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Research article

Unified existence results for nonlinear fractional boundary value problems

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Abstract: In this work, we focus on investigating the existence of solutions to nonlinear fractional boundary value problems (FBVPs) with generalized nonlinear boundary conditions. By extending the framework of the technique based on well-ordered coupled lower and upper solutions, we guarantee the existence of solutions in a sector defined by these solutions. One notable aspect of our study is that the proposed approach unifies the existence results for the problems that have previously been discussed separately in the literature. To substantiate these findings, we have added three illustrative examples.

Keywords: lower and upper solutions; fractional differential equations; nonlinear boundary conditions; Hilfer fractional derivative; periodic boundary conditions; anti-periodic boundary conditions
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1. Introduction

Fractional calculus (FC) is a dynamic field dealing with non-integer orders of differentiation and integration, offering an extended framework beyond traditional calculus. It finds applications in diverse areas like viscoelastic materials, diffusion processes, and fractals (see [1, 2]). Fractional-order derivatives (FOD) lack a universally accepted definition due to challenges in extending differentiation to non-integer orders, leading to various definitions, including Riemann-Liouville, Grunwald-Letnikov, Caputo, and Atangana-Baleneau, among others (see [3, 4]). Overall, this diversity in defining various

FOD operators reflects the richness and adaptability of the theory of fractional calculus. It provides researchers with a toolkit to select the most suitable approach for their specific applications, taking into account the mathematical properties, physical interpretations, and practical considerations relevant to the problems at hand.

Among the various types of FOD operators, the Hilfer operator stands out due to its unifying interpolatory role. While the Caputo and Riemann-Liouville FOD operators have their own distinct mathematical properties, the Hilfer FOD operator seamlessly connects them by adjusting a single parameter (see [3, 5, 6]). It provides a more nuanced and useful modeling framework for a wide range of scientific and engineering applications by integrating the key features of the Caputo and Riemann-Liouville derivatives. For numerous applications of the Hilfer operator, we direct the reader to [7–11]. It has been used, for instance, in the study of fractional-time evolution equations and in regular variation in thermodynamics and physics, as referenced in [12, pp. 87, 429].

The investigation of the existence and uniqueness of solutions to FBVPs, that comprise various kinds of FOD operators and initial/boundary conditions (BCs), has been tackled by several mathematicians (see [13–21]) and the references cited therein. Notable contributions from previous studies regarding the existence of FBVPs in the Hilfer sense encompass various cases. For instance, research by Suphawat et al. [14] proved the existence and uniqueness of the solutions to the following boundary value problems (BVPs) with Hilfer fractional derivatives and a nonlocal condition.

$$\begin{aligned} {}^{H}D^{p,q}y(s) &= \Psi(s, y(s)), \;\; s \in I = [a, b]\,, \; b > a > 0, \\ y(a) = 0, y(b) &= \sum_{k=1}^{m} c_k J_a^{s_k} t(\chi_k), \;\; \chi_k \in I, c_k \in \mathbb{R}, \end{aligned}$$

here, ${}^{H}D^{p,q}$ is the Hilfer derivative of order between 0 and 1 and the parameter q is such that $0 \le q \le 1$, and $\Psi : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $J_a^{s_k} t(.)$ is the Riemann-Liouville fractional integral (RLFI) of order s_k. Suphawat et al. [14] used the Banach contraction principle, as well as the approach of Hölder, Wang and Boyd, Leray-Schauder and Krasnoselskii's to prove the existence.

In [22], Krasnoselskii's fixed-point (FP) theorem was used to prove the existence of solutions to an Initial Value Problem(IVP) with nonlocal conditions given as

$$\begin{aligned} {}^{\rho}D^{\nu,q}u(s) &= f(s, u(s)), \;\; s \in I = (a, T], \\ {}^{\rho}I_{a^+}^{1-\gamma}u(a) &= \sum_{k=1}^{n} b_k u(\sigma_k), \;\; \sigma_k \in I, \rho > 0, b_k \in \mathbb{R}, \end{aligned}$$

involving Hilfer-Katugampola FOD ${}^{\rho}D^{\nu,q}$ having order between 0 and 1. Here, $0 \le q \le 1$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The nonlocal condition contains the generalized fractional-order integral ${}^{\rho}I_{a^+}^{1-\gamma}(.)$ of positive order that is $1 - \gamma > 0$, and σ_k fulfills $0 < a < \sigma_1 \le \sigma_2 \le .... \le \sigma_n < T$ for $n = 1, 2, ..., k$.

Through the use of Hilfer fractional derivatives, Vivek et al. [20] investigated the uniqueness and existence for an implicit differential equation (DE) with condition involving non locality described as

$$\begin{aligned} D^{\alpha,\beta}v(s) &= g(s, v(s), D^{\alpha,\beta}v(s)), \;\; s \in A = [0, c]\,, \; c > 0, \\ I^{1-\gamma}v(0) &= \sum_{i=1}^{m} b_i v(\varsigma_i), \;\; \varsigma_i \in A, \gamma = \alpha + \beta - \alpha\beta, \end{aligned}$$

in this manuscript, Schuader and Banach FP theorems have been used by the authors to demonstrate the desired results. Additionally, stability outcomes were examined. In [23], the authors conducted a research containing pantograph fractional differential equations (FDE) involving the Hilfer derivative subject to non-local integral boundary conditions. They studied both single and multi-valued cases of these equations. Their work aimed to prove that solutions satisfying the given conditions exist and are unique.

