{H^\alpha(\mathbb{R}^N),\lambda} - \frac{c^*}{k\overline{v}_1^{(k)}}||u||^k_{H^\alpha(\mathbb{R}^N),\lambda}.$$

Hence, there exists $||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0 > 0$ small enough, we can get that there exists a constant $\varrho_0 > 0$ such that

$$\inf\left\{\mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \; with \; ||u||_{H^\alpha(\mathbb{R}^N),\lambda} = r_0\right\} > \varrho_0$$

for all $\lambda \geq \gamma_N$, with $2 < k < 2^*_\alpha$. which is what we wanted to prove. □

Lemma 3.3. *If assumptions $(V_1)-(V_3),(F_1)$ and $(F_2)-(F_3)$ are satisfied. As in Lemma 3.1 $r_0>0$. Then there exists $v_0\in E$ with $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ such that $\mathcal{J}_{\lambda,a}(v_0)<0$, where λ and a are positive numbers*.

Proof. We set $u\in E\backslash\{0\}$ with $u>0$ and define $u_n(x)=n^{-\frac{N}{k}}u(\frac{x}{n})$. It is easy to compute that

$$\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx=n^{N-2\alpha-\frac{2N}{k}}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx$$

and

$$\int_{\mathbb{R}^N}Q(x)u_n^kdx=n^{-N}\int_{\mathbb{R}^N}Q(x)u^k(\frac{x}{n})dx=\int_{\mathbb{R}^N}Q(nx)u^k(x)dx.$$

Hence, Using (F_2) and Fatou's lemma, we have

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^k}{\int_{\mathbb{R}^N}Q(x)u_n^kdx}&=\frac{n^{-(N-\frac{k(N-2\alpha)}{2})}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^k}{\int_{\mathbb{R}^N}Q(nx)u^kdx}=\frac{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2k}}{n^{N-\frac{k(N-2\alpha)}{2}}\int_{\mathbb{R}^N}Q(x)u^kdx}\\&\leq\frac{\Omega_0^\mu\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|dx}{Cn^{N-\frac{k(N-2\alpha)}{2}-\mu}\int_{|x|\leq\Omega_0}u^k(x)dx}\to 0\ as\ n\to\infty\end{aligned}$$

It is following that

$$\inf_{u\in E}\frac{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{\frac{k}{2}}}{\int_{\mathbb{R}^N}Q(x)|u|^kdx}=0.$$

Therefore, for every $a>0$, there exists $\phi_k\in E\backslash\{0\}$ with $\phi_k>0$ such that

$$am_\infty(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\phi_k|^2dx)^k-\int_{\mathbb{R}^N}Q(x)\phi_k^kdx<0.$$

Applying the above inequality, and by $(F_1),(F_2)-(F_3)$ and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathcal{J}_{a,\lambda}(t\phi_k)}{t^k}&=\lim_{t\to+\infty}\frac{1}{2t^{k-2}}\left(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\phi_k|^2dx+\int_{\mathbb{R}^N}\lambda V(x)\phi_k^2dx\right)\\&\quad+\lim_{t\to+\infty}\left[\frac{a\widehat{m}(t^2\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\phi_k|^2dx)}{(2t^k\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^k}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\phi_k|^kdx-\int_{\mathbb{R}^N}\frac{F(x,t\phi_k)}{t^k\phi_k^k}\phi_k^kdx\right]\\&\leq\frac{1}{k}\left((am_\infty\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\phi_k|^2dx)^k-\int_{\mathbb{R}^N}Q(x)\phi_k^kdx\right)<0\end{aligned}$$

Which means that $\mathcal{J}_{a,\lambda}(t\phi_k)\to-\infty$ as $t\to+\infty$. Hence, there exists $v_0\in E$ with $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ such that $\mathcal{J}_{a,\lambda}(v_0)<0$. Thus, we complete the proof. □

Now, we can obtain the following lemma and then finally yield the convergence result.

Lemma 3.4. *If assumptions $(V_1)-(V_3)$ and $(F_2)-(F_3)$ are satisfied. As in Lemma 3.1 $r_0>0$. Then there exist $\widetilde{a}_*>0$ and $v_0\in E$ with $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ such that $\mathcal{J}_{a,\lambda}(v_0)<0$ for all $0<a<\widetilde{a}_*$ and $\lambda>0$.*

Proof. By Lemma 3.3, using (F_2), for $\phi_k \in E\backslash\{0\}$ let $\phi_k > 0$ satisfy $\int_{\mathbb{R}^N} Q(x)\phi_k^k dx > 0$. Then from $(F_2)-(F_3)$ and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathcal{J}_{\lambda,0}(t\phi_k)}{t^k} =& \lim_{t\to\infty}\frac{1}{2t^{k-2}}\left(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx+\int_{\mathbb{R}^N}\lambda V(x)\phi_k^2dx\right)\\ &-\lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{F(x,t\phi_k)}{t^k\phi_k^k}dx\\ \le& -\frac{1}{k}\int_{\mathbb{R}^N}Q(x)\phi_k^k dx<0\end{aligned}$$

where $\mathcal{J}_{\lambda,0}(u)=\mathcal{J}_{\lambda,a}(u)$ with $a=0$. Hence $\mathcal{J}_{\lambda,0}(t\phi_k)\to-\infty$ as $t\to\infty$, then there exists $v_0\in E$ with $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ such that $\mathcal{J}_{\lambda,0}(v_0)<0$. Since $\mathcal{J}_{\lambda,a}\to\mathcal{J}_{\lambda,0}(v_0)$ as $a\to0^+$, we get that there exists $\widetilde{a}_*>0$ such that $\mathcal{J}_{\lambda,a}(v_0)<0$ for all constants $0<a<\widetilde{a}_*$ and $\lambda>0$. □

Lemma 3.5. *If assumptions* $(V_1)-(V_3),(F_1)$. (F_6) *and* (F_3) *are satisfied. Let* $r_0>0$ *be as Lemma 3.2. Then for each* $0<a<\frac{1}{m_\infty\widetilde{\mu}_0^{(k)}}$, *there exists* $v_0\in E$ *with* $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ *such that* $\mathcal{J}_{a,\lambda}(v_0)<0$ *for every* $\lambda>0$.

Proof. It follows from (1.5) that for each $0<a<\frac{1}{m_\infty\widetilde{\mu}_0^k}$, there exists $\varphi_k\in H^\alpha(\mathbb{R}^N)$ with $\psi_k>0$ such that

$$\widetilde{\mu}_0^k\le\frac{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k}{\int_{\mathbb{R}^N}Q(x)\varphi_k^kdx}<\frac{1}{am_\infty},$$

which implies that

$$am_\infty(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k-\int_{\mathbb{R}^N}Q(x)\varphi_k^kdx<\frac{1}{\widetilde{\mu}_0^k}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k-\int_{\mathbb{R}^N}Q(x)\varphi_k^kdx\le0.$$

Using this, together with conditions $(F_1),(F_6),(F_3)$ and Lebesgue's dominated convergence theorem, yields

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathcal{J}_{\lambda,a}(t\varphi_k)}{t^k}=&\lim_{t\to\infty}\frac{1}{t^{2k-2}}\left(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx+\int_{\mathbb{R}^N}\lambda V(x)\varphi_k^2dx\right)\\ &+\lim_{t\to\infty}\left[\frac{a\widehat{m}(t^2\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)}{2t^k(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k-\int_{\mathbb{R}^N}\frac{F(x,t\varphi_k)}{t^k\varphi_k^k}dx\right]\\ \le&\frac{am_\infty(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k}{k}-\frac{1}{k}\int_{\mathbb{R}^N}Q(x)\varphi_k^kdx\\ =&\frac{1}{k}\left(am_\infty(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}\varphi_k|^2dx)^k-\int_{\mathbb{R}^N}Q(x)\varphi_k^k\right)\le0\end{aligned}$$

This implies that $\mathcal{J}_{\lambda,a}(t\varphi_k)\to-\infty$ as $t\to\infty$. Hence, for each $0<a<\frac{1}{m_\infty\widetilde{\mu}_0^k}$, there exists $v_0\in E$ with $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ such that $\mathcal{J}_{a,\lambda}(v_0)<0$ for all $\lambda>0$. we complete the proof.

Lemma 3.6. *If assumptions* $(V_1)-(V_3),(F_6)$ *and* (F_3) *are satisfied.* $r_0>0$ *is defined as in the proof of Lemma 3.2. Then there exists constants* $\overline{a}_*>0$ *and* $v_0\in E$ *with* $\|v_0\|_{H^\alpha(\mathbb{R}^N),\lambda}>r_0$ *such that* $\mathcal{J}_{\lambda,a}(v_0)<0$

for each $0 < a < \overline{a}_*$ *and* $\lambda > 0$.