$$
\begin{aligned}
{}^{H}D^{\alpha,\beta}x(s) &= f(s, x(s), x(\lambda(s))), \;\; s \in T = [a, b], a \geq 0, \;\; b > a, \\
x(a) = 0, \quad & Ax(b) + BJ^{\gamma}x(\eta) = d, \qquad \eta \in (a, b),
\end{aligned}
$$

here, ${}^{H}D^{\alpha,\beta}$ is the Hilfer derivative of order between 1 and 2 and β is the parameter such that $0 \leq \beta \leq 1$, and $f : T \times \mathbb{R} \to \mathbb{R}$ is continuous function and $J^{\gamma}(.)$ is the RLFI of order $\gamma > 0$, $A, B, d \in \mathbb{R}$ and $\lambda \in (0, 1)$.Banach FP theorem is used to ensure the uniqueness of solution and Leray-Schauder type and Krasnoselskii's FP theorems are used to acquire existence results for single-valued cases. Furthermore, multi-valued convex and non-convex problems have been addressed by using the Bohnenblust-Karlin FP theorem, Martelli's FP theorem, and contractive multi-valued maps FP theorem. In [24], existence and uniqueness for the most latest generlized variant that is (k, Ψ) of Hilfer derivative has been studied. In this paper, existence and uniqueness has been investigated by determining the equivalent fractional integral equation to the nonlinear (k, Ψ) Hilfer fractional differential equation of the form

$$
\begin{aligned}
{}^{k,H}D^{\eta,\nu;\Psi}y(t) &= f(t, y(t)), t \in (a, b], 0 < \eta < k; 0 \leqslant \nu \leqslant 1, \\
{}^{k}J_{a^+}^{k-\xi_k;\Psi}y(a) &= y_a \in \mathbb{R}, \xi_k = \eta + \nu(k - \eta),
\end{aligned}
$$

where ${}^{k,H}D^{\eta,\nu;\Psi}(.)$ is the (k, Ψ) hilfer derivative of order η and type ν, ${}^{k}J_{a^+}^{k-\xi_k;\Psi}(.)$ is the (k, Ψ) RLFI of order $k - \xi_k$, and $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Additionally, another existence result for Hilfer FDE, which investigates the existence and uniqueness of global solutions in the space of weighted continuous functions, is presented in [11]. The study of a nonlinear FDE with a nonlocal initial condition using a Hilfer generalized proportional fractional derivative is discussed in [25]. In [17], researchers examine the existence, uniqueness, and stability of ψ-Hilfer fractional derivative random differential equations, employing Schauder's fixed-point theorem and Banach's contraction principle for the respective existence and uniqueness analyses.

The studies mentioned above have investigated existence and uniqueness of Hilfer FDE in the presence of diverse initial and boundary conditions. Expanding upon the research in [11], our primary contribution is centered on proving the existence of solutions for the nonlinear FDE that incorporates the Hilfer fractional derivative alongside generalized boundary conditions. The significance of our proposed results is the development of a unified framework in order to ensure the existence of nonlinear FBVPs with various BCs including initial, anti periodic boundary conditions (APBCs), and periodic boundary conditions (PBCs). Whereas the researchers handled these BCs separately, but our method provides a general approach that cover them as particular cases. To achieve this unification, we extend the lower and upper solutions approach (LUSA), whose foundation relies on well-ordered lower and upper solutions in conjunction with specific FP theorems.

We begin by considering the following Hilfer derivative FDE

$$
D^{\mu,\nu}h(y) = g(y, h(y)), \;\; y \in [0, 1], \tag{1.1}
$$

corresponding to the following generalized nonlinear boundary conditions (GNBC)

$$v(h(0), h(1)) = 0, \tag{1.2}$$

here, the functions $v : \mathbb{R}^2 \to \mathbb{R}$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are both continuous, and $D^{\mu,\nu}$ is the generalized derivative introduced by Hilfer of order $0 < \mu < 1$, and type ν (see [12]).

$$\frac{1}{\Gamma(\mu)} \int_0^y (y-\tau)^{\mu-1} h(\tau) d\tau =_{RL} I^{\mu} h(y) \tag{1.3}$$

is the RLFI operator of order μ (see [4]).

The nonlinear FBVPs, (1.1) and (1.2) covering special cases, such as, when $v(y_1, y_2) = y_1 - s$ with $s \in \mathbb{R}$, then (1.1) and (1.2) is the fractional initial value problem, with

$$h(0) = s. \tag{1.4}$$

If $v(y_1, y_2) = y_1 - y_2$, then (1.1) and (1.2) is the periodic FBVPs, with

$$h(0) = h(1). \tag{1.5}$$

If $v(y_1, y_2) = y_1 + y_2$, then (1.1) and (1.2) is the anti-periodic FBVPs, with

$$h(0) = -h(1). \tag{1.6}$$

The structure of the proposed study is as follows: In Section 2, we review some previous results and definitions that will be used in the proof of the main results. To investigate the existence of solutions to ((1.1) and (1.4), (1.1) and (1.5), and (1.1) and (1.6)), a generalized result is proposed in Section 3. In Section 4, we solve examples with nonlinear periodic, anti-periodic and some other types of boundary conditions to show how the theoretical principles derived in Section 3 can be applied. Section 5 provides a final overview of the proposed study.

2. Fundamental results

Several FC related definitions and results will be discussed in this section that will serve as building blocks for proving the key results.

Let us designate the set of continuous functions on $[0,1]$ as $C[0,1]$, the set of absolutely continuous functions as $AC[0,1]$, and the set of n times continuously differentiable functions as $C^n[0,1]$. The set of all Lebesgue integrable functions on the interval $(0,1)$ is denoted by $L^p(0,1)$, $p \geq 1$.