Proof. We can get the result as in the proof of Lemma 3.4, so we omit it here. □

Proof of Theorem 1.1

By Lemma 3.1 and 3.3 and the mountain pass theorem [32], we get that for every $\lambda \geq \gamma_N$ and $a > 0$, there exists a sequence $\{u_n\} \subset E_\lambda$ such that

$$\mathcal{J}_{a,\lambda}(u_n) \to \alpha_{a,\lambda} \text{ and } 1 + \|u_n\|_{H^\alpha(\mathbb{R}^N,\lambda)}\|\mathcal{J}'_{a,\lambda}\|_{E_\lambda^{-1}} \to 0 \tag{4.0}$$

as $n \to \infty$ where $0 \leq \alpha_{a,\lambda} \leq \alpha_{a,0} \leq \mathcal{D}_a$. By the following some lemmas we able to get our main result.

Lemma 4.1. *If assumptions* $(V_1)-(V_3)$, (F_1) *and* (F_3) *are satisfied. Then for every constant* $a > 0$ *and* $\lambda \geq \gamma_N$, *we can admits an bounded sequence* $\{u_n\}$ *in* E_λ *as in the define of* (4.0).

Proof. Using (F_3), for any $t > 0$ we obtain that

$$F(x,t) - \frac{1}{k}f(x,t)t = \int_0^t \left(\frac{f(x,s)}{s^{k-1}} - \frac{f(x,t)}{t^{k-1}}\right)s^{k-1}ds \leq 0. \tag{4.1}$$

For $n \to \infty$, note that by (F_1) and (4.0) – (4.1) we have

$$\begin{aligned}
\alpha_{\lambda,a} + 1 \geq & \mathcal{J}_{\lambda,a}(u_n) - \frac{1}{k}\langle \mathcal{J}'_{\lambda,a}(u_n), u_n\rangle \\
= & \frac{k-2}{2k}\left(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx + \lambda V(x)u_n^2dx\right) \\
& + \frac{a}{2}\left[\widehat{m}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx) - \frac{2}{k}m(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx\right] \\
& - \int_{\mathbb{R}^N}[F(x,u_n) - \frac{1}{k}f(x,u_n)u_n]dx \\
\geq & \frac{(k-2)min\{b,1\}}{2k}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2 + \lambda V(x)u_n^2dx
\end{aligned}$$

we also deduce that $\{u_n\}$ is bounded in E_λ for each $a > 0$ and $\lambda \geq \gamma_n$. □

Lemma 4.2. *If assumptions* $(V_1)-(V_3)$, (F_5) *with* $\sigma \geq \frac{2\alpha}{N-2\alpha}$ *and* $(F_2)-(F_3)$ *are satisfied when* $N \geq 3$. *Then for all* $0 < a < \widetilde{a}_*$ *and*

$$\lambda > \widetilde{\Lambda}_0 := \begin{cases} \frac{|Q|_\infty}{c_0}max\{(\frac{|Q|_\infty}{am_0S_\alpha^{2_\alpha^*}}), \frac{2_\alpha^*-k}{2_\alpha^*-2}\}, \ if \ \sigma = \frac{2\alpha}{N-2\alpha} \\ \frac{|Q|_\infty(2_\alpha^*-k)}{c_0(2_\alpha^*-2)}, \ if \ \sigma > \frac{2\alpha}{N-2\alpha}; \end{cases}$$

we can admits an bounded sequence $\{u_n\}$ *in* E_λ *as in the define of* (4.0).

Proof. (i) $\sigma = \frac{2\alpha}{N-2\alpha}$: Note that $2(\delta+1) = 2_\alpha^*$. By using reduction to absurdity. Set $\|u_n\|_{H^\alpha(\mathbb{R}^N),\lambda} \to \infty$ as $n \to \infty$. This proof is divided into three cases:

Case I: $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx \to \infty$ and

$$\frac{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2(\sigma+1)}} = \lambda c_0 S_\alpha^{-2_\alpha^*}(\frac{|Q|_\infty}{\lambda c_0})^{\frac{2_\alpha^*-2}{k-2}}. \tag{4.2}$$

Using (4.0), we obtain that

$$\frac{\langle \mathcal{J}'_{a,\lambda}(u_n), u_n\rangle}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2(\sigma+1)}} = o(1),$$

Applying (F_5) and (3.1), we get that

$$\begin{aligned}
o(1) =& \frac{am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} + \frac{b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \\
&- \frac{\int_{\mathbb{R}^N} f(x,u_n)u_n dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \\
\geq& am_0 + \frac{b}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2\sigma}} + \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}}
\end{aligned} \tag{4.3}$$

In addition, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^k dx \leq S_\alpha^{-\frac{2_\alpha^*(k-2)}{2_\alpha^*-2}} (\lambda c_0)^{-\frac{2_\alpha^*-k}{2_\alpha^*-2}} (\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx)^{\frac{2_\alpha^*-k}{2_\alpha^*-2}} (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{\frac{2_\alpha^*(k-2)}{2_\alpha^*-2}} \\
&+ |\{V < c_0\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}} S_\alpha^{-k} (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2k}
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx - |Q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \\
&\geq \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \left[1 - |Q|_\infty (\lambda c_0)^{-\frac{2_\alpha^*-k}{2_\alpha^*-2}} (\frac{S_\alpha^{-2_\alpha^*} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2_\alpha^*}}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx})^{\frac{k-2}{2_\alpha^*-2}}\right] \\
&- \frac{|Q|_\infty |\{V < c_0\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}}}{S_\alpha^k (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)-k}}
\end{aligned} \tag{4.4}$$

To proceed, note that by (4.2) − (4.4) we have

$$\begin{aligned}
o(1) \geq& am_0 + \frac{b}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2\sigma}} - \frac{|Q|_\infty |\{V < c_0\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}}}{S_\alpha^k (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)-k}} \\
&+ \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \cdot [1 - |Q|_\infty (\lambda c_0)^{-\frac{2_\alpha^*-k}{2_\alpha^*-2}} (\frac{S_\alpha^{-2_\alpha^*} (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2_\alpha^*}}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx})^{\frac{k-2}{2_\alpha^*-2}}] \\
=& am_0 + \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \cdot [1 - |Q|_\infty (\lambda c_0)^{-\frac{2_\alpha^*-k}{2_\alpha^*-2}} (\frac{S_\alpha^{-2_\alpha^*} (\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2_\alpha^*}}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx})^{\frac{k-2}{2_\alpha^*-2}}] \\
&+ o(1) \\
\geq& am_0 + o(1)
\end{aligned}$$

But it's a paradox.
Case II: $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx \to \infty$ and

$$\frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} < \lambda c_0 S_\alpha^{-2^*_\alpha} (\frac{|Q|_\infty}{\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}}. \tag{4.5}$$

Using (2.4) and (2.6) leads to

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |u_n|^k dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} &\le S_\alpha^{-2^*_\alpha} (\frac{|Q|_\infty}{\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}} + \frac{S_\alpha^{-k} |\{V < c_0\}|^{\frac{2^*_\alpha - k}{2^*_\alpha}}}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\delta+1)-k}} \\ &= S_\alpha^{-2^*_\alpha} (\frac{|Q|_\infty}{\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}} + o(1)\end{aligned} \tag{4.6}$$

In view of (4.3) and (4.6) we have

$$\begin{aligned}o(1) &= \frac{\langle \mathcal{J}_{a,\lambda}(u_n), u_n \rangle}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \\ &\ge am_0 + \frac{b}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2\sigma}} + \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} - \frac{|Q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} \\ &\ge am_0 + \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx)^{2(\sigma+1)}} - |Q|_\infty S_\alpha^{-2^*_\alpha} (\frac{|Q^+|_\infty}{\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}} + o(1) \\ &\ge am_0 - |Q|_\infty S_\alpha^{-2^*_\alpha} (\frac{|Q^+|_\infty}{\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}} + o(1)\end{aligned}$$

This a contradicts with

$$\lambda > \frac{|Q|_\infty}{c_0} (\frac{|Q|_\infty}{am_0 S_\alpha^{2^*_\alpha}})^{\frac{k-2}{2^*_\alpha} - k}.$$

Case III: $\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx \to \infty$ and $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx \le C_*$ for some $C_* > 0$ and for every n. Using (2.4), (2.2) and (F_5) it conclude that

$$\begin{aligned}o(1) =& \frac{am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} + \frac{b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \\ &- \frac{\int_{\mathbb{R}^N} f(x, u_n) u_n dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \\ \ge& \frac{b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} + 1 - \frac{|Q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}\end{aligned} \tag{4.7}$$

By (2.4) and the Young inequality we also get that

$$\int_{\mathbb{R}^N} |u_n|^k dx \le \frac{2^*_\alpha - k}{2^*_\alpha - 2} \left(\frac{1}{\lambda} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx + |\{V < c_0\}|^{\frac{2}{N}} S_\alpha^{-2} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx \right)$$

$$+\frac{k-2}{2_\alpha^*-2}S_\alpha^{-2_\alpha^*}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}. \tag{4.8}$$