Definition 1. *We take into account the weighted space of continuous functions*

$$C_\delta[0,1] = \{g : (0,1] \to R : x^\delta g(x) \in C[0,1]\}, \quad 0 \leq \delta < 1,$$

and

$$C_\delta^n[0,1] = \{g \in C^{n-1}[0,1] : g^{(n)} \in C_\delta[0,1]\}, \quad n \in \mathbb{N},$$
$$C_\delta^0[0,1] = C_\delta[0,1],$$

with the norms defined as

$$\|g\|_{C_\delta} = \|x^\delta g(x)\|_C,$$

and

$$\|g\|_{C^n_\delta} = \sum_{k=0}^{n-1} \|g^{(k)}\|_C + \|g^{(n)}\|_{C_\delta}.$$

The following characteristics are met by these spaces:

$$C_0[0,1] = C[0,1].$$
$$C^n_\delta(0,1) \subset AC^n[0,1].$$
$$C_{\delta_1}[0,1] \subset C_{\delta_2}[0,1], \quad 0 \leqslant \delta_1 < \delta_2 < 1.$$

Lemma 2.1. *([11, Lemma 15]) If $\mu_1 \geq 0$, and $\mu_2 \geq 0$, and $g \in L^1(0,1)$ then the semigroup property stated as follows is valid for almost every point $y \in [0,1]$:*

$$(I^{\mu_1} I^{\mu_2} g)(y) \doteq (I^{\mu_1+\mu_2} g)(y).$$

Specifically, when $g \in C_\delta[0,1]$ equality is satisfied for all $y \in (0,1]$, and when $g \in C[0,1]$, equality holds for all $y \in [0,1]$.

Lemma 2.2. *([11, Lemma 11]) Let $\mu > 0$ and $0 \leq \delta < 1$, then I^μ is bounded from $C_\delta[0,1]$ into $C_\delta[0,1]$.*

Lemma 2.3. *([11, Lemma 17]) Let $0 < \mu < 1, 0 \leq \delta < 1$. If $g \in C_\delta[0,1]$ and $I^{1-\mu} g \in C^1_\delta[0,1]$, then*

$$I^\mu D^\mu g(y) = g(y) - \frac{I^{1-\mu} g(0)}{\Gamma(\mu)} y^{\mu-1}$$

holds true for all $y \in (0,1]$.

Lemma 2.4. *([11, Lemma 12]) Let $\mu > 0$ and $0 \leq \delta < 1$. If $\delta \leq \mu$, then I^μ is bounded from $C_\delta[0,1]$ into $C[0,1]$.*

Lemma 2.5. *([11, Lemma 13]) Let $0 \leq \delta < 1$ and $g \in C_\delta[0,1]$. Then*

$$I^\mu g(0) := \lim_{y \to 0^+} I^\mu g(y) = 0, \qquad 0 \leq \delta < \mu.$$

Proof. By Lemma 2.4, we have $I^\mu g \in C[0,1]$. Since $g \in C_\delta[0,1]$ then $y^\delta g(y) \in C[0,1]$ and

$$|y^\delta g(y)| < M, \qquad y \in [0,1]$$

for some positive constant M. Therefore

$$|I^\mu g(y)| < M[I^\mu t^{-\delta}](y)$$

and

$$|I^\mu g(y)| \leq M \frac{\Gamma(1-\delta)}{\Gamma(\mu+1-\delta)} (y)^{\mu-\delta}.$$

Since $\mu > \delta$, the R.H.S $\to 0$ as $y \to 0$. □

Definition 2. *We call a function* $\phi \in C^1[0,1]$ *as a lower solution of* (1.1) *if it fulfills the following inequality*

$$D^{\mu,\nu}\phi(y) \leq g(y,\phi(y)),\ y \in [0,1]. \tag{2.1}$$

In a similar manner, a function $\psi \in C^1[0,1]$ *serves as an upper solution of* (1.1)*, provided that it meets*

$$D^{\mu,\nu}\psi(y) \geq g(y,\psi(y)),\ y \in [0,1]. \tag{2.2}$$

In light of the foregoing, let us assume

$$\phi(y) \leq \psi(y)\ , y \in [0,1]. \tag{2.3}$$

For $h_1, h_2 \in C[0,1]$ with $h_1(y) \leq h_2(y)$ for all $y \in [0,1]$, the set below is characterized by

$$[h_1,h_2] = \{u \in C[0,1] : h_1(y) \leq u(y) \leq h_2(y),\ \text{for all } y \in [0,1]\}.$$

Definition 3. *Two functions* ϕ *and* $\psi \in C^1[0,1]$ *satisfying the relation* $\phi(y) \leq \psi(y)$ *are considered as coupled lower and upper solutions (CLUSs) for problem* (1.1) *and* (1.2)*, provided they adhere to the inequalities* (2.1) *and* (2.2) *as well as the subsequent set of inequalities:*

$$\begin{cases} \max\{v(\phi(0),\phi(1)), v(\phi(0),\psi(1))\} \leq 0, \\ \min\{v(\psi(0),\psi(1)), v(\psi(0),\phi(1))\} \geq 0. \end{cases} \tag{2.4}$$

Remark 1. *The conventional existence requirements for boundary value problems consisting PBCs and APBCs can be consolidated into a single framework by imposing monotonicity presumptions on the arguments of the boundary function* (1.2) *and employing* (2.3) *and* (2.4)*. The inequalities* (2.3) *and* (2.4) *provide the standard existence criterion for the periodic BVPs, (*(1.1) *and* (1.5)*), if in the second argument the boundary function v is non-increasing, we have*

$$\phi(0) \leq \phi(1),\ \textit{and}\ \psi(0) \geq \psi(1).$$

Inequalities (2.3) *and* (2.4) *validate the following conventional conditions for existence of the anti-periodic BVPs (*(1.1) *and* (1.6)*), if in the second argument the boundary function v is non-decreasing, we have*

$$\phi(0) \leq -\psi(1),\ \textit{and}\ \psi(0) \geq -\phi(1).$$

Definition 4. *The definition of the left-sided fractional derivative operator of order* $0 < \mu < 1$ *and type* $0 \leq \nu \leq 1$ *is as follows:*

$$D^{\mu,\nu} = I^{\nu(1-\mu)} D I^{(1-\mu)(1-\nu)}. \tag{2.5}$$

From this definition, we get the following observations.
(1) We can write the operator $D^{\mu,\nu}$ as

$$D^{\mu,\nu} = I^{\nu(1-\mu)} D I^{(1-\delta)} = I^{\nu(1-\mu)} D^{\delta}, \quad \delta = \mu + \nu - \mu\nu.$$

(2) Between the Caputo derivative and Riemann-Liouville derivative, $D^{\mu,\nu}$ works as an interpolator, because

$$D^{\mu,\nu} = \begin{cases} D^{\mu}, & \text{when } \nu = 0, \\ I^{1-\nu}D, & \text{when } \nu = 1. \end{cases}$$

(3) The parameter δ satisfies

$$0 < \delta \leq 1, \quad \delta \geq \mu, \quad \delta > \nu, \quad 1 - \delta < 1 - \nu(1 - \mu).$$

The following spaces will be introduced as

$$C^{\mu,\nu}_{1-\delta}[0, 1] = \{g \in C_{1-\delta}[0, 1], D^{\mu,\nu}g \in C_{1-\delta}[0, 1]\},$$

and

$$C^{\delta}_{1-\delta}[0, 1] = \{g \in C_{1-\delta}[0, 1], D^{\delta}g \in C_{1-\delta}[0, 1]\}.$$

Since $D^{\mu,\nu}g = I^{\nu(1-\mu)}D^{\delta}g$, It follows from Lemma 2.2, that

$$C^{\delta}_{1-\delta}[0, 1] \subset C^{\mu,\nu}_{1-\delta}[0, 1].$$

The subsequent Lemma can be readily derived from the semigroup property outlined in Lemma 2.1

Lemma 2.6. *([11, Lemma 20]) Let* $0 < \mu < 1$, $0 \leq \nu \leq 1$, *and* $\delta = \mu + \nu - \mu\nu$. *If* $g \in C^{\delta}_{1-\delta}[0, 1]$, *then*

$$I^{\delta}D^{\delta}g = I^{\mu}D^{\mu,\nu}g,$$

and

$$D^{\delta}I^{\mu}g = D^{\nu(1-\mu)}g,$$
$$I^{\delta}D^{\mu}g = I^{\nu(1-\mu)}g.$$

Proof. Since from (2.5), we have

$$\begin{aligned} D^{\mu,\nu}g &= I^{\nu(1-\mu)}DI^{(1-\mu)(1-\nu)}g \\ I^{\mu}D^{\mu,\nu}g &= I^{\mu}I^{\nu(1-\mu)}DI^{(1-\mu)(1-\nu)}g \\ &= I^{\mu+\nu-\mu\nu}DI^{1-\mu-\nu+\mu\nu}g \\ &= I^{\delta}DI^{1-\delta}g \\ &= I^{\delta}D^{\delta}g. \end{aligned}$$

Similarly,

$$\begin{aligned} D^{\delta}I^{\mu}g &= D^{\mu+\nu-\mu\nu}I^{\mu}g \\ &= D^{\mu+\nu-\mu\nu-\mu}g \\ &= D^{\nu(1-\mu)}g, \end{aligned}$$

and

$$\begin{aligned} I^{\delta}D^{\mu}g &= I^{\mu+\nu-\mu\nu}D^{\mu}g \\ &= I^{\mu+\nu-\mu\nu-\mu}g \\ &= I^{\nu(1-\mu)}g. \end{aligned}$$

□

Lemma 2.7. *([11, Lemma 21]) Let $g \in L^1(0,1)$. If $D^{\nu(1-\mu)}g$ exists and lies in $L^1(0,1)$, then*

$$D^{\mu,\nu}I^{\mu}g = I^{\nu(1-\mu)}D^{\nu(1-\mu)}g.$$

Proof.

$$\begin{aligned} D^{\mu,\nu}I^{\mu} &= I^{\nu(1-\mu)}DI^{(1-\nu)(1-\mu)}I^{\mu}g \\ &= I^{\nu(1-\mu)}DI^{(1-\nu(1-\mu))} \\ &= I^{\nu(1-\mu)}D^{\nu(1-\mu)}. \end{aligned}$$

□

Lemma 2.8. *([11, Lemma 22]) Let $0 < \mu < 1$, $0 \le \nu \le 1$, and $\delta = \mu + \nu - \mu\nu$. If $g \in C_{1-\delta}[0,1]$ and $I^{1-\nu(1-\mu)}g \in C^1_{1-\delta}[0,1]$, then $D^{\mu,\nu}I^{\mu}g$ exists in $(0,1]$ and*

$$D^{\mu,\nu}I^{\mu}g(x) = g(x), \quad x \in (0,1].$$

Proof. From Lemmas 2.3, 2.5 and 2.7, we have

$$I^{\nu(1-\mu)}D^{\nu(1-\mu)}g(x) = g(x) - \frac{I^{1-\nu(1-\mu)}g(0)}{\Gamma(\nu(1-\mu))}(x)^{\nu(1-\mu)-1} = g(x), \quad x \in (0,1].$$

□

Following this, we introduce a highly valuable lemma that holds a pivotal role in establishing the main result. Define

$$C_{\circ}[0,1] = \{h \in C[0,1] : h(0) = 0\}.$$

Lemma 2.9. *Let $A : C[0,1] \to C_{\circ}[0,1] \times \mathbb{R}$ be a linear operator. Then the inverse $A^{-1} : C_{\circ}[0,1] \times \mathbb{R} \to C[0,1]$ exists if and only if $Ah(y) = 0 \Rightarrow h(y) = 0$.*

Proof. First we assume that $Ah(y) = 0 \Rightarrow h(y) = 0$, and show that A^{-1} exists. Let $Ah(y_1) = Ah(y_2)$, where $y_1, y_2 \in [0,1]$. Since A is linear, we have

$$\begin{aligned} &A(h(y_1) - h(y_2)) = 0 \Rightarrow h(y_1) - h(y_2) = 0, \text{ (by hypothesis)} \\ &\Rightarrow h(y_1) = h(y_2). \end{aligned}$$

Since for the linear operators, if $Ah(y_1) = Ah(y_2) \Rightarrow h(y_1) = h(y_2)$, then there exists a mapping $A^{-1} : C_{\circ}[0,1] \times \mathbb{R} \to C[0,1]$ which maps every $h_{\circ}(y) \in C_{\circ}[0,1] \times \mathbb{R}$ onto that $h(y) \in C[0,1]$ for which $Ah(y) = h_{\circ}(y)$.