By (4.8) and the fact of $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx \le C_*$ for all n, we obtain that

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}|u_n|^kdx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx} &\le \frac{2_\alpha^*-k}{(2_\alpha^*-2)\lambda c_0}+\frac{(2_\alpha^*-k)|\{V<c_0\}|^{\frac{2}{N}}S_\alpha^{-2}C_*^2+(k-2)S_\alpha^{-2_\alpha^*}C_*^{2_\alpha^*}}{(2_\alpha^*-2)\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}\\ &=\frac{2_\alpha^*-k}{(2_\alpha^*-2)\lambda c_0}+o(1)\end{aligned} \tag{4.9}$$

In view of (4.7) and (4.9), we can get that

$$o(1)\ge\frac{b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}+1-\frac{|Q|_\infty\int_{\mathbb{R}^N}|u_n|^pdx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}\ge 1-\frac{|Q|_\infty(2_\alpha^*-k)}{(2_\alpha^*-2)\lambda c_0}+o(1)$$

This contradicts with

$$\lambda>\frac{|Q|_\infty(2_\alpha^*-k)}{(2_\alpha^*-2)c_0}$$

(ii) $\sigma>\frac{2\alpha}{N-2\alpha}$: Clearly, $2(\sigma+1)>2_\alpha^*$. We argue indirectly. Set $\|u_n\|_{H^\alpha(\mathbb{R}^N),\lambda}\to\infty$ as $n\to\infty$. This proof is divided into two cases:

Case IV: $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx\to\infty$. It follows from (2.2) − (3.1) and condition (F_5) that

$$\begin{aligned}o(1)=&\frac{am(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}+\frac{b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx+\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\\ &-\frac{\int_{\mathbb{R}^N}f(x,u_n)u_ndx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\\ \ge& am_0(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2(\delta+1)-2_\alpha^*}+\frac{b}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}-2}\\ &+\frac{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx-|Q|_\infty\int_{\mathbb{R}^N}|u_n|^kdx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\end{aligned} \tag{4.10}$$

By (4.8), we deduce that

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx-|Q|_\infty\int_{\mathbb{R}^N}|u_n|^kdx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\\ \ge&1-\frac{|Q|_\infty(2_\alpha^*-k)}{\lambda c_0(2_\alpha^*-2)}\frac{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\\ &-\frac{|Q|_\infty S_\alpha^{-2}(2_\alpha^*-k)|\{V<c_0\}|^{\frac{2}{N}}}{(2_\alpha^*-2)(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*-2}}-|Q|_\infty S_\alpha^{-2_\alpha^*}(\frac{k-2}{2_\alpha^*-2})\\ \ge&-\frac{|Q|_\infty S_\alpha^{-2}(2_\alpha^*-k)|\{V<c_0\}|^{\frac{2}{N}}}{(2_\alpha^*-2)(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*-2}}-|Q|_\infty S_\alpha^{-2_\alpha^*}(\frac{k-2}{2_\alpha^*-2})\end{aligned}$$

Using this, together with (4.10), leads to

$$\begin{aligned}
&am_0(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2(\sigma+1)-2_\alpha^*}+\frac{b}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}-2}\\
&+\frac{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx-|Q|_\infty\int_{\mathbb{R}^N}|u_n|^kdx}{(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}\\
\geq &am_0(\int_{R^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2(\sigma+1)-2_\alpha^*}-\frac{|Q|_\infty S_\alpha^{-2}(2_\alpha^*-k)|\{V<c_0\}|^{\frac{2}{N}}}{((2_\alpha^*-2)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)^{2_\alpha^*}}-|Q|_\infty S_\alpha^{-2_\alpha^*}(\frac{k-2}{2_\alpha^*-2})\\
&\to\infty\ as\ n\to\infty
\end{aligned}$$

Since $2(\sigma+1)>2_\alpha^*$. We get a contradiction.
Case V: $\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx\to\infty$ and $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx\leq C^*$ for some $C^*>0$ and for all n. It follows from $(2.2)-(3.1)$ and condition (F_5) that

$$\begin{aligned}
o(1)=&\frac{am(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}+\frac{b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx+\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}\\
&-\frac{\int_{\mathbb{R}^N}f(x,u_n)u_ndx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}\\
\geq&\frac{b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}+1-\frac{|Q|_\infty\int_{\mathbb{R}^N}|u_n|^kdx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}
\end{aligned}\tag{4.11}$$

By (4.9) and (4.11) one has

$$o(1)\geq\frac{b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}+1-\frac{|Q|_\infty\int_{\mathbb{R}^N}|u_n|^kdx}{\int_{\mathbb{R}^N}\lambda V(x)u_n^2dx}\geq1-\frac{|Q|_\infty(2_\alpha^*-k)}{\lambda c_0(2_\alpha^*-2)}+o(1),$$

This contradicts with

$$\lambda>\frac{|Q|_\infty(2_\alpha^*-k)}{c_0(2_\alpha^*-2)}$$

We the conclude that the sequence $\{u_n\}$ is bounded in E_λ for all constants $0<a<\widetilde{a}_*$ and $\lambda>\widetilde{\gamma}_0$. The proof is complete. □

Lemma 4.3. *If assumptions $(V_1)-(V_3),(F_1)-(F_3)$ are satisfied, $N\geq1$. Then for every $\mathcal{D}>0$ there exists constant $\widetilde{\gamma}=\widetilde{\gamma}(a,\mathcal{D})\geq\gamma_N>0$ such that $\mathcal{J}_{\lambda,a}$ satisfies the $(C)_\alpha$-condition in E_λ for numbers $\lambda>\widetilde{\gamma}$ and $\alpha<\mathcal{D}$.*

Proof. We may suppose that $\{u_n\}$ is a C_α-sequence with $\alpha<\mathcal{D}$. Note that Lemma 4.1, we can get that $\{u_n\}$ is bounded in E_λ. Then there exist a subsequence $\{u_n\}$ and $u_0\in E_\lambda$ such that $u_n\rightharpoonup u_0$ weakly in E_λ and $u_n\to u_0$ strongly in $L^r_{loc}(\mathbb{R}^N)$ for $2\leq r<2_\alpha^*$. From this there follows $u_n\to u_0$ strongly in E_λ. Let $v_n=u_n-u_0$. Apply (V_1), we have

$$\int_{\mathbb{R}^N}v_n^2dx\leq\frac{1}{\lambda c_0}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2+\lambda V(x)u_n^2dx+o(1).\tag{4.12}$$

Based on the above results and the Hölder inequalities, for each $\lambda > \gamma_N$, Let's verify the following conclusion: Case $(i)N = 1, 2$:
Case (i) $N = 1, 2$:

$$\int_{\mathbb{R}^N} |v_n|^r dx \leq \left(\int_{\mathbb{R}^N} v_n^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} v_n^{2(r-1)} dx\right)^{\frac{1}{2}}$$
$$\leq \left[\frac{1}{\lambda c_0}(1 + \beta_N^{\frac{8}{N}} |\{V < c_0\}|^{\frac{2}{N}})^{r-1}\right]^{\frac{1}{2}} S^{\,1-r}_{\alpha,2(r-1)} \|v_n\|^r_{H^\alpha(\mathbb{R}^N),\lambda} + o(1)$$

Case $(ii)N \geq 3$:

$$\int_{\mathbb{R}^N} |v_n|^r dx \leq \left(\int_{\mathbb{R}^N} v_n^2 dx\right)^{\frac{2^*_\alpha - k}{2^*_\alpha - 2}} \left(\int_{\mathbb{R}^N} v_n^{2(r-1)} dx\right)^{\frac{r-2}{2^*_\alpha - 2}}$$
$$\leq \left(\frac{1}{\lambda c_0}\right)^{\frac{2^*_\alpha - r}{2^*_\alpha - 2}} S_\alpha^{-\frac{2^*_\alpha (r-2)}{2^*_\alpha - 2}} \|v_n\|^r_{H^\alpha(\mathbb{R}^N),\lambda} + o(1)$$

Hence, we claim that

$$\Upsilon_{\lambda,r} := \begin{cases} \left[\frac{1}{\lambda c_0}(1 + \beta_N^{\frac{8}{N}} |\{V < c_0\}|^{\frac{2}{N}})^{r-1}\right]^{\frac{1}{2}} S^{\,1-r}_{\alpha,2(r-1)} & \text{if } N = 1, 2, \\ \left(\frac{1}{\lambda c_0}\right)^{\frac{2^*_\alpha - r}{2^*_\alpha - 2}} S_\alpha^{-\frac{2^*_\alpha (r-2)}{2^*_\alpha - 2}}, & \text{if } N \geq 3; \end{cases}$$

It is easy see that $\Upsilon_{\lambda,r} \to 0$ as $\lambda \to \infty$. Hence we obtain

$$\int_{\mathbb{R}^N} |v_n|^r dx \leq \Upsilon_{\lambda,r} \|v_n\|^r_\lambda + o(1). \tag{4.13}$$

As in the proof of [33], we can get

$$\int_{\mathbb{R}^N} F(x, v_n) dx = \int_{\mathbb{R}^N} F(x, u_n) dx - \int_{\mathbb{R}^N} F(x, u_0) dx + o(1) \tag{4.14}$$

and

$$\sup_{\|h\|_\lambda = 1} \int_{\mathbb{R}^N} [f(x, v_n) - f(x, u_n) + f(x, u_0)] h(x) dx = o(1).$$

Then by (2.1), (4.14) and Brezis-Lieb Lemma [34], we infer that

$$\begin{aligned} \mathcal{J}_{a,\lambda}(u_n) - \mathcal{J}_{a,\lambda}(u_0) =& \frac{a}{2}\left[\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) - \widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx)\right] \\ &+ \frac{b}{2}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx - \int_{\mathbb{R}^N} F(x, v_n) dx + o(1) \end{aligned}$$

Furthermore, from the boundedness of the sequence $\{u_n\}$ in E_λ we get that there is a number $\chi_0 > 0$ satisfying $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx \to \chi_0$ as $n \to \infty$, this implies that for each $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$o(1) = \langle \mathcal{J}'_{a,\lambda}(u_n), \phi \rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \lambda V(x) u_n \phi dx + \left(am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) + b \right) \int_{\mathbb{R}^N} (-\triangle)^{\frac{\alpha}{2}} u_n (-\triangle)^{\frac{\alpha}{2}} \phi dx \\
&\quad - \int_{\mathbb{R}^N} f(x, u_n) \phi dx \\
&\quad \to \int_{\mathbb{R}^N} \lambda V(x) u_0 \phi dx + (am(\chi_0) + b) \int_{\mathbb{R}^N} (-\triangle)^{\frac{\alpha}{2}} u_0 (-\triangle)^{\frac{\alpha}{2}} \phi dx - \int_{\mathbb{R}^N} f(x, u_0) \phi dx
\end{aligned}
$$

as $n \to \infty$. which means that

$$
\int_{\mathbb{R}^N} \lambda V(x) u_0^2 dx + (am(\chi_0) + b) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0^2 dx - \int_{\mathbb{R}^N} f(x, u_0) u_0 dx = 0. \tag{4.15}
$$

Observe that

$$
o(1) = \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx + \left(am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) + b \right) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx - \int_{\mathbb{R}^N} f(x, u_n) u_n dx. \tag{4.16}
$$

By (4.15) and (4.16) we obtain

$$
\begin{aligned}
o(1) = &\int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx + \left(am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx \right) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx - am(\chi_0) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx \\
&+ b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} v_n|^2 dx - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \\
&\int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx + \left(am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_n|^2 dx) + b \right) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} v_n|^2 dx - \int_{\mathbb{R}^N} f(x, v_n) v_n dx
\end{aligned} \tag{4.17}
$$

In particular, by (4.15) together with (F_1) and (F_3) we have

$$
\begin{aligned}
\mathcal{J}_{\lambda,a}(u_0) =& \frac{a}{2} \widehat{m} \left(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx \right) + \frac{1}{2} \left(b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_0^2 dx \right) \\
&- \int_{\mathbb{R}^N} F(x, u_0) u_0 dx - \frac{1}{k} (\int_{\mathbb{R}^N} \lambda V(x) u_0^2 dx + (am(\chi_0) + b) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx \\
&- \int_{\mathbb{R}^N} f(x, u_0) u_0 dx) \\
\geq & \frac{a}{2} \left[\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx) - \frac{2}{k} m(\chi_0) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx \right] \\
\geq & \frac{a}{k} \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx \left[m(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx) - m(\chi_0) \right].
\end{aligned}
$$

Thus there is a number κ satisfying $\kappa = 0$ when $m(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx) \geq m(\chi_0)$ or $\kappa < 0$ when $m(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u_0|^2 dx) < m(\chi_0)$, we obtain

$$
\mathcal{J}_{a,\lambda}(u_0) \geq \kappa.
$$

Based on the above results and (4.12), (4.17), (F_1) and (F_3) we have

$$
\mathcal{D} - \kappa \geq \alpha - \mathcal{J}_{\lambda,a}(u_0) = \mathcal{J}_{\lambda,a}(u_n) - \mathcal{J}_{\lambda,a}(u_0) + o(1)
$$

$$
\begin{aligned}
=&\frac{k-2}{k}(b)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2dx+\int_{\mathbb{R}^N}\lambda V(x)v_n^2dx)-\int_{\mathbb{R}^N}(F(x,v_n-\frac{1}{k}f(x,v_n)v_n\\
&+\frac{a}{2}[\widehat{m}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)-\widehat{m}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_0|^2dx)]\\
&-\frac{2}{k}m(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2dx+o(1)\\
\geq&\frac{(k-2)min\{1,b\}}{2k}\|v_n\|_{H^\alpha(\mathbb{R}^N),\lambda}+o(1)
\end{aligned}
$$

We can infer that there is a number $\widetilde{\mathcal{D}}=\widetilde{\mathcal{D}}(\mathcal{D}>0)$ such that

$$
\|v_n\|^2_{H^\alpha(\mathbb{R}^N),\lambda}\leq\frac{2k\widetilde{\mathcal{D}}}{(k-2)min\{1,b\}}+o(1). \tag{4.18}
$$

By (3.1), (4.13), (4.17) and (4.18) we have

$$
\begin{aligned}
o(1)=&\int_{\mathbb{R}^N}\lambda V(x)v_n^2dx+am(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx)\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2dx\\
&+b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2dx-\int_{\mathbb{R}^N}f(x,v_n)v_ndx\\
\geq&min\{1,b\}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2+\lambda V(x)v_n^2dx-|Q|_\infty\Upsilon_{\lambda,k}\left(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2+\lambda V(x)v_n^2dx\right)^{2k}\\
\geq&min\{1,b\}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}v_n|^2+\lambda V(x)v_n^2dx-|Q|_\infty\Upsilon_{\lambda,k}\left[\frac{2k\widetilde{D}}{(k-2)min\{1,b\}}\right]^{\frac{k}{2}}+o(1),
\end{aligned}
$$

which means that there exists $\widetilde{\gamma}:=\widetilde{\gamma}(a,\mathcal{D})\geq\gamma_N$ such that for $\lambda>\widetilde{\gamma}$, $v_n\to0$ strongly in E_λ. which is what we wanted to prove. □

Lemma 4.4. *If assumptions* $(V_1)-(V_3),(F_4),(F_5)$ *with* $\sigma\geq\frac{2\alpha}{N-2\alpha}$ *and* $(F_2)-(F_3)$ *are satisfied,* $N\geq3$. *Then for every* $\mathcal{D}>0$ *there exists numbers* $\widetilde{\gamma}_1=\widetilde{\gamma}_1(a,\mathcal{D})\geq\gamma_N>0$ *such that* $\mathcal{J}_{a,\lambda}$ *satisfies the* $(C)_\alpha-$ *condition in* E_λ *for* $\alpha<\mathcal{D}$, $\lambda>\widetilde{\gamma}_1$. □

Lemma 4.5. *If assumptions* $(V_1),(V_3),(F_4),(F_5)$ *and* $(F_2),(F_5)$ *are satisfied,* $N\geq1$. *Then for any* $a>0$ *and* $\lambda>\widetilde{\gamma}$, *there exists a critical point* $u_\lambda\in E_\lambda$ *of* $\mathcal{J}_{\lambda,a}(u)$ *such that* $\mathcal{J}_{\lambda,a}(u_\lambda)>0$.
Proof. Apply Lemma 4.3 and $0<\varrho_0\leq\alpha_{a,\lambda}\leq\alpha_{0,a}(\Omega)$ for each $\lambda\geq\gamma_N$, $\mathcal{J}_{a,\lambda}$ satisfies the $(C)_{\alpha\lambda,a}$-condition in E_λ for each $a>0$ and $\lambda>\widetilde{\gamma}$. In other word, we can get a subsequence $\{u_n\}$ and $u_\lambda\in E_\lambda$ such that $u_n\to u_\lambda$ strongly in E_λ. This means that u_λ is a nontrivial critical point of $\mathcal{J}_{a,\lambda}$ such that $\mathcal{J}_{a,\lambda}(u_\lambda)=\alpha_{a,\lambda}>0$. □

Lemma 4.6. *If assumptions* $(V_1)-(V_3),(F_4),(F_5)$ with $\sigma\geq\frac{2\alpha}{N-2\alpha}$ *and* $(F_2),(F_3)$ *are satisfied,* $N\geq3$. *Then there exists a number* $\widetilde{\gamma}_2\geq max\{\widetilde{\gamma}_0,\widetilde{\gamma}_1\}$ *such that for any* $0<a<\widetilde{a}_*$ *and* $\lambda>\widetilde{\gamma}_2$, *there exists a critical point* $u^2_{\lambda,a}\in E_\lambda$ *of* $\mathcal{J}_{a,\lambda}(u)$ *such that* $\mathcal{J}_{\lambda,a}(u^2_{\lambda,a})>0$.
Proof. For this proof, we can easily prove our results by applying Lemma 4.4 and 4.7 of conclusions. Here, we omit its proof. □