Conversely, suppose that A^{-1} exists and show that $Ah(y) = 0 \Rightarrow h(y) = 0$. As A^{-1} exists, so $A(h(y_1)) = A(h(y_2)) \Rightarrow h(y_1) = h(y_2)$ with $h(y_2) = 0 \Rightarrow A(h(y_1)) = A0 = 0$, which finally implies $h(y_1) = 0$. □

3. Main results

In this section, we present a comprehensive result for investigating the existence of solutions of nonlinear FBVPs with GNBCs, as represented by Eq (1.2). The result we present here highlights its ability to unify the criteria for the existence of particular FBVPs, including those mentioned in ((1.1) and (1.5)) and ((1.1) and (1.6)), as specific instances.

Theorem 3.1. *Assume that functions $\phi, \psi \in C^1[0,1]$ are characterized as CLUSs for the nonlinear FBVPs* (1.1) *and* (1.2) *satisfying* (2.3)*. Moreover the functions*

$$\chi_\phi(y) := v(\phi(0), y),$$
$$\chi_\psi(y) := v(\psi(0), y)$$

are monotone in $[\phi(1), \psi(1)]$*. Then FBVPs* (1.1) *and* (1.2) *yield a minimum of one solution that lies within the sector defined by CULSs ϕ and ψ keeping their order preserved.*

Proof. Associated to (1.1) and (1.2), consider the following modified FBVPs

$$\begin{cases} D^{\mu,\nu} h(y) + \lambda h(y) = G^*(y, h(y)), \ y \in [0,1], \ \lambda > 0, \\ h(0) = v^*(h(0), h(1)), \end{cases} \tag{3.1}$$

where

$$G^*(y, h(y)) = \begin{cases} g(y, \psi(y)) + \lambda\psi(y), & \text{if } \psi(y) < h, \\ g(y, h(y)) + \lambda h, & \text{if } \phi(y) \le h \le \psi(y), \\ g(y, \phi(y)) + \lambda\phi(y), & \text{if } h < \phi(y), \end{cases} \tag{3.2}$$

and

$$\begin{cases} v^*(x_1, x_2) = p(0, x_1) - v(p(0, x_1), p(1, x_2)), \\ p(t, x_1) = \max\{\phi(t), \min\{x_1, \psi(t)\}\}. \end{cases} \tag{3.3}$$

For the sake of clarity, we will prove the result in steps.

Step 1. Finding the solution of (3.1) is similar to find the FP of the operator, $A^{-1}B : C[0,1] \to C[0,1]$ that is defined by means of the composition formed from the subsequent mappings.

$$\begin{cases} A : C[0,1] \to C_\circ[0,1] \times \mathbb{R}, \\ B : C[0,1] \to C_\circ[0,1] \times \mathbb{R}, \end{cases}$$

which can be described as

$$\begin{cases} [Ah](y) = \left(h(y) - h(0) + \left(\lambda_{RL} I^\lambda h\right)(s), h(0)\right), \\ [Bh](y) = \left(\left({}_{RL} I^\mu G^*(y, h(y))\right)\right), v^*(y(0), y(1))\right). \end{cases} \tag{3.4}$$

It follows that $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous due to its continuity and boundedness on the domain $[0,1] \times \mathbb{R}$. As a result, G^* is uniformly continuous and bounded. Furthermore, Riemann-Liouville fractional integral and v^* are continuous. Consequently, $\{By : y \in C[0,1]\}$ class becomes uniformly bounded. This class is equicontinuous as well because for every $\epsilon > 0$, there exist a $\delta > 0$, such that

$$|Bh(y_1) - Bh(y_2)| < \epsilon, \forall\ h \in B \text{ whenever } |h(y_1) - h(y_2)| < \delta\ \forall\ y_1, y_2 \in [0, 1].$$

Suppose that $y_1 \leq y_2$, then we have

$$\begin{aligned}
|Bh(y_1) - Bh(y_2)| &= \left|\Big(({}_{RL}I^{\mu}(G^*(y_1, h(y_1)), v^*(y(0), y(1)) - ({}_{RL}I^{\mu}(G^*(y_2, h(y_2)), v^*(y(0), y(1))\Big)\right| \\
&= \left|\Big({}_{RL}I^{\mu}(G^*(y_1, h(y_1)) - ({}_{RL}I^{\mu}(G^*(y_2, h(y_2}\Big)\right| \\
&= \left|\frac{1}{\Gamma(\mu)}\left(\int_0^{y_1}(y_1-\tau)^{\mu-1}(G^*\tau, h(\tau))d\tau - \int_0^{y_2}(y_2-\tau)^{\mu-1}(G^*\tau, h(\tau))d\tau\right)\right| \\
&\leq \frac{1}{\Gamma(\mu)}\Big(\Big|\int_0^{y_1}(y_1-\tau)^{\mu-1}G^*(\tau, h(\tau))d\tau\Big| - \int_0^{y_2}\Big|(y_2-\tau)^{\mu-1}G^*(\tau, h(\tau))d\tau\Big|\Big) \\
&= \frac{1}{\Gamma(\mu)}\Big(\Big|\int_0^{y_1}(y_1-\tau)^{\mu-1}G^*(\tau, h(\tau))d\tau\Big| - (\int_0^{y_1} + \int_{y_1}^{y_2})\Big|(y_2-\tau)^{\mu-1}G^*(\tau, h(\tau))d\tau\Big|\Big) \\
&= \frac{1}{\Gamma(\mu)}\Big((\Big|\int_0^{y_1}((y_1-\tau)^{\mu-1} - (y_2-\tau)^{\mu-1})G^*(\tau, h(\tau))d\tau\Big| - \int_{y_1}^{y_2}\Big|(y_2-\tau)^{\mu-1}G^*(\tau, h(\tau))d\tau\Big|\Big) \\
&\to 0, \quad as\ y_1 \to y_2.
\end{aligned}$$

Since the class of functions $\{By : y \in C[0, 1]\}$ preserves uniform boundedness and equicontinuity on $[0, 1]$, by virtue of the Arzelá-Ascoli theorem, we know that this class exhibits relative compactness. Therefore, the mapping B described in (3.4) is a compact operator on $C[0, 1]$. Furthermore, the continuity and existence of A^{-1} implies the continuity and compactness of $A^{-1}B$. Now, by the application of the Schauder FP theorem, the presence of at least one fixed point is guaranteed. Therefore, the problem given by Eq (3.1) has a minimum of one solution $h(y) \in C[0, 1]$, which serves as a FP of the composite operator $A^{-1}B$.