Lemma 4.7. *If assumptions* $(V_1)-(V_3),(F_5)$ *with* $\sigma\geq\frac{2\alpha}{N-2\alpha}$ *and* $(F_2),(F_3)$ *are satisfied,* $N\geq3$. *Then*

$\mathcal{J}_{\lambda,a}$ has a bounded below on E_λ for all $a > 0$ and

$$\lambda > \widetilde{\gamma}_3 = \begin{cases} max\{\gamma_N, \frac{2|Q|_\infty}{c_0 k}(\frac{2(\sigma+1)|Q|_\infty}{m_0 a k S_\alpha^{2^*_\alpha}})^{\frac{k-2\alpha}{2^*_\alpha}}\}, & \text{if } \sigma = \frac{2\alpha}{N-2\alpha}, \\ \gamma_N, & \text{if } \sigma > \frac{2\alpha}{N-2\alpha}; \end{cases}$$

Moreover, suppose

$$\lambda > \widetilde{\gamma}_4 := max\{\widetilde{\gamma}_3, \frac{2|Q|_\infty(2^*_\alpha - k)}{c_0 k(2^*_\alpha - 2)}\},$$

then there exists $\widetilde{C}_a > \varrho_0^{\frac{1}{2}}$ such that $\mathcal{J}_{a,\lambda}(u) \geq 0$ for any $u \in E_\lambda$ with $\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} \geq \widetilde{C}_a$.

Proof. Assume $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx < \varrho_0^{\frac{1}{2}}$, then using (2.4), (3.1) and the Young inequality we have

$$\begin{aligned} \mathcal{J}_{\lambda,a}(u) \geq& \frac{min\{b,1\}}{2}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x)u^2 dx - \frac{|Q|_\infty}{k}\int_{\mathbb{R}^N} |u|^k dx \\ \geq& \frac{min\{b,1\}}{2}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x)u^2 dx \\ &- \frac{|Q|_\infty}{kS_\alpha^{k}}|\{V < c_0\}|^{\frac{2^*_\alpha - k}{2^*_\alpha}}\|u\|_\lambda^{\frac{2(2^*_\alpha - k)}{2^*_\alpha - 2}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{\frac{2^*_\alpha(k-2\alpha)}{2^*_\alpha - 2}} \\ \geq& \frac{2^*_\alpha(k-2)min\{b,1\}}{2p(2^*_\alpha - 2)}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 + \lambda V(x)u^2 dx \\ &- \frac{k-2}{k(2^*_\alpha - 2)min\{1,b\}^{\frac{2^*_\alpha - k}{k-2}}}(|Q|_\infty S_\alpha^{-k}|\{V < c\}|^{\frac{2^*_\alpha - k}{2^*_\alpha}})^{\frac{2^*_\alpha - 2}{k-2}}\varrho_0^{\frac{2^*_\alpha}{2}} \end{aligned}$$

Hence $\mathcal{J}_{\lambda,a}(u)$ has a bounded below on E_λ for any $a > 0$ and $\lambda > \gamma_N$.
As $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq \varrho_0^{\frac{1}{2}}$, then we can divide it into two parts for discussion: $(i)\sigma = \frac{2\alpha}{N-2\alpha}$ and $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq \varrho_0^{\frac{1}{2}}$: Then we will also discuss this situation in two different ways:
Case one: $\int_{\mathbb{R}^N} \lambda V(x)u^2 dx \geq \lambda c_0 S_\alpha^{-2^*_\alpha}(\frac{2|Q|_\infty}{k\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2^*_\alpha}$. By (F_5), (2.4), (2.1) and (3.2), for any $\lambda > 0$ using Young and Sobolev inequalities we have

$$\begin{aligned} \mathcal{J}_{\lambda,a} \geq& \frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) + \frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x)u^2 dx) - \frac{|Q|_\infty}{k}\int_{\mathbb{R}^N} |u|^k dx \\ \geq& \frac{m_0 a}{2(\sigma+1)}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2(\sigma+1)}\frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x)u^2 dx) \\ &- \frac{|Q|_\infty}{k}(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x)u^2 dx + S_\alpha^{-2}|\{V < c_0\}|^{\frac{2}{N}}\frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x)u^2 dx))^{\frac{2^*_\alpha - k}{k-2\alpha}} \\ &(S_\alpha^{-1}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{\frac{2^*_\alpha(k-2\alpha)}{2^*_\alpha - 2}} \\ \geq& \frac{m_0 a}{2^*_\alpha}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2^*_\alpha} + \frac{b}{2}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx - \frac{|Q|_\infty|\{V < c_0\}|^{1-\frac{k}{2^*_\alpha}}}{kS_\alpha^{k}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k \end{aligned}$$

Which means that $\mathcal{J}_{\lambda,a}(u)$ have a bounded below on E for any $a > 0$ and $\lambda > \gamma_N$.
Case B: $\int_{\mathbb{R}^N} \lambda V(x)u^2 dx < \lambda c_0 S_\alpha^{-2^*_\alpha}(\frac{2|Q|_\infty}{k\lambda c_0})^{\frac{2^*_\alpha - 2}{k-2}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2^*_\alpha}$. From (2.4) we obtain that

$$\int_{\mathbb{R}^N} |u|^k dx$$

$$\leq (\frac{1}{\lambda c_0}\int_{\mathbb{R}^N}\lambda V(x)u^2dx + S_\alpha^{-2}|\{V<c_0\}|^{\frac{2}{N}}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{\frac{2_\alpha^*-k}{2_\alpha^*-2}}(S_\alpha^{-1}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{\frac{2_\alpha^*(k-2)}{2_\alpha^*-2}}$$
$$\leq S_\alpha^{-2_\alpha^*}(\frac{2|Q|_\infty}{k\lambda c_0})^{\frac{2_\alpha^*-k}{k-2}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2_\alpha^*} + S_\alpha^{-k}|\{V<c\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^k$$

Based on the inequality above and using (F_5) again, we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,a}(u) \geq & \frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx) + \frac{1}{2}(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^N}\lambda V(x)u^2dx) - \frac{|Q|_\infty}{k}\int_{\mathbb{R}^N}|u|^kdx\\ \geq & \frac{m_0a}{2(\delta+1)}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2(\sigma+1)} + \frac{1}{2}(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^N}\lambda V(x)u^2dx)\\ & - \frac{|Q|_\infty}{k}[S_\alpha^{-2_\alpha^*}(\frac{2|q|_\infty}{k\lambda c_0})^{\frac{2_\alpha^*-k}{k-2}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2_\alpha^*} + S_\alpha^{-k}|\{V<c_0\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^k]\\ \geq & \left[\frac{m_0a}{2(\sigma+1)} - \frac{|Q|_\infty}{kS_\alpha^{2_\alpha^*}}(\frac{2|Q|_\infty}{p\lambda c_0})^{\frac{2_\alpha^*-k}{k-2}}\right](\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2_\alpha^*}\\ & - \frac{|Q|_\infty}{kS_\alpha^{2_\alpha^*}}|\{V<c_0\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^k\end{aligned}$$

which means that if

$$\lambda > max\{\gamma_N, \frac{2|Q|_\infty}{kc_0}(\frac{2(\delta+1)|Q|_\infty}{m_0akS_\alpha^{2_\alpha^*}})^{\frac{k-2}{2_\alpha^*-k}}\},$$

thus $\mathcal{J}_{\lambda,a}(u)$ has a bounded below on E_λ for any $a>0$ and there exists $C_a>0$ such that $\mathcal{J}_{\lambda,a}(u)\geq 0$ for all $u\in E_\lambda$ with $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx \geq C_a$.

$(ii)\sigma > \frac{2\alpha}{N-2\alpha}$ and $\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx \geq \varrho_0^{\frac{1}{2}}$: Using (F_5), (2.4), (3.1) and the Young inequality we can get

$$\begin{aligned}\mathcal{J}_{\lambda,a}(u) \geq & \frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx) + \frac{1}{2}(b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^N}\lambda V(x)u^2dx) - \int_{\mathbb{R}^N}F(x,u)dx\\ \geq & \frac{m_0a}{2(\sigma+1)}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2(\sigma+1)} + \frac{min\{b,1\}}{2}\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2 + \lambda V(x)u^2dx - \frac{|Q|_\infty}{k}\int_{\mathbb{R}^N}|u|^kdx\\ \geq & \frac{m_0a}{2(\sigma+1)}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2(\sigma+1)}\\ & - \frac{k-2}{k(2_\alpha^*-2)(min\{1,b\})^{\frac{2_\alpha^*-k}{2_\alpha^*-2}}}(|Q|_\infty S_\alpha^{-k}|\{V<c\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}})^{\frac{2_\alpha^*-k}{k-2}}(\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx)^{2_\alpha^*}\end{aligned}$$

we can infer that $\mathcal{J}_{\lambda,a}(u)$ is bounded below on E for all $a>0$ and $\lambda>\gamma_N$, in view of $\sigma>\frac{2\alpha}{N-2\alpha}$. Furthermore, for any $a>0$, there exists

$$C_a > t_{\overline{B}} := [\frac{2(\sigma+1)(k-2)(|Q|_\infty S_\alpha^{-k}|\{V<c\}|^{\frac{2_\alpha^*-k}{2_\alpha^*}})^{\frac{2_\alpha^*-k}{k-2}})}{km_0a(2_\alpha^*-2)(min\{1,b\})^{\frac{2_\alpha^*-k}{k-2}}}]^{\frac{1}{2\sigma+2-2_\alpha^*}}$$

such that $\mathcal{J}_{\lambda,a}(u)\geq 0$ for any $u\in E_\lambda$ with $\int_{R^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2dx \geq \overline{C}_a = max\{\varrho_0^{\frac{1}{2}}, C_a\}$. Then we will verify that there exists a number $\widehat{C}_a > \overline{C}_a$.

$\mathcal{J}_{\lambda,a}(u)\geq 0$ for all $u\in E_\lambda$ with $\|u\|_\lambda \geq \widehat{C}_a$. Set

$$\widetilde{C_a} = [\overline{R}_a^2 + 2\beta_0(\overline{C}_a)(1-\frac{1}{\lambda c_0})(\frac{2(2_\alpha^*-k)|Q|_\infty}{k(2_\alpha^*} - 2)^{-1}]^{\frac{1}{2}}, \tag{4.19}$$

where

$$\beta_0(\overline{C}_a) = \frac{(2^*_\alpha - k)|Q|_\infty}{k(2^*_\alpha - 2}|\{V < c_0\}|^{\frac{2}{N}} S_\alpha^{-2}\overline{C}_a^2 + \frac{(k-2)|Q|_\infty}{k(2^*_\alpha - 2)} S_\alpha^{-2^*_\alpha}\overline{C}_a^{2^*_\alpha}$$

For $u \in E_\lambda$ with $\|u\|_\lambda \geq \widetilde{C}_a$. If $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq \overline{C}_a$, then the result is obtained. If $\varrho_0^{\frac{1}{2}} \leq \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx < \overline{C}_a$, thus we can verify that $\mathcal{J}_{\lambda,a}(u) \geq 0$ when

$$\int_{\mathbb{R}^N} \lambda V(x) u^2 dx \geq 2\beta_0(\overline{C}_a)(1 - \frac{2(2^*_\alpha - k)|Q|_\infty}{\lambda c_0 k(2^*_\alpha - 2)}^{-1}.$$

In fact, applying (4.8) we can infer that

$$\begin{aligned}
\mathcal{J}_{\lambda,a}(u) \geq& \frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) + \frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx) - \frac{|Q|_\infty}{k}\int_{\mathbb{R}^N} |u|^k dx \\
\geq& \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x) u^2 dx - \frac{(k-2)|Q|_\infty}{(2^*_\alpha - 2)k} S_\alpha^{-2^*_\alpha}\overline{C}_a^{2^*_\alpha} \\
&- \frac{(2^*_\alpha - p)|Q|_\infty}{k(2^*_\alpha - 2)}(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x) u^2 dx + |\{V < c_0\}|^{\frac{2}{N}} S_\alpha^{-2}\overline{C}_a^2) \\
\geq& \frac{1}{2}[1 - \frac{2(2^*_\alpha - k)|Q|_\infty}{\lambda c_0 k(2^*_\alpha - 2)}]\int_{\mathbb{R}^N} \lambda V(x) u^2 dx - \beta_0(\overline{C}_a^2) \\
\geq& 0
\end{aligned}$$

Therefore, we get that there exists a number $\widetilde{C}_a > 0$ defined as (4.19) such that $\mathcal{J}_{\lambda,a}(u) > 0$ for any $u \in E_\lambda$ with $\|u\|_\lambda \geq \widetilde{C}_a$. which is what we wanted to prove. □

Lemma 4.8. *If assumptions* $(V_1) - (V_3), (F_4), (F_5)$ *with* $\sigma \geq \frac{2\alpha}{N-2\alpha}$ *and* $(F_2), (F_3)$ *are satisfied,* $N \geq 3$. *Then for any* $a > 0$ *and* $\lambda > \widetilde{\gamma}_4$ *there holds*

$$\widetilde{\theta}_a =: inf\{\mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \ with \ \|u\|_{H^\alpha(\mathbb{R}^N),\lambda} < \widetilde{C}_a\} < 0.$$

Proof. This proof can be obtained from Lemma 3.4 and Lemma 4.5, we omit its proof. □

Lemma 4.9. *If assumptions* $(V_1) - (V_3), (F_4), (F_5)$ *with* $\sigma \geq \frac{2\alpha}{N-2\alpha}$ *and* $(F_2), (F_3)$ *are satisfied,* $N \geq 3$. *Then there exists a number* $\widetilde{\gamma}_5 \geq max\{\widetilde{\gamma}_1, \widetilde{\gamma}_4\}$ *such that for each* $a > 0$ *and* $\lambda \geq \widetilde{\gamma}_5$, *and there exists a critical point* $u^1_{\lambda,a} \in E_\lambda$ of $\mathcal{J}_{\lambda,a}$ *such that* $\mathcal{J}_{\lambda,a}(u^1_{\lambda,a}) = \widetilde{\vartheta}_a < 0$.

Proof. Using Lemma 4.8 and the Ekeland variational principle, we may assume that $\{u_n\} \subset E_\lambda$ is a bounded minimizing sequence with $\|u_n\|_{H^\alpha(\mathbb{R}^N),\lambda} < \widetilde{C}_a$ such that $\mathcal{J}_{\lambda,a}(u_n) \to \widetilde{\vartheta}_a$ and $(1 + \|u\|_{E_\lambda}\|\mathcal{J}'_{\lambda,a}(u_n)\|_{E_\lambda^{-1}} \to 0$ as $n \to \infty$. By Lemma 4.4, there exist a subsequence $\{u_n\}$ and $u^1_{\lambda,a} \in E_\lambda$ such that $u_n \to u^1_{\lambda,a}$ strongly in E_λ. This implies that u_λ is a nontrivial critical point of $\mathcal{J}_{\lambda,a}$ satisfying $\mathcal{J}_{\lambda,a}(u^1_{\lambda,a}) = \widetilde{\vartheta}_a < 0$.

Proof of Theorem 1.1 and Theorem 1.2:

The proof of Theorem 1.1 directly follows from Theorem 4.5. Applying Theorems 4.6 Theore4.9, there exists a positive constant $\widetilde{\gamma}_* \geq max\{\widetilde{\gamma}_2, \widetilde{\gamma}_5\}$ such that for any constant $0 < a < a_*$ and $\lambda > \widetilde{\gamma}_*$, Equation 1.1 admits two nontrivial positive solutions $u^1_{\lambda,a}$ and $u^2_{\lambda,a}$ satisfying $\mathcal{J}(u^1_{\lambda,a}) < 0 < \mathcal{J}_{\lambda,a}(u^2_{\lambda,a})$. Furthermore, $u^1_{\lambda,a}$ is a ground state solution of Eq 1.1. So we complete the proof of Theorem 1.2. □

Proofs of Theorem 1.3 and Theorem 1.4:

By Lemmas 3.2 and 3.5 and the mountain pass theorem [32] (see Theorem 1.15), we get that for every $\lambda \geq \gamma_N$ and $0 < a < \frac{1}{m_\infty \overline{\mu}_1^{(k)}}$ there exists a sequence $\{u_n\} \subset E_\lambda$ such that

$$\mathcal{J}_{a,\lambda}(u_n) \to \alpha_{a,\lambda} > 0 \text{ and } (1 + \|u_n\|_{H^\alpha_{(\mathbb{R}^N),\lambda}})\|\mathcal{J}'_{a,\lambda}\|_{E_\lambda^{-1}} \to 0, \text{ as } n \to \infty, \tag{5.1}$$

where $0 < \eta \leq \alpha_{a,\lambda} \leq \alpha_{0,a}(\Omega) < \mathcal{D}_\alpha$.

Lemma 5.1. *If assumptions* $(V_1) - (V_3), (F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied,* $N \geq 3$. *Then for any* $0 < a < \frac{1}{m_\infty \widetilde{\mu}_0^{(k)}}$, *the sequence* $\{u_n\}$ *defined in* (5.1) *is bounded in* E_λ *for all* $\lambda \geq \gamma_N$.