Step 2. We claim that if $h(y) \in C[0, 1]$ is a solution of (3.1),then necessarily it will lie inside the sector bounded by the functions ϕ and ψ which are well-ordered CLUSs, such that $\phi(y) \leq h(y) \leq \psi(y)$, $y \in [0, 1]$. Suppose on contrary that $h(y) \nleq \psi(y)$, then $h - \psi$ takes on a positive maximum at some $c \in [0, 1]$. We will now discuss three possible scenarios:

Case 1. If $c \in (0, 1]$.

In this case there exists $b \in (0, c)$, in such a way that $0 \leq h(y) - \psi(y) \leq h(c) - \psi(c)$, for all $y \in [b, c]$. However, in light of Lemmas 2.6–2.8 and Eq (2.2), there is a contradiction. Since

$$\begin{aligned}
\psi(c) - \psi(b) &\leq h(c) - h(b) \\
&= {}_{RL}I^{\nu(1-\mu)}D^{\nu(1-\mu)}h(s) \\
&= {}_{RL}I^{\delta}D^{\delta}h(s) \\
&= {}_{RL}I^{\mu}D^{\mu,\nu}h(s) = {}_{RL}\ I^{\mu}\left([g(y, \psi(y)) - \lambda(h(y) - \psi(y))]\right) \\
&< {}_{RL}I^{\mu}D^{\mu,\nu}\psi(s) = \psi(c) - \psi(b).
\end{aligned}$$

Case 2. If $c = 0$, χ_{ψ} is monotone nonincreasing, and $h(0) - \psi(0) > 0$. Then as a result of (2.4), we have

$$h(0) \quad = \quad v^*(h(0), h(1))$$

$$\begin{aligned}
&= \psi(0) - v(\psi(0), p(1, h(1))) \\
&\leq \psi(0) - v(\psi(0), \psi(1)) \\
&\leq \psi(0).
\end{aligned}$$

Therefore, we have a contradiction to $h(0) - \psi(0) > 0$.

Case 3. If $c = 0$, χ_ψ is monotone nondecreasing, and $h(0) - \psi(0) > 0$, then with the help of (2.4) it implies that

$$\begin{aligned}
h(0) &= v^*(h(0), h(1)) \\
&= \psi(0) - v(\psi(0), p(1, h(1))) \\
&\leq \psi(0) - v(\psi(0), \phi(1)) \\
&\leq \psi(0).
\end{aligned}$$

Hence, it again contradicts to $h(0) - \psi(0) > 0$. Consequently, we have $h(y) \leq \psi(y)$ for all $y \in [0, 1]$. On the same fashion, we can show that, $\phi(y) \leq h(y)$ for all $y \in [0, 1]$.

Step 3. The boundary conditions (1.2) must also be satisfied by the solution $h(y) \in C[0, 1]$. So we claim that, $\phi(0) \leq h(0) - v\,(h(0), h(1)) \leq \psi(0)$. On contrary now assume that, $h(0) - v\,(h(0), h(1)) \nleq \psi(0)$. Then

$$\begin{aligned}
h(0) &= v^*(h(0), h(1)) \\
&= p(0, h(0)) - v\,(p(0, h(0)), p(1, h(1))) \\
&= \psi(0).
\end{aligned}$$

If $\chi_\psi(y) := v(\psi(0), y)$ is monotone nonincreasing, then we have

$$\begin{aligned}
&h(0) - v\,(h(0), h(1)) \\
&= \psi(0) - v\,(\psi(0), h(1)) \\
&\leq \psi(0) - v\,(\psi(0), \psi(1)) \\
&\leq \psi(0),
\end{aligned}$$

here we get a contradiction. A similar contradiction holds if we assume that $\chi_\psi(y) := v(\psi(0), y)$ is monotone nondecreasing. Thus, $\phi(0) \leq h(0) - v\,(h(0), h(1)) \leq \psi(0)$ holds. Therefore, the problem (1.1) and (1.2) has a minimum of one solution, such that $\phi(y) \leq h(y) \leq \psi(y)$, for all $y \in [0, 1]$. □

4. Examples

Some examples are given in this section to justify the validity and credibility of the theoretical conclusions drawn. In one case, nonlinear PBCs are studied, and in the other, nonlinear APBCs are analysed.