Proof. Adopting the method of proof to the contrary. Let $\|u_n\|_{H^\alpha(\mathbb{R}^N),\lambda} \to \infty$ as $n \to \infty$. We will divide it into two cases for proof:
Case one: $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \to \infty$. By (2.2), (3.3), together with (F_5) and the Nirenberg inequality we obtain

$$\begin{aligned}
o(1) =& \frac{am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k} + \frac{b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k} \\
& - \frac{\int_{\mathbb{R}^N} f(x, u_n) u_n dx}{(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k} \\
& \geq am_0(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2(\delta+1)-k} - \frac{c^*(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k}{\overline{v}_1^{(k)}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k} \\
=& am_0(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2(\delta+1)-k} - \frac{c^*}{\overline{v}_1^{(k)}} \\
& \to \infty
\end{aligned}$$

as $n \to \infty$ since $2(\sigma + 1) > k$. This is a contradiction.
Case two: $\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx \to \infty$ and $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \leq C_*$ for some $C_* > 0$ and for all n. Using (2.2), (3.3) and (F_5) we have

$$\begin{aligned}
o(1) =& \frac{am(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} + \frac{b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \\
& - \frac{\int_{\mathbb{R}^N} f(x, u_n) u_n dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \\
\geq & 1 - \frac{c^*(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k}{\overline{v}_1^{(k)} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \\
\geq & 1 - \frac{c^* C_0^k}{\overline{v}_1^{(k)} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} = 1 + o(1)
\end{aligned}$$

contrary to assumption. we conclude that the sequence $\{u_n\}$ is bounded in E_λ for any $0 < a < \frac{1}{m_\infty \widetilde{\mu}_0^{(k)}}$ and $\lambda \geq \gamma_N$. This concludes the proof. □

Lemma 5.2. *If assumptions* $(V_1)-(V_3),(F_1),(F_6)$ *and* (F_3) *are satisfied,* $N \geq 3$. *Then for every* $\mathcal{D} > 0$ *there exists constants* $\overline{\gamma} = \overline{\gamma}(a, \mathcal{D}) \geq \gamma_N > 0$ *such that* $\mathcal{J}_{a,\lambda}$ *satisfies the* $(C)_\alpha$*-condition in* E_λ *for any* $\alpha < \mathcal{D}$ *and* $\lambda > \overline{\gamma}$. □

Lemma 5.3. *If assumptions* $(V_1)-(V_3),(F_4),(F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied,* $N \geq 3$. *Then for every* $\mathcal{D} > 0$ *there exists* $\overline{\gamma}_* = \overline{\gamma}_*(a, \mathcal{D}) \geq \gamma_N > 0$ *such that* $\mathcal{J}_{a,\lambda}$ *satisfies the* $(C)_\alpha$*-condition in* E_λ *for all* $\alpha < \mathcal{D}$ *and* $\lambda > \overline{\gamma}_*$. □

Lemma 5.4. *If assumptions* $(V_1)-(V_3),(F_1),(F_6),(F_3)$ *are satisfied,* $N \geq 3$. *Then for any* $0 < a \leq \frac{1}{m_\infty \overline{\mu}_0^{(k)}}$ *there exists a critical point* $u_\lambda \in E_\lambda$ *of* $\mathcal{J}_{\lambda,a}$ *such that* $\mathcal{J}_{\lambda,a}(u_\lambda) > 0$ *for all* $\lambda > \overline{\gamma}$.

Proof. This proof is similar to Theorem 4.3, we can prove the result by applying (5.1), Lemma 4.1 and Lemma 5.2. □

Lemma 5.5. *If assumptions* $(V_1)-(V_3),(F_4),(F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied,* $N \geq 3$. *Then for each* $0 < a < \overline{a}_*$ *and* $\lambda > \overline{\gamma}_*$, *then for every* $0 < a \leq \frac{1}{m_\infty \overline{\mu}_0^{(k)}}$ *the energy functional* $\mathcal{J}_{\lambda,a}$ *admits a nontrivial critical point* $u^2_{\lambda,a} \in E_\lambda$ *such that* $\mathcal{J}_{\lambda,a}(u^2_{\lambda,a}) > 0$. □

Proof. This proof is similar to Theorem 4.3, we can prove the result by applying Lemma 5.1 and Lemma 5.3.

Lemma 5.6. *If assumptions* $(V_1)-(V_3),(F_4),(F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied,* $N \geq 3$. *Then the energy functional* $\mathcal{J}_{\lambda,a}$ *is bounded below on* E_λ *for all* $a > 0$ *and* $\lambda > 0$. *Furthermore, there exists* $\overline{C}_a > 0$ *such that* $\mathcal{J}_{\lambda,a}(u) \geq 0$ *for all* $u \in E_\lambda$ with $\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} \geq \overline{C}_a$.

Proof. For $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx < \varrho_1^{\frac{1}{2}}$, then using (2.1) and (3.5) we have

$$\begin{aligned}\mathcal{J}_{\lambda,a}(u) \geq &\frac{1}{2}(b \int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx) \\ &- \frac{c^*}{k\overline{v}_1^{(k)}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k \\ \geq &\frac{b}{2}\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx - \frac{c^*}{k\overline{v}_1^{(k)}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k\end{aligned}$$

This means that $\mathcal{J}_{\lambda,a}$ is bounded on E_λ for any constants $a > 0$ and $\lambda > 0$.
For $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq \varrho_1^{\frac{1}{2}}$, then by (F_5) with $\sigma > \frac{k-2\alpha}{2\alpha}$, (2.1) and (3.5) we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,a}(u) \geq &\frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) + \frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx) \\ &- \frac{c^*}{k\overline{v}_1^{(k)}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k \\ \geq &\frac{m_0 a}{2(\sigma+1)}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^{2(\delta+1)} - \frac{c^*}{k\overline{v}_1^{(k)}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k\end{aligned}$$

This means that $\mathcal{J}_{\lambda,a}(u)$ is bounded below on E for any constants $a > 0$ and $\lambda > 0$, since $\sigma > \frac{k}{2\alpha} - 1$. Furthermore, for every $a > 0$, there exists $C_a > t_{\overline{B}} := (\frac{2(\sigma+1)c^*}{k\overline{v}_1^{(k)} m_0 a})^{\frac{1}{2(\sigma+1)-k}}$ such that $\mathcal{J}_{a,\lambda}(u) \geq 0$ for all $u \in E_\lambda$ with $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq C_a$.

Thus we will prove that there exists a constant $\overline{C}_a > 0$ such that $\mathcal{J}_{\lambda,a}(u) \geq 0$ for all $u \in E_\lambda$ with

$\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} \geq \overline{C}_a$. $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx \geq C_a$, So the results are verified. If $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx < C_a$, then we can infer that $\mathcal{J}_{\lambda,a} \geq 0$ when $\int_{\mathbb{R}^N} \lambda V(x) u^2 dx \geq \frac{2c^*}{k\overline{\nu}_1^{(k)}} C_a^k$. In fact, we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,a}(u) \geq & \frac{a}{2}\widehat{m}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx) + \frac{1}{2}(b\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx) \\ & - \frac{c^*}{k\overline{\nu}_1^{(k)}}(\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 dx)^k \\ \geq & \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x) u^2 dx - \frac{c^*}{k\overline{\nu}_1^{(k)}} C_a^k\end{aligned}$$

Therefore, we get that there exists a number $\overline{C}_a > 0$ such that $\mathcal{J}_{\lambda,a}(u) \geq 0$ for all $u \in E_\lambda$ with $\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} \geq \overline{C}_a$. which is what we wanted to prove. □

Lemma 5.7. *If assumptions* $(V_1) - (V_3), (F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied*, $N \geq 3$. *Then for any* $a > 0$ and $\lambda > 0$ *one has*

$$\overline{\vartheta} =: inf\{\mathcal{J}_{\lambda,a}(u) : u \in E_\lambda \ with \ \|u\|_{H^\alpha(\mathbb{R}^N),\lambda} < \overline{C}_a\}) < 0. \tag{5.2}$$

Proof. We can directly obtain this proof from Lemma 3.6 and 5.6. Hence, here we omit its proof. □

Lemma 5.8. *If assumptions* $(V_1) - (V_3), (F_4), (F_5)$ *with* $\sigma > \frac{k-2\alpha}{2\alpha}$, (F_6) *and* (F_3) *are satisfied*, $N \geq 3$. *Then for any* $a > 0$ *and* $\lambda > \overline{\gamma}_*$, $\mathcal{J}_{\lambda,a}(u)$ *has a nontrivial critical point* $u^1_{\lambda,a} \in E_\lambda$ *such that* $\mathcal{J}_{\lambda,a}(u^1_{\lambda,a}) = \overline{\vartheta} < 0$, *where* $\overline{\vartheta}$ *is as in* (5.2).