Example 1. *A nonlinear FBVPs is given as*

$$D^{\frac{1}{2},\frac{1}{2}} h(y) = -7h^3(y) + \lambda \cos(4y), \lambda > 0, \ y \in [0, 1], \tag{4.1}$$

having APBCs involving non linearity

$$v((h(0), h(1)) = h^5(0) + h^5(1). \tag{4.2}$$

The function given by

$$\phi(y) = -4\lambda \tag{4.3}$$

serve as the lower solution of the problem (4.1)*, satisfying* (2.1) *as follows:*

$$\begin{gathered}
D^{\frac{1}{2},\frac{1}{2}}\phi(y) = I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}\phi(y),\\
D^{\frac{1}{2},\frac{1}{2}}(-4\lambda) = I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda),\\
I^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda) = I^{\frac{1}{4}}(-4\lambda),\\
I^{\frac{1}{4}}(-4\lambda) = \frac{1}{\Gamma(0.25)}\int_0^y (y-t)^{-0.75}(-4\lambda)\mathrm{d}t = -4.41306\lambda y^{0.25},\\
DI^{\frac{1}{4}}(-4\lambda) = D(-4.41306\lambda y^{0.25}) = -1.103263\lambda y^{-0.75},\\
I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda) = I^{\frac{1}{4}}(-1.103265\lambda y^{-0.75}),\\
I^{\frac{1}{4}}(-1.103265\lambda y^{-0.75}) = \frac{-1.103263\lambda}{\Gamma(0.25)}\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = -0.3042971\lambda\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t,\\
\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = \int_0^y y^{-0.75}(1-\tfrac{t}{y})^{-0.75}t^{-0.75}\mathrm{d}t = y^{-0.75}\int_0^y (1-\tfrac{t}{y})^{-0.75}t^{-0.75}\mathrm{d}t.
\end{gathered}$$

By using the substitution $\frac{t}{y} = u$ *and using the definition of Beta function, we have*

$$\begin{gathered}
\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = \frac{\Gamma(0.25).\Gamma(0.25)}{\Gamma(0.25+0.25)} = 7.4162985y^{-0.5}\\
I^{\frac{1}{4}}(-1.103263\lambda y^{-0.75}) = -2.2567581\lambda y^{-0.5}
\end{gathered}$$

so

$$D^{\frac{1}{2},\frac{1}{2}}\phi(y) = -2.2567581\lambda y^{-0.5} < g(y, -4\lambda) = 448\lambda^3 + \lambda\cos(4y),\ y \in [0, 1],$$

and by working in a similar manner, we get to know that the function $\psi(y) = 4\lambda$ *is an upper solution of the problem* (4.1)*, because it satisfies* (2.2) *as*

$$D^{\frac{1}{2},\frac{1}{2}}\psi(y) = 2.2567581\lambda y^{-0.5} > g(y, 4\lambda) = -448\lambda^3 + \lambda\cos(4y),\ y \in [0, 1],$$

For problems (4.1) *and* (4.2)*, the functions given by* -4λ *and* 4λ *are CLUSs since they meet condition given in (*2.4*). Also when v demonstrates monotone nondecreasing behaviour in the second variable, then as a result the inequalities listed below are fulfilled.*

$$\begin{cases} v\,(\phi(0), \psi(1)) \le 0,\\ v\,(\psi(0), \phi(1)) \ge 0. \end{cases}$$

Likewise, when v demonstrates a monotone nonincreasing trend in its second variable, then the succeeding inequalities are validated.

$$\begin{cases} v\,(\phi(0), \phi(1)) \le 0,\\ v\,(\psi(0), \psi(1)) \ge 0. \end{cases}$$

Moreover the following functions

$$\chi_\phi(y) := v(\phi(0), y),$$

and

$$\chi_\psi(y) := v(\psi(0), y)$$

are monotone on $[-4\lambda, 4\lambda]$.

Thus, we have confirmed that all the conditions of the Theorem 3.1 hold true. So it follows that the problem (4.1) and (4.2) will have one solution at least, $\phi(y) \leq h(y) \leq \psi(y)$, for all $y \in [0, 1]$.

Example 2. *Given a nonlinear FBVPs*

$$D^{\frac{1}{3},\frac{1}{4}}h(y) = -5h^5(y) + \lambda\sin(5y), \lambda > 0,\ y \in [0, 1], \tag{4.4}$$

having PBCs involving non linearity

$$v((h(0), h(1)) = h^3(0) - h^3(1). \tag{4.5}$$

The functions given by

$$\phi(y) = -7\lambda,\ \textit{and}\ \psi(y) = 7\lambda \tag{4.6}$$

serve as the lower and upper solutions of the problem (4.4)*, satisfying* (2.1) *and* (2.2)*, as*

$$\begin{gathered}
D^{\frac{1}{3},\frac{1}{4}}\phi(y) = I^{\frac{1}{4}(1-\frac{1}{3})}DI^{(1-\frac{1}{3})(1-\frac{1}{4})}\phi(y),\\
D^{\frac{1}{3},\frac{1}{4}}(-4\lambda) = I^{\frac{1}{4}(1-\frac{1}{3})}DI^{(1-\frac{1}{3})(1-\frac{1}{4})}(-7\lambda),\\
I^{(1-\frac{1}{3})(1-\frac{1}{4})}(-7\lambda) = I^{\frac{1}{2}}(-7\lambda),\\
I^{\frac{1}{2}}(-7\lambda) = \frac{1}{\Gamma(0.5)}\int_0^y (y-t)^{-0.5}(-7\lambda)\mathrm{d}t = -7.898654\lambda y^{0.5},\\
DI^{\frac{1}{2}}(-7\lambda) = D(-7.898654\lambda y^{0.5}) = -3.949327\lambda y^{-0.5},\\
I^{\frac{1}{4}(1-\frac{1}{3})}DI^{(1-\frac{1}{3})(1-\frac{1}{4})}(-7\lambda) = I^{\frac{1}{6}}(-3.949327\lambda y^{-0.5}),\\
I^{\frac{1}{6}}(-3.949327\lambda y^{-0.5}) = \frac{-3.949327\lambda}{\Gamma(\frac{1}{6})}\int_0^y (y-t)^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t = -0.709505\lambda\int_0^y (y-t)^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t,\\
\int_0^y (y-t)^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t = \int_0^y y^{-\frac{5}{6}}(1-\tfrac{t}{y})^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t = y^{-\frac{5}{6}}\int_0^y (1-\tfrac{t}{y})^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t.
\end{gathered}$$