Proof. Applying Lemma 5.7 and the Ekeland variational principle, we can get that there exists a minimizing bounded sequence $\{u_n\} \subset E_\lambda$ with $\|u\|_{H^\alpha(\mathbb{R}^N),\lambda} < \overline{C}_a$ such that $\mathcal{J}_{\lambda,a}(u_n) \to \overline{\vartheta}$ and $\mathcal{J}'_{\lambda,a}(u_n) \to 0$ as $n \to \infty$. Thus by Lemma 5.3 which means that there is a subsequence $\{u_n\}$ and $u^1_{\lambda,a} \in E_\lambda$ with $\|u^1_{\lambda,a}\|_{H^\alpha(\mathbb{R}^N),\lambda} < \overline{C}_a$ such that $u_n \to u^1_{\lambda,a}$ strongly in E_λ. Which infer that$\mathcal{J}'_{\lambda,a}(u^1_{\lambda,a}) = 0$ and $\mathcal{J}_{\lambda,a}(u^1_{\lambda,a}) = \overline{\vartheta} < 0$, which is what we wanted to prove. □

Proof of Theorem 1.3 and 1.4.

From Lemma 5.4, we can directly proof Theorem 1.3. By virtue of Lemma 5.5 and 5.8, for any $0 < a < \overline{a}_*$ and $\lambda > \overline{\gamma}_*$, Equation (1.1) admits two nontrivial positive solutions $u^1_{\lambda,a}$ and $u^2_{\lambda,a}$ satisfying $\mathcal{J}_{\lambda,a}(u^1_{\lambda,a}) < 0 < \mathcal{J}_{\lambda,a}(u^2_{\lambda,a})$. Specifically, $u^1_{\lambda,a}$ is a ground state solution of Eq (1.1). Therefore, We have completed the proof of Theorem 1.4. □

4. Conclusions and discussion of the results

By the condition, f satisfies $\lim\limits_{|t|\to\infty} f(x,t)/|t|^{k-1} = Q(x)$ uniformly in $x \in \mathbb{R}^N$ for each $2 < k < 2^*_\alpha$, $(2^*_\alpha = \frac{2N}{N-2\alpha})$. We investigated the effects of functions m and Q on the solution. By applying the variational method, we obtain the existence of multiple solutions. Furthermore, it is worth mentioning that the ground state solution has also been obtained. We find that the number of differently solutions are obtained when the assumptions about m and f are different. The main contribution of this paper is to establish a multiplicity theorem in which the main method is based on the variational method. It is worth noting that we have not yet provided multiple solutions for the critical case, and we will continue to study this case.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is partially supported by the Fundamental Research Funds for the National Natural Science Foundation of China 11671403 and Guizhou University of Finance and Economics of 2022KY.

Conflict of interest

The authors declare that they have no competing interests.

References

1. G. Kirchhoff, *Mechanik, Teubener, Leipzig,* 1983.
2. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.-Theor.,* **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011
3. A. Alghamdi, S. Gala, M. Ragusa, Z. Zhang, A Regularity Criterion for the 3D Density-Dependent MHD Equations, *B. Braz. Math. Soc.,* **52** (2021), 241–251. https://doi.org/10.1007/s00574-020-00199-5
4. F. Faraci, C. Farkas, On a critical Kirchhoff-type problem, *Nonlinear Analysis*, **192** (2020), 111679. https://doi.org/10.1016/j.na.2019.111679
5. A. Fiscella, Infinitely many solutions for a critical Kirchhoff type problem involving a fractional operator, *Differ. Int. Equ.,* **29** (2016), 513–530. https://doi.org/10.57262/die/1457536889
6. E. Nezza, G. Palatucci, E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. Math.,* **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
7. A. Cotsiolis, N. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.,* **295** (2004), 225–236. https://doi.org/10.1016/j.jmaa.2004.03.034
8. P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2785–2806. https://doi.org/10.1007/s00526-015-0883-5
9. R. Servadei, E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, **367** (2015), 66–102. https://doi.org/10.1090/S0002-9947-2014-05884-4
10. H. Brezis, T. Kato, Remarks on the Schrödinger operator with singular complex potential, *J. Math. Pures Appl.*, **58** (1979), 137–151.
11. G. Figueiredo, N. Ikoma, J. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearitie, *Arch. Ration. Mech. Anal.*, **213** (2014), 931–979. https://doi.org/10.1007/s00205-014-0747-8

12. C. Alves, V. Ambrosio, T. Isernia, Existence, multiplicity and concentration for a class of fractional p-q Laplacian problems in $\mathbb{R}^N$, *Comm. Pure. Appl. Anal.*, **184** (2019), 2009–2045.

13. D. Goel, K. Sreenadh, Kirchhoff equations with Hardy-Littlewood-Sobolev critical nonlinearity, *Nonlinear Analysis*, **186** (2019), 172–186. https://doi.org/10.1016/j.na.2019.01.035

14. P. Lions, The concentration compactness principle in the calculus of variations: The locally compact case, part 2, *Ann. Inst. H. Poincar. Anal. Non Linaire*, **1** (1984), 223–283. https://doi.org/10.1016/s0294-1449(16)30422-x

15. P. Lions, The concentration compactness principle in the calculus of variations: the locally compact case, part 1, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire,* **1** (1984), 109–145. https://doi.org/10.1016/s0294-1449(16)30428-0

16. G. Figueiredo, N. Ikoma, J. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, *Arch. Ration. Mech. Anal.*, **213** (2014), 931–979. https://doi.org/10.1007/s00205-014-0747-8

17. P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.*, **108** (1992), 247–262. https://doi.org/10.1007/BF02100605

18. L. Zhang, X. Tang, S. Chen, Multiple solutions for fractional Kirchhoff equation with critical or supercritical nonlinearity, *App, Math, Lett.*, **119** (2021), 107204. https://doi.org/10.1016/j.aml.2021.107204

19. M. Xiang, B. Zhang, D. Repovš, Existence and multiplicity of solutions for fractional Schrödinger-Kirchhoff equations with Trudinger-Moser nonlinearity, *Nonlinear Analysis.* **186** (2019), 74–98. https://doi.org/10.1016/j.na.2018.11.008

20. D. Yafaev, Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$, *Nonlinear Anal.-Real,* **21** (2014), 76–86. https://doi.org/10.1016/j.nonrwa.2014.06.008

21. M. Xiang, D. Hu, B. Zhang, Y. Wang, Multiplicity of solutions for variable-order fractional Kirchhoff equations with nonstandard growth, *J. Math. Anal. Appl.*, **501** (2021), 124269. https://doi.org/10.1016/j.jmaa.2020.124269

22. V. Ambrosio, T. Lsernia, A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity, *Comm. Cont. Math.*, **20** (2018), 1750054. https://doi.org/10.1142/S0219199717500547

23. W. Chen, Y. Gui, Multiple solutions for a fractional p-Kirchhoff problem with Hardy nonlinearity, *Nonlinear Analysis*, **188** (2019), 316–338. https://doi.org/10.1016/j.na.2019.06.009

24. S. Peng, A. Xia, Multiplicity and concentration of solutions for nonlinear fractional elliptic equations with steep potentia, *Commun. Pur. Appl. Anal.*, **17** (2018), 1201–1207. https://doi.org/10.3934/cpaa.2018058

25. J. Zuo, D. Choudhuri, D. Repovš, Multiplicity and boundedness of solutions for critical variable-order Kirchhoff type problems involving variable singular exponent, *J. Math. Anal. Appl.*, **514** (2022), 126264. https://doi.org/10.1016/j.jmaa.2022.126264

26. D. Choudhuri, Existence and Hölder regularity of infinitely many solutions to a p-Kirchhoff type problem involving a singular nonlinearity without the Ambrosetti-Rabinowitz (AR)

condition, *Zeitschrift für Angewandte Mathematik und Physik (Z.A.M.P.)*, **72** (2021), 36. https://doi.org/10.1097/01.BACK.0000753296.84688.86

27. T. Bartsch, Z. Wang, Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Part. Differ. Equ.*, **20** (1995), 1725–1741. https://doi.org/10.1080/03605309508821149
28. R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032
29. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **189** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
30. H. Hajaiej, X. Yu, Z. Zhai, Fractional Gagliardo–Nirenberg and Hardy inequalities under Lorentz norms, *J. Math. Anal. Appl.*, **396** (2012), 569–577. https://doi.org/10.1016/j.jmaa.2012.06.054
31. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm Pur. Appl. Math.*, **60** (2006), 67–112. https://doi.org/10.1002/cpa.20153
32. M. Millem, *Minimax Theorems*, Birkhäuser, Berlin, 1996. https://doi.org/10.1007/978-1-4612-4146-1
33. J. Sun, T. Wu, On the nonlinear Schrödinger-Poisson system with sign-changing potential, *Z. Angew. Math. Phys.*, **66** (2015), 1649–1669. https://doi.org/10.1007/s00033-015-0494-1
34. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *J. Differ. Equ.,* **60** (2006), 67–112. https://doi.org/10.1002/cpa.20153