By using the following substitution $\frac{t}{y} = u$ *and using the definition of Beta function, we have*

$$\begin{gathered}
\int_0^y (y-t)^{-\frac{5}{6}}t^{-\frac{1}{2}}\mathrm{d}t = \frac{\Gamma(\frac{1}{2}).\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2}+\frac{1}{6})} = 7.285953y^{-\frac{1}{3}},\\
I^{\frac{1}{6}}(-3.949327\lambda y^{-0.5}) = -5.169420\lambda y^{-\frac{1}{3}}
\end{gathered}$$

so

$$D^{\frac{1}{3},\frac{1}{4}}\phi(y) = -5.169420\lambda y^{-\frac{1}{3}} < g(y, -7\lambda) = 84035\lambda^5 + \lambda\sin(5y),\ y \in [0, 1],$$

and by working in a similar manner, we get

$$D^{\frac{1}{3},\frac{1}{4}}\psi(y) = 5.169420\lambda y^{-\frac{1}{3}} > g(y, 7\lambda) = -84035\lambda^5 + \lambda\sin(5y),\ y \in [0, 1],$$

by employing the same procedures as in Example 1, it can be confirmed that all the conditions outlined in Theorem 3.1 are satisfied. As a result, it can be concluded that the problem defined by Eqs (4.4) *and* (4.5) *possesses at least one solution, such that* $\phi(y) \leq h(y) \leq \psi(y)$*, for all* $y \in [0, 1]$.

Example 3. *A nonlinear FBVPs is given as*

$$D^{\frac{1}{2},\frac{1}{2}}h(y) = -7h^3(y) + \lambda\cos(4y), \lambda > 0,\ y \in [0,1], \tag{4.7}$$

having non linear boundary conditions

$$v((h(0), h(1)) = h(0)h^4(1) - h(0)h(1). \tag{4.8}$$

The function given by

$$\phi(y) = -4\lambda \tag{4.9}$$

serve as the lower solution of the problem (4.8)*, satisfying* (2.1) *as follows:*

$$\begin{gathered}
D^{\frac{1}{2},\frac{1}{2}}\phi(y) = I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}\phi(y),\\
D^{\frac{1}{2},\frac{1}{2}}(-4\lambda) = I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda),\\
I^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda) = I^{\frac{1}{4}}(-4\lambda),\\
I^{\frac{1}{4}}(-4\lambda) = \tfrac{1}{\Gamma(0.25)}\int_0^y (y-t)^{-0.75}(-4\lambda)\mathrm{d}t = -4.41306\lambda y^{0.25},\\
DI^{\frac{1}{4}}(-4\lambda) = D(-4.41306\lambda y^{0.25}) = -1.103263\lambda y^{-0.75},\\
I^{\frac{1}{2}(1-\frac{1}{2})}DI^{(1-\frac{1}{2})(1-\frac{1}{2})}(-4\lambda) = I^{\frac{1}{4}}(-1.103265\lambda y^{-0.75}),\\
I^{\frac{1}{4}}(-1.103265\lambda y^{-0.75}) = \tfrac{-1.103263\lambda}{\Gamma(0.25)}\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = -0.3042971\lambda\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t,\\
\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = \int_0^y y^{-0.75}(1-\tfrac{t}{y})^{-0.75}t^{-0.75}\mathrm{d}t = y^{-0.75}\int_0^y (1-\tfrac{t}{y})^{-0.75}t^{-0.75}\mathrm{d}t.
\end{gathered}$$

By using the substitution $\frac{t}{y} = u$ and using the definition of Beta function, we have

$$\begin{gathered}
\int_0^y (y-t)^{-0.75}t^{-0.75}\mathrm{d}t = \tfrac{\Gamma(0.25).\Gamma(0.25)}{\Gamma(0.25+0.25)} = 7.4162985y^{-0.5},\\
I^{\frac{1}{4}}(-1.103263\lambda y^{-0.75}) = -2.2567581\lambda y^{-0.5}
\end{gathered}$$

so

$$D^{\frac{1}{2},\frac{1}{2}}\phi(y) = -2.2567581\lambda y^{-0.5} < g(y,-4\lambda) = 448\lambda^3 + \lambda\cos(4y),\ y \in [0,1],$$

and by working in a similar manner, we get to know that the function $\psi(y) = 4\lambda$ is an upper solution of the problem (4.8)*, because it satisfies* (2.2) *as*

$$D^{\frac{1}{2},\frac{1}{2}}\psi(y) = 2.2567581\lambda y^{-0.5} > g(y,4\lambda) = -448\lambda^3 + \lambda\cos(4y),\ y \in [0,1],$$

For problems (4.7) *and* (4.8)*, the functions given by -4λ and 4λ are CLUSs since they meet condition given in* (2.4)*. Also when v demonstrates monotone nondecreasing behaviour in the second variable, then as a result the inequalities listed below are fulfilled.*

$$\begin{cases} v\left(\phi(0), \psi(1)\right) \le 0,\\ v\left(\psi(0), \phi(1)\right) \ge 0.\end{cases}$$

Likewise, when v demonstrates a monotone nonincreasing trend in its second variable, then the succeeding inequalities are validated.

$$\begin{cases} v\left(\phi(0), \phi(1)\right) \leq 0, \\ v\left(\psi(0), \psi(1)\right) \geq 0. \end{cases}$$

Moreover the following functions

$$\chi_{\phi}(y) := v(\phi(0), y)$$

and

$$\chi_{\psi}(y) := v(\psi(0), y)$$

are monotone on $[-4\lambda, 4\lambda]$.

Thus, we have confirmed that all the conditions of the Theorem 3.1 hold true. So it follows that the problem (4.7) and (4.8) will have one solution at least, $\phi(y) \leq h(y) \leq \psi(y)$, for all $y \in [0, 1]$.

5. Conclusions

In our research, we extend the LUSA to address nonlinear FBVPs with GNBC. The approach we propose is generic, as it brings together the existence results of specific fractional-order problems that were previously treated in literature as separate entities, covering periodic and antiperiodic FBVPs as specific instances. We exhibit the practical applicability of our newly formed theoretical findings by examining three illustrative examples.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